March 19, 1959

Professor Horton W. Bittenger and
Professor Robert E. Glover
Department of Civil Engineering
Colorado State University
Fort Collins, Colorado

Gentlemen:

We have received a report from our reviewer on your manuscript,
GROUND-WATER HYDRAULICS, a copy of which I am sending along.

I have discussed the project and the review with our Editorial
Committee. The reviewer's comments seem to indicate that a book
based on the present manuscript would be a useful manual for an
engineer in the field but would not be readily adapted to use as
a college text. Since there are only a few college courses in
this subject, the college market would be rather limited to begin
with, and so we would not be in a position to sponsor your book
in this department.

I then sent the manuscript and the review to our Industrial and
Business Book Department and urged them to consider publishing it
as a "field manual." They, however, are rather pessimistic about
the size of the industrial market, so I regret to say that our
overall estimate of the sales potential is too small to warrant
our making you a contractual proposal.

Thank you very much for considering McGraw-Hill as a possible
publisher. Under separate cover, I am returning your manuscript
to you, and I wish you the best of luck in its publication.

Sincerely yours,

Malcolm Johnson
Editor

 jm/enc.
I have read the partial manuscript of GROUND-WATER HYDRAULICS by Glover and Bittinger over several times and have pondered it considerable length. As you know, I have felt the need for a modern text on ground water and I hoped that this might be a partial answer. Since its scope is limited to the hydraulics of ground water, it does not cover many aspects of ground-water hydrology. However, a good treatment of ground-water hydraulics would be a considerable step forward.

Technically, the presentation is quite satisfactory. That is, the mathematics seem to be ok and the examples are carefully worked out. I would hope that the authors would use commonly accepted symbols, for example Q instead of F for flow and S instead of V for the storage coefficient. I presume that the authors intend to add credits. The derivation in Chapter I is basically similar to those of Theis and Jacob—the main difference being that the author's u is the square root of the u commonly used.

The style of the manuscript is characteristic of the manuals produced by many of the government agencies. While these do doubt serve a useful purpose for intra-agency use and in-service training they do not fit the picture of the ideal college text. It seems to me that a text on this subject could profitably start by a discussion of Darcy's law which is the fundamental equation of flow in permeable media. The physical characteristics of the medium—permeability, porosity, etc. should also be discussed in an introduction. Finally, throughout the text there should be careful statements of the assumptions on which the derivation is based.

In short, the material submitted and the table of contents suggest a manual which the technician can use to solve any of several specific problems by mechanically following the examples (and trusting that his problem does not violate any of the unstated assumptions). It fails to present the fundamentals of ground-water hydraulics in a logical manner and thus falls short of the ideal for a textbook.
Mr. Robert E. Glover  
1936 South Lincoln Street  
Denver, Colorado  

Dear Mr. Glover:

We are pleased to grant permission to you to include Bureau of Reclamation data in the textbook on ground-water hydraulics which you and Mr. M. W. Bittenger plan to prepare. We understand from your letter of December 26, that the book will give proper acknowledgment to the Bureau as the source of such information.

As we informed you during your recent visit to this office, we plan to assemble and issue as one of our Technical Memoranda Series, the memoranda on ground-water movement which bear your name as author or co-author, as well as memoranda on the same subject prepared by other Bureau engineers. We believe the information and formulas developed in the assembled memoranda will be of value to all Bureau offices and individuals engaged in ground-water and drainage development work.

Sincerely yours,

[Signature]

Grant Bloodgood  
Assistant Commissioner  
and Chief Engineer
Mr. Grant Bloodgood  
Assistant Commissioner and Chief Engineer  
Bureau of Reclamation  
Building 53 - Denver Federal Center  
Denver, Colorado  

Attention 209  

Dear Sir:  

A text to deal with the transient phases of ground water hydraulics seems to be needed and it is the intention of Mr. M.W. Bittenger of the Colorado State University and I to prepare one.  

Many important developments were made at the Bureau of Reclamation to provide data for the design of drainage systems which would be both adequate and economical and we would like to include an account of these in our text. We wish, therefore, to request permission to use some of these developments and would, of course, see that appropriate credit is given if this permission is granted. We believe that it would be useful to Bureau engineers to have these developments collected into a text and that it would help to secure credit to the Bureau for these developments if this were done.  

This request applies to data contained in Bureau memoranda which bear my name as author or co-author. If other data should be desirable, for some reason, it would be made the subject of a separate request.  

Sincerely yours  

Robert E. Glover
\[ y = \frac{\Phi}{2\pi KD} \int_{0}^{r} \frac{e^{-\frac{r^2}{4\pi t}}}{\sqrt{4\pi t}} \, dr \]

\[ \frac{\partial y}{\partial t} = \frac{\Phi}{2\pi KD} e^{-\frac{r^2}{4\pi t}} \frac{r}{\sqrt{4\pi t}} = -\frac{\Phi}{2\pi KD} \frac{e^{-\frac{r^2}{4\pi t}}}{r} \]

\[ F = -2\pi KD \frac{\partial y}{\partial t} \]

\[ F = -2\pi KD \frac{r}{\Phi} \frac{\partial y}{\partial t} = \frac{2\pi KD \frac{\Phi}{r} e^{-\frac{r^2}{4\pi t}}}{r} \]

\[ F = \Phi \frac{\Phi}{r^2} e^{-\frac{r^2}{4\pi t}} \]

\[ 2\pi KD \frac{dh}{dy} = F = \frac{\Phi}{r^2} e^{-\frac{r^2}{4\pi t}} \]

\[ h \frac{dh}{dy} = \frac{\Phi}{2\pi KD} \frac{e^{-\frac{r^2}{4\pi t}}}{r} = \frac{\Phi}{2\pi KD} \frac{e^{-\frac{r^2}{4\pi t}}}{r} \]

By integration

\[ \frac{D^2 - h^2}{2} = \frac{\Phi}{2\pi KD} \int_{0}^{r} \frac{e^{-\frac{r^2}{4\pi t}}}{\sqrt{4\pi t}} \, dr \]

Let

\[ u = \frac{r}{\sqrt{4\pi t}} \]

\[ du = \frac{dr}{\sqrt{4\pi t}} \]

Then

\[ \frac{D^2 - h^2}{2} = \frac{\Phi}{2\pi KD} \int_{0}^{u} \frac{e^{-u^2}}{\sqrt{4\pi t}} \, du \]

\[ \frac{1}{\sqrt{4\pi t}} \int_{0}^{u} e^{-u^2} \, du \]

\[ h^2 = D^2 - 2Dy \]

\[ h^2 = D^2 - 2D \cdot \Phi \]

\[ h^2 = D^2 (1 - \frac{\Phi}{D}) \]
\[ h = 11 - \frac{24}{b} \]

\[ Q = 2.0 \quad \frac{F^2}{Z_e} \quad K = 0.01 \quad \frac{Pr}{D} = 100 \quad T = 0.2 \]

\[ \frac{KD}{D} = 0.5 \quad 4x = 2 \quad KD = 0.1 \]

\[ \frac{Q}{2\pi KD^2} = \frac{2.0}{(6.283)(10)} = 0.0319 \]

\[ \frac{Q}{2\pi KD (0.1)} = 0.319 \]

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For \( t = 10 \text{ days} \), \( t = 864000. \)

\[ \frac{1.0}{1728000} \cdot \frac{1.0}{1315} = 0.00076 \]

From chart

\[ \frac{y}{Q} = 8.0 \quad y = (8.0)(3.19) = 25.5 \]

\[ 100 - 25.5 = 74.5 \]

This is a good check.
If

\[ Y = \frac{\Phi}{-2\pi kD} \int_{-\infty}^{\infty} e^{-\frac{u^2}{2u}} \, du, \]

and

\[ u = \frac{1}{\sqrt{4\pi t}} \quad \frac{de}{dt} = \frac{-r}{2\sqrt{4\pi t} + t} = -\frac{u}{2t}. \]

\[
\frac{\partial Y}{\partial t} - \frac{\partial Y}{\partial u} \frac{de}{dt} = \frac{t}{\sqrt{4\pi D}} \frac{e^{-u^2}}{u} \frac{u}{2t}.
\]

\[ F = \int_{-\infty}^{\infty} \frac{2\sqrt{rT}}{\sqrt{\pi}} \frac{\partial Y}{\partial u} du. \]

\[ F = \int_{0}^{\infty} 2\pi u \sqrt{4\pi t + u^2} \, du \sqrt{4\pi t} \left( \frac{\Phi}{2\pi kD} \right) e^{-u^2/2t}. \]

\[ F = 2\Phi \int_{0}^{\infty} u e^{-u^2} \, du. \]

\[ F = \Phi \left[ -e^{-u^2} \right]_u^{\infty} = \Phi e^{-u^2}. \]
In general

If \( u = \frac{r}{1 + \lambda t} \)

\[
\frac{2u}{\dot{t}} = \frac{2u}{2(\lambda + 2)\dot{t}} = \frac{-u}{\dot{t}}
\]

\[
\frac{\dot{r}}{\dot{t}} = \frac{\dot{r}}{\dot{t}} \cdot \frac{\lambda}{\dot{t}}
\]

\[
\frac{r}{\dot{t}} = \frac{u}{\sqrt{\lambda + 2}}
\]

\[
F = \int_{-\infty}^{\infty} 2\pi r \, dr \cdot v \, \frac{\dot{r}}{\dot{t}}
\]

Dimension: OK

\[F = -\int_{-\infty}^{\infty} 2\pi r \lambda \dot{t} \, dr \, v \, \frac{\dot{r}}{\dot{t}}
\]

\[F = -\int_{-\infty}^{\infty} 4\pi r^2 \lambda \, dr \, v \, \frac{\dot{r}}{\dot{t}}
\]

If

\[
\frac{y}{\frac{\dot{c}}{2\pi k}} = \frac{y}{\frac{\dot{c}}{2\pi k}}
\]

\[
\frac{\dot{y}}{\dot{u}} = \frac{\dot{y}}{\dot{u}} \cdot \frac{\dot{c}}{\dot{u}}
\]

Then

\[
F = -2\pi \int_{-\infty}^{\infty} u^2 \frac{\dot{y}}{\dot{u}} \, du
\]

\[
\frac{F}{\dot{c}} = -2\int_{-\infty}^{\infty} u^2 \frac{\dot{y}}{\dot{u}} \, du
\]
\[ F = -2 \pi K R (0-y) \frac{\partial y}{\partial t} \]

If \[ u = \frac{r}{14 \pi t} \quad \frac{\partial y}{\partial t} = \frac{2y}{\partial u} \frac{du}{dt} = \frac{1}{14 \pi t} \frac{2y}{u} \]

Then:

\[ F = -2 \pi K (0-y) \frac{r}{14 \pi t} \frac{3y}{u} \]

or

\[ F = -2 \pi K (0-y) u \frac{3y}{u} \]

or

\[ -2 (0-y) \frac{3y}{u} = \frac{F}{\pi K u} \]

By integration, if \( y = 0 \) when \( u = \infty \)

\[ (0-y)^2 = D^2 - \int_{0}^{\infty} \frac{F}{\pi K u} \]

or if

\[ \frac{1}{u^2} = \frac{\phi}{2 \pi K D^2} \]

\[ \left(1 - \frac{y}{D}\right)^2 = 1 - 2 \phi \int_{0}^{\infty} \frac{\phi}{u} \frac{du}{\sqrt{u}} \]

Suppose, as from page 3.

\[ \frac{F}{\phi} = e^{-u^2} \quad \text{Then} \quad \int_{0}^{\infty} \frac{e^{-u^2}}{\sqrt{u}} \frac{du}{\sqrt{u}} = \int_{0}^{\infty} \frac{e^{-u^2}}{u} \]

And for a first approximation

\[ \left(1 - \frac{y}{D}\right)^2 = 1 - 2 \phi \int_{0}^{\infty} \frac{e^{-u^2}}{u} \]

or

\[ \frac{y}{D} = 1 - \sqrt{1 - 2 \phi \int_{0}^{\infty} \frac{e^{-u^2}}{u} \} \]
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Signed by: [Signature]

June 30, 1958
Drawdown Produced
by Pumping a Well in an Unconfined Aquifer.
by Robert E. Glover and Morton W. Bittinger

Abstract
Drawdown computations for wells in unconfined aquifers are usually made on the basis that the drawdown is small compared to the original saturated depth of the aquifers. This is done as a concession to mathematical difficulties. Because pumping rates are often such as to produce drawdowns which are not small, in this sense, a closer approximation is needed. The present development conforms to the Dupuit-Forschheimer concepts as applied to that part of the aquifer which remains saturated, thereby removing the restrictions imposed by the assumption that the drawdowns are small. The first approximation solution can be plotted as a single line upon a graph having the dimensionless parameters $y/(Q/2 \pi K D)$ and $r/\sqrt{4 \alpha t}$ as ordinate and abscissa. The improved solution plots as a series of lines on such a graph. These lines are identified by the parameter $(Q/2 \pi K D^2)$. The method of using the graph is the same as before but is now possible to add a family of curves which indicate the ratio of the drawdown to the original saturated depth.
Definitions

The following symbols and terminology are used:

$$\alpha = \frac{K D}{V}$$

$$D = \text{The original saturated depth of an unconfined aquifer. (ft)}$$

$$F = \text{The flow through a cylindrical surface of radius } r \text{ and height } (D - y) \text{ (ft}^3/\text{sec}).$$

$$K = \text{The permeability of the aquifer (ft/sec).}$$

$$P = \text{A derivative}$$

$$Q = \text{The flow of the well, here considered to be constant. (ft}^3/\text{sec})$$

$$r = \text{A radius drawn from the center of the well. (ft)}$$

$$S = \left(\frac{Q}{2 \pi K D^2}\right) \text{ (dimensionless)}$$

$$t = \text{Time (seconds)}$$

$$u = \frac{r}{\sqrt{4 \alpha t}} \text{ (dimensionless)}$$

$$V = \text{The drainable voids in the aquifer expressed as a ratio to the gross volume (dimensionless)}$$

$$y = \text{The drawdown of the water table caused by pumping the well (feet)}$$

$$\pi = 3.14159$$

$$e = 2.71828$$

$$\psi = \frac{y}{2 \pi R} \text{ (dimensionless)}$$
Introduction

The requirement that the rise or fall of the ground water surface in the annulus bounded by the cylindrical surfaces \( r \) and \( r + dr \) should be compatible with the difference between the flows entering and leaving it is expressed by the differential equation:

\[
\alpha (D - y) \left( -\frac{\partial^2 Y}{\partial r^2} + \frac{1}{r} \frac{\partial Y}{\partial r} \right) - \alpha \left( \frac{\partial Y}{\partial r} \right)^2 = D \frac{\partial Y}{\partial t} \quad (1)
\]

This relationship is derived on the basis that the aquifer is unconfined at the top but rests at the bottom on a impermeable bed. The flow toward the well is assumed to be proportional to the permeability, the surface gradient and the area available below the water table. All of the water pumped out of the well is assumed to come from storage \( D \) in the aquifer.

If \( y \) is neglected as being small compared to \( D \) then the equation is reduced to the form

\[
\alpha \left( \frac{\partial^2 Y}{\partial r^2} + \frac{1}{r} \frac{\partial Y}{\partial r} \right) = \frac{\partial Y}{\partial t} \quad (2)
\]

The substitution of variable

\[
u = \frac{r}{\sqrt{4\alpha t}} \quad (3)
\]

into either of the partial equations (1) or (2) will reduce them to an ordinary differential equation in \( y \) and \( u \).

Equation 2 is reduced to the form:

\[
\frac{d^2 Y}{du^2} + \left( \frac{1}{u} + \frac{2u}{u} \right) \frac{dy}{du} = 0 \quad (4)
\]

If

\[
p = \frac{dy}{du} \quad (5)
\]
This equation becomes of the first order
\[
\frac{dp}{du} + \left(1 + 2u\right) p = 0 \tag{6}
\]
An integrating factor is: \(ue^{u^2}\) and a solution is:
\[
pue^{u^2} = C_1 \tag{7}
\]
An integration, subject to the condition that the aquifer is to be of infinite lateral extent and that the flow of the well is to be \(Q\) is:
\[
y = \frac{Q}{2\pi KD} \int_0^\infty \frac{e^{-\beta^2}}{\beta} \, d\beta \tag{8}
\]
\[
\frac{r}{\sqrt{4\alpha t}}
\]
* The integral which appears in this equation is a form of the exponential integral since
\[
\int_0^\infty \frac{e^{-\beta^2}}{\beta} \, d\beta = -\frac{1}{2} Ei(-u^2)
\]
and the substitution
\[
\beta^2 = v \text{ makes the transformation}
\]
\[
\int_0^\infty \frac{e^{-\beta^2}}{\beta} \, d\beta = \frac{1}{2} \int_0^\infty \frac{e^{-v}}{v^2} \, dv
\]
This solution constitutes the first approximation. The presence of \(D\) in equation 1 implies that this quantity will be involved in the second approximation. It will be seen that the quantity \(y/D\) becomes essential.
Accounting for the effect of drawdown

An iteration procedure, based upon simple physical concepts provides an effective means for obtaining a second approximation. Suppose we have an approximate expression for the flow $F$ as a function of radius and time. Then we could compute an improved drawdown curve from the relationship which accounts for the effect of drawdown

$$F = -2\pi r K (D - y) \frac{\partial y}{\partial r} \quad (9)$$

or, in terms of the variable $u$

$$-2(D - y) \frac{\partial y}{\partial u} = \frac{F}{\pi K u} \quad (10)$$

By integration, subject to the requirement that $y$ should approach zero as $u$ approaches infinity

$$(D - y)^2 = D^2 - \int_{u_1}^{\infty} \frac{F du}{\pi K u} \quad (11)$$

If

$$S = \frac{Q}{2\pi K D^2} \quad (12)$$

This relation can be put in the form

$$\frac{V}{D} = -\sqrt{1 - 2S \int_{u_1}^{\infty} \frac{F}{Q} \frac{du}{u}} \quad (13)$$
With the improved values of $y$ so obtained we may compute an improved $F$ function from the relation

$$F = \int_{\infty}^{\infty} 2 \pi r \, dr \, V \frac{\partial y}{\partial t} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (14)$$

In terms of the variable $u$ this becomes:

$$F = -\int_{u_1}^{\infty} 4 \pi KD u^2 \frac{\partial y}{\partial u} \, du \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (15)$$

with

$$\psi = \frac{y}{\sqrt{\frac{\partial y}{\partial u}}} \quad \text{or} \quad \psi = \frac{(-\sqrt{\frac{\partial y}{\partial u}})}{2 \pi KD} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (16)$$

Equation (15) becomes:

$$\frac{F}{Q} = -2 \int_{u_1}^{\infty} u^2 \frac{\partial \psi}{\partial u} \, du \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (17)$$

The integrals appearing in equations (13) and (17) can be evaluated by graphical integration procedures. This makes it possible to avoid the difficulties of formal integrations since the required integrals can be evaluated by means of a planimeter. The process converges very rapidly.

An excellent starting point can be obtained from the first approximation. This yields

$$\frac{F_1}{Q} = e^{-u^2} \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (18)$$
When this first approximation flow is substituted into equation (18) it becomes:

$$\frac{y_2}{D} = 1 - \sqrt{1 - 2s} \int_{u}^{\infty} \frac{e^{-u^2}}{u} du \quad \ldots \ldots (19)$$

A check of this second approximation by the iteration procedure described shows that little improvement can be made. The chart of figure 1 was therefore computed directly from this expression. A comparison of second approximation and third approximation values obtained by the iteration procedure are shown in Table 1.

Table 1.

Comparison of second and third approximation computations for $S = 0.1$

<table>
<thead>
<tr>
<th>$u$</th>
<th>$y$</th>
<th>$\frac{y}{D}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.0482$</td>
<td>10.00</td>
<td>1.000</td>
</tr>
<tr>
<td>$0.05$</td>
<td>9.16</td>
<td>0.916</td>
</tr>
<tr>
<td>$0.06$</td>
<td>7.90</td>
<td>0.790</td>
</tr>
<tr>
<td>$0.07$</td>
<td>7.28</td>
<td>0.728</td>
</tr>
<tr>
<td>$0.08$</td>
<td>6.82</td>
<td>0.682</td>
</tr>
<tr>
<td>$0.09$</td>
<td>6.46</td>
<td>0.646</td>
</tr>
<tr>
<td>$0.10$</td>
<td>6.18</td>
<td>0.618</td>
</tr>
<tr>
<td>$0.10$</td>
<td>2.23</td>
<td>0.223</td>
</tr>
</tbody>
</table>

The second approximation drawdowns are greater than those of the first approximation because the reduced areas available for flow are accounted for. The $F/Q$ values computed for the third approximation attain values slightly in excess of unity because these increased drawdowns imply an increased drainage of the aquifer. Since the increase of drawdown is greatest near the well this increase is small. For the extreme case considered, where $S = 0.1$, the $F/Q$ values reach only 1.041. The third approximation computation of Table 1 was based upon $F/Q$ values corrected back to unity.
It will be noted that the third approximation drawdowns are slightly reduced, as compared to those of the second approximation. The use of the second approximation for computing drawdowns near the well will, therefore, be somewhat on the safe side since the actual drawdowns should be less than those estimated.

**Example**

**Conditions.**

A farmer installed an irrigation well in an aquifer, having a saturated thickness of 60 feet at the time of installation. This well has an effective diameter of 2.5 feet and the aquifer properties are

\[ K = 0.00040 \text{ ft/sec and } V = 0.15. \]

The well casing completely penetrates the aquifer and the pump is installed 10 feet above the bottom of the well. This pump has a capacity of 0.5 cubic feet per second and operated satisfactorily under the original conditions. After a long period of drought, however, heavy demands on the groundwater supply has lowered the water table 20 feet and he finds that, under these conditions his pump begins to surge after a short period of operation. How much must his pump capacity be reduced so that he can pump for a 2 month period without unwatering his pump?

**Solution**

Under these conditions \( D = 40 \text{ feet} \)

and with slide rule accuracy

\[
\alpha = \frac{KD}{V} = \frac{(0.00040)(40)}{0.15} = 0.1067
\]

\[
2\pi KD^2 = (6.2832)(0.00040)(40)^2 = 4.02 \text{ ft}^3/\text{sec}
\]

Two months is equivalent to about 5256000 seconds then

\[
\frac{r}{\sqrt{4\alpha t}} = \frac{1.25}{(4)(0.1067)5256000} = \frac{1.25}{1505} = 0.00083
\]

To prevent unwatering of the pump he must hold \( y/D \) to 0.75 or less.

From the chart for

\[
\frac{r}{\sqrt{4\alpha t}} = 0.00083 \quad \frac{y}{D} = 0.75 \quad \text{read } S = .070
\]

Then his reduced pumping rate is

\[
Q = (2\pi KD^2)S = (4.02)(.070) = 0.28 \text{ ft}^3/\text{sec}
\]
A computation based directly upon formula 19 would proceed in the following way: In equation 19 transpose the $1$ and square both sides to obtain

$$\frac{y^2}{D} = 0.75 - \left(1 - \frac{y_2}{D}\right) = -0.25 - \left(1 - \frac{y_2}{D}\right)^2 = 0.0625$$

Then solve the resulting expression to obtain

$$2S \int_0^\infty \frac{e^{-u^2}}{u} \, du = 1 - \left(1 - \frac{y_2}{D}\right)^2 = 0.93750$$

with

$$u = \frac{r}{\sqrt{4 \pi \alpha t}} = 0.00083470$$

From tables (2), or from the series

$$\int_0^\infty \frac{e^{-u^2}}{u} \, du = -0.288668 - \log e u + \frac{u^2}{2} - \frac{u^4}{8} + \ldots$$

obtain:

$$\int_0^\infty \frac{e^{-u^2}}{u} \, du = 6.77983 \text{ and } 2 \int_0^\infty \frac{e^{-u^2}}{u} \, du = 13.55966$$

then solve for $S$

$$S = \frac{0.93750}{13.55966} = 0.069137$$
with
\[ 2 \pi KD^2 = 4.02125 \quad S = \frac{Q}{2 \pi KD^2} \]
and the relation
\[ Q = S (2 \pi KD^2) = (0.069137)(4.02125) = 0.27802 \text{ ft}^3/\text{sec} \]
or, in recognition of the limitations of the second approximation
\[ Q = 0.28 \text{ ft}^3/\text{sec} \]

**Comments**

It was expected that the quantity \( D \) would be involved in the second approximation and a scrutiny of figure 1 will show that since the ratio \( y/D \) appears as an essential part of the improved solution, this expectation is realized. The dotted lines show equal values of the ratio \( y/D \) which are readily computed with the help of equation 19. These lines should prove useful in the manner indicated in example.

The curves are not drawn beyond \((y/D) = 1\) since this ratio indicates that the drawdown has reached the bottom of the aquifer.

At extreme drawdowns it should be expected that some discrepancy between observed and computed values will be found since the vertical velocity components and their effects are neglected in the present development. It is believed, however, that the evaluation of drawdown effects in the manner described will provide useful information when drawdowns are no longer small when compared to the original saturated depth.
References

The following references were consulted:


(A tabulation of the function \( \int_{x}^{\infty} \frac{e^{-\beta}}{\beta} \, d\beta \) is given in appendix F).


(4) Jahnke E. and Emde F., Tables of Functions with Formulas and Curves- Dover Pub. 1945.


[Signature]

July 1, 1958
Determination of a criterion constant.

First sentence of abstract.

The development in abstract.

The equation which expresses how important is \( \frac{y}{u} \) ?

Explanation

Example

Conditions

Problem.

Solution.

Chart of Fig 1

Reduced pumping rate

Equation.

In equation 19 from page 1 and...

For the example

References.
p-1 "present development"

2. units of u = ?

the eqn satisfying the requirement that ___ is ___

3. "...p% becomes important" (really how important)

6. "increased drainage"?

should the F/Q values be shown?

7. example

conditions: (intake) to the pump installed

problem: how much ___ so that he can pump (continue)

--- unwatering his pump.

solution:

pump

from figure 1. for ___

then his (reduced) pumping rate is:

p-8 use of "formulas" instead of equation?

"transpose the 1..."?

p-9 "dotted lines"?

"...in the example..."

p-10 "references consulted" should they be referenced in the text.

some discrepancy, {but not great}?

fig. 1 - showed 10% 0.1 curved intersect. -- reason for redefine...
\[ \alpha = \frac{K}{V} \]

\[ S = \frac{Q}{2\pi K D^2} \]
Second approximation

\[
\frac{y}{b} = 1 - \sqrt{1 - 2S \int \frac{e^{-u^2}}{u} \, du}
\]

\[
S = \frac{a}{2TKD^2}
\]

\[
\begin{array}{lcc}
0.001 & 8.9217 \\
0.002 & 8.2286 \\
0.003 & 7.8231 \\
0.004 & 7.5354 \\
0.005 & 7.3123 \\
0.010 & 6.6191 \\
0.020 & 5.7260 \\
0.030 & 5.5205 \\
0.040 & 5.2329 \\
0.050 & 5.0097 \\
0.100 & 4.3166 \\
0.200 & 3.6236 \\
0.300 & 3.2184 \\
0.400 & 2.9311 \\
0.500 & 2.7084 \\
1.000 & 2.0190 \\
1.500 & 1.4066 \\
2.000 & 0.9594 \\
2.500 & 0.7044 \\
3.000 & 0.5221 \\
3.500 & 0.2452 \\
1.000 & 0.1097 \\
1.500 & 0.0173 \\
2.000 & 0.00189 \\
2.500 & 0.0001352 \\
3.000 & 0.000106
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & 1.0 & 10.00 & 6.680 & .000025 \\
13.04 & 8.89 & 7.80 & 4.87 & & & \\
6.308 & 5.415 & .0001 & & & & \\
4.759 & 4.342 & .0004 & & & & \\
4.030 & 3.745 & .0009 & & & & \\
3.568 & 3.630 & .0016 & & & & \\
3.230 & 3.060 & .0025 & & & & \\
2.300 & 1.555 & .010 & & & & \\
1.444 & 1.414 & .040 & & & & \\
\end{array}
\]

See later computation

R.E. Glaedt

July 1, 1958