[redacted], American Geophysical Union Volume 34, Number 3 June 1953

ON TIDES IN ESTUARIES AND AROUND SMALL ISLANDS

Chia-Shun Yih

Abstract--Tides in estuaries and around small islands are studied in this paper. Under the assumption that the width and the mean depth of the estuary can be adequately expressed as power functions of the longitudinal distance from a certain point upstream, and that the depth of the ocean varies as a power function of the radial distance from the island, analytical solutions can be found by very simple transformations.

Tides in estuaries--The problem of determining the variation of the amplitude of tide in a gradually widening and deepening estuary while a periodic motion is maintained at sea is of some interest. With sufficient latitude one can assume that the width b and the average depth h of the estuary vary as certain arbitrary powers of x, which is measured along the estuary downstream from a certain point. Denoting by s, b_S, and h_S respectively the values of x, b, and h at the mouth of the estuary, one can write

$$b/b_S = (x/s)^m = \xi^m \qquad (1)$$

$$h/h_S = (x/s)^n = \xi^n \qquad (2)$$

where

$$\xi = x/s \qquad (3)$$

The periodic motion maintained at the mouth of the estuary can be described by

$$\eta_S = C \cos(\sigma t + \epsilon) \qquad (4)$$

where η is the deviation of the water surface from its equilibrium position and η_S is its value at x = s, t is the time, and C, σ, and ϵ are respectively the amplitude, frequency, and phase angle of the periodic motion. The differential equation for η at any point in the estuary is [LAMB, 1945, p. 274]

$$\partial^2\eta/\partial t^2 = (g/b)(\partial/\partial x)(h\, b\, \partial\eta/\partial x) \qquad (5)$$

where g is the gravitational acceleration. In virtue of (1), (2), and (3), (5) can be written as

$$(s^2/g\, h_S)\, \partial^2\eta/\partial t^2 = \xi^{-m}(\partial/\partial\xi)(\xi^{m+n}\, \partial\eta/\partial\xi) \qquad (6)$$

If one takes

$$\eta = C \cos(\sigma t + \epsilon)\, X(\xi) \qquad (7)$$

then (6) becomes

$$\xi^{-m}(d/d\xi)(\xi^{m+n}\, dX/d\xi) + (s^2\sigma^2/g\, h_S)\, X = 0 \qquad (8)$$

For convenience one writes

$$\lambda^{2-n} = s^2\sigma^2/g\, h_S \qquad \xi_1 = \lambda\xi \qquad (9)$$

Then (8) becomes

$$(d/d\xi_1)(\xi_1^{m+n}\, dX/d\xi_1) + \xi_1^m X = 0 \qquad (10)$$

To seek a solution of (10) in terms of familiar functions, one tries the transformations

$$X = \xi_1^p f(\zeta) \qquad \zeta = \xi_1^q \qquad (11)$$

Then

$$d\,X/d\,\xi_1 = p\,\xi_1^{p-1}\,f\,(\zeta) + q\,\xi_1^{p+q-1}\,f'\,(\zeta) \qquad (12)$$

$$d^2\,X/d\,\xi_1^{\,2} = p\,(p-1)\,\xi_1^{p-2}\,f\,(\zeta) + q\,(2\,p + q - 1)\,\xi_1^{p+q-2}\,f'\,(\zeta) + q^2\,\xi_1^{p+2q-2}\,f''\,(\zeta) \qquad (13)$$

where the primes denote differentiation with respect to ζ. Substituting (11) to (13) into (10), one obtains

$$f'' + [(2\,p + m + n + q - 1)/q\,\zeta]\,f' + \left[(\zeta^{\frac{-n-2q+2}{q}}/q^2) + p\,(p + m + n - 1)/q^2\,\zeta^2\right] f = 0 \qquad (14)$$

Demanding

$$p = (1 - m - n)/2 \qquad q = (2 - n)/2 \qquad (15)$$

one obtains Bessel's differential equation

$$f'' + f'/\zeta + (1/q^2 - p^2/q^2\,\zeta^2)\,f = 0 \qquad (16)$$

the solutions of which are $J_\nu\,(\zeta/q)$ and $J_{-\nu}\,(\zeta/q)$ where $\nu = |p/q|$. Of these two solutions, only one will give an $X\,(\xi_1)$ which is finite at $\xi_1 = 0$ together with $d\,X/d\,\xi_1$, unless ν is an integer, when they will coincide. One will assume $n \leq 1$, so that q is positive. Remembering $\zeta = \xi_1{}^q$, and that [WHITTAKER and WATSON, 1945, p. 359]

$$J_\nu\,(\zeta/q) = \zeta^\nu (a_o + a_2\,\zeta^2 + \ldots)$$

$$J_{-\nu}\,(\zeta/q) = \zeta^{-\nu}\,(b_o + b_2\,\zeta^2 + \ldots)$$

where a_o, a_2, b_o, b_2, etc., are numerical constants depending on ν, one easily sees that with p negative (so that $\nu q = -p$),

$$\xi_1^p\,J_\nu\,(\zeta/q) = \xi_1^{p+\nu q}\,(a_o + a_2\,\zeta^2 + \ldots) = a_o + a_2\,\zeta^2 + \ldots$$

$$\xi_1^p\,J_{-\nu}\,(\zeta/q) = \xi_1^{2p}\,(b_o + b_2\,\zeta^2 + \ldots)$$

$$(d/d\,\xi_1)\,[\xi_1^p\,J_\nu\,(\zeta/q)] = \xi_1^{q-1}\,(2\,a_2\,\zeta + 4\,a_4\,\zeta^3 + \ldots) = \xi_1^{1-n}\,(2\,a_2 + 4\,a_4\,\zeta^2 + \ldots)$$

$$(d/d\,\xi_1)\,[\xi_1^p\,J_{-\nu}\,(\zeta/q)] = 2\,p\,\xi_1^{2p-1}\,(b_o + b_2\,\zeta^2 + \ldots) + \xi_1^{2p+q-1}\,(2\,b_2\,\zeta + 4\,b_4\,\zeta^3 + \ldots)$$

Thus when p is negative, $\xi_1^p\,J_\nu\,(\zeta/q)$ and its derivative with respect to ξ_1 are finite at $\xi_1 = 0$ (recalling $n \leq 1$), whereas, with $2\,p - 1 = -(m + n)$, the quantity

$$(d/d\,\xi_1)\,[\xi_1^p\,J_{-\nu}\,(\zeta/q)]$$

is definitely infinite at $\xi_1 = 0$. Thus $A\,\xi_1^p\,J_\nu\,(\zeta/q)$ should be chosen to be the solution. Similarly, when p is positive, $B\,\xi_1^p\,J_{-\nu}\,(\zeta/q)$ should be used.

The constants A and B are determined from (4) (the boundary condition at $x = s$ where $\xi = 1$, $\xi_1 = \lambda$, $\zeta = \lambda^q$) to be

$$A = 1/\lambda^p\,J_\nu\,(\lambda^q/q) \qquad B = 1/\lambda^p\,J_{-\nu}\,(\lambda^q/q) \qquad (17)$$

Thus one concludes that, for negative p

$$\eta = C\,A\,(\lambda\,\xi)^{\,p}\,J_\nu\,(\zeta/q)\,\cos\,(\sigma\,t + \epsilon) \qquad (18)$$

and for positive p

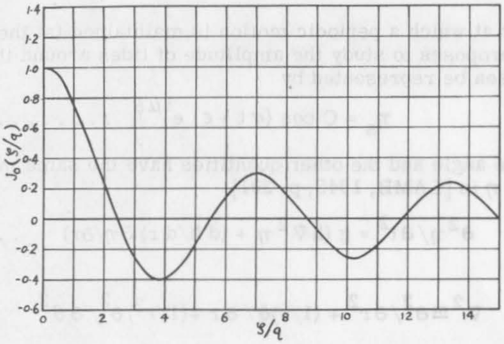
$$\eta = C\,B\,(\lambda\,\xi)^{\,p}\,J_{-\nu}\,(\zeta/q)\,\cos\,(\sigma\,t + \epsilon) \qquad (19)$$

where A and B are given by (17) and ζ is given by (9) and (11) to be $(\lambda\,\xi)^{\,q}$. When ν is an integer, the solutions given by (18) and (19) are identical.

As an example one takes the case m + n - 1, p =0. The solution is

$$\eta = C\,A\,J_o\,(\zeta/q)\,\cos\,(\sigma\,t + \epsilon)$$

In order to visualize the variation of the function $J_0(\zeta/q)$, it is plotted in Figure 1 against ζ/q. It is seen that at certain points of the estuary the function is a maximum in absolute value, and these maximum values increase in the upstream direction, so that the tide is augmented in the



neighborhood of these points. It is also seen that at certain other points of the estuary the function $J_0(\zeta/q)$ is zero. These are the nodal points where the elevation of the water surface is not affected by the tide. Similar conclusions can be drawn for the cases in which p is not equal to zero.

The points at which the amplitude is a maximum are determined by

$$d\, J_0(\zeta/q)/d\,\zeta = 0$$

or [WHITTAKER and WATSON, 1945, p. 360, formula (B)]

$$J_1(\zeta/q) = 0$$

the roots of which are [JAHNKE and EMDE, 1945, p. 168]

$$\zeta_1/q = 0 \qquad \zeta_2/q = 3.832 \qquad \zeta_3/q = 7.016 \qquad \zeta_4/q = 10.173 \quad \ldots \qquad \ldots\ldots \text{(20)}$$

the spacing of subsequent roots being approximately π. For cases in which $p \neq 0$, the points at which the amplitude is a maximum can be similarly located.

The points of zero amplitude (the nodes) are determined by

$$J_0(\zeta/q) = 0$$

the roots of which are [JAHNKE and EMDE, p. 168]

$$\zeta_1/q = 2.405 \qquad \zeta_2/q = 5.520 \qquad \zeta_3/q = 8.624 \quad \ldots \qquad \ldots\ldots\ldots\ldots \text{(21)}$$

the spacing of subsequent roots being approximately π. For cases in which $p \neq 0$, the nodes can be similarly located. Of course, the actual number of nodes as well as that of maximum amplitudes is limited by the length of the estuary for a certain frequency and a certain h_s. If the value of λ calculated from (9) is too small, it may happen that there are no nodes at all, and that the only maximum amplitude occurs at $\zeta = 0$ (or $x = 0$). This happens if the estuary is not extremely long. Thus for an ordinary estuary there is an augmentation of the amplitude of tide over its entire length.

Another point of interest is the phenomenon of resonance. There are many values of λ for which $J_0(\lambda^q/q)$ is zero. In fact, they are given by (21) on changing ζ to λ^q. For a given estuary there are many values of σ corresponding to these values of λ according to (9). These special values of σ will cause A to become infinite and can be called the natural or characteristic frequencies of the given channel. When the frequency at the sea comes near the natural frequencies of the channel, a state of resonance is approached. In such cases the given solution fails, and one is compelled to take friction into consideration. For an estuary of moderate length, σ is usually so small that even the first of these natural frequencies is not reached.

Tides around small islands--Supposing that r is measured radially from an island which is small compared with the wave length of the tide, and that the depth h of the sea is a power function of r

$$h/h_s = (r/s)^n \qquad (22)$$

where s is the value of r at which a periodic motion is maintained (at the open sea) and h_s is the value of h at r = s, one proposes to study the amplitude of tides around the island. Let the periodic motion at the open sea be represented by

$$\eta_s = C \cos(\sigma t + \epsilon)\, e^{i\mu\theta} \qquad (23)$$

where θ is the coordinate angle and the other quantities have the same meanings as before. The differential equation for η is [LAMB, 1945, p. 291]

$$\partial^2\eta/\partial t^2 = g\,(h\nabla^2\eta + (dh/dr)\,\partial\eta/\partial r) \qquad (24)$$

where

$$\nabla^2 \equiv \partial^2/\partial r^2 + (1/r)\partial/\partial r + (1/r^2)\partial^2/\partial\theta^2$$

and g is again the gravitational acceleration.

With $\rho = r/s$ one can write (24) as

$$(s^2/g\,h_s)\,\partial^2\eta/\partial t^2 = (\rho^n\,\nabla_\rho^2\,\eta + n\rho^{n-1}\,\partial\eta/\partial\rho) \qquad (25)$$

where

$$\nabla_\rho^2 \equiv \partial^2/\partial\rho^2 + (1/\rho)\,\partial/\partial\rho + (1/\rho^2)\,\partial^2/\partial\theta^2$$

Taking

$$\eta = C\cos(\sigma t + \epsilon)\,R(\rho)\,e^{i\mu\theta}$$

one has

$$\rho^n\,[R'' + (1/\rho)\,R' - (\mu^2/\rho^2)\,R] + n\rho^{n-1}\,R' + \lambda^{2-n}\,R = 0$$

where

$$\lambda^{2-n} = s^2\sigma^2/g\,h_s \qquad (26)$$

Writing ρ_1 for $\lambda\rho$, one has

$$d^2R/d\rho_1^2 + (2/\rho_1)\,dR/d\rho_1 - (\mu^2/\rho_1^2)\,R + (1/\rho_1^n)\,R = 0 \qquad (27)$$

To seek a solution in terms of familiar functions, one tries the transformations

$$R = \rho_1^{\,q}\,f(\zeta) \qquad \zeta = \rho_1^{\,q}$$

substitution of which into (27) yields, in a way similar to that of deriving (14)

$$d^2f/d\zeta^2 + [(2p + q + n)/q\,\zeta]\,df/d\zeta + \left\{[\zeta^{(-n+2-2q)/q}/q^2] - [\mu^2 - p(p+n)]/q^2\,\zeta^2\right\}f = 0$$

Demanding

$$p = -n/2 \qquad q = (2-n)/2 \qquad (28)$$

one obtains the Bessel equation

$$d^2f/d\zeta^2 + (1/\zeta)\,df/d\zeta + [1/q^2 - (\mu^2 + n^2/4)/q^2\zeta^2]\,f = 0 \qquad (29)$$

the solutions of which are $J_\nu(\zeta/q)$ and $J_{-\nu}(\zeta/q)$ where

$$\nu = \sqrt{(\mu^2/q^2) + (n^2/4\,q^2)} \qquad (30)$$

It may be noted that since values of n greater than two are unlikely, q will be considered as positive. Since p = - n/2 is always negative, J_ν should always be chosen so that R will not have a pole at $\zeta = 0$. It is interesting to note that since $J_\nu(\zeta/q)$ varies as

$$\zeta^\nu = \rho_1^{(\mu^2+n^2/4)^{1/2}}$$

near the origin,

$$R = \rho_1^{\;p} \; J_\nu(\zeta/q) = \rho_1^{-n/2} \; J_\nu(\zeta/q)$$

varies there as

$$\rho_1^{\;(\mu^2 + n^2/4)^{1/2} - n/2}$$

which vanishes at $\eta = 0$ for $\mu \neq 0$. Consequently, only the effect of the gravest mode corresponding to $\mu = 0$ is felt at the island.

In view of (23) (since $\rho_1 = \lambda\rho$ and $\rho = 1$ at $r = s$) the solution is thus

$$\eta = [C/J_\nu(\lambda^q/q)]\, \rho^{-n/2} J_\nu(\zeta/q) \cos(\sigma t + \epsilon)\, e^{i\mu\theta} \quad \ldots\ldots\ldots\ldots \quad (31)$$

This solution can be discussed in a way similar to that for the case of the estuary. It is topographically unlikely that any nodal circle exists for the low frequency of the ordinary tide. For the gravest mode ($\mu = 0$) usually there will be only one point of maximum amplitude, which is at the origin. Thus for this mode (which is the most likely to occur), the amplitude of tide increases with decreasing distance from the island.

For higher modes ($\mu = 1, 2, 3, \ldots$), there are nodal lines radiating from the island dividing the whole angle into equal segments.

It is obvious that (31) can be generalized to satisfy the following boundary condition at $r = s$

$$\eta_s = C \cos(\sigma t + \epsilon)\, F(\theta)$$

where $F(\theta)$ has a period 2π. All that is necessary is to expand $F(\theta)$ in a Fourier series, and to use as the solution the corresponding series consisting of terms like the one on the right side of (31).

Concluding remarks--Under the assumptions on which the calculations of this paper are based, it can be concluded from the foregoing that there is an augmentation of the amplitude of tide over certain portions of the estuary if it is extremely long, that this augmentation is over the entire length of the estuary if it is of ordinary length, that at the island only the effect of the gravest mode of the tide (which is the most likely to occur) is felt, and that for this mode the amplitude of tide generally increases with decreasing distance from the island.

It is worth noticing that if the periodic motion at the mouth of the estuary or in the open sea for the case of the island is described by a general periodic function of time, the corresponding problems can be solved by superposition and applying the Fourier series, utilizing the results obtained in the foregoing.

Acknowledgment--The writer wishes to acknowledge support of this project by the Office of Naval Research and the Civil Engineering Department of the Colorado Agricultural and Mechanical College, and the assistance rendered by the staff of the Iowa Institute of Hydraulic Research in preparing the manuscript.

References

JAHNKE, E., and F. EMDE, Table of functions, Dover Publications, 1945.
LAMB, H., Hydrodynamics, Dover, 1945.
WHITTAKER, E. I., and G. N. WATSON, Modern analysis, MacMillan, 1945.

Iowa Institute of Hydraulic Research
State University of Iowa
Iowa City, Iowa

(Communicated manuscript received April 2, 1952, and as revised, January 28, 1953; open for formal discussion until November 1, 1953.)