DISSERTATION

MODULI SPACES OF RATIONAL GRAPHICALLY STABLE CURVES

Submitted by

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ABSTRACT

MODULI SPACES OF RATIONAL GRAPHICALLY STABLE CURVES

We use a graph to define a new stability condition for the algebraic and tropical moduli spaces of rational curves. Tropically, we characterize when the moduli space has the structure of a balanced fan by proving a combinatorial bijection between graphically stable tropical curves and chains of flats of a graphic matroid. Algebraically, we characterize when the tropical compactification of the compact moduli space agrees with the theory of geometric tropicalization. Both characterization results occur only when the graph is complete multipartite.

ACKNOWLEDGEMENTS

This work would not be possible without the kind and patient guidance of my advisor, Renzo Cavalieri. He has been an excellent teacher of both the mathematical and non-mathematical aspects of becoming a mathematician. I would also like to acknowledge Dhruv Ranganathan, for suggesting stability induced by a graph; and Vance Blankers, Adam Afandi, Levi Heath, Jae Hwang Lee, Andreas Gross, and my committee for their interest, encouragement, and many enlightening conversations.

DEDICATION

To Mackenzie and Rowan.

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Chapter 1

Introduction

A strong trend in modern algebraic geometry is the study of *moduli (parameter) spaces*. Broadly, a moduli space parameterizes geometric objects. An important and well-studied moduli space is $\mathcal{M}_{0,n}$, the moduli space of smooth rational curves with n marked points. The space $\mathcal{M}_{0,n}$ is not compact, which is undesirable for algebraic geometers because of the many applications that require such a condition. A 'nice' compactification of $\mathcal{M}_{0,n}$ brings along with it a modular interpretation, that is, compact spaces containing $\mathcal{M}_{0,n}$ as a dense open subset have a boundary (equal to the complement of $\mathcal{M}_{0,n}$) that parameterizes n-marked algebraic curves that may not be smooth. The most notable compactification, $\overline{\mathcal{M}}_{0,n}$, is due to Deligne and Mumford. The boundary of their compactification is comprised of nodal curves with finite automorphism group called stable curves. It is interesting to know what alternate compactifications exist and how the boundary combinatorics differs in each case. Another important family of compactifications, $\overline{\mathcal{M}}_{0,w}$, alters the original stability condition by assigning a weight to each marked point. The moduli space of weighted stable curves was established by Hassett in the context of the log minimal model program.

Tropical mathematics offers tools to investigate the structure of the boundary of compact moduli spaces by relating complex algebraic varieties to piecewise linear objects. A strength of tropical geometry is that it allows us to look at a "linear" skeleton of a potentially complicated variety, reducing algebro-geometric questions to those of combinatorics. For instance, the tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$ is a cone complex which parameterizes leaf-labelled metric trees. The combinatorial relation between algebraic moduli spaces and tropical moduli spaces is that the cones of $\mathcal{M}_{0,n}^{\text{trop}}$ are in bijection with the boundary strata of $\overline{\mathcal{M}}_{0,n}$.

In this thesis, we define a new family of stability conditions determined by the combinatorics of a graph Γ, called graphic stability. I investigate how graphic stability is applied in both the algebraic and tropical moduli spaces and how the two moduli spaces relate to each other. We begin with the moduli space of tropical curves.

The structure of $\mathcal{M}_{0,n}^{\text{trop}}$ is obtained by gluing positive orthants of $\mathbb{R}^{n-3}$ corresponding to trivalent trees. Speyer-Sturmfels [27] give an embedding of this cone complex (in the context of phylogenetic trees) into a real vector space as a balanced fan where each top-dimensional cone is assigned weight 1. In [1], Ardila-Klivans study phylogenetic trees and show that the fan structure of $\mathcal{M}_{0,n}^{\text{trop}}$ has a refinement which coincides with the Bergman fan of the cycle matroid of K_{n-1}, the complete graph on $n-1$ vertices. As a generalization of Ardila-Klivans, it is shown by Cavalieri-Hampe-Markwig-Ranganathan in [3] that the fan associated to the moduli space of rational heavy/light weighted stable tropical curves, $\mathcal{M}_{0,w}^{\text{trop}}$, and the Bergman fan of a graphic matroid have the same support.

The first chapter involving original work (Chapter 3) introduces *rational graphically stable tropical curves* (Definition 3.2.1) and writes $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ to denote the moduli space of these curves. We define this moduli space so that if we begin with a graph that is also a reduced weight graph (Definition 2.13 of [3]) we recover the corresponding weighted moduli space. We also add the extra condition of *radial alignment* to $\mathcal{M}_{0,n}^{\text{trop}}$ to define two new families of moduli spaces parameterizing rational radially aligned stable (resp. graphically stable) tropical curves denoted $\mathcal{M}_{0,n}^{\text{trad}}$ (resp. $\mathcal{M}_{0,\Gamma}^{\text{trad}}$). Radial alignment refers to an ordered partition on the vertices of the combinatorial type of a tropical curve which results in the Bergman fan refinement. The first main result of this paper classifies tropical moduli spaces given by graphic stability.

Theorem 3.2.15 The balanced fan underlying the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ is naturally identified with the Bergman fan of the cycle matroid of Γ if and only if Γ is a complete multipartite graph.

Algebraically, we define a compactification of $\mathcal{M}_{0,n}$ using graphic stability called the moduli space of rational graphically stable pointed curves, denoted $\overline{\mathcal{M}}_{0,\Gamma}$. Taking the interior, $\mathcal{M}_{0,\Gamma}$, to be smooth Γ-stable curves, this new moduli space has many characteristics that we would expect from a modular compactification $\mathcal{M}_{0,n}$, namely its boundary is a divisor with simple normal crossings.

I am also able to build an embedding of $\mathcal{M}_{0,\Gamma}$ into a torus using the Plücker embedding of the Grassmannian.

For a smooth subvariety of a torus with a simple normal crossings compactification, the theory of geometric tropicalization relates the combinatorics of the boundary to a balanced fan in a real vector space. Using this theory we show that the tropicalization of $\overline{\mathcal{M}}_{0,\Gamma}$ is identified with a projection of the tropical moduli space $\mathcal{M}_{0,n}^{\text{trop}}$, and therefore the Bergman fan $\mathcal{B}'(\Gamma)$.

Proposition 4.2.8 The geometric tropicalization of $\overline{\mathcal{M}}_{0,\Gamma}$ using the embedding in Lemma 4.2.6 is $\text{trop}(\mathcal{M}_{0,\Gamma}) = \text{pr}_\Gamma(\mathcal{M}_{0,n}^{\text{trop}}) = \mathcal{B}'(\Gamma)$.

However, the tropicalization doesn't necessarily line up with the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\text{trop}}$. The obstruction is a lack of injectivity in the tropicalization map (this is mimicked in the combinatorial case in Chapter 3). Specifically, the divisorial valuation map $\pi_\Gamma : \Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}) \to N_\mathbb{R}$ may not be injective and this fact is highlighted in Equation (4.5). The main results of this work is a classification result stating exactly when the tropical compactification of $\overline{\mathcal{M}}_{0,\Gamma}$ agrees with the theory of geometric tropicalization for rational graphically stable curves.

Theorem 4.2.14 The cone complex $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ is embedded as a balanced fan in a real vector space by π_Γ if and only if Γ is a complete multipartite graph. For such Γ, there is a torus embedding

$$\mathcal{M}_{0,\Gamma} \hookrightarrow T^{\binom{n}{2}-n-N} = T_\Gamma$$

whose tropicalization $\text{trop}(\mathcal{M}_{0,\Gamma})$ has underlying cone complex $\mathcal{M}_{0,\Gamma}^{\text{trop}}$. Furthermore, the tropical compactification of $\mathcal{M}_{0,\Gamma}$ is $\overline{\mathcal{M}}_{0,\Gamma}$, i.e, the closure of $\mathcal{M}_{0,\Gamma}$ in the toric variety $X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ is $\overline{\mathcal{M}}_{0,\Gamma}$.

The motivation for this paper comes from the theory of tropical compactifications, geometric tropicalization, and log geometry. From work of Tevelev [28] and Gibney-Maclagan [10] it has been shown that there is an embedding of $\mathcal{M}_{0,n}$ into the torus of a toric variety $X(\Sigma)$ where the tropicalization of $\mathcal{M}_{0,n}$ is a balanced fan $\Sigma \cong \mathcal{M}_{0,n}^{\text{trop}}$. This embedding is special in the sense that the closure of $\mathcal{M}_{0,n}$ in $X(\Sigma)$ is $\overline{\mathcal{M}}_{0,n}$. Cavalieri et al. [3] show a similar embedding can be constructed for weighted moduli spaces when the weights are heavy/light. In [24], Ranganathan-(Santos-Parker)-Wise describe radial alignments of genus 1 tropical curves and show how this extra data can be used for desingularization. The subdivision given by radial alignments has been studied before in [1] and [7] but we use a rephrasing in order to relate it to log geometry and the results of Ranganathan et al. In the future, we plan on proving a tropicalization statement for $\overline{\mathcal{M}}_{0,\Gamma}$ when Γ is a general graph using log geometry and radial alignment.

The dissertation is organized as follows. Chapter 2 walks through the basic material needed for this manuscript. This chapter is separated into two sections: the first introduces the combinatorial theory while the second introduces algebraic and tropical moduli spaces. Section 2.1 begins by axiomatically defining a matroid and discussing common terminology. Given a matroid, we define the Bergman fan as a polyhedral cone complex that coincides with the order complex of the lattice of flats of the matroid. This cone complex is a balanced fan which lives in a real vector space. Finally, we restrict our attention to the cycle matroid and discuss relevant graph theory. In Section 2.2, we introduce the moduli space of smooth rational pointed curves. We define the Deligne-Mumford compactification by introducing stable pointed curves. Next we introduce tropical moduli spaces independently from their algebraic counterpart as cone complexes parameterizing metric trees.

Chapter 3 deals heavily in combinatorics. In Section 3.1, we describe how the support of $\mathcal{M}_{0,n}^{\text{trop}}$ coincides with the Bergman fan of the complete graph on $n-1$ vertices. To obtain this subdivision solely in terms of tropical curves, we define radial alignment for a tropical curve by imposing a weak ordering on the vertices of the curve by their distance from the root vertex. We also provide a several of original examples to build intuition for these tropical moduli spaces. Section 3.2 contains original work motivated by [3]. First, we define a new tropical moduli space using graphic stability. We also investigate the projections of $\mathcal{M}_{0,n}^{\text{trad}}$ and $\mathcal{B}'(K_{n-1})$ given by forgetting the coordinates of rays corresponding unstable curves and show that the fans coincide with $\mathcal{B}'(\Gamma)$. We also relate these projections to the work of Shaw [25]. Later, we investigate an obstruction that stops $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ from being embedded as a balanced fan. Finally, we prove our main result that states $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ is equal to $\mathcal{B}'(\Gamma)$ as balanced fans when Γ is a complete multipartite graph.

Chapter 4 delves into the notions of tropical compactification and geometric tropicalization. Section 4.1 briefly introduces the concept of geometric tropicalization in the general sense before diving into the specifics for $\mathcal{M}_{0,n}$. Notably, we describe a torus embedding of $\mathcal{M}_{0,n}$ via the Plücker map and describe a tropicalization map using divisorial valuations. Section 4.2 relies heavily on the foundations built in the previous section. Subsection 4.1 contains the proof that $\overline{\mathcal{M}}_{0,\Gamma}$ is not only a modular compactification of $\mathcal{M}_{0,n}$ but indeed a simple normal crossings compactification of the locus of smooth Γ-stable curves , $\mathcal{M}_{0,\Gamma}$. To invoke geometric tropicalization, we also need a torus embedding $\mathcal{M}_{0,\Gamma}$. Subsection 4.2 begins by identifying the interior of the moduli space with the quotient of an open set of the Grassmannian and thus creating the necessary torus embedding. We notice that the divisorial valuation map, which furnishes the combinatorics of the boundary with a fan structure, does not have the desired underlying cone complex, $\mathcal{M}_{0,\Gamma}^{\text{trop}}$. Indeed, we achieve this compatibility only when Γ is complete multipartite. After the main theorem, we end this section with an example where the graph is not complete multipartite. In this case, the toric variety does not have enough boundary strata to contain the modular compactification.

Chapter 2

Preliminaries

2.1 Combinatorial Theory

In this section we begin with a introduction to matroid theory. Next, we define the Bergman fan of a generic matroid using its lattice of flats. Then, we narrow our focus to the main family of matroids we work with in this paper, the cycle matroid. The end of this section includes a key example (Example 2.1.4) and a lemma that gives an alternate description of complete multipartite graphs (Lemma 2.1.6). This example and lemma are referenced many times throughout this thesis.

2.1.1 Matroid Theory

The concept of a matroid was independently developed in 1930's by Whitney [34], van der Waerden [32], and Nakasawa [21]. Whitney's original paper looked at the similarities between linear independence and graph theoretic independence. Similarly, van der Waerden was also interested in generalizing the notion of independence by comparing linear independence and algebraic independence.

Over the next 30 years the following key results arose. In the 30's Birkhoff made the connection that one of the rank axioms ((R3$'$) specifically) is the semimodular condition for lattices [2] and Mac Lane wrote an article on the relations to projective geometry [20]. The 1940's saw expansions by Rado with work on transversality [22] and infinite matroids [23] and Dilworth who wrote more on lattice theory [6]. It wasn't until the late 50's/early 60's when matroid theory took off. Much of this due to the results of Rado, Tutte, Edmonds, and Lehman. Highlighting some results of Tutte are the categorization of binary and regular [29], and graphic [30] matroids.

Since that time matroids have been a target study for linear algebra, graph theory, optimization, block designs, combinatorial algebraic geometry and more. We begin as Whitney did, the axiomatic definition in terms of independence.

A *matroid* is a tuple $M = (E, I)$ where E is a finite set (called the *ground set*) and I is a collection of subsets of E such that **(I1)**–**(I3)** are satisfied.

(I1) $\emptyset \in I$

(I2) If $X \in I$ and $Y \subseteq X$, then $Y \in I$

(I3) If $U, V \in I$ with $|U| = |V| + 1$, then there exists $x \in U \setminus V$ such that $V \cup x \in I$.

The elements of I are called *independent sets* and thusly call **(I1)**, **(I2)**, and **(I3)** the *independence axioms*. If a subset of E is not independent, then we call it *dependent*. More commonly **(I3)** is known as the *exchange property*. Other resources tend to restrict the definition of a matroid to just the latter two properties.

Let $M = (E, I)$ be a matroid. A *base* B of M is a maximal independent subset of E. A *circuit* C of M is a minimal dependent set. Minimal and maximal refer to the size of the circuit or base. Denote 2^E as the power set of E. The *rank function* of a matroid is $\text{rk} : 2^E \to \mathbb{Z}$ defined by

$$\text{rk}(A) = \max(|X| : X \subseteq A, X \in I).$$

In the case where $A \in I$, then $\text{rk}(A) = |A|$.

We are also interested in the notion of a flat or subspace. A subset $F \subseteq E$ is a *flat* (also called a *subspace* or *closed*) of $M(\Gamma)$ if for all $x \in E \setminus F$,

$$\text{rk}(F \cup x) = \text{rk}(F) + 1.$$

In other words, F is a flat if there are no elements that can be added to F without increasing the rank of F. Define the *closure operator* to be a function $\text{cl} : 2^E \to 2^E$ such that $\text{cl}(A)$ is the set of elements that satisfy the following property: If $x \in E$ and $A \subset E$, then $\text{rk}(A \cup x) = \text{rk}(A)$. It turns out that cl$(A)$ is the smallest flat containing A. Ordered by rank we may associate a partially ordered set (poset) to the flats of a matroid. This poset forms a lattice called the *lattice of flats*. Next, we give some equivalent axiomatic definitions of a matroid as presented by Welsh [33], the first comprehensive book on matroid theory.

Base Axiom: A non-empty collection $\mathscr{B}$ of subsets of E is the set of bases of a matroid on E iff it satisfies the following condition:

(B1) For $B_1, B_2 \in \mathscr{B}$ and for $x \in B_1 \setminus B_2$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \cup y) \setminus x \in \mathscr{B}$.

Rank Axioms 1: A function rk : $2^E \to \mathbb{Z}$ is the rank function of a matroid E if and only if for $X \subseteq E$, and $y, z \in E$:

(R1) $\text{rk}(\emptyset) = 0$;

(R2) $\text{rk}(X) \leq \text{rk}(X \cup y) \leq \text{rk}(X) + 1$;

(R3) if $\text{rk}(X \cup y) = \text{rk}(X \cup z) = \text{rk}(X)$ then $\text{rk}(X \cup y \cup z) = \text{rk}(X)$.

Rank Axioms 2: A function rk : $2^E \to \mathbb{Z}$ is the rank function of a matroid E if and only if for any subsets X, Y of E:

(R1′) $0 \leq \text{rk}(X) \leq |X|$;

(R2′) $X \subseteq Y \Rightarrow \text{rk}(X) \leq \text{rk}(Y)$;

(R3′) $\text{rk}(X \cup Y) + \text{rk}(X \cap Y) \leq \text{rk}(X) + \text{rk}(Y)$.

Closure Axioms: A function cl : $2^E \to 2^E$ is the closure operator of a matroid on E if and only if for $X, Y \subset E$ and $x, y \in E$:

(S1) $X \subseteq \text{cl}(X)$

(S2) $Y \subseteq X \Rightarrow \text{cl}(Y) \subseteq \text{cl}(X)$

(S3) $\mathrm{cl}(X) = \mathrm{cl}(\mathrm{cl}(X))$

(S4) if $y \notin \mathrm{cl}(X)$ but $y \in \mathrm{cl}(X \cup x)$, then $x \in \mathrm{cl}(X \cup y)$.

Circuit Axioms: A collection $\mathscr{C}$ of subsets of E is the set of circuits of a matroid on E if and only if the following conditions are satisfied:

(C1) If $X \neq Y \in \mathscr{C}$, then $X \nsubseteq Y$.

(C2) If C_1, C_2 are distinct members of $\mathscr{C}$ and $z \in C_1 \cap C_2$ then there exists $C_3 \in \mathscr{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus z$.

2.1.2 The Bergman Fan

Given any matroid M with ground set E we define the *Bergman fan* which is a polyhedral fan $\mathcal{B}(M) \subseteq \mathbb{R}^{|E|}$. The Bergman fan definition we use is a non-conventional one given to us by Ardila and Klivans [1]. They show that $\mathcal{B}(M)$ is a polyhedral cone complex that coincides with the order complex of the lattice of flats of M. An *order complex* of a poset P is defined to be the simplicial complex whose vertices are the elements of P and whose faces are chains of elements of P. Define $\rho_F = -\Sigma_{e \in F} v_e$, where v_e is a standard basis vector of $\mathbb{R}^{|E|}$. Given a chain of flats (COF) $\mathcal{F}$ in M

$$\emptyset \subsetneq F_1 \subsetneq \cdots \subsetneq F_r \subsetneq F_{r+1} = E,$$

we let $C_{\mathcal{F}}$ be the cone in $\mathbb{R}^{|E|}$ spanned by the rays $\rho_{F_1}, \ldots, \rho_{F_{r+1}}$, with lineality space spanned by ρ_E.

Remark 2.1.1. Any Bergman fan contains the vector $(1, 1, \ldots, 1)$ as a ray. So rather than studying $\mathcal{B}(M)$ we quotient out the lineality space L, spanned by the vector $(1, 1, \ldots, 1)$, to get

$$\mathcal{B}'(M) := \mathcal{B}(M)/L.$$

Thus we identify a COF $\mathcal{F}$ by its nontrivial flats $F_1, \ldots, F_r$ and denote r to be its *length*. Note that a COF of length r corresponds to a cone of dimension r in the Bergman fan. We call this polyhedral structure the *chains-of-flats* subdivision of $\mathcal{B}'(M)$, also known as the *fine* subdivision.

Example (simple non-graphic matroid) small uniform matroid, 3 lines through origin,

2.1.3 The Cycle Matroid

Now we introduce the matroid associated to a finite simple connected graph $\Gamma = (V, E)$ where V is the ordered vertex set and $E = E(\Gamma)$ is the edge set. We define $e_{ij} \in E$ to be an edge between vertices v_i and v_j. A graph is *complete* if each pair of distinct vertices has an edge between them. The complete graph on n vertices is denoted K_n. A *clique* is a subgraph that is complete, denoted K_S, where S is the set of vertices with edges between them. A disjoint union of complete graphs is called a *cluster graph*.

Often called the *cycle matroid*, this matroid is given by $M(\Gamma) = (E(\Gamma), I)$ where I is the collection of all forests of Γ. A *circuit* is a path in which the initial and terminal vertices are the same and no other vertices repeat. The *rank* of a set of edges E' is the number of edges in a spanning forest of $\Gamma_{E'}$, the subgraph induced by E'. Alternatively, the rank of a subgraph $G \subset \Gamma$ may be computed by $n - k$ where n is the number of non-isolated vertices in G and k is the connected components of among non-isolated vertices of G. An *isolated vertex* is a vertex that is not a part of an edge.

We restrict our attention to $\Gamma = K_n$ to examine flats. A *flat* of $M(K_n)$ is a cluster graph, $\coprod_{j=1}^{k} K_{I_j}$. For a subgraph G of K_n, whose connected components are given by vertex sets $V_1, \ldots, V_k$. Then the *closure of* G is the flat $\text{cl}(G) = \coprod_{j=1}^{k} K_{V_j}$. Closure of a graph can be seen as *completing* each connected component. See Figure 2.1 for the lattice of flats for $M(K_4)$.

Example 2.1.2. The Bergman fan of the cycle matroid associated to the complete graph on 4 vertices, $B'(K_4)$, is the refinement of the cone over the Peterson graph where 3 2D cones are

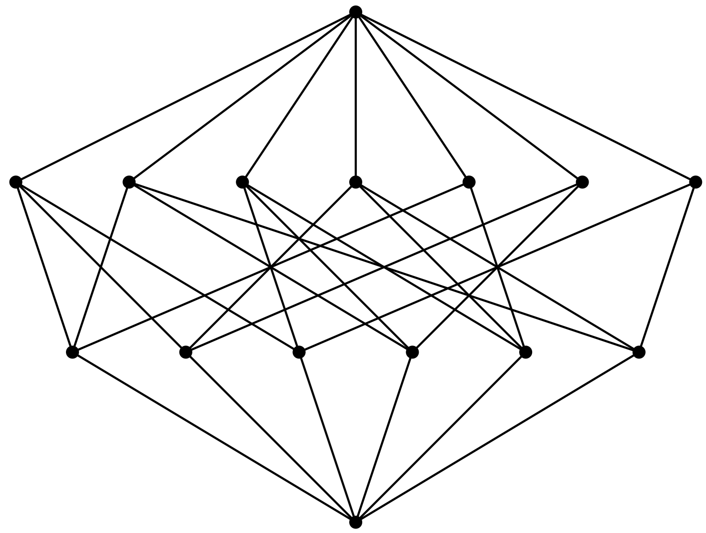


Figure 2.1: Lattice of flats of $M(K_4)$

subdivided; see Figure 2.2b. Label the lattice of flats of K_4 in the following way:

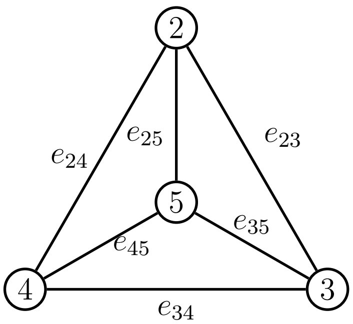
$$\begin{aligned}
&\text{Rank 1}: \; F_1 = \{e_{23}\},\; F_2 = \{e_{24}\},\; F_3 = \{e_{34}\},\; F_4 = \{e_{35}\},\; F_5 = \{e_{45}\},\; F_6 = \{e_{25}\} \\
&\text{Rank 2 connected}: \; F_7 = \{e_{23}, e_{24}, e_{34}\},\; F_8 = \{e_{23}, e_{35}, e_{25}\},\; F_9 = \{e_{34}, e_{35}, e_{45}\}, \\
&\qquad\qquad\qquad\qquad F_{10} = \{e_{24}, e_{45}, e_{25}\} \\
&\text{Rank 2 disconnected}: \; F_{11} = \{e_{23}, e_{45}\},\; F_{12} = \{e_{24}, e_{35}\},\; F_{13} = \{e_{25}, e_{34}\}.
\end{aligned}$$

See Figure 2.3 for a visual representation of the flats F_1, F_7, and F_{11}.

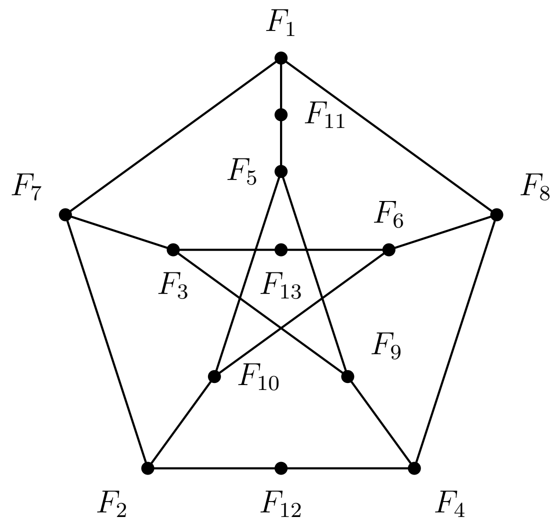
One natural operation on a graph is to delete edges. The cycle matroid respects this operation in the sense that a subgraph induces a submatroid, called the *restriction matroid*. Rather than deleting edges, we may think of restricting the edge set to a subset of edges.

Lemma 2.1.3. (Theorem 1 from [33], Chapter 4.2) Let G be a subgraph of K_n and denote $M(K_n) = (E, I)$. Let $I|G = \{X | X \subseteq E(G) \text{ and } X \in I\}$ be the restriction of forests of K_n to the edge set of G. Then $I|G$ is the set of independent sets of $M(G)$.

Next we write some notation and a technical lemma used for Proposition 3.2.12. The proof is purely graph theoretic so we prove it here.

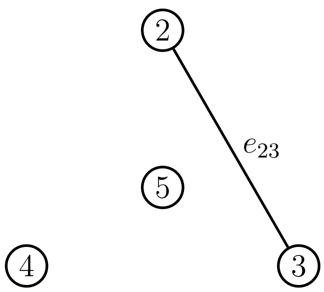


(a) K_4

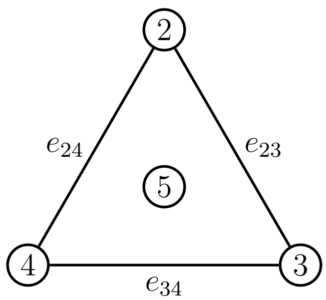


(b) A slice of the Bergman fan $\mathcal{B}'(K_4)$ with rays labeled by their corresponding flats.

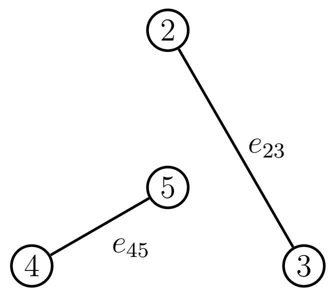
Figure 2.2



(a) Flat F_1



(b) Flat F_7



(c) Flat F_{11}

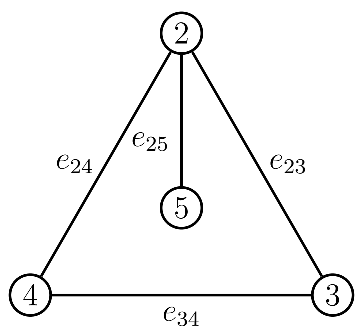
Figure 2.3: Some flats of K_4

The forests of G can be obtained by intersecting a forest of Γ with G. Denote the closure operators for $M(K_n)$ and $M(G)$ as cl_{K_n} and cl_G, respectively. They are related by

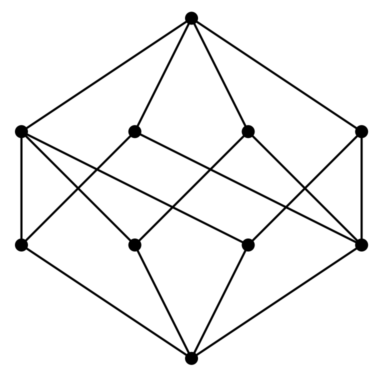
$$\mathrm{cl}_G(A) = \mathrm{cl}_{K_n}(A) \cap G. \tag{2.1}$$

Unlike the closure operator, there is no ambiguity between the rank functions on $M(G)$ and $M(K_n)$ so we will denote both as $\mathrm{rk}(A)$.

Example 2.1.4. Suppose the graph $\widetilde{\Gamma}$ is obtained by removing the edges e_{35} and e_{45} from K_4, as labeled in Figure 2.2a. Analyzing its lattice of flats, we see that we obtain a sublattice of the lattice of flats of $M(K_4)$, see Figure 2.4b. The graph from this example will play a key role in the main results of this paper and we will refer back to it many times.



(a) The graph $\widetilde{\Gamma}$ in Example 2.1.4



(b) Lattice of flats of $M(\widetilde{\Gamma})$

Figure 2.4

Lemma 2.1.5. Let Γ be a simple graph, not necessarily connected, and let G be a subgraph of Γ. Then $\text{rk}(\Gamma) = \text{rk}(G)$ if and only if G and Γ share a common spanning forest.

Proof. The backwards direction follows from the definition of rank so let us assume that G and Γ have the same rank. Let T' be a spanning forest of G. Then there exists T a spanning forest of Γ such that $T \cap G = T'$. By assumption we know that $\text{rk}(T) = \text{rk}(T')$ and therefore they have the same number of edges. Since T' is a subgraph of T, they must be the equal. $\square$

Here we define the complete multipartite graph and discuss some facts about it. A *multipartite graph* (or k-partite graph) is a graph on $n = \sum_{i=1}^{k} n_i$ vertices, partitioned into k sets (called *independent sets*) such that no two vertices from the same set are adjacent. A *complete* multipartite graph is a multipartite graph such that every pair of vertices in different sets are adjacent, such a graph is denoted $K_{n_1,\ldots,n_k}$. Alternatively, we may obtain a complete multipartite graph by removing the disjoint cliques on vertices given by the independent sets. Thus the complement of a complete multipartite graph is a cluster graph.

The following lemma describes the key characterization of a complete multipartite graph that many of our results rely on.

Lemma 2.1.6. Let G be a graph. The following are equivalent:

1. G is a complete multipartite graph.

2. If e_{ij} is an edge of G, then for any vertex v_k, either e_{ik} or e_{jk} is an edge of G.

3. There do not exist 3 vertices whose induced subgraph has exactly 1 edge.

Proof. The complement of a complete multipartite graph is a cluster graph. A graph is a cluster graph if and only if it has no three-vertex induced path. This property is complementary to condition 2 and 3. $\square$

Example 2.1.7. The graph $\widetilde{\Gamma}$ from Example 2.1.4 is not complete multipartite as it contains the edge e_{34} but not e_{35} nor e_{45}.

Notation 2.1.8. For the rest of the paper we write Γ to represent the graph and the cycle matroid of Γ and use $|E(\Gamma)|$ for the number of edges in Γ.

2.2 Moduli Spaces

This document focuses on moduli spaces of rational pointed curves. For a more in-depth treatment of these moduli spaces and moduli space in a broader sense see [18] and [13]. This section introduces both the algebraic and tropical moduli space of rational curves and describes their structures.

2.2.1 Algebraic Moduli Spaces

The moduli space $\mathcal{M}_{0,n}$ parameterizes isomorphism classes of smooth, genus 0 curves with n marked points. A point of $\mathcal{M}_{0,n}$ is an isomorphism class of n ordered, distinct marked points on

$\mathbb{P}^1$ which we denote $(p_1, \ldots, p_n)$. Two points $(\mathbb{P}^1, p_1, \ldots, p_n), (\mathbb{P}^1, q_1, \ldots, q_n) \in \mathcal{M}_{0,n}$ are equal if there is $\Phi \in \text{Aut}(\mathbb{P}^1)$ such that $\Phi(p_i) = (q_i)$, for all i. Using cross ratios, we may assign any n-tuple $(p_1, \ldots, p_n)$ to $(0, 1, \infty, \Phi_{CR}(p_4), \ldots, \Phi_{CR}(p_n))$ where Φ_{CR} is the unique automorphism of $\mathbb{P}^1$ sending p_1, p_2, and p_3 to 0, 1, and ∞. The first two nontrivial cases occur when $n = 3$ and $n = 4$. As varieties, $\mathcal{M}_{0,3}$ is a point, as we send (p_1, p_2, p_3) to $(0, 1, \infty)$ and $\mathcal{M}_{0,4} = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ because the fourth point is free to vary as long as it doesn't coincide with the other 3 markings. In general, this shows that $\mathcal{M}_{0,n}$ is an $n-3$ dimensional space and

$$\mathcal{M}_{0,n} = \overbrace{\mathcal{M}_{0,4} \times \cdots \times \mathcal{M}_{0,4}}^{n-3 \text{ times}} \setminus \{\text{all diagonals}\}.$$

From the $n = 4$ example, we can see that $\mathcal{M}_{0,n}$ is not compact in general. The most notable compactification, $\overline{\mathcal{M}}_{0,n}$, is due to Deligne and Mumford which allows nodal curves with finite automorphism group; such curves are called *stable* curves [5], [17].

Definition 2.2.1. A rational marked curve $(C, p_1, \ldots, p_n)$ is *stable* if

- C is a connected curve of arithmetic genus 0, whose only singularities are nodes
- $(p_1, \ldots, p_n)$ are distinct points of $C \setminus Sing(C)$
- The only automorphisms of C that preserve the marked points is the identity.

These nodal curves arise as the limit of a family of a smooth curve where a number of points come together, e.g., $p_1 \mapsto p_2$. In Figure 2.5, we see an example of a nodal curve in $\overline{\mathcal{M}}_{0,4}$ where the marked points p_3, and p_4 have come together. This curve also arises if p_1 and p_2 come together. The *dual graph* or *combinatorial type* of a stable curve in $\overline{\mathcal{M}}_{0,n}$, is defined by assigning a vertex to each component, an edge to each node, and a half-edge to each marked point, as shown in Figure 2.5. An alternative definition of stability can be posed in terms of dual graphs.

Definition 2.2.2. A rational marked curve $(C, p_1, \ldots, p_n)$ is *stable* if it's dual graph is a tree where each vertex has valence greater than 2.

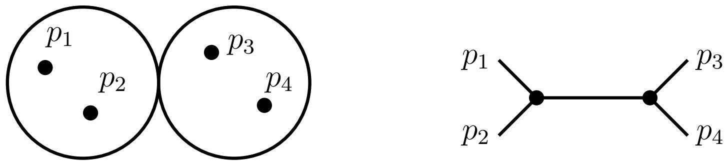


Figure 2.5: A marked algebraic curve and it's dual graph

We define the *boundary of* $\overline{\mathcal{M}}_{0,n}$ to be $\partial\overline{\mathcal{M}}_{0,n} = \overline{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n}$ and consists of all points corresponding to nodal stable curves. We call the closure of a codimension 1 strata a *boundary divisor*. The boundary is stratified by nodal curves of a given topological type with an assignment of marks to each component. In other words, $\partial\overline{\mathcal{M}}_{0,n}$ is stratified by dual graphs of stable nodal pointed curves.

When $n = 4$, $\partial\overline{\mathcal{M}}_{0,4}$ consists of 3 points corresponding to nodal curves where points p_1 and p_2, p_1 and p_3, and p_1 and p_4 have come together, respectively. The boundary of $\overline{\mathcal{M}}_{0,5}$ consists of 2 types of boundary strata, as shown in Figure 2.6.

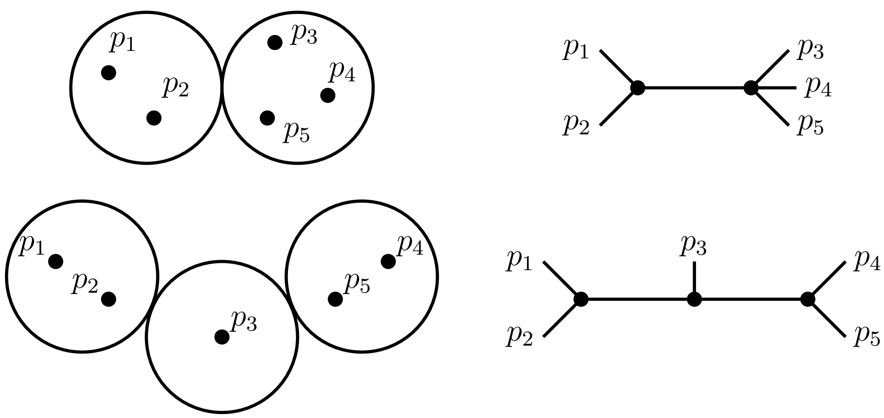


Figure 2.6: Topological types of all boundary strata in $\partial\overline{\mathcal{M}}_{0,5}$ and their dual graphs.

2.2.2 Tropical Moduli Spaces

Consider the space of genus 0, n-marked abstract tropical curves $\mathcal{M}_{0,n}^{\text{trop}}$. Points of $\mathcal{C} \in \mathcal{M}_{0,n}^{\text{trop}}$ are in bijection with metrized trees with *bounded edges* having finite length and n unbounded labeled edges called *ends*. By forgetting the lengths of the bounded edges of $\mathcal{C}$ we get a tree with labeled ends called the *combinatorial type* of $\mathcal{C}$. The space $\mathcal{M}_{0,n}^{\text{trop}}$ naturally has the structure of a cone complex obtained by gluing several copies of $\mathbb{R}_{\geq 0}^{n-3}$ via appropriate face morphisms, one for each trivalent combinatorial type.

The space $\mathcal{M}_{0,n}^{\text{trop}}$ may be embedded into a real vector space as a balanced, weighted, pure-dimensional polyhedral fan as in [9]. A *weighted fan* (X, ω) is a fan X in $\mathbb{R}^n$ where each cone σ has a positive integer weight associated to it, denoted $\omega(\sigma)$. A weighted fan is *balanced* if for all cones τ of codimension one, the weighted sum of primitive normal vectors of the top-dimensional cones $\sigma_i \supset \tau$ is 0, i.e.,

$$\sum_{\sigma_i \supset \tau} \omega(\sigma_i) \cdot u_{\sigma_i/\tau} = 0 \in V/V_\tau$$

where $u_{\sigma_i/\tau}$ is the primitive normal vector, V is the ambient real vector space, and V_τ is the smallest vector space containing the cone τ. See [9, Construction 2.3] for a construction of the primitive normal vector.

For a curve $\mathcal{C}$, define dist(i, j) as the sum of lengths of all bounded edges between the ends marked by i and j. Then the vector

$$d(\mathcal{C}) = (\text{dist}(i, j))_{i<j} \in \mathbb{R}^{\binom{n}{2}}/\Phi(\mathbb{R}^n) = Q_n$$

identifies $\mathcal{C}$ uniquely, where $\Phi : \mathbb{R}^n \to \mathbb{R}^{\binom{n}{2}}$ by $x \mapsto (x_i + x_j)_{i<j}$.

The combinatorial type of an abstract n-marked tropical curve $\mathcal{C}$ with one bounded edge *splits* the set of ends $[n]$ into $I \sqcup I^c$ where we adopt the convention that $1 \in I^c$. We denote the ray corresponding to $\mathcal{C}$ by $d(\mathcal{C}) = \rho_I = \rho_{I^c}$. In [16], Kerber and Markwig prove the following relation

$$\sum_{S \in V_1} \rho_S = 0 \in Q_n \tag{2.2}$$

where $V_1 = \{I \mid 1 \notin I, |I| = 2\}$. They also show that for $I \subset [n] \setminus \{1\}$

$$\sum_{S \in \binom{I}{2}} \rho_S = \rho_I + \Phi(x) \in \mathbb{R}^{\binom{n}{2}} \tag{2.3}$$

where $\binom{I}{2}$ is the set of all size-2 subsets of a set I and $x \in \mathbb{R}^n$.

Remark 2.2.3. Equation (2.2) means that any set of $\binom{n-1}{2} - 1$ of combinatorial types of curves with one bounded edge and a trivalent vertex not containing the end 1 corresponds to a basis of Q_n. Equation (2.3) gives us the unique way to write any ray of Q_n as a linear combination of our basis.

A combinatorial type of a tropical curve $\mathcal{C}$ with d bounded edges has d splits, $I_1, \ldots, I_d$, where a *split* I_j is defined by the combinatorial type one obtains by contracting all but the jth bounded edge of $\mathcal{C}$. The cone corresponding to the combinatorial type of $\mathcal{C}$ is the span of rays $\rho_{I_1}, \ldots, \rho_{I_d}$.

Chapter 3

Tropical Moduli Spaces of Graphically Stable Curves

3.1 Radially aligned tropical curves

This section defines the radially aligned tropical curve and discusses how the moduli space parameterizing them relates to the Bergman fan of K_{n-1}. It is shown in [1, Section 4] and [8, Example 7.2] that the supports of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{B}'(K_{n-1})$ coincide. Ardila-Klivans and separately Feichtner in [7, Remark 3.4] describe that $\mathcal{B}'(K_{n-1})$ is a refinement of $\mathcal{M}_{0,n}^{\text{trop}}$. We can define the Bergman fan refinement solely in terms of tropical curves.

Define the *root vertex* of a tropical curve $\mathcal{C}$ to be the vertex containing the end with marking 1, and we denote it $\mathcal{V}_0$. Given a labeling of the non-root vertices of $\mathcal{C}$, $\mathcal{V}_1, \ldots, \mathcal{V}_d$, we define ℓ_i to be the distance from the root vertex to $\mathcal{V}_i$. We set $\ell_0 = 0$.

Definition 3.1.1. A *radially aligned tropical curve* $\mathcal{C}$ is a tropical curve with the additional data of a weak ordering on the vertices given by $\{\ell_i\}_{i=0}^{d}$. Define $\mathcal{M}_{0,n}^{\text{trad}}$ as the parameter space of genus 0, n-marked radially aligned abstract tropical curves. Similar to before, we get the *radially aligned combinatorial type* by forgetting the lengths but keeping the weak ordering on the vertices.

Remark 3.1.2. A weak ordering of a set can be viewed as an ordered partition. Meaning a partition of the vertices into disjoint subsets together with a total ordering on the subsets. Thus the number of cones of $\mathcal{M}_{0,n}^{\text{trad}}$ can be counted using ordered Bell numbers or Fubini numbers, this fact highlighted in Example 3.1.4.

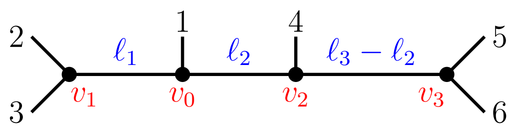
Although the supports of $\mathcal{M}_{0,n}^{\text{trop}}$ and $\mathcal{M}_{0,n}^{\text{trad}}$ are the same, $\mathcal{M}_{0,n}^{\text{trad}}$ is a refinement of $\mathcal{M}_{0,n}^{\text{trop}}$, called the *radially aligned subdivision*. The next two examples illustrate particular 3-dimensional cones of $\mathcal{M}_{0,n}^{\text{trop}}$ that become subdivided in the radially aligned subdivision.

Example 3.1.3. Consider the combinatorial type $\mathcal{C} \in \mathcal{M}_{0,6}^{\text{trop}}$ with splits $I_1 = \{2,3\}$, $I_2 = \{4,5,6\}$, $I_3 = \{5,6\}$; see Figure 3.1a. In $\mathcal{M}_{0,6}^{\text{trop}}$, the combinatorial type of such a curve corresponds to a single 3-dimensional cone with faces consisting of three 2-dimensional cones, and three rays. The 2-dimensional faces correspond to the combinatorial types obtained by shrinking the length of a bounded edge to 0. The rays correspond to contracting 2 bounded edges. In $\mathcal{M}_{0,6}^{\text{trad}}$, the radially aligned subdivision yields three distinct isomorphism classes, i.e., three 3-dimensional cones. By contracting the various bounded edges, there are seven 2-dimensional cones and five rays; see Figure 3.1b. The weak orderings are compiled in the 15 strings of inequalities listed below:

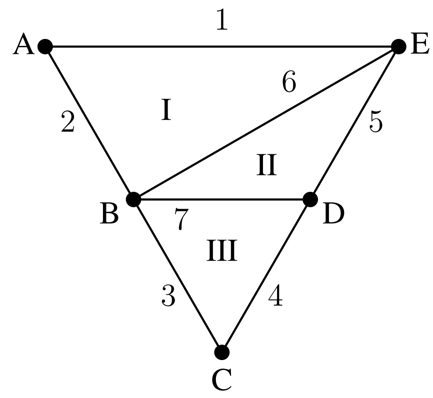
	$0 = \ell_1 < \ell_2 < \ell_3$	
$0 = \ell_1 < \ell_2 = \ell_3$	$0 < \ell_1 < \ell_2 = \ell_3$	
$0 < \ell_1 = \ell_2 = \ell_3$	$0 < \ell_2 = \ell_3 < \ell_1$	$0 < \ell_1 < \ell_2 < \ell_3$
$0 = \ell_2 = \ell_3 < \ell_1$	$0 = \ell_2 < \ell_3 < \ell_1$	$0 < \ell_2 < \ell_1 < \ell_3$
$0 = \ell_2 < \ell_1 = \ell_3$	$0 = \ell_2 < \ell_1 < \ell_3$	$0 < \ell_2 < \ell_3 < \ell_1$
$0 = \ell_2 = \ell_1 < \ell_3$	$0 < \ell_2 = \ell_1 < \ell_3$	
	$0 < \ell_2 < \ell_1 = \ell_3$	

The number of strict inequalities is the same as the dimension of the corresponding cone, i.e., the columns, from left to right, correspond to rays, 2D cones, and 3D cones.

Example 3.1.4. Now consider the combinatorial type of a curve $\mathcal{C} \in \mathcal{M}_{0,7}^{\text{trop}}$ with splits $I_1 = \{2, 3\},\ I_2 = \{4, 5\},\ I_3 = \{6, 7\}$; see Figure 3.2a. Similar to Example 3.1.3, in $\mathcal{M}_{0,7}^{\text{trop}}$, this combinatorial type corresponds to a single 3-dimensional cone with faces consisting of three 2-dimensional cones, and three rays. The radially aligned subdivision yields six 3-dimensional cones, twelve 2-dimensional cones, and seven rays; see Figure 3.2b. If we also consider the 0-dimensional cone which is the intersection of all of these cones there are 26 in total. We may also obtain 26 by doubling the ordered Bell number on a set of three elements. The factor of 2 is due to having a distinguished least element of $\ell_0 = 0$.

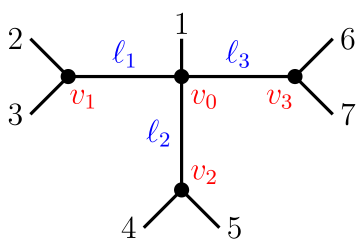


(a) Tropical curve of $\mathcal{M}_{0,6}^{\text{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5,6\},\ I_3 = \{5,6\}$

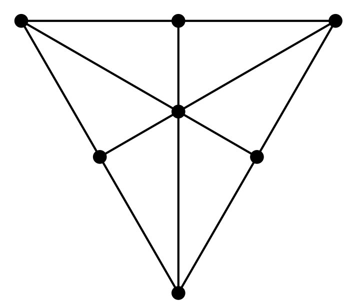


(b) A slice of the cone of $\mathcal{M}_{0,6}^{\text{trad}}$. Rays are labeled by letters A-E, 2D cones are labeled by numbers 1-7, and 3D cones are labeled by numerals I, II, and III.

Figure 3.1



(a) A tropical curve of $\mathcal{M}_{0,7}^{\text{trop}}$ with splits $I_1 = \{2,3\},\ I_2 = \{4,5\},\ I_3 = \{6,7\}$



(b) A slice of the cone of $\mathcal{M}_{0,7}^{\text{trad}}$

Figure 3.2

Lemma 3.1.5. The polyhedral cone complexes $\mathcal{M}_{0,n}^{\text{trad}}$ and $\mathcal{B}'(K_{n-1})$ are equal. In particular, there is a bijection Ψ between chains of flats of K_{n-1} and radially aligned combinatorial types of $\mathcal{M}_{0,n}^{\text{trad}}$.

Rather than presenting a tedious combinatorial proof of this lemma, we illustrate in an example the strategy that is used to construct the necessary explicit bijection.

Example 3.1.6. The radially aligned tropical curve in $\mathcal{M}_{0,8}^{\text{trad}}$, as pictured in Figure 3.3, corresponds to the following chain of flats of length 3, $\mathcal{F}$

$$K_{\{4,5\}} \sqcup K_{\{6,7\}} \subset K_{\{4,5,6,7\}} \sqcup K_{\{2,3\}} \subset K_{\{4,5,6,7\}} \sqcup K_{\{2,3,8\}}.$$

Starting with tropical curve, we recover the chain of flats by shrinking a circle centered at the root vertex and recording a new flat each time the circle passes over a vertex of the tropical curve.

Starting with a chain of flats, we recover a tropical curve by examining the chain of flats in the descending direction. Each time a label disappears or a flat splits up, we add structure to the tropical curve. For instance, having two components in the 3rd flat means that there are two bounded edges coming out of the root vertex.

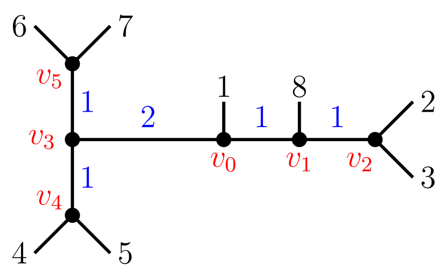


Figure 3.3: A tropical curve in $\mathcal{M}_{0,8}^{\text{trad}}$

Remark 3.1.7. Using the bijection from Lemma 3.1.5 we get an isomorphism of vector spaces, also denoted by Ψ,

$$\begin{tikzcd} \mathcal{M}_{0,n}^{\text{trad}} \arrow[r, "="', "\Psi"'] \arrow[d, hook] \arrow[dr, phantom, "\square"] & \mathcal{B}'(K_{n-1}) \arrow[d, hook] \\ Q_n \arrow[r, "\cong", "\Psi"'] & \mathbb{R}^{|E(K_{n-1})|}/L \end{tikzcd}$$

which respects the cone complex structures of $\mathcal{M}_{0,n}^{\text{trad}}$ and $\mathcal{B}'(K_{n-1})$. Therefore we may write $\mathcal{M}_{0,n}^{\text{trad}} = \mathcal{B}'(K_{n-1})$ as polyhedral fans.

Example 3.1.8. The Bergman fan $B'(K_4)$ is a refinement of the cone complex of $\mathcal{M}_{0,5}^{\text{trop}}$. Label the flats of $B'(K_4)$ as in Example 2.1.2. Consider the top-dimensional cone σ in $\mathcal{M}_{0,5}^{\text{trop}}$ corresponding to a combinatorial type that has a root vertex $\mathcal{V}_0$ with two bounded edges and adjacent vertices $\mathcal{V}_1$ and $\mathcal{V}_2$ with ends marked by $I_1 = \{2,3\}$ and $I_2 = \{4,5\}$. An abstract tropical curve $\mathcal{C}$ with this combinatorial type has edge lengths $\ell_1, \ell_2 \in \mathbb{R}^+$; see Figure 3.4.

In $B'(K_4)$, and therefore $\mathcal{M}_{0,5}^{\text{trad}}$, we see that this cone is subdivided into $\sigma_1 = \text{cone}(\rho_{F_1}, \rho_{F_{11}})$ and $\sigma_2 = \text{cone}(\rho_{F_{11}}, \rho_{F_5})$ with their intersection being a ray $\rho = \rho_{F_{11}}$. The ray ρ corresponds to $\mathcal{C}$

where $\ell_1 = \ell_2$ and σ_i is the cone corresponding to the abstract tropical curve $\mathcal{C}$ where $\ell_i > \ell_j$.

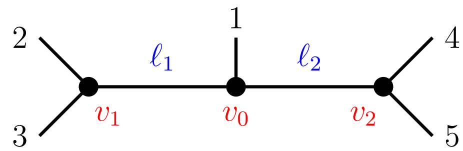


Figure 3.4: A tropical curve of $\mathcal{M}_{0,5}^{\text{trop}}$ with splits $I_1 = \{2,3\}$ and $I_2 = \{4,5\}$.

3.2 Moduli spaces of rational graphically stable tropical curves as Bergman fans

The main point of section 3.2.1 is to define Γ-stability and set up Theorem 3.2.15. We note that Lemma 3.2.7 and Lemma 3.2.8 are corollaries of Shaw's Proposition 2.22 in [25] but they are useful in building up the context in this paper.

3.2.1 The image of $\mathcal{M}_{0,n}^{\text{trad}}$ equals $\mathcal{B}'(\Gamma)$

As a generalization of [3] we define the space of graphically stable tropical curves and investigate its ability to be embedded as a balanced fan. In particular, we explore the relationship between $\mathcal{M}_{0,\Gamma}^{\text{trop}}$, $\mathcal{M}_{0,\Gamma}^{\text{trad}}$, and $\mathcal{B}'(\Gamma)$. Let Γ be a simple connected graph whose nodes are in bijection with ends $2, \ldots, n$ of $\mathcal{C}$.

Definition 3.2.1. A stable tropical curve $\mathcal{C}$ with n ends is Γ-*stable*, if at each non-root vertex with exactly 1 bounded edge there exists an edge $e_{ij} \in E(\Gamma)$ where i and j are ends adjacent to the vertex.

Define $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ to be the parameter space of all rational n-marked Γ-stable abstract tropical curves. Similarly, we define $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ to be the parameter space of rational n-marked Γ-stable radially aligned abstract tropical curves.

These spaces are well-defined as cone complexes but not necessarily as balanced fans.

Definition 3.2.2. We define the *contraction morphism*

$$c_\Gamma : \mathcal{M}_{0,n}^{\mathrm{trop}} \longrightarrow \mathcal{M}_{0,\Gamma}^{\mathrm{trop}} \tag{3.1}$$

which successively contracts bounded edges adjacent to Γ-unstable vertices.

Example 3.2.3. Let Γ be a path of length 2. Then $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ is exactly the tropical moduli space of weighted stable tropical curves $\mathcal{M}_{0,\mathcal{A}}^{\mathrm{trop}}$ with weight data $\mathcal{A} = (1, 1, 1/2, 1/2)$. The fan associated to this moduli space sits in $\mathbb{R}$ and contains a node at the origin and two rays pointing in opposite directions.

The previous example shows that the set of moduli spaces of weighted stable tropical curves and the set of moduli spaces of Γ-stable tropical curves have an intersection. The next two examples show that neither is contained in the other.

Example 3.2.4. Let Γ be the complete bipartite graph obtained from K_4 by removing the edges e_{25} and e_{34}. In this case, $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ is not isomorphic to a tropical moduli space with weighted points. If this graph could be described using a weight vector, without loss of generality, we can suppose that $w_2 < w_5$ and $w_3 < w_4$. But then $w_2 + w_3 \leq 1$ so e_{23} should also be removed from the graph.

Example 3.2.5. Consider the weight data $\mathcal{A} = (1, 1, 1/2, 1/2, 1/2)$. A likely choice of a corresponding graph would be the graph Γ obtained by removing the set of edges $\{e_{34}, e_{35}, e_{45}\}$ from K_4. However, that graph corresponds to the weight data $\mathcal{A}' = (1, 1, \varepsilon, \varepsilon, \varepsilon)$. We can see the difference by looking at the combinatorial type of a curve with split $I = \{3, 4, 5\}$. It is $\mathcal{A}$-stable but not Γ-stable nor $\mathcal{A}'$-stable.

To relate the theory of Bergman fans to these new graphically stable moduli spaces we need to understand what Γ-stability means in terms of chain of flats of K_{n-1}.

Definition 3.2.6. A flat F of K_{n-1} is Γ-*stable* if the combinatorial type $\mathcal{C}_F$ is Γ-stable.

Recall that a flat of Γ can be thought of as a flat of K_{n-1} restricted to the edge set of Γ, i.e., a flat of Γ is $F \cap \Gamma$ where F is a flat of K_{n-1}. Consider a ray ρ with splits $I_1, \ldots, I_d$ and its corresponding cluster graph $F_\rho = \coprod_{j=1}^{d} K_{I_j}$. Then ρ is Γ-unstable if and only if there exists I_j such that $K_{I_j} \cap \Gamma$ has no edges. In this matroidal notion, Γ-stability can be thought of as deletion of cliques of K_{n-1}.

Now consider the map

$$\mathrm{pr}_\Gamma : \mathbb{R}^{|E(K_{n-1})|}/L \longrightarrow (\mathbb{R}^{|E(K_{n-1})|}/L)/S \tag{3.2}$$

where L is the lineality space spanned by the vector $(1, 1, \ldots, 1)$ and $S = \mathrm{span}\{v_e \mid e \notin \Gamma\}$ is the span of basis vectors corresponding to edges not in Γ. Note that pr_Γ is the natural projection map that forgets the coordinates corresponding to edges that are not in Γ.

Simultaneously, define

$$\widetilde{\mathrm{pr}}_\Gamma : Q_n \longrightarrow Q_n/U \tag{3.3}$$

where U is the linear span of Γ-unstable rays of $\mathcal{M}_{0,n}^{\mathrm{trop}}$. As described in Remark 2.2.3, we associate to a basis of Q_n a set of combinatorial types of curves with splits I of size 2. A split of size 2 corresponds to an edge of Γ. Thus U is generated by combinatorial types of curves corresponding to the edges removed from Γ.

Lemma 3.2.7. The fans $\mathrm{pr}_\Gamma(\mathcal{B}'(K_{n-1}))$, $\widetilde{\mathrm{pr}}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trad}})$, and $\widetilde{\mathrm{pr}}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trop}})$ have the same support. Furthermore $\mathrm{pr}_\Gamma(\mathcal{B}'(K_{n-1})) = \widetilde{\mathrm{pr}}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trad}})$ as fans.

Proof. It is clear from the discussion leading up to this lemma that the diagram in Figure 3.5 is commutative, and since Ψ respects the fan structures, we obtain an isomorphism Ψ' that also respects the fan structures. $\square$

Note that the support of $\mathcal{B}'(\Gamma)$ is a subset of $\mathbb{R}^{|E(\Gamma)|}/L$, where L is the lineality space spanned by the all ones vector. It is a straightforward computation to see that the dimension of $(\mathbb{R}^{|E(K_{n-1})|}/L)/S$ is the same as the dimension of $\mathbb{R}^{|E(\Gamma)|}/L$. There is a natural isomorphism between these two

$$
\begin{tikzcd}
Q_n \arrow[r, "\cong", "\Psi"'] \arrow[d, "\widetilde{\mathrm{pr}}_\Gamma"] \arrow[dr, phantom, "\square"] & \mathbb{R}^{|E(K_{n-1})|}/L \arrow[d, "\mathrm{pr}_\Gamma"] \\
Q_n/U \arrow[r, "\cong", "\Psi'"'] & \mathbb{R}^{|E(K_{n-1})|}/L/S
\end{tikzcd}
$$

Figure 3.5: Diagram for Proposition 3.2.7

spaces given by underlying matroidal structure. In other words, the standard basis vectors of each space are given by edges in their respective graphs, and the vectors in S correspond to precisely the edges not in Γ.

Lemma 3.2.8. Let Γ be a connected graph on $n-1$ vertices. Then $\mathrm{pr}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{rad}}) = \mathcal{B}'(\Gamma)$.

Proof. Corollary [25, Proposition 2.22]. $\square$

3.2.2 $\mathcal{B}'(\Gamma)$ equals $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ for Γ complete multipartite

In general, the cone complex structures of $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$ do not coincide. Both $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ and $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ are well-defined as cone complexes but may not be able to be embedded into Q_n/U as balanced fans. Geometrically, the obstruction is that these fans may contain cones which are adjacent to only 1 maximal cell, and thus cannot be balanced.

To investigate the relationship between $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$, consider the locus in $\mathcal{M}_{0,n}^{\mathrm{trad}}$ of Γ-stable curves given by the section ι, the natural inclusion map. We define $\Psi_\Gamma = \mathrm{pr}_\Gamma \circ (\Psi \circ \iota)$ as the map of cone complexes in the following diagram.

$$
\begin{tikzcd}
\mathcal{M}_{0,n}^{\mathrm{trad}} \arrow[r, "=", "\Psi"'] \arrow[d, bend right, "c_\Gamma"'] & \mathcal{B}'(K_{n-1}) \arrow[d, "\mathrm{pr}_\Gamma"] \\
\mathcal{M}_{0,\Gamma}^{\mathrm{trad}} \arrow[u, bend right, "\iota"'] \arrow[r, dashed, "\Psi_\Gamma"'] & \mathcal{B}'(\Gamma)
\end{tikzcd}
$$

The map Ψ_Γ induces a map (denoted by the same name) between Γ-stable radially aligned combinatorial types and chains of flats of Γ. Note that $\Psi \circ \iota$ induces a bijection between the set

Γ-stable radially aligned combinatorial types and Γ-stable flats of K_{n-1}. Hence statements about Ψ_Γ are equivalent to statements about pr_Γ restricted to Γ-stable flats.

By showing Ψ_Γ is a bijective map between the set of Γ-stable radially aligned combinatorial types and the set of chains of flats of Γ, we obtain an induced bijection of the fans $\mathcal{M}_{0,\Gamma}^{\mathrm{trad}}$ and $\mathcal{B}'(\Gamma)$. The next lemma shows that surjectivity of this map follows from the fact that flats of Γ are flats of K_{n-1} restricted to the edge set of Γ.

Lemma 3.2.9. The map Ψ_Γ is surjective.

Proof. Consider the chain of flats $\mathcal{F}$ of Γ given by $F_1 \subset \cdots \subset F_r$ where F_i has k_i connected components. Write the vertex set of each connected component of F_i as I_j^i. Construct the chain of flats $\mathcal{G}$ of K_{n-1} as

$$\mathcal{G} : \coprod_{j=1}^{k_1} K_{I_j^1} \subset \cdots \subset \coprod_{j=1}^{k_r} K_{I_j^r}.$$

Then we have $\mathrm{pr}_\Gamma(\mathcal{G}) = \mathcal{F}$, and thus Ψ_Γ is surjective. □

An interesting aspect of the map Ψ_Γ is that it is not always injective. The obstruction is highlighted in the following example.

Example 3.2.10. Let $\widetilde{\Gamma}$ be the subgraph of K_4 with edges e_{35} and e_{45} removed as in Example 2.1.4; see Figure 2.4a. In $\mathcal{M}_{0,n}^{\mathrm{trop}}$, there are now 8 combinatorial types with 1 bounded edge and 9 combinatorial types with 2 bounded edges that are Γ-stable. This means that $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trop}}$, as a cone complex, has 8 rays and 9 2-dimensional cones and $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trad}}$ has 9 rays and 10 2-dimensional cones, as described in Example 3.1.8.

It is important to note that as cone complexes $\mathcal{B}'(\widetilde{\Gamma})$ is not equal to $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trop}}$ nor $\mathcal{M}_{0,\widetilde{\Gamma}}^{\mathrm{trad}}$; see Figures 3.6a and 3.6b. The obstruction lies in the ray $\rho = \rho_{\{3,4,5\}}$ and the cone $\sigma = \mathrm{cone}(\rho_{\{3,4,5\}}, \rho_{\{3,4\}})$. Let $\mathcal{C}_\rho$ and $\mathcal{C}_\sigma$ be their corresponding combinatorial types. Geometrically, ρ is adjacent to only 1 graphically stable maximal cell, meaning there is no way to embed ρ and σ into a vector space as a balanced fan.

Write the lattice of flats of $\widetilde{\Gamma}$ with the same labels as in Example 3.1.8:

$$\text{Rank 1}: \ F_1 = \{e_{23}\},\ F_2 = \{e_{24}\},\ F_3 = \{e_{34}\},\ F_6 = \{e_{25}\}$$
$$\text{Rank 2 connected}: \ F_7 = \{e_{23}, e_{24}, e_{34}\},\ F_8 = \{e_{23}, e_{25}\},\ F_{10} = \{e_{24}, e_{25}\}$$
$$\text{Rank 2 disconnected}: \ F_{13} = \{e_{25}, e_{34}\}$$

The flat corresponding to $\rho_{\{3,4,5\}}$ in K_4, i.e., $(\Psi \circ \iota)(\mathcal{C}_\rho)$, is $K_{\{3,4,5\}} = F_9$. When restricting the edge set,

$$K_{\{3,4,5\}} \cap \widetilde{\Gamma} = K_{\{3,4\}} = F_3.$$

Similarly, $(\Psi \circ \iota)(\mathcal{C}_\sigma)$ is the chain of flats $K_{\{3,4\}} \subset K_{\{3,4,5\}}$, and this chain of flats reduces to the single flat $K_{\{3,4\}}$ when restricting the edge set. That is to say, there are 3 combinatorial types of $\mathcal{M}^{\text{trad}}_{0,\widetilde{\Gamma}}$ whose cones all coincide in $\mathcal{B}'(\widetilde{\Gamma})$, namely

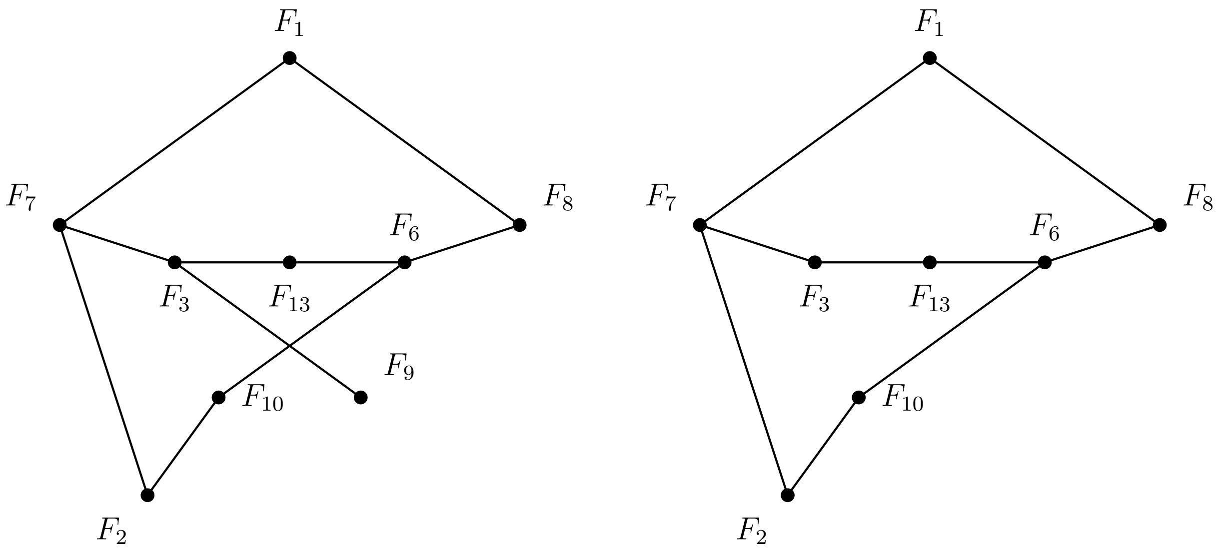
$$\Psi_{\widetilde{\Gamma}}(\mathcal{C}_\rho) = \Psi_{\widetilde{\Gamma}}(\mathcal{C}_\sigma) = \Psi_{\widetilde{\Gamma}}(\mathcal{C}_{\rho_{\{3,4\}}}) = \rho_{F_3}.$$

The map $\Psi_{\widetilde{\Gamma}}$ takes the cone complex depicted in Figure 3.6a to the one in Figure 3.6b. Here we can clearly see the obstruction in $\mathcal{M}^{\text{trad}}_{0,\widetilde{\Gamma}}$ and how it is collapsed in $\mathcal{B}'(\widetilde{\Gamma})$.

Remark 3.2.11. In Example 3.2.10, $\text{pr}_{\widetilde{\Gamma}}$ (restricting the edge set) is not injective on cones corresponding to the flats which dropped in rank. In particular, the obstruction was a K_3 subgraph which had 2 of its 3 edges deleted. So in order for $\text{pr}_{\widetilde{\Gamma}}$ to be injective we can only allow a graph if it has the property that if one deletes 2 edges of a K_3 subgraph, then the third edge must also be deleted. This is equivalent to the third characterization in Lemma 2.1.6.

The following series of results build to the central result of the paper, Theorem 3.27.

Proposition 3.2.12. The map pr_Γ is injective on cones corresponding to Γ-stable flats if and only if for any Γ-stable flat F, $\text{rk}(F) = \text{rk}(\text{pr}_\Gamma(F))$.



(a) A slice of the cone complex of $\mathcal{M}^{\mathrm{trad}}_{0,\widetilde{\Gamma}}$ with rays labeled by their corresponding flats.

(b) A slice of the cone complex of $\mathcal{B}'(\widetilde{\Gamma})$ with rays labeled by their corresponding flats.

Figure 3.6

Proof. First assume that pr_Γ is injective. Let F be a Γ-stable flat of K_{n-1} and let T be a spanning forest of $\mathrm{pr}_\Gamma(F) = F \cap \Gamma$. By way of contradiction, suppose that $\mathrm{rk}(F \cap \Gamma) < \mathrm{rk}(F)$. Consider $\mathrm{cl}_{K_{n-1}}(T)$ as a flat of K_{n-1}. Then we have

$$\mathrm{pr}_\Gamma(F) = F \cap \Gamma = \mathrm{cl}_\Gamma(T) := \mathrm{cl}_{K_{n-1}}(T) \cap \Gamma = \mathrm{pr}_\Gamma(\mathrm{cl}_{K_{n-1}}(T)).$$

But since $\mathrm{rk}(\mathrm{cl}_{K_{n-1}}(T)) = \mathrm{rk}(T) < \mathrm{rk}(F)$, $\mathrm{cl}_{K_{n-1}}(T) \neq F$. This contradicts the injectivity of pr_Γ on Γ-stable flats.

Now we prove the backwards direction. Suppose that for any Γ-stable flat F, $\mathrm{rk}(F) = \mathrm{rk}(\mathrm{pr}_\Gamma(F))$. Let F and G be Γ-stable flats of K_{n-1} with $\mathrm{pr}_\Gamma(F) = \mathrm{pr}_\Gamma(G)$. By our hypothesis, we deduce that $\mathrm{rk}(F) = \mathrm{rk}(G)$. By Lemma 2.1.5, F and G share a spanning forest; call it T. Then by definition, $\mathrm{cl}_{K_{n-1}}(T) = F$ and $\mathrm{cl}_{K_{n-1}}(T) = G$. This proves pr_Γ is injective, completing the proof. □

Lemma 3.2.13. Suppose C is a clique of K_{n-1} and that Γ is a complete multipartite graph labeled by the same $n-1$ vertices. Then $\mathrm{rk}(C) = \mathrm{rk}(\mathrm{pr}_\Gamma(C))$ or $\mathrm{rk}(\mathrm{pr}_\Gamma(C)) = 0$.

Proof. Suppose that $\text{rk}(\text{pr}_\Gamma(C)) \neq 0$, i.e., $\text{pr}_\Gamma(C)$ is not the empty graph. Fix an edge e_{ij} between vertices v_i and v_j. By Lemma 2.1.6, for any other vertex v_k, either e_{ik} or e_{jk} exists. So there is a path between any two vertices of $\text{pr}_\Gamma(C)$ going through the edge e_{ij}. This means that $\text{pr}_\Gamma(C)$ is connected and any spanning tree contains all $n-1$ vertices, proving the lemma. □

Lemma 3.2.14. The map pr_Γ is injective on cones corresponding to Γ-stable flats if and only if Γ is a complete multipartite graph.

Proof. First we prove the backwards direction. Let F be a Γ-stable flat of K_{n-1}. Note that F is a disjoint union of cliques, C_i. By assumption, the image of each clique, under pr_Γ, is not empty. Using Lemma 3.2.13, we have

$$\text{rk}(F) = \sum_{i=1}^{k} \text{rk}(C_i) = \sum_{i=1}^{k} \text{rk}(C_i \cap \Gamma) = \text{rk}(F \cap \Gamma) = \text{rk}(\text{pr}_\Gamma(F)).$$

By Proposition 3.2.12, pr_Γ is injective on cones corresponding to Γ-stable flats.
Now suppose that pr_Γ is injective on cones corresponding to Γ-stable flats. It is enough to prove the equivalent statement from Lemma 2.1.6. Let v_i and v_j be vertices of Γ such that e_{ij} is an edge of Γ. Fix another vertex v_k. Consider the flat $F = K_{\{v_i,v_j,v_k\}}$. We know that

$$K_{\{v_i,v_j\}} \subset \text{pr}_\Gamma(F).$$

Since $\text{rk}(\text{pr}_\Gamma(F)) = \text{rk}(F) = 2$, either e_{ik} or e_{jk} must exist as edges in Γ. □

Lemma 3.2.13 and Lemma 3.2.14 show that when Γ is a complete multipartite graph, the cone complex $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ will not contain a ray which is adjacent to only 1 maximal cell, and thus can be embedded as a balanced fan.

Theorem 3.2.15. The cone complex underlying $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ has the structure of a balanced fan and is naturally identified with $\text{pr}_\Gamma(\mathcal{M}_{0,n}^{\text{rad}}) = \mathcal{B}'(\Gamma)$ if and only if Γ is a complete multipartite graph.

Proof. By Lemma 3.2.14 and Proposition 3.2.12, pr_Γ induces a bijection between the set of Γ-stable flats of K_{n-1} and flats Γ only when Γ is a complete multipartite graph. In this case Ψ_Γ is a bijection between Γ-stable radially aligned combinatorial types of $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ and flats of Γ. Thus the map Ψ_Γ induces an isomorphism of cone complexes on the ambient vector spaces. We finish the proof by noting that $\mathcal{M}_{0,\Gamma}^{\text{trad}}$ is a balanced fan by the fact that it has the same structure as the balanced fan $\text{pr}_\Gamma(\mathcal{M}_{0,n}^{\text{rad}})$. $\square$

Chapter 4

Algebraic Moduli Spaces of Graphically Stable Curves

Two theories, developed simultaneously, arise when dealing with tropicalizations of subvarieties of tori: *tropical compactification* and *geometric tropicalization*. The former describes a situation where the tropical variety determines a good choice of compactification. Specifically, the tropical compactification of $U \subset \mathbb{T}^r$ is its closure $\overline{U}$ in a toric variety $X(\Sigma)$ with $|\Sigma| = \text{trop}(U)$. The latter explores the converse statement, how a nice compactification determines its tropicalization. In this chapter, we use geometric tropicalization to compute the tropicalization of certain moduli spaces. Throughout this process, we discover a combinatorial criterion that allows us to precisely state when $\mathcal{M}_{0,\Gamma}^{\text{trop}} = \text{trop}(\mathcal{M}_{0,\Gamma})$, and thus when the tropical compactification of $\mathcal{M}_{0,\Gamma}$ in the toric variety $X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ is $\overline{\mathcal{M}}_{0,\Gamma}$.

4.1 Geometric Tropicalization of $\mathcal{M}_{0,n}$

Let U be a smooth subvariety of a torus $\mathbb{T}^r$ and Y be a smooth compactification containing U as a dense open subvariety. The boundary of Y, $\partial Y = Y \setminus U$, is *divisorial* if it is a union of codimension-1 subvarieties of Y. We say $(Y, \partial Y)$ is a *simple normal crossings (snc) pair* when the boundary of Y behaves locally like an arrangement of coordinate hyperplanes. In other words, ∂Y is an *snc divisor* if a non-empty intersection of k irreducible boundary divisors is codimension k and the intersection is transverse. The *boundary complex of* Y, $\Delta(\partial Y)$, is a simplicial complex whose vertices are in bijection with the irreducible divisors of the boundary divisor ∂Y, and whose k-cells correspond to a non-empty intersection of k boundary divisors. The cells containing a face τ correspond to the boundary strata that lie in the closure of τ's stratum.

Let $\phi_1, \ldots, \phi_r \in \mathcal{O}^*(U)$. The ϕ_i's define a morphism $\vec{\phi}$ from U to a torus $\mathbb{T}^r$, sending $u \in U$ to $(\phi_1(u), \ldots, \phi_r(u))$. When there are enough invertible functions, this map is an embedding. Given an irreducible boundary divisor $D \subset \partial Y$ we can compute the order of vanishing of each ϕ_i on D, $\mathrm{ord}_D(\phi_i)$, yielding an r-dimensional integer vector $\vec{v}_D = (\mathrm{ord}_D(\phi_1), \ldots, \mathrm{ord}_D(\phi_r))$ living inside of the cocharacter lattice of $\mathbb{T}^r$, $N_{\mathbb{T}^r} \subseteq N_{\mathbb{R}} = \mathbb{R}^r$. Let $\pi : \Delta(\partial Y) \to N_{\mathbb{R}}$ be the map defined by sending a vertex v_i to $\vec{v}_{D_i}$ and extending linearly on every simplex. Geometric tropicalization says precisely that the support of the tropical fan is the cone over this complex and this result is independent of our choice of compactification Y, i.e., $\mathrm{trop}(U) = \mathrm{cone}(\mathrm{Im}(\pi))$. As we will see later, π is not necessarily injective so $\mathrm{trop}(U)$ may not be the cone over $\Delta(\partial Y)$.

Tevelev [28, Theorem 5.5] first writes the geometric tropicalization of $\overline{\mathcal{M}}_{0,n}$ by combining results of [15, 27]. This result is generalized by Gibney and Maclagan [10, Theorem 5.7]. They use the fact that $\mathcal{M}_{0,n}$ can be embedded into a torus of dimension $\binom{n}{2} - n$ using the Plücker embedding of the Grassmannian $G(2,n)$ into $\mathbb{P}^{\binom{n}{2}-1}$. The Plücker embedding is given by sending a $2 \times n$ matrix representing a choice of basis for a subspace V to its vector of 2×2 minors called the Plücker coordinates. Let $G^0(2,n)$ be the open set of $G(2,n)$ consisting of points given by nonvanishing Plücker coordinates, i.e., the two-planes that do not pass through the intersection of any pair of coordinate hyperplanes. Let $\mathrm{Mat}^0(2,n)$ be the set of $2 \times n$ matrices having all nonzero 2×2 minors. An n-tuple of distinct points of $\mathbb{P}^1$, $([x_1 : y_1], \ldots, [x_n : y_n])$, may be encoded into a $2 \times n$ matrix where each point is a column of the matrix. Thus, $([x_1 : y_1], \ldots, [x_n : y_n])$ gets sent to $(x_{12} : \cdots : x_{n-1n}) \in \mathbb{P}^{\binom{n}{2}-1}$ where $x_{ij} = x_i y_j - x_j y_i$. The locus of n points on smooth curves $\mathcal{M}_{0,n}$ can be identified with a quotient of $G^0(2,n)$ by an $n-1$ dimensional torus, T^{n-1}, representing the torus action given by scaling the columns of the $2 \times n$ matrix mod the diagonal. The image of $\mathcal{M}_{0,n}$ via the Plücker embedding is an $(n-3)$ dimensional subspace inside of an $\left(\binom{n}{2} - n\right)$ dimensional torus, $T^{\binom{n}{2}-n} \cong T^{\binom{n}{2}-1}/T^{n-1}$ in $\mathbb{P}^{\binom{n}{2}-1}$ as depicted in Figure 4.1.

The boundary of $\overline{\mathcal{M}}_{0,n}$ has $2^{n-1} - n - 1$ irreducible divisors D_I, indexed by $I \sqcup I^c = [n]$ such that $2 \leq |I| \leq n-2$. We impose the convention that $1 \in I^c$. For $2 \leq i < j \leq n$, $(i,j) \neq (2,3)$, the ratios x_{ij}/x_{23} are regular functions on $\mathcal{M}_{0,n}$ and act as a choice of coordinates on the torus $T^{\binom{n}{2}-n}$. Let $\pi_n : \Delta(\partial\overline{\mathcal{M}}_{0,n}) \to N_{\mathbb{R}}$ be the divisorial valuation map assigning the

$$\begin{tikzcd}
\mathrm{Mat}^0(2,n) \arrow[r] \arrow[d, two heads] & G^0(2,n) \arrow[r, hook, "\mathrm{Pl}"] \arrow[d, two heads] & T^{\binom{n}{2}-1} \arrow[d, two heads] \\
\mathcal{M}_{0,n} \arrow[r, "\cong"] & G^0(2,n)/T^{n-1} \arrow[r, hook, "\mathrm{Pl}"] & T^{\binom{n}{2}-1}/T^{n-1}
\end{tikzcd}$$

Figure 4.1: Torus embedding of $\mathcal{M}_{0,n}$ via Plücker map.

vector $\vec{v}_{D_I} = (\mathrm{ord}_{D_I}(x_{24}/x_{23}), \ldots, \mathrm{ord}_{D_I}(x_{n-1n}/x_{23}))$ to a divisor D_I where

$$\mathrm{ord}_{D_I}(x_{ij}/x_{23}) = \begin{cases} 1 & \{2,3\} \not\subset I \text{ and } \{i,j\} \subset I; \\ -1 & \{2,3\} \subset I \text{ and } \{i,j\} \not\subset I; \\ 0 & \text{else.} \end{cases}$$

This means

$$\vec{v}_{D_I} = \begin{cases} \sum_{i,j \in I} \vec{e}_{ij} & \{2,3\} \not\subset I; \\ -\sum_{i \notin I \text{ or } j \notin I} \vec{e}_{ij} & \{2,3\} \subset I. \end{cases}$$

For explicit details see [10] and [19]. The standard basis vectors of $N_{\mathbb{R}} = \mathbb{R}^{\binom{n}{2}-n}$ equal the image of the divisors $D_{\{ij\}}$, $(i,j) \neq (2,3)$, and $\vec{v}_{D_{\{23\}}} = -\vec{1}$ is the negative all ones vector. For a divisor D_I with $|I| \geq 3$, the vector $\vec{v}_{D_I}$ is the sum over all vectors corresponding to divisors $D_{\{i,j\}}$ where $\{i,j\} \subset I$, i.e.,

$$\vec{v}_{D_I} = \sum_{\{i,j\} \subset I} \vec{v}_{D_{\{i,j\}}}. \tag{4.1}$$

Comparing the algebraic Plücker embedding to the tropical distance coordinates we realize that the distance coordinates from Section 2.2.2 can be recovered from the tropicalization of the Plücker coordinates, for details see [11, Section 3.1]. By similar comparison, the torus action on $\mathcal{M}_{0,n}$ corresponds to the lineality space $\Phi(\mathbb{R}^n)$.

Example 4.1.1. For $\mathcal{M}_{0,5}$, we have an embedding into $T^{\binom{5}{2}-5} = T^5$ with coordinates x_{24}/x_{23}, $x_{25}/x_{23},\ x_{34}/x_{23},\ x_{35}/x_{23}$, and x_{45}/x_{23}. In the boundary of $\overline{\mathcal{M}}_{0,5}$, there are 10 irreducible divisors and they are labeled by their index sets in Figure 4.2. The rays of trop$(\mathcal{M}_{0,5})$, i.e., the images of the divisors in $\mathbb{R}^5$ via π_5 are:

$$\vec{v}_{D_{\{2,4\}}} = \vec{e}_1 \qquad \vec{v}_{D_{\{3,5\}}} = \vec{e}_4$$
$$\vec{v}_{D_{\{2,5\}}} = \vec{e}_2 \qquad \vec{v}_{D_{\{4,5\}}} = \vec{e}_5$$
$$\vec{v}_{D_{\{3,4\}}} = \vec{e}_3 \qquad \vec{v}_{D_{\{2,3\}}} = -\vec{1}$$
$$\vec{v}_{D_{\{2,3,4\}}} = (0,-1,0,-1,-1) \qquad \vec{v}_{D_{\{2,4,5\}}} = (1,1,0,0,1)$$
$$\vec{v}_{D_{\{2,3,5\}}} = (-1,0,-1,0,-1) \qquad \vec{v}_{D_{\{3,4,5\}}} = (0,0,1,1,1)$$

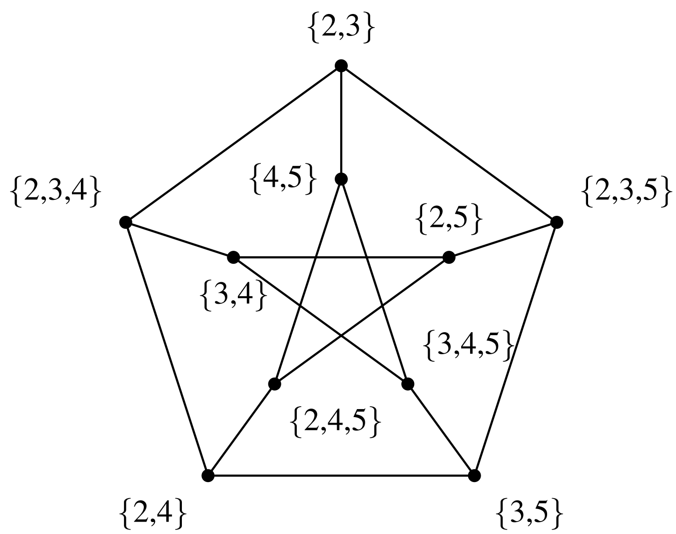


Figure 4.2: The boundary complex of $\overline{\mathcal{M}}_{0,5}$.

We may define $\mathcal{M}_{0,n}^{\text{trop}}$ alternatively as the cone over $\Delta(\partial\overline{\mathcal{M}}_{0,n})$. Geometric tropicalization states precisely that cone(Im(π_n)) $=$ trop$(\mathcal{M}_{0,n})$. The following theorem, due to Tevelev and Gibney-Maclagan, states that $\mathcal{M}_{0,n}^{\text{trop}} =$ trop$(\mathcal{M}_{0,n})$.

Theorem 4.1.2 ([28], [10]). The geometric tropicalization of $\overline{\mathcal{M}}_{0,n}$ via the embedding

$$\mathcal{M}_{0,n} \hookrightarrow T^{\binom{n}{2}-n}$$

gives the fan $\text{trop}(\mathcal{M}_{0,n})$ whose underlying cone complex is identified with $\mathcal{M}_{0,n}^{\text{trop}}$. Furthermore, the tropical compactification of $\mathcal{M}_{0,n}$ in the toric variety $X(\mathcal{M}_{0,n}^{\text{trop}})$ is $\overline{\mathcal{M}}_{0,n}$.

It follows from the previous theorem that π_n is injective and thus induces a bijective map of cone complexes from $\mathcal{M}_{0,n}^{\text{trop}}$ to $\text{trop}(\mathcal{M}_{0,n})$. This is an important fact that we will come back to later when discussing Γ-stability.

Hassett [14] describes a weighted variation on the moduli space of curves that assigns rational weights to each marked point, now referred to as *Hassett spaces*. He introduces these spaces in the context of the log minimal model program. A *weight data* is a vector $w = (w_1, \ldots, w_n) \in ((0,1] \cap \mathbb{Q})^n$. For a weight data, he defines the moduli space of weighted stable rational marked curves $\overline{\mathcal{M}}_{0,w}$. In this space, marked points are not necessarily distinct: a subset of points are allowed to coincide if the sum of their weight is less than 1. Let C be a tree of $\mathbb{P}^1$'s with marked points $p_1, \ldots, p_n$. The curve C is w-*stable*, if for each component T of C, we have

$$\sum_{i;p_i \in T} w_i + \#\text{nodes} > 2. \tag{4.2}$$

Ulirsch [31] shows $\mathcal{M}_{0,n} \subset \overline{\mathcal{M}}_{0,w}$ is in general not a simple normal crossing compactification, but the locus of smooth weighted stable curves $\mathcal{M}_{0,w} \subset \overline{\mathcal{M}}_{0,w}$ is a simple normal crossing compactification. Cavalieri et al. [3, Theorem 3.9] describe the geometric tropicalization for Hassett spaces. A weight data w is *heavy/light* if each weight w_i is either 1 or ϵ where $\epsilon < 1/n$. Using the same Plücker embedding above, they show that $\mathcal{M}_{0,w}$ can also be embedded into a torus and prove the following theorem.

Theorem 4.1.3 ([3]). Let w be heavy/light. Consider the embedding

$$\mathcal{M}_{0,w} \hookrightarrow T_w = \text{Pr}_w(T^{\binom{n}{2}}/T^n).$$

The closure of $\mathcal{M}_{0,w}$ in the compactification of T_w defined by the fan $\mathcal{M}_{0,w}^{\text{trop}}$ is isomorphic to $\overline{\mathcal{M}}_{0,w}$. The tropicalization of $\mathcal{M}_{0,w}$ with respect to this embedding is $\mathcal{M}_{0,w}^{\text{trop}}$.

4.2 Geometric Tropicalization of $\mathcal{M}_{0,\Gamma}$

This section is the culmination of this dissertation. We define the family of algebraic moduli spaces parameterizing rational graphically stable curves, $\overline{\mathcal{M}}_{0,\Gamma}$, and investigate its geometric tropicalization. The main theorem classifies *all* graphically stable moduli spaces in which the tropical compactification of $\overline{\mathcal{M}}_{0,\Gamma}$ agrees with the theory of geometric tropicalization.

4.2.1 The Moduli Space of Rational Graphically Stable Curves

In [26], Smyth gives a complete classification of all modular compactifications of $\mathcal{M}_{0,n}$ (Theorem 1.9). In this section, we prove that $\overline{\mathcal{M}}_{0,\Gamma}$ is a modular compactification of $\mathcal{M}_{0,n}$ by showing that Γ-stability, as in Definition 3.2.1, is an extremal assignment over $\mathcal{M}_{0,n}$. In addition, we show that the pair $(\overline{\mathcal{M}}_{0,\Gamma}, \partial\overline{\mathcal{M}}_{0,\Gamma})$ is an snc pair. We begin with the definition of a Γ stable curve and $\overline{\mathcal{M}}_{0,\Gamma}$.

Definition 4.2.1. Let Γ be a simple connected graph on vertices, $2, \ldots, n$. A rational stable n-marked curve $(C, p_1, \ldots, p_n)$ is Γ-*stable*, if at each component not containing p_1 with exactly 1 node there exists an edge $e_{ij} \in E(\Gamma)$ where p_i and p_j are points on the component.

Define $\overline{\mathcal{M}}_{0,\Gamma}$ to be the parameter space of all rational Γ-stable n-marked curves with the interior $\mathcal{M}_{0,\Gamma}$ to be all smooth rational Γ-stable n-marked curves.

Consider the assignment defined by

$$\mathcal{Z}(C) = \{Z \subset C \mid |Z \cap Z^c| = 1,\ p_1 \notin Z,\ \{e_{ij} \in E(\Gamma) | \, p_i, p_j \in Z\} = \emptyset\} \qquad (4.3)$$

If we call a subcurve $Z \subset C$ satisfying $|Z \cap Z^c| = 1$ a *tail*, then the assignment $\mathcal{Z}$ is defined by picking out all tails of $(C, p_1, \ldots, p_n)$ not containing p_1 that have no edges in Γ between vertices corresponding to the marked points on the tail. For the purposes of this document, we require Γ

to be a simple connected graph on $n-1$ vertices, in which case Equation (4.3) is an extremal assignment. The first axiom holds, since Γ contains an edge. In genus 0, the automorphism group of the dual graph is trivial so the second axiom holds. Indeed, the third axiom is satisfied as degenerating a curve does not change the stability of tails.

By definition, the only components contracted by $\mathcal{Z}$-stability are exactly those which are contracted by Γ-stability. Every tail is contracted to a point of singularity type $(0,1)$ which is a smooth point. Therefore, $\overline{\mathcal{M}}_{0,\Gamma} = \overline{\mathcal{M}}_{0,n}(\mathcal{Z})$ (as defined in [26]) and so $\overline{\mathcal{M}}_{0,\Gamma}$ is a modular compactification of $\mathcal{M}_{0,n}$.

Remark 4.2.2. To the author's understanding, Γ only needs to contain a single edge to be an extremal assignment. However, many of the tropicalization statements would not hold so we use a connected graph.

Lemma 4.2.3. The boundary $\partial\overline{\mathcal{M}}_{0,\Gamma} = \overline{\mathcal{M}}_{0,\Gamma} \setminus \mathcal{M}_{0,\Gamma}$ is a divisor with simple normal crossings.

Proof. Just like in the case of $\overline{\mathcal{M}}_{0,n}$, the boundary $\partial\overline{\mathcal{M}}_{0,\Gamma}$ is divisorial, meaning it is a union of divisors of $\overline{\mathcal{M}}_{0,\Gamma}$. In addition, the boundary strata are parameterized by dual graphs. Each edge of a dual graph corresponds to a node in its associated complex curve, where locally, each node is given by an equation $xy = t_i$ when $t_i = 0$. Since a boundary stratum of codimension k is the intersection of k divisors, each divisor acts as a coordinate hyperplane $t_i = 0$. Therefore, $\partial\overline{\mathcal{M}}_{0,\Gamma}$ behaves locally like an arrangement of coordinate hyperplanes. $\square$

4.2.2 Tropicalization of $\mathcal{M}_{0,\Gamma}$

Fix Γ to be a simple connected graph on vertices $2,\ldots,n$ containing the edge e_{23}. The goal of this section is to walk through the process of geometric tropicalization for the case of Γ stability and study the tropical compactification of $\mathcal{M}_{0,\Gamma}$. We begin by investigating the projection of the Plücker embedding of $\mathcal{M}_{0,n}$. Graphic stability defines a projection map that will give us a torus embedding using the remaining Plücker coordinates. Next we examine the divisorial valuation map from the boundary complex of $\overline{\mathcal{M}}_{0,\Gamma}$ into the cocharacter lattice of the torus. This tropicalization yields a fan which coincides with the tropical moduli space $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ if and only if Γ is complete multipartite.

We may set up an embedding for $\mathcal{M}_{0,\Gamma}$ similar to how we did for $\mathcal{M}_{0,n}$. Let $\text{Mat}^{\Gamma}(2,n)$ be the set of $2 \times n$ matrices where the ij^{th} minor is nonzero whenever $e_{ij} \in E(\Gamma)$. Define $G^{\Gamma}(2,n)$ as the subspace of the Grassmannian $G(2,n)$ given by the matrices in $\text{Mat}^{\Gamma}(2,n)$. By Γ stability, the minors which are allowed to be zero exactly coincide with the pairs of points that are allowed to collide. Thus, we obtain the following lemma.

Lemma 4.2.4. The open space $\mathcal{M}_{0,\Gamma}$ is identified with a quotient of $G^{\Gamma}(2,n)$.

Proof. We demonstrate $\mathcal{M}_{0,\Gamma} \cong G^{\Gamma}(2,n)/T^{n-1}$ by showing the points added to $\mathcal{M}_{0,n}$ to get $\mathcal{M}_{0,\Gamma}$ correspond to exactly the points added to $G^{0}(2,n)/T^{n-1}$ to get $G^{\Gamma}(2,n)/T^{n-1}$. Each space adds points corresponding to the sets of disjointed subsets $I \subset [n] \setminus \{1\}$ such that there are no edges in Γ_I (the subgraph of Γ induced by the vertices marked by I). In other words, each space adds points corresponding to the clusters (disjoint unions of cliques) of Γ^c. Given a cluster of Γ^c, the moduli space $\mathcal{M}_{0,n}$ adds a configuration of n points on $\mathbb{P}^1$ where a collection of points coincides according to the each clique. Given a cluster of Γ^c, the quotiented Grassmannian subset $G^{0}(2,n)/T^{n-1}$ adds a $2 \times n$ matrix that has a zero 2×2 minors according to the each clique. $\square$

Lemma 4.2.4 allows investigate the image of $\mathcal{M}_{0,\Gamma}$ via the Plücker embedding.

Definition 4.2.5. Let Pr_{Γ} be the rational map from $\mathbb{P}^{\binom{n}{2}-1}$ to $\mathbb{P}^{\binom{n}{2}-1-N}$ dropping all the Plücker coordinates x_{ij} for which e_{ij} is not an edge of Γ, where N is the number of edges removed from K_{n-1} to obtain Γ. Precisely, $N = \binom{n-1}{2} - E(\Gamma)$.

The projection map Pr_{Γ} is regular on the torus $T^{\binom{n}{2}-1}$ and also on $\text{Pl}(\mathcal{M}_{0,\Gamma})$. After projecting, all the remaining Plücker coordinates are non-zero and so the projection of $\text{Pl}(\mathcal{M}_{0,\Gamma})$ in $\mathbb{P}^{\binom{n}{2}-n-N}$ lives inside of a torus. The $(n-1)$ dimensional torus action on $\mathcal{M}_{0,n}$ (given by scaling the columns of the $2 \times n$ matrix mod the diagonal), acts on $\mathcal{M}_{0,\Gamma}$ in the same way. Therefore, $\mathcal{M}_{0,\Gamma}$ is embedded into an $\left(\binom{n}{2} - N\right)$ dimensional torus. We prove this fact in Lemma 4.2.6 and the diagram in Figure 4.3 summarizes the above conversation.

$$\begin{tikzcd}
\mathrm{Mat}^{\Gamma}(2,n) \arrow[r] \arrow[d, two heads] & G^{\Gamma}(2,n) \arrow[r, hook, "\mathrm{Pl}"] \arrow[d, two heads] & \mathrm{Pl}(\mathcal{M}_{0,\Gamma}) \arrow[r, phantom, "\subset"] \arrow[d, "\mathrm{Pr}_{\Gamma}"] & \mathbb{P}^{\binom{n}{2}-1-N} \arrow[d, dashed, two heads, "\mathrm{Pr}_{\Gamma}"] \\
\mathcal{M}_{0,\Gamma} \arrow[r, "\cong"] & G^{\Gamma}(2,n)/T^{n-1} \arrow[r, hook] & T^{\binom{n}{2}-1-N}/T^{n-1} \arrow[r, phantom, "\subset"] & \mathbb{P}^{\binom{n}{2}-1-N}
\end{tikzcd}$$

Figure 4.3: Torus embedding of $\mathcal{M}_{0,\Gamma}$ via Plücker map.

Lemma 4.2.6. The open part $\mathcal{M}_{0,\Gamma}$ can be embedded into the torus $\mathrm{Pr}_{\Gamma}\left(T^{\binom{n}{2}-n}\right) = T^{\binom{n}{2}-n-N}$ using the Plücker coordinates.

Proof. Let $(\mathbb{P}^1, (x_1 : y_1), \ldots, (x_n : y_n))$ be a Γ-stable curve in $\mathcal{M}_{0,\Gamma}$. As in the $\mathcal{M}_{0,n}$ case, this marked curve corresponds to, up to equivalence, a point $(x_{12} : \cdots : x_{n-1n}) \in \mathbb{P}^{\binom{n}{2}-1}$ where $x_{ij} = x_i y_j - x_j y_i$. Some coordinates may be zero, specifically x_{ij} is allowed to be 0 when $e_{ij} \notin E(\Gamma)$. By definition, $\mathrm{Pr}_{\Gamma}(x_{12} : \cdots : x_{n-1n}) = (x_{ij})_{i<j} \in \mathbb{P}^{\binom{n}{2}-1-N}$ for $(i,j) = (1,j)$ where $2 \leq j \leq n$ and $e_{ij} \in E(\Gamma)$. Denote $\vec{x} = \mathrm{Pr}_{\Gamma}(x_{12} : \cdots : x_{n-1n})$. Since each coordinate is now nonzero, $\vec{x}$ lies in the torus $T^{\binom{n}{2}-1-N}$. It is not hard to see that the same $(n-1)$ dimensional torus, that was quotiented out in the $\mathcal{M}_{0,n}$ case, also acts on $T^{\binom{n}{2}-1-N}$. Finally, we see this map must injective on $\mathcal{M}_{0,\Gamma}$ because the Plücker map is injective on $\mathcal{M}_{0,\Gamma}$ and the projection map is injective its image. $\square$

Lemma 4.2.3 and Lemma 4.2.6 give us a boundary which is a simple normal crossings divisor and a torus embedding of the interior. This allows us to study the tropical compactification $\mathcal{M}_{0,\Gamma}$ and geometric tropicalization of $\overline{\mathcal{M}}_{0,\Gamma}$. The boundary of $\overline{\mathcal{M}}_{0,\Gamma}$ is divisorial in the same way that $\partial\overline{\mathcal{M}}_{0,n}$ is, except that there are fewer irreducible divisors (this number is easily computable but annoying to write). The ratios x_{ij}/x_{23}, for $2 \leq i < j \leq n$, $(i,j) \neq (2,3)$ and $e_{ij} \in E(\Gamma)$, are regular functions on $\overline{\mathcal{M}}_{0,\Gamma}$ and act as a choice of coordinates on the torus $T^{\binom{n}{2}-n-N}$ (we lose N functions). Define the divisorial valuation map $\pi_{\Gamma} : \Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}) \to N_{\mathbb{R}}$ identically to π_n but restricted to the remaining non-zero coordinates,

$$\pi_\Gamma(D_I) = \vec{v}_{D_I} = \begin{cases} \displaystyle\sum_{i,j \in I} \vec{e}_{ij} & \{2,3\} \not\subset I \text{ and } e_{ij} \in E(\Gamma); \\ \displaystyle -\sum_{i,j \notin I} \vec{e}_{ij} & \{2,3\} \subset I \text{ and } e_{ij} \in E(\Gamma). \end{cases}$$

Similar to before, the standard basis vectors of $T^{\binom{n}{2}-n-N}$ are $\vec{v}_{D_{\{ij\}}}$, where $e_{ij} \in E(\Gamma) \setminus \{e_{23}\}$, and $\vec{v}_{D_{\{23\}}} = -\vec{1}$. There is a similar relation to Equation (4.1): For a divisor D_I with $|I| \geq 3$,

$$\vec{v}_{D_I} = \sum_{\{i,j\} \subset I;\ e_{ij} \in E(\Gamma)} \vec{v}_{D_{\{i,j\}}}. \tag{4.4}$$

Lemma 4.2.7. The tropicalization of the map Pr_Γ agrees with the projection pr_Γ from Equation (3.2).

Proof. The basis of $T^{\binom{n}{2}-n}$ given by x_{ij}/x_{23} for $2 \leq i < j \leq n$, $(i,j) \neq (2,3)$. These coordinates are in bijection with divisors $D_{\{i,j\}}$. The tropicalization of representatives of such divisors are the basis elements of $\mathbb{R}^{\binom{n}{2}-n}$. Both projections, Pr_Γ and pr_Γ, forget coordinates corresponding edges deleted from K_{n-1} to obtain Γ. The discussion above confirms that the tropicalization of the basis elements of $T^{\binom{n}{2}-n-N}$ coincide with the basis elements of $\mathbb{R}^{\binom{n}{2}-n-N}$. $\square$

Proposition 4.2.8. The geometric tropicalization of $\overline{\mathcal{M}}_{0,\Gamma}$ using the embedding in Lemma 4.2.6 is $\mathrm{trop}(\mathcal{M}_{0,\Gamma}) = \mathrm{pr}_\Gamma(\mathcal{M}_{0,n}^{\mathrm{trop}}) = \mathcal{B}'(\Gamma)$.

Proof. Geometric tropicalization requires a simple normal crossings compactification and a torus embedding. These two conditions are satisfied by Lemma 4.2.3 and Lemma 4.2.6. By Lemma 4.2.7 the divisorial valuations of the boundary divisors yield the rays of this fan. Theorem 2.5 from [4] states that the weight of each top-dimensional cone $\sigma \subset \mathrm{trop}(\mathcal{M}_{0,\Gamma})$ is equal to the intersection number, with multiplicity, of the divisors corresponding to the rays of σ. A non-empty intersection of $n-3$ codimension one curves is a single point with multiplicity 1. This weighting convention is exactly the same as in the definition of the Bergman fan, concluding our proof. $\square$

Unlike π_n, the map π_Γ may not be injective. Using Equation (4.1), we demonstrate the simplest case of non-injectivity: Consider the divisor $D_{\{i,j,k\}}$ in $\overline{\mathcal{M}}_{0,n}$ and it's image via π_n

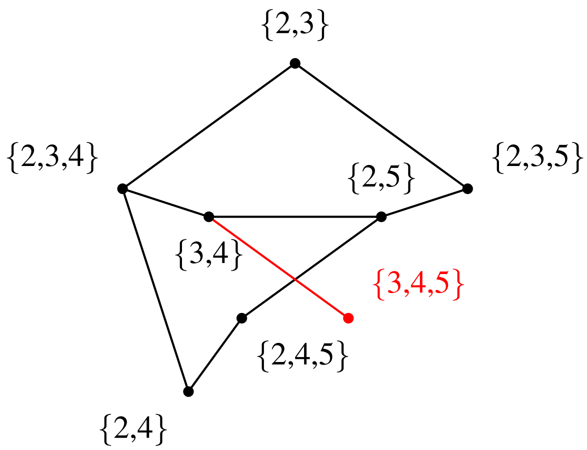
$$\vec{v}_{D_{\{i,j,k\}}} = \vec{v}_{D_{\{i,j\}}} + \vec{v}_{D_{\{i,k\}}} + \vec{v}_{D_{\{j,k\}}}. \tag{4.5}$$

If exactly two of the vectors on the right correspond to Γ-unstable divisors, then π_Γ cannot be injective. This cannot happen when Γ is complete multipartite.

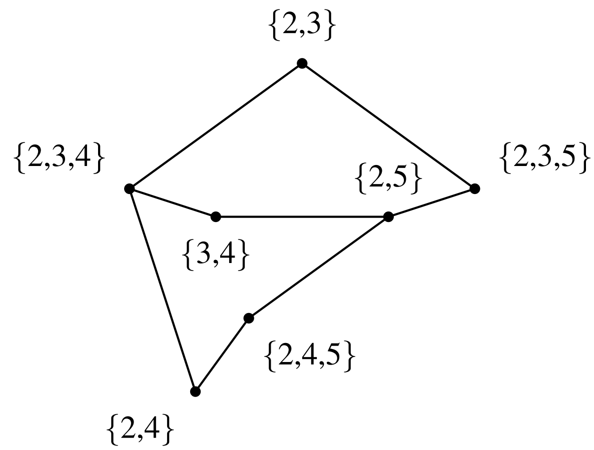
Example 4.2.9. Let $\widetilde{\Gamma}$ be the subgraph of K_4 with edges e_{35} and e_{45} removed as in Example 2.1.4; see Figure 2.4a. Then we have $\mathcal{M}_{0,\widetilde{\Gamma}} \hookrightarrow T^{\binom{5}{2}-5-2} = T^3$ with coordinates $x_{24}/x_{23},\ x_{25}/x_{23}$, and x_{34}/x_{23}. In $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$, there are 8 irreducible boundary divisors, labeled in Figure 4.4a. Comparing the cone complexes of $\mathcal{M}_{0,\widetilde{\Gamma}}^{\text{trop}}$ and $\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$, we can see that the cones associated to the boundary strata $D_{\{3,4\}}$, $D_{\{3,4,5\}}$, and $D_{\{3,4\}} \cap D_{\{3,4,5\}}$ in $\mathcal{M}_{0,\widetilde{\Gamma}}^{\text{trop}}$ are all mapped to the ray given by $D_{\{3,4\}}$ in $\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$. Explicitly, $\pi_{\widetilde{\Gamma}} : \Delta(\partial\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}) \to \mathbb{R}^3$ where the divisors have been mapped to the following primitive vectors:

$$\begin{aligned}
&\vec{v}_{D_{\{2,4\}}} = (1,0,0) && \vec{v}_{D_{\{2,5\}}} = (0,1,0) && \vec{v}_{D_{\{3,4\}}} = (0,0,1)\\
&\vec{v}_{D_{\{2,3\}}} = (-1,-1,-1) && \vec{v}_{D_{\{2,3,4\}}} = (0,-1,0) && \vec{v}_{D_{\{2,3,5\}}} = (-1,0,-1)\\
&\vec{v}_{D_{\{2,4,5\}}} = (1,1,0) && \color{red}{\vec{v}_{D_{\{3,4,5\}}} = (0,0,1)} &&
\end{aligned}$$

Example 4.2.10. Let $\Gamma = K_{2,2}$ be the complete bipartite graph obtained by removing edges e_{25} and e_{34} from K_4, as shown in Figure 4.5a. Then we have $\mathcal{M}_{0,K_{2,2}} \hookrightarrow T^{\binom{5}{2}-5-2} = T^3$ with coordinates $x_{24}/x_{23},\ x_{35}/x_{23}$, and x_{45}/x_{23}. In $\overline{\mathcal{M}}_{0,K_{2,2}}$, there are 8 irreducible boundary divisors, labeled in Figure 4.5b. Explicitly, $\pi_\Gamma : \Delta(\partial\overline{\mathcal{M}}_{0,K_{2,2}}) \to \mathbb{R}^3$ where the divisors have been mapped to the



(a) A slice of the cone complex $\mathcal{M}^{\mathrm{trop}}_{0,\widetilde{\Gamma}}$ with rays labeled by their divisor index set.



(b) A slice of the cone complex trop$(\mathcal{M}_{0,\widetilde{\Gamma}})$ with rays labeled by their divisor index set.

Figure 4.4

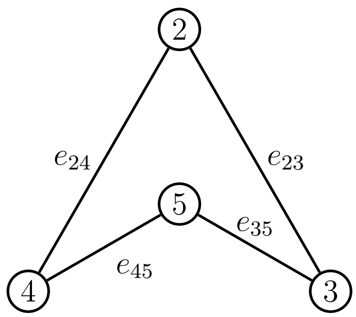
following primitive vectors:

$$\vec{v}_{D_{\{2,4\}}} = (1,0,0) \qquad \vec{v}_{D_{\{3,5\}}} = (0,1,0) \qquad \vec{v}_{D_{\{4,5\}}} = (0,0,1)$$
$$\vec{v}_{D_{\{2,3\}}} = (-1,-1,-1) \qquad \vec{v}_{D_{\{2,3,4\}}} = (0,-1,-1) \qquad \vec{v}_{D_{\{2,3,5\}}} = (-1,0,-1)$$
$$\vec{v}_{D_{\{2,4,5\}}} = (1,0,1) \qquad \vec{v}_{D_{\{3,4,5\}}} = (0,1,1)$$

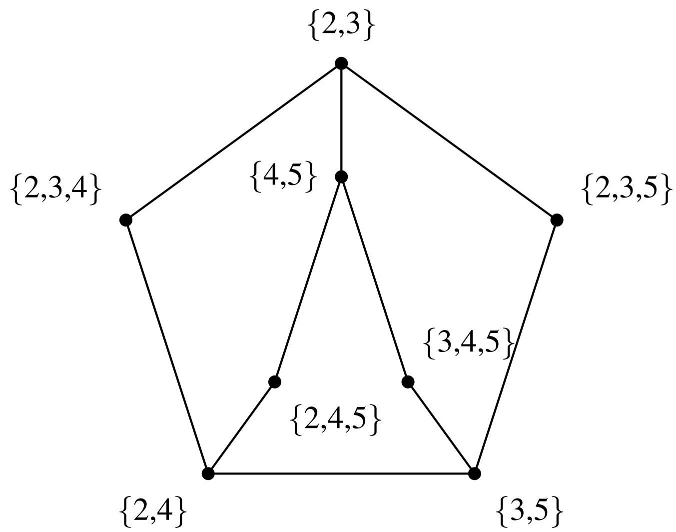
Lemma 4.2.11. The divisorial valuation map π_Γ is injective if and only if Γ is complete multipartite.

Proof. We begin by proving the forwards direction by contradiction. Suppose π_Γ is injective and Γ is not complete multipartite. Using Lemma 2.1.6, fix three vertices v_i, v_j, and v_k where $e_{ij} \in E(\Gamma)$ but $e_{ik}, e_{jk} \notin E(\Gamma)$. We have the following contradiction

$$\pi_\Gamma(D_{\{i,j,k\}}) = \vec{v}_{D_{\{i,j,k\}}} = \vec{v}_{D_{\{i,j\}}} = \pi_\Gamma(D_{\{i,j\}}).$$



(a) The graph $K_{2,2}$ in Example 4.2.10



(b) A slice of the cone complex $\mathrm{trop}(\mathcal{M}_{0,K_{2,2}}) = \mathcal{M}_{0,K_{2,2}}^{\mathrm{trop}}$ with rays labeled by their divisor index set.

Figure 4.5

For the backwards direction, assume Γ is complete multipartite. Let D_I and D_J be two Γ-stable divisors such that $\vec{v}_{D_I} = \vec{v}_{D_J}$. Thus we have

$$\sum_{\{i,j\}\subset I;\ e_{ij}\in E(\Gamma)} \vec{e}_{ij} = \sum_{\{i,j\}\subset J;\ e_{ij}\in E(\Gamma)} \vec{e}_{ij}.$$

This implies that I and J induce the same subgraph of Γ denoted $\Gamma_I = \Gamma_J$. If $I \neq J$, then there exists $i \in I \setminus J$. But $v_i \in \Gamma$ must be isolated in Γ_I. So there exists $j, k \in I \cup J$ such that $e_{ij}, e_{ik} \notin \Gamma_I$ but $e_{jk} \in \Gamma_I$ which contradicts Lemma 2.1.6 and concludes the proof. $\square$

Corollary 4.2.12. The cone complex $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}$ is embedded as a fan in a real vector space by π_Γ if and only if Γ is complete multipartite.

Proof. The map π_Γ induces a map of cone complexes from $\mathcal{M}_{0,\Gamma}^{\mathrm{trop}} = \mathrm{cone}(\Delta(\partial\overline{\mathcal{M}}_{0,\Gamma}))$ to $\mathrm{trop}(\mathcal{M}_{0,\Gamma}) = \mathrm{cone}(\mathrm{Im}(\pi_\Gamma))$ which is an isomorphism if and only if Γ is complete multipartite by Lemma 4.2.11. $\square$

Lemma 4.2.13. The units of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ are generated by cross ratios, i.e. forgetful morphisms to $\mathcal{M}_{0,4}$.

Proof. The space $\mathcal{M}_{0,n}$ can be viewed as the subset of $(\mathbb{C}^* \setminus \{1\})^{n-3}$ minus the hyperplanes $x_i - x_j = 0$. The functions which don't vanish on $\mathcal{M}_{0,n}$ are rational functions that have zeros and poles on the hyperplanes, i.e. monomials in x_i, $x_i - 1$, and $x_i - x_j$. We can write any monomial function as a product of cross ratios:

$$\frac{(P_1 - P_2)(P_3 - P_4)}{(P_1 - P_3)(P_2 - P_4)}.$$

For x_i, let $P_1 = x_i$, $P_2 = 0$, $P_3 = \infty$, and $P_4 = 1$.

For $x_i - 1$, let $P_1 = x_i$, $P_2 = 1$, $P_3 = \infty$, and $P_4 = 0$.

For $x_i - x_j$, take a product of x_i and $P_1 = x_i$, $P_2 = x_j$, $P_3 = 0$, and $P_4 = \infty$.

Consider the embedding of $\mathcal{M}_{0,n}$ into $\mathcal{M}_{0,\Gamma}$ in the diagram below where ϕ is a unit of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$.

$$\begin{tikzcd}
\mathcal{M}_{0,n} \arrow[r, hook] \arrow[dr, "\tilde{\phi}"'] & \mathcal{M}_{0,\Gamma} \arrow[d, "\phi"] \\
& \mathbb{C}^*
\end{tikzcd}$$

From arguments above, $\tilde{\phi}$ must be a product of cross ratios. Indeed, ϕ is also a product of cross ratios because $\mathcal{M}_{0,n}$ is dense in $\mathcal{M}_{0,\Gamma}$. $\square$

Theorem 4.2.14. The cone complex $\mathcal{M}_{0,\Gamma}^{\text{trop}}$ is embedded as a balanced fan in a real vector space by π_Γ if and only if Γ is a complete multipartite graph. For such Γ, there is a torus embedding

$$\mathcal{M}_{0,\Gamma} \hookrightarrow T^{\binom{n}{2}-n-N} = T_\Gamma$$

whose tropicalization $\text{trop}(\mathcal{M}_{0,\Gamma})$ has underlying cone complex $\mathcal{M}_{0,\Gamma}^{\text{trop}}$. Furthermore, the tropical compactification of $\mathcal{M}_{0,\Gamma}$ is $\overline{\mathcal{M}}_{0,\Gamma}$, i.e, the closure of $\mathcal{M}_{0,\Gamma}$ in the toric variety $X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ is $\overline{\mathcal{M}}_{0,\Gamma}$

Proof. As in [3], we wish to show the map $\overline{\mathcal{M}}_{0,\Gamma} \to X(\mathcal{M}_{0,\Gamma}^{\text{trop}})$ is an embedding. According to [12, Lemma 2.6 (4) and Theorem 2.10], this occurs when the following two conditions hold. For a stratum S, let M_S be $\mathcal{O}^*(S)/k^*$ and $M^S_{\mathcal{M}_{0,\Gamma}}$ be the sublattice of $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})/k^*$ generated by units having zero valuation on S.

1. For each boundary divisor D containing S, there is a unit $u \in \mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ with valuation 1 on D and valuation 0 on other boundary divisors containing S.

2. S is very affine and the restriction map $\mathcal{M}^S_{\mathcal{M}_{0,\Gamma}} \to \mathcal{M}_S$ is surjective.

We note that condition (1) occurs if and only if Γ is a complete multipartite graph, but condition (2) does not force Γ to be complete multipartite.

For condition (1), recall that the general element of a boundary divisor D_I has exactly one node and may be described by I, the set of marked points on a component. Observe that units in $\mathcal{O}^*(\mathcal{M}_{0,\Gamma})$ are generated by forgetful morphisms to $\mathcal{M}_{0,4}$ using *cross ratios* as in [28, Section 5]. Such a forgetful morphism has valuation 1 on D if the image of the general element of D is nodal and valuation 0 on D if the image of the general element of D is smooth. We show forgetful morphism with that property exists if and only if Γ is a complete multipartite graph.

We prove the forwards direction by way of contradiction. Assume Γ is not complete multipartite. Using Lemma 2.1.6, fix three vertices v_i, v_j, and v_k where $e_{ij} \in E(\Gamma)$ but $e_{ik}, e_{jk} \notin E(\Gamma)$. Consider the divisors $D_{\{i,j,k\}}$ and $D_{\{i,j\}}$ whose intersection yields the stratum of the dual graph in Figure 4.6. Every forgetful morphism that has valuation 1 on the general element of $D_{\{i,j,k\}}$ must keep i and j, otherwise the image of the general element of $D_{\{i,j,k\}}$ is smooth. However, any such morphism also has valuation 1 on $D_{\{i,j\}}$, a contradiction.

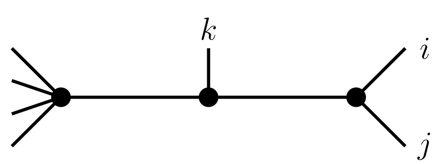


Figure 4.6: Dual graph of the stratum contained in $D_{\{i,j\}}$ and $D_{\{i,j,k\}}$.

Now suppose Γ is complete multipartite. Fix a stratum S and a divisor D_I containing S, as shown in Figure 4.7. Note that a forgetful morphism u which remembers the markings $\{1, a, b, c\}$ has valuation 1 on the general element of D_I if and only if (without loss of generality $I \cap \{1, a, b, c\} =$

$\{a, b\}$) $e_{ab} \in E(\Gamma)$, and valuation 0 otherwise. So, we must pick points $a, b \in I$ such that $e_{ab} \in E(\Gamma)$ and for any other divisor D_J containing S, $|J \cap \{1, a, b, c\}| \neq 2$.

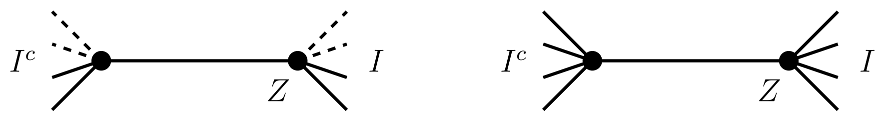


Figure 4.7: Dual graphs of the $\overline{S}$ and divisor D_I from the proof of Main Theorem where dashed edges represent potential extra components

Consider the vertex Z of the dual graph of $\overline{S}$ as illustrated in Figure 4.7. If Z has exactly one bounded edge adjacent to it, we are done by Γ-stability. Otherwise, consider the connected components on the right of Z obtained by deleting the bounded edges adjacent to Z. We may partition I by the markings on Z and by these components. Pick a nonempty part A of I and let $B = I \setminus A$.

Assume by way of contradiction that there are no markings $a \in A,\ b \in B$ such that $e_{ab} \in \Gamma$. Since Γ is complete multipartite, the vertices in Γ given by $A \sqcup B$ are part of the same independent set. Thus, for any $b_1, b_2 \in B$, $e_{b_1 b_2} \notin E(\Gamma)$, contradicting the Γ-stability of $\overline{S}$. A choice of markings $a \in A$ and $b \in B$ guarantees that for any other divisor D_J containing S, $|J \cap \{1, a, b, c\}| \neq 2$.

For condition (2), a stratum S is very affine because it can viewed as a product of $\mathcal{M}_{0,\Gamma}$'s. Each component of S contains at least one node so that point may act as the 'special' point 1 and the marked points behave under another Γ-stability condition since any subgraph of a complete multipartite graph is complete multipartite. Finally, since the boundary of $\overline{\mathcal{M}}_{0,\Gamma}$ is a simple normal crossings divisor as in the case of $\overline{\mathcal{M}}_{0,w}$, the surjectivity of the restriction map follows the same proof outline as in [3]. The local structure of $\partial\overline{\mathcal{M}}_{0,\Gamma}$ is an intersection of coordinate hyperplanes and restricting the coordinates is surjective. $\square$

Remark 4.2.15. Many statements remain true when Γ is not complete multipartite. The geometric tropicalization of $\overline{\mathcal{M}}_{0,\Gamma}$ using the embedding in Lemma 4.2.6 still equals $\mathrm{pr}_\Gamma(\mathcal{M}_{0,\Gamma}^{\mathrm{trop}}) = \mathrm{trop}(\mathcal{M}_{0,\Gamma})$. However, we have seen that not all cones are mapped injectively. On the algebraic side, we still have a map from $\overline{\mathcal{M}}_{0,\Gamma}$ to the toric variety $X(\text{trop}(\mathcal{M}_{0,\Gamma}))$, but it does not map all boundary strata injectively.

Example 4.2.16. Let $\widetilde{\Gamma}$ be the subgraph of K_4 with edges e_{35} and e_{45} removed as in Example 2.1.4; see Figure 2.4a. Then Figure 4.4a depicts the boundary of $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$ and a slice of $\mathcal{M}_{0,\widetilde{\Gamma}}^{\text{trop}}$ while Figure 4.4b depicts the boundary of the closure of $\mathcal{M}_{0,\widetilde{\Gamma}}$ in $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ and a slice of $\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$.

There are only 8 2D cones and 7 rays in $\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}})$ while the cone over $\partial\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$ has 9 2D cones and 8 rays. This means $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ isn't large enough to contain $\overline{\mathcal{M}}_{0,\widetilde{\Gamma}}$. In other words, the locus of smooth curves $\mathcal{M}_{0,\widetilde{\Gamma}}$ is missing the limit as the marked points 3, 4, and 5 come together. The modular compactification of $\mathcal{M}_{0,\widetilde{\Gamma}}$ assigns a $\mathbb{P}^1$ to the limit but $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ doesn't have enough coordinates to include a $\mathbb{P}^1$. Rather, this limit gets closed with a single point in $X(\text{trop}(\mathcal{M}_{0,\widetilde{\Gamma}}))$ (which is the intersection of two smooth curves where the marked points 3 and 4, and 4 and 5, have come together).

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