II

Computations for Spreading of a Ground Water Mound

by

Robert E. Glover
The height of the mound under a strip of width $W$ at the time $t$ due to a continuous supply at the rate $R$ (ft/see) is

$$h = \frac{R}{2} \int_0^t \left( \frac{2}{\pi} \int_0^{u_2} e^{-u_2^2} \, du \right) \, dt$$

where

$$u_2 = \frac{x + \frac{W}{2}}{4x(t+\xi)} \quad \text{and} \quad u_1 = \frac{x - \frac{W}{2}}{1 + 4x(t+\xi)}$$

If

$$\xi = \frac{t}{t}$$

$$\frac{h}{Rt} = \frac{1}{2} \int_0^1 \left( \frac{2}{\pi} \int_0^{u_2} e^{-u_2^2} \, du \right) \, d\xi$$

or

$$\frac{h}{Rt} = \frac{1}{2} \int_0^1 \left( \frac{2}{\pi} \int_0^{u_2} e^{-u_2^2} \, du \right) \, d\xi - \frac{1}{2} \int_0^1 \left( \frac{2}{\pi} \int_0^{u_1} e^{-u_1^2} \, du \right) \, d\xi$$

with

$$u_2 = \frac{(x + \frac{W}{2})}{4x + (1-\xi)} \quad \text{and} \quad u_1 = \frac{(x - \frac{W}{2})}{1 + 4x(t+\xi)}$$

If

$$\beta = (1-\xi) \quad \text{d}\beta = -\,d\xi$$

$$\frac{h}{Rt} = \frac{1}{2} \int_0^1 \left( \frac{2}{\pi} \int_0^{u_2} e^{-u_2^2} \, du \right) \, d\beta - \frac{1}{2} \int_0^1 \left( \frac{2}{\pi} \int_0^{u_1} e^{-u_1^2} \, du \right) \, d\beta$$
An evaluation can then be made by an integration by parts, with:

\[ u_m = \frac{2}{\pi} \int_0^1 e^{-u_2^2} du, \quad du_m = -\frac{2}{\pi} e^{-\frac{(x+y)^2}{4\alpha^2}} \alpha dx ds. \]

Then

\[ I_m = R \int_0^1 \left( \frac{2}{\pi} \int_0^1 e^{-u_2^2} du \right) ds = R \int_0^1 \left( \frac{2}{\pi} \int_0^1 e^{-\frac{u_2^2}{4\alpha^2}} du \right) ds. \]

Since

\[ u_2 = \left( \frac{x+y}{2} \right), \quad s = \frac{(x+y)^2}{4\alpha^2}, \quad ds = -\frac{2(x+y)^2}{4\alpha^2} dx. \]

And, by substitution

\[ I_m = R \int_0^1 \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \int_0^1 \frac{e^{-\frac{u_2^2}{4\alpha^2}}}{4\alpha^2} du \right) ds. \]

\[ I_m = R \int_0^1 \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \int_0^1 \frac{e^{-\frac{u_2^2}{4\alpha^2}}}{4\alpha^2} du \right) ds. \]

\[ I_m = R \int_0^1 \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \int_0^1 \frac{e^{-\frac{u_2^2}{4\alpha^2}}}{4\alpha^2} du \right) ds. \]

A similar integration must be made for the lower limit \( u_1 \) to obtain

\[ I_n = R \int_0^1 \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \left( \frac{2}{\pi} \right) \int_0^1 \frac{e^{-\frac{u_2^2}{4\alpha^2}}}{4\alpha^2} du \right) ds. \]

And, finally

\[ h = I_m - I_n. \]
An evaluation can be made by an integration by parts. With:

\[ u_m = 2 \int \frac{u^2}{u_1} e^{-u^2} du \]
\[ dU_m = 2 \int \frac{u^2}{u_1} e^{-u^2} du + \frac{u_1 e^{-u^2}}{u_1} du \]
\[ dU_m = -du \]
\[ U_m = -\frac{u_1}{2} \]

Then, for the first term:

\[ I_m = -\int_0^t \left( \frac{2}{\pi} \int_0^{u_1} u^2 e^{-u^2} du \right) ds = t \left[ \frac{2}{\pi} \int_0^{u_1} e^{-u^2} du + \frac{2(\sqrt{\pi})}{\pi} \right] \left[ \frac{e^{-u^2}}{u^2} du \right]_{u_1}^{\infty} \]

The integral for the second term is:

\[ I_n = -\int_0^t \left( \frac{2}{\pi} \int_0^{u_1} u^2 e^{-u^2} du \right) ds = t \left[ \frac{2}{\pi} \int_0^{u_1} e^{-u^2} du + \frac{2(\sqrt{\pi})}{\pi} \right] \left[ \frac{e^{-u^2}}{u^2} du \right]_{u_1}^{\infty} \]

And finally:

\[ h = \frac{R}{2} (I_m - I_n) \]
\[ u_m = \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \, du \quad \text{and} \quad du_m = \frac{2}{\pi} e^{-u^2} \, du \]

\[ u_2 = \frac{(x + \frac{\xi}{2})}{4 \xi} \quad \text{and} \quad u_2 = \frac{(x + \frac{\xi}{2})}{4 \xi} \quad \xi = \frac{(x + \frac{\xi}{2})}{4 \xi u_2} \]

\[ d\xi = -\frac{2(x + \frac{\xi}{2})}{4 \xi u_2^2} \, du_2 \quad \text{and} \quad du_2 = \frac{(x + \frac{\xi}{2})}{4 \xi} \, d\xi = -\frac{4 \xi u_2}{2(4 \xi)^3} \, d\xi \]

\[ du_m = -\frac{2}{\pi} \frac{(x + \frac{\xi}{2})^2}{2(4 \xi)^3} \cdot \frac{e^{-u^2}}{u_2^2} \, d\xi = -\frac{2}{\pi} \cdot u_2 \cdot \frac{e^{-u^2}}{u_2^2} \, d\xi \]

\[ \mathcal{I}_m = \frac{R}{2} \int_0^t \left( \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \, du \right) \, d\xi = \frac{R}{2} \left[ \frac{2}{\pi} \int_0^t e^{-u^2} \, du \right] \]

\[ = \frac{R}{2} \left[ \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \, du + 2 \frac{(x + \frac{\xi}{2})}{4 \xi} \int_0^{\infty} \frac{e^{-u^2}}{u^2} \, du \right] \]

\[ = \frac{R}{2} \left[ \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \, du + \frac{2}{4 \xi} \right] \]

\[ + \frac{2}{\pi} \int_0^{\infty} \frac{e^{-u^2}}{u^2} \, du \]

\[ = \frac{R}{2} \left[ \frac{2}{\pi} \int_0^{\infty} e^{-u^2} \, du + \frac{2}{4 \xi} \right] \]

\[ \text{REG, Feb 15, 1961} \]
For
\[
\frac{1}{2\pi} \int_0^1 \left( \frac{u_2}{1 + u_2^2} \right) du \quad d\beta
\]

By \( p \rightarrow \infty \), let
\[
u_m = \frac{2}{\pi} \int_0^1 \frac{u_2}{1 + u_2^2} du
\]
\[
d\nu_m = d\beta
\]
\[
u_m = \beta
\]

\[
du_m = \frac{2}{\pi} \frac{u_2}{2(4\chi + \beta)} e^{-\frac{(X+\frac{u_2}{2})^2}{2(4\chi + \beta)^2}} d\beta
\]

By substitution
\[
\frac{1}{2\pi} \int_0^1 \left( \frac{u_2}{1 + u_2^2} \right) du \quad d\beta = \left[ \left( \frac{2}{\pi} \right) \int_0^1 e^{-u_2^2} du \right] + \left[ \left( \frac{2}{\pi} \right) \int_0^1 \frac{u_2 e^{-u_2^2}}{u_2} d\beta \right]
\]

Since
\[
u_2 = \frac{(X + \frac{u_2}{2})}{2(4\chi + \beta)}
\]
\[
u_2^2 = \frac{(X + \frac{u_2}{2})^2}{2(4\chi + \beta)}
\]
\[
\beta = \frac{\left( \frac{X + \frac{u_2}{2}}{2} \right)^2}{2(4\chi + \beta)}
\]

\[
d\beta = -\frac{2(X + \frac{u_2}{2})}{4\chi + \beta} du_2
\]

Thus
\[
\frac{1}{2\pi} \int_0^1 \left( \frac{u_2}{1 + u_2^2} \right) du \quad d\beta = \left[ \left( \frac{2}{\pi} \right) \int_0^1 e^{-u_2^2} du \right] + \left[ \left( \frac{2}{\pi} \right) \int_0^1 \frac{u_2 e^{-u_2^2}}{u_2} d\beta \right]
\]

\[
= \left[ \left( \frac{2}{\pi} \right) \int_0^1 e^{-\frac{(X + \frac{u_2}{2})^2}{2(4\chi + \beta)^2}} du \right] + \left[ \left( \frac{2}{\pi} \right) \int_0^1 \frac{u_2 e^{-\frac{(X+\frac{u_2}{2})^2}{2(4\chi + \beta)^2}}}{u_2} d\beta \right]
\]
\[ M = t \left[ \frac{2}{\pi} \int_0^1 e^{-u^2} \, du + \frac{2(x+\frac{y}{2})^2}{4\pi} \int_0^\infty e^{-u^2} \, du \right] \]

Let
\[ u_2 = \frac{(x+\frac{y}{2})}{\sqrt{4\pi t}} \]
\[ \frac{du_2}{dt} = -\frac{4x}{2} \cdot \frac{u_2^3}{(x+\frac{y}{2})^2} \]

Then
\[ M = t \left[ \frac{2}{\pi} \int_0^1 e^{-u^2} \, du + \frac{2u_2^2}{\pi} \int_0^\infty e^{-u^2} \, du \right] \]
\[ \frac{\partial M}{\partial t} = t \left[ -\frac{2}{\pi} \cdot \frac{4x}{2} \cdot \frac{u_2^3}{(x+\frac{y}{2})^2} + \frac{2u_2^2}{\pi} \cdot \frac{4x}{2} \cdot \frac{u_2^3}{(x+\frac{y}{2})^2} - \frac{e^{-u_2^2}}{u_2^2} \right] + \left[ \frac{2}{\pi} \int_0^1 e^{-u^2} \, du + \frac{2u_2^2}{\pi} \int_0^\infty e^{-u^2} \, du \right] \]

Since
\[ t = \frac{(x+\frac{y}{2})^2}{4\pi u_2^2} \]
\[ -\frac{4x+\frac{y}{2}}{2} \cdot \frac{u_2^3}{(x+\frac{y}{2})^2} \cdot \frac{4u_2}{\pi} \int_0^\infty e^{-u^2} \, du = -\frac{2u_2}{\pi} \int_0^\infty e^{-u^2} \, du \]

And
\[ \frac{\partial M}{\partial t} = \frac{2}{\pi} \int_0^1 e^{-u^2} \, du \]

Okie, Red.
Feb 16, 1961.
The integral \( \frac{2}{\pi}\int_{1}^{\infty} e^{-u^2} du \) is a form of the probability integral. It has been extensively tabulated in terms of the upper limit \( \beta \). The integral \( \int_{\beta}^{\infty} \frac{e^{-u^2}}{u^2} du \) has been tabulated by M. W. Bittenger. This table may be found in reference 2. The integral may be evaluated by using formula 21. Fig 6 may also be found useful.

When \( x \) is less than \( W/2 \) some negative limits will appear in the formula for \( F(x) \). If the lower limit of \( \int e^{-u^2} du \) is negative, the upper limit will also be negative. These negative limits will make the integrals negative.
For a rectangular plate, \( d(\theta_0) = w \, dw + v \, dv \)

\[
h = R \int \left( \frac{1}{\pi} \int_{\theta_1}^{\theta_2} e^{-u^2 du} \right) \left( \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \right) \, dn
\]

By parts with

\[
w = \int_{\theta_1}^{\theta_2} e^{-u^2 du} \quad dv = \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \, dn
\]

Then

\[
dw = \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \, dn
\]

\[
v = \int_{\theta_1}^{\theta_2} e^{-u^2 du} \, dn
\]

\[
h = R \left[ \frac{1}{\pi} \int_{\theta_1}^{\theta_2} e^{-u^2 du} \right] \left( \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \right) - \frac{1}{\pi} \int_{\theta_1}^{\theta_2} e^{-u^2 du} \left( \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \right) \, dn \]

\[
\quad - \frac{1}{\pi} \int_{\theta_1}^{\theta_2} e^{-u^2 du} \left( \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \right) \, dn
\]

\[
+ \frac{1}{\pi} \int_{\theta_1}^{\theta_2} e^{-u^2 du} \left( \frac{1}{\pi} \int_{\theta_3}^{\theta_4} e^{-u^2 du} \right) \, dn
\]

This will not permit use of the product law.
Example. From Warren plat 60' wide 600' long.
Refer to Barchie letter of Aug 30 1960

Take \( D = 30 \) ft.

\[
K = \frac{1}{3600} = 0.000278 \text{ ft/sec} \approx 17/4 h
\]

\[
K = 0.008340 \text{ ft/sec} \quad V = 0.07 \quad \alpha = \frac{KD}{V} = 0.1191 \frac{ft}{sec}
\]

Suppose water is applied to the surface at the rate of 0.72 feet per day. Then \( R = \frac{0.72}{0.07} = 10.28 \) ft/day

\[
R = \frac{10.28}{86400} = 0.000119 \text{ ft/sec}
\]

Time of application 10 days \( \approx 864000 \) seconds

\[
W = \frac{60}{\sqrt{(4)(0.1191)(86400)}} = \frac{60}{619.11} = 0.0962
\]

From the chart for \( \frac{W}{K} = 0 \).

\[
\frac{h}{R} = 0.1 \quad R^2 = (0.0001190)(86400) = 102.82 \text{ feet}
\]

\[
h = (0.1)(102.82) = 10.28 \text{ ft}
\]

By working backward for aquifer constant.

\[
\frac{h}{R} = \frac{10.28}{10282} = 0.1 \quad \text{from the chart for} \quad \alpha = 0 \quad \frac{W}{4\alpha R} \approx 0.1
\]

\[
\frac{W}{4\alpha R} = 0.1 \quad \alpha = \frac{(40)^2}{(4)(0.1)(86400)} = 0.1041 \frac{\text{ft}^2}{\text{sec}}
\]

\[
\alpha = \frac{KD}{V} \quad KD = \alpha V = (0.1041)(0.07) = 0.007287 \text{ ft} \quad \text{sec}
\]

With \( D = 30 \) ft.

\[
K = \frac{0.007287}{30} = 0.000243 \text{ ft/sec}
\]
Computation of height of a mound due to a continuous recharge over the width \( W \). Water table present.

For \( x = 0 \): \[ \frac{W}{4\pi t} \left( \int_0^\infty \frac{e^{-u^2}}{u^2} \, du \right) = \frac{W^2}{2\pi t} \sum_{n=0}^\infty \frac{e^{-u^2}}{u^2} \, du \]

\( \frac{W}{2\pi t} \sum_{n=0}^\infty \frac{e^{-u^2}}{u^2} \, du \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.25</td>
</tr>
<tr>
<td>1.0</td>
<td>0.50</td>
</tr>
<tr>
<td>1.5</td>
<td>0.75</td>
</tr>
<tr>
<td>2.0</td>
<td>1.00</td>
</tr>
<tr>
<td>2.5</td>
<td>1.25</td>
</tr>
<tr>
<td>3.0</td>
<td>1.50</td>
</tr>
<tr>
<td>3.5</td>
<td>1.75</td>
</tr>
<tr>
<td>4.0</td>
<td>2.00</td>
</tr>
<tr>
<td>4.5</td>
<td>2.25</td>
</tr>
<tr>
<td>5.0</td>
<td>2.50</td>
</tr>
</tbody>
</table>

For a circular plot - From equation 16 (USDA)

\[ \frac{W}{\sqrt{4\pi t}} \int_0^\infty \frac{e^{-u^2}}{u^2} \, du \]

\[ \frac{W^2}{2\pi t} \sum_{n=0}^\infty \frac{e^{-u^2}}{u^2} \, du \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( h )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.01</td>
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</tr>
<tr>
<td>1.5</td>
<td>0.09</td>
</tr>
<tr>
<td>2.0</td>
<td>0.13</td>
</tr>
<tr>
<td>2.5</td>
<td>0.17</td>
</tr>
<tr>
<td>3.0</td>
<td>0.21</td>
</tr>
<tr>
<td>3.5</td>
<td>0.25</td>
</tr>
<tr>
<td>4.0</td>
<td>0.29</td>
</tr>
<tr>
<td>4.5</td>
<td>0.33</td>
</tr>
<tr>
<td>5.0</td>
<td>0.37</td>
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</tbody>
</table>

Check by use of Formula 16.

\[ \frac{W}{\sqrt{4\pi t}} \left( \int_0^\infty \frac{e^{-u^2}}{u^2} \, du \right) = \frac{W^2}{2\pi t} \sum_{n=0}^\infty \frac{e^{-u^2}}{u^2} \, du \]

\[ \frac{W}{\sqrt{4\pi t}} = 0.63662 \]

For the square recharge area - By \( h \) product law, at the center.

<table>
<thead>
<tr>
<th>( h )</th>
<th>( \frac{h}{Rt} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.00</td>
</tr>
<tr>
<td>0.5</td>
<td>0.21</td>
</tr>
<tr>
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<td>1.89</td>
</tr>
<tr>
<td>5.0</td>
<td>2.10</td>
</tr>
</tbody>
</table>

Check from Fig. 6.

Feb 3, 1961
Graphical check:

From formula 3. (USDA) The height of the mound $h$ due to an initial height $H$ over 20 x width $W$ is

$$h = H \frac{1}{2} \int_0^{u_2} e^{-u_2} du = H \frac{2}{3\pi} \int_0^{u_2} e^{-u_2} du,$$

$$= H \frac{2}{3\pi} \int_0^{u_2} e^{-u_2} du - H \frac{2}{3\pi} \int_0^{u_1} e^{-u_2} du,$$

Where

$$u_2 = \frac{x + \frac{W}{2}}{V4\pi t}, \quad u_1 = \frac{x - \frac{W}{2}}{V4\pi t},$$

The height of a mound at the time $t$ due to a continuous recharge at the rate $R$, beginning at $t=0$ is

$$h = \frac{R}{2} \int_0^t \left( \frac{2}{3\pi} \int_0^{u_2} e^{-u_2} du \right) dh,$$

Where now,

$$u_2 = \frac{x + \frac{W}{2}}{V4\pi (t-n)}, \quad u_1 = \frac{x - \frac{W}{2}}{V4\pi (t-n)}.$$

If $\frac{n}{t} = \xi$

$$\frac{h}{Rt} = \frac{1}{2} \int_0^1 \left( \frac{2}{3\pi} \int_0^{u_2} e^{-u_2} du \right) d\xi.$$
Suppose $x = 0$

\[
\frac{W}{2^{14x+1}} = 0.5
\]

\[
W = 2 \left( \int e^{-u^2} du \right)
\]

\[
\xi = \frac{n}{t} (1-\xi)
\]

<table>
<thead>
<tr>
<th>$n$</th>
<th>0.0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
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<tbody>
<tr>
<td>$t$</td>
<td>1.00</td>
<td>0.90</td>
<td>0.80</td>
<td>0.70</td>
<td>0.60</td>
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<td>0.40</td>
<td>0.30</td>
<td>0.20</td>
<td>0.10</td>
<td>0.00</td>
</tr>
</tbody>
</table>

\[
W = 0.52050
\]

\[
0.5
\]

\[
2 \left[ \begin{array}{c}
72.45 \\
36.22 \\
50.30 \\
100.56
\end{array} \right]
\]

\[
50.28 \text{ m}^2
\]

\[
36.22 \approx 0.7204
\]

\[
\text{Should be } 72.01 \text{ OK.}
\]

\[
\xi (1-\xi) \left( \frac{1}{n-\xi} \right)
\]

\[
0.6
\]

\[
0.8
\]

\[
0.6039
\]

\[
0.7421
\]

\[
0.7668
\]

\[
0.7939
\]

\[
0.8241
\]

\[
0.8563
\]

\[
0.8903
\]

\[
0.9264
\]

\[
0.9610
\]

\[
0.9886
\]

\[
0.9985
\]

\[
1.0000
\]
For \( x = 0 \)

\[
\begin{array}{c|c|c|c}
\hline
\theta & \frac{1}{1 - \theta} & (0.2) & (0.4) \\
\hline
0.0 & 1.000 & 0.200 & 2.227 \\
0.1 & 1.054 & 0.211 & 2.346 \\
0.2 & 1.118 & 0.224 & 2.484 \\
0.3 & 1.196 & 0.239 & 2.646 \\
0.4 & 1.292 & 0.258 & 2.848 \times 10^{-1} \\
0.5 & 1.414 & 0.283 & 3.144 \\
0.6 & 1.581 & 0.316 & 3.450 \\
0.7 & 1.825 & 0.365 & 3.943 \\
0.8 & 2.236 & 0.447 & 4.727 \\
0.9 & 3.162 & 0.632 & 6.286 \\
1.0 & 0.000 & & \\
\hline
\end{array}
\]

\[
\left(\frac{\omega}{2\pi f_{c4}}\right) = 0.410, 0.428, 0.449, 0.472, 0.501, 0.535, 0.576, 0.628, 0.698, 0.793, 0.926 \\
\]

\[
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\end{array}
\]

**Note:** Red figures are from table.

**RCG**

**Feb 15 1961**
\[
M\left(\frac{\chi}{4\hbar^2}\right) = \frac{2}{\pi} \left( \int_{0}^{\infty} e^{-u^2} du + \int_{0}^{\infty} \frac{u^2}{e^{u^2}} du \right) \approx \frac{\pi}{4} \cdot 0.63661978
\]

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KCE  
Feb 15, 1961
Computation for a sheet flow, recharge at the constant rate $R_j$, water table present.

Make chart for the cases

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<tr>
<td>3.00</td>
<td>0.99</td>
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</table>
For $x = \frac{W}{4}$

\[
U_3 = \frac{(x - \frac{W}{2})}{14tx^4} = -\frac{W^5}{14tx^4}
\]

\[
U_4 = \frac{(x + \frac{W}{2})}{14tx^4} = \frac{3W^3}{14tx^4}
\]

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<tr>
<th>$\frac{W}{14tx^4}$</th>
<th>$U_4$</th>
<th>$U_3$</th>
<th>$M(U_4)$</th>
<th>$M(U_3)$</th>
<th>$(\frac{h}{R+})$</th>
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For $x = \frac{W}{2}$

\[
U_3 = 0 \quad U_4 = \frac{W}{14tx^4}
\]

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<th>$(\frac{h}{R+})$</th>
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<td>4.990</td>
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<td>4.996</td>
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</table>
For $X = 0.75\ W$  

\[
\begin{array}{cccc}
0.1 & 0.551 & 1.25 & 2.478 \\
0.4 & 1.079 & 2.50 & 4.509 \\
0.6 & 1.291 & 1.75 & 3.772 \\
0.8 & 1.450 & 1.25 & 3.920 \\
1.0 & 1.576 & 1.75 & 4.992 \\
1.2 & 1.630 & 2.00 & 9.998 \\
1.8 & 1.727 & 2.25 & 9.999 \\
2.0 & 1.720 & 2.50 & 1.000 \\
3.0 & 8.684 & 3.75 & 0.0658 \\
\end{array}
\]

For $X = 1.0\ W$  

\[
\begin{array}{cccc}
0.1 & 1.079 & 0.296 & 0.941 \\
0.4 & 2.064 & 0.517 & 1.554 \\
0.6 & 3.773 & 0.791 & 2.068 \\
0.8 & 4.511 & 0.919 & 2.212 \\
1.0 & 7.201 & 0.992 & 1.714 \\
1.2 & 8.459 & 0.999 & 1.034 \\
1.6 & 8.888 & 1.000 & 0.560 \\
2.0 & 9.432 & 1.000 & 0.284 \\
3.0 & 9.920 & 1.000 & 0.040 \\
\end{array}
\]

For $X = 1.25\ W$  

\[
\begin{array}{cccc}
0.1 & 1.582 & 0.175 & 3.376 \\
0.4 & 2.961 & 0.350 & 5.767 \\
0.6 & 4.512 & 0.760 & 8.459 \\
0.8 & 6.777 & 1.105 & 9.518 \\
1.0 & 8.910 & 1.300 & 9.878 \\
1.2 & 10.984 & 1.750 & 9.997 \\
1.4 & 12.919 & 2.150 & 9.995 \\
1.6 & 14.728 & 2.550 & 9.999 \\
2.0 & 16.920 & 3.000 & 1.000 \\
3.0 & 9.998 & 5.250 & 0.001 \\
\end{array}
\]
Check if \( h = \frac{W}{2} \int_0^t \left( \frac{2}{14xW} \int_0^x e^{-u^2} du \right) dx \)

For an instantaneous reaction:

\[ h = R \int_0^t \left( \frac{2}{14xW} \int_0^x e^{-u^2} du \right) dt \]

For a continuous reaction at no rate \( R \):

\[ h = \frac{W}{4xW} = 1.00 \quad \frac{W^2}{4xW} = 1.00 \quad t = \frac{W^2}{4x} \]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{W^2}{4x} )</th>
<th>( \frac{W}{4x(1-x)} )</th>
<th>( \frac{5W}{14x(1-x)} )</th>
<th>( \frac{5W}{14x(1-x)} )</th>
<th>( \frac{2}{14x} \int_0^x e^{-u^2} du )</th>
<th>( \frac{2}{14x} \int_0^x e^{-u^2} du )</th>
<th>Difference</th>
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</thead>
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<td>0.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>0.1</td>
<td>0.9</td>
<td>1.11</td>
<td>1.05</td>
<td>0.526</td>
<td>1.578</td>
<td>0.430</td>
<td>9.744</td>
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<td>0.571</td>
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</table>

\[ \frac{h}{Rt} = 0.2716 \]

If \( \frac{h}{Rt} = \) then the above integral becomes

\[ h = \frac{R}{2} \int_0^t \left( \frac{2}{14xW} \int_0^x e^{-u^2} du \right) ds \]

\[ \frac{h}{Rt} = \frac{0.2716}{0.2} = 1.358 \]

By formula \( \frac{h}{Rt} = 0.1356 \)
Check at \( x = W \)

\[
\frac{W}{14x^2} = 2.00
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( \frac{W}{14x^2(1-x)} )</th>
<th>( \frac{0.5W}{14x^2(1-x)} )</th>
<th>( \frac{U_3}{15W} )</th>
<th>( \frac{U_4}{15W} )</th>
<th>( \frac{2}{W} \int_0^{U_3} \frac{e^{u^2}}{u} du )</th>
<th>( \frac{2}{W} \int_0^{U_4} \frac{e^{u^2}}{u} du )</th>
<th>Difference</th>
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</table>

\[
\frac{h}{R_t} = \frac{0.5683}{2} = 0.2841
\]

By formula, \( \frac{h}{R_t} = 0.284 \)

OK
For $X = 1.5W$

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<th>$M(U_4)$</th>
<th>($h$)</th>
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For $X = 1.75$

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<th>$M(U_4)$</th>
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For $X = 2.0W$

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<th>$U_4$</th>
<th>$M(U_4)$</th>
<th>($h$)</th>
</tr>
</thead>
<tbody>
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</table>
For $x = 0.4 W$

<table>
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<tr>
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<th>$M(u_4)$</th>
<th>$(\frac{h}{R^+})$</th>
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<td>0.5172</td>
<td>1.0000</td>
<td>7.582</td>
</tr>
</tbody>
</table>

For $x = 0.6 W$

<table>
<thead>
<tr>
<th>$W$</th>
<th>$M(u_3)$</th>
<th>$M(u_4)$</th>
<th>$(\frac{h}{R^+})$</th>
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<tbody>
<tr>
<td>0.1</td>
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<td>1.6</td>
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<td>2.931</td>
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<td>1.8</td>
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<td>0.5172</td>
<td>1.0000</td>
<td>2.414</td>
</tr>
</tbody>
</table>
Line source - perching layer present  
Steady state case

The flow through the height $h$ is:

$$ F = K h \frac{dh}{dx} \quad h \frac{dh}{dx} = \frac{F}{K} \quad \frac{h^2}{2} = \frac{1}{K} \int_0^h F \, dx $$

The continuity condition is:

$$ \frac{2F}{dx} \, dx - \rho \left( \frac{h}{m} + 1 + \frac{c}{m} \right) \, dx = \nabla \frac{\partial h}{\partial t} \, dx $$

$$ \frac{2F}{\partial x} = \frac{\rho}{m} h + \rho (1 + \frac{c}{m}) + \nabla \frac{\partial h}{\partial t} $$

Let

$$ F = \rho \cdot \frac{Q}{p} \quad h = m \cdot n \quad x = m \cdot \xi \quad t = \frac{T m}{p} \Theta $$

Then the two relations become:

$$ \frac{Q}{p} \cdot \xi = \frac{K \cdot m \cdot n}{\rho} \left( \frac{1}{m} \cdot \frac{\partial n}{\partial \xi} \right) \quad \frac{\partial n}{\partial \xi} = \frac{Q}{K \cdot m} \cdot \frac{\partial h}{\partial t} \quad \frac{n^2}{2} = \frac{Q}{K \cdot m} \int_0^\xi \rho \cdot \partial s $$

Dimension etc.

$$ \frac{\partial}{\partial \xi} \left( \frac{\partial n}{\partial \xi} \right) = \frac{1}{m} \cdot m \cdot n + \rho (1 + \frac{c}{m}) + \rho \cdot \frac{\partial n}{\partial t} $$

$$ \frac{\partial n}{\partial \xi} = \frac{b m}{q} \cdot n + \frac{b m}{q} (1 + \frac{c}{m}) + \frac{b m}{q} \cdot \frac{\partial n}{\partial \xi} $$
Let \( F = \frac{4}{6} \)  
\( h = \frac{5}{p} n \)  
\( x = \frac{5}{p} \)  
\( t = \frac{v_0^2}{p^2} \)

Then, by substitution:

\[
y' = k h \frac{\partial h}{\partial x} \quad y = \frac{k}{p} \frac{\partial n}{\partial \xi} = \frac{k}{p} \frac{\partial n}{\partial \xi} \quad \frac{\partial x}{\partial x} = \frac{v_0}{k} \sqrt{q} \frac{ds}{d\xi}
\]

\[
\frac{\partial w}{\partial x} = k h \frac{\partial h}{\partial \xi} \quad \frac{\partial w}{\partial \xi} = \frac{\partial n}{\partial \xi} \quad t = \frac{v_0^2}{p^2} \theta
\]

And

\[
\frac{\partial F}{\partial x} = \frac{p}{m} h + \frac{1}{p} \frac{\partial}{\partial x} (1 + \frac{e}{m}) + \frac{\partial}{\partial x} \frac{\partial h}{\partial x}
\]

Becomes

\[
p \frac{\partial y}{\partial \xi} = \frac{k}{m} \frac{\partial n}{\partial \xi} + \frac{1}{p} (1 + \frac{e}{m}) + \frac{v_0^2}{p^2} \frac{\partial n}{\partial \theta} \frac{\partial \theta}{\partial t}
\]

\[
\frac{\partial y}{\partial \xi} = \frac{k}{mp} n + (1 + \frac{e}{m}) + \frac{v_0^2}{p^2} \frac{\partial n}{\partial \theta} \frac{\partial \theta}{\partial t}
\]

\[
\frac{v_0}{p^2} \frac{d\theta}{dt} = 1 \quad \frac{d\theta}{dt} = \frac{p^2}{v_0^2} \quad \theta = \frac{p^2}{v_0} t
\]

Then

\[
\frac{\partial y}{\partial \xi} = \left( \frac{8}{mp} \right) n + (1 + \frac{e}{m}) + \frac{2v}{\theta}
\]

\[
\left( \frac{8}{mp} \right) \quad \left( \frac{p}{k} \right) \quad \left( \frac{e}{m} \right)
\]
Suppose
\[ R = \frac{q}{g}, \quad h = (m+e) \eta, \quad \chi = \frac{m \eta^2}{2}. \]

Then
\[ \frac{\partial F}{\partial \chi} = \frac{p}{m} h + \frac{b}{m} (m+e) + \frac{\eta}{n} \frac{\partial h}{\partial \eta}. \]

Becomes
\[ \frac{d}{d\xi} \frac{\partial y}{\partial \chi} = \frac{b}{m} \frac{(m+e) p n + \frac{b}{m} (m+e) + \frac{\eta}{n} \frac{\partial h}{\partial \eta}}{d\xi}. \]

If \( \frac{d}{dt} \frac{v(m+e)}{dt} = \frac{(m+e) p}{m} \), then \( \frac{d}{dt} = \frac{b}{2 m} + \frac{\eta}{n} \frac{\partial h}{\partial \eta} \sum \).

2
\[ \frac{d}{d\xi} \frac{\partial y}{\partial \chi} = \frac{(m+e) p}{n} \sum \left( n + 1 + \frac{\eta}{n} \right) \]

\[ \frac{\eta^2}{2} = \frac{q}{(m+e) p} \frac{k(m+e)}{2} \int_0^5 \psi \, d\xi. \]
Try
\[ F = 4 \varphi, \quad h = (m+e) \eta, \quad x = (m+e) \xi, \quad t = \frac{\nu m}{\rho} \theta \]

Then
\[ F = 4 \varphi \frac{\partial h}{\partial x} \]
becomes
\[ F = 4 \varphi \frac{\partial h}{\partial x} = \frac{K (m+e) \eta}{(m+e)} \frac{\partial \eta}{\partial x} = K(m+e) \eta \frac{\partial \eta}{\partial x} \]

By integration with respect to \( s \)
\[ \frac{n^2}{2} = \frac{\eta^2}{2} \frac{\partial \eta}{\partial x} \int_0^s \varphi \eta \, ds = \beta_1 \int_0^s \varphi \eta \, ds \]

The relation
\[ \frac{\partial F}{\partial x} = \frac{b}{m} h + \frac{p}{m} (m+e) + \frac{\nu \partial h}{\partial t} \]

Becomes
\[ \frac{\partial \eta}{\partial x} \frac{\partial h}{\partial s} = \frac{b}{m} (m+e) + \frac{p}{m} (m+e) + \frac{\partial (m+e)}{\partial \theta} \frac{\partial h}{\partial m} \]

Or
\[ \frac{\partial \eta}{\partial s} = \frac{(m+e)^2 b}{m q^2} \left[ n + \frac{\partial n}{\partial \theta} \right] = \frac{1}{\beta_1 \beta_2} \left[ n + 1 + \frac{\partial n}{\partial \theta} \right] \]

Use This
\[ \frac{n^2}{2} = \frac{m \varphi}{(m+e)^2} \frac{p (m+e)}{K \eta} \int_0^s \varphi \, ds = \beta_1 \int_0^s \varphi \, ds \]

\[ (\frac{\rho}{(m+e)}) = \frac{1}{\beta_1 \beta_2} \]

\[ \frac{p (m+e)}{K \eta} = \frac{1}{\beta_1 \beta_2} \]

\[ (\frac{\rho}{(m+e)}) = \frac{1}{\beta_1 \beta_2} \]