

Financial Risk Management

MSBA IN FINANCIAL RISK MANAGEMENT





Introduction to Monte Carlo Simulation

Why Simulation?

“Informed decision-making comes from a long tradition of guessing and then blaming others for **inadequate results.**”

Let's stop **guessing.**



What is Monte Carlo Simulation?

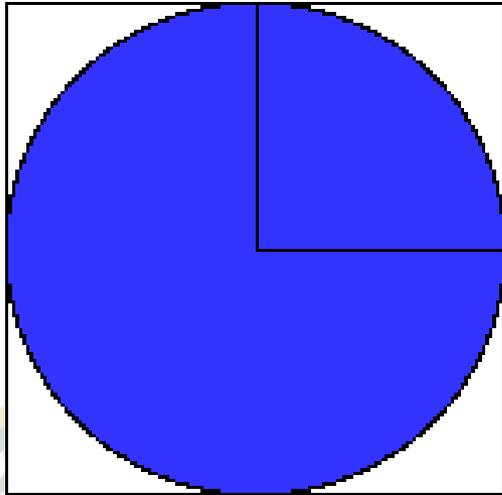
- Monte Carlo methods randomly select values to create scenarios of a problem by performing statistical sampling experiments.
- These values are taken from within a fixed range and selected to fit a probability distribution [e.g. bell curve, linear distribution, etc.].
- This is like rolling a dice. The outcome is always within the range of 1 to 6 and it follows a linear distribution - there is an equal opportunity for any number to be the outcome.

Why Monte Carlo?

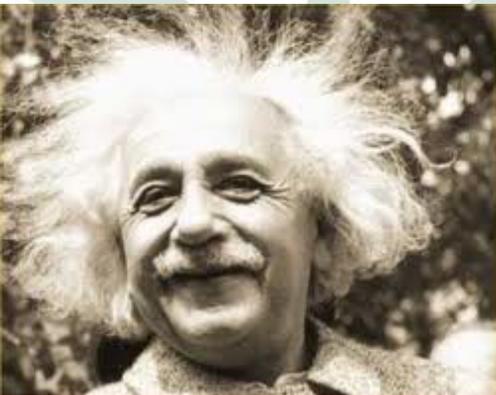
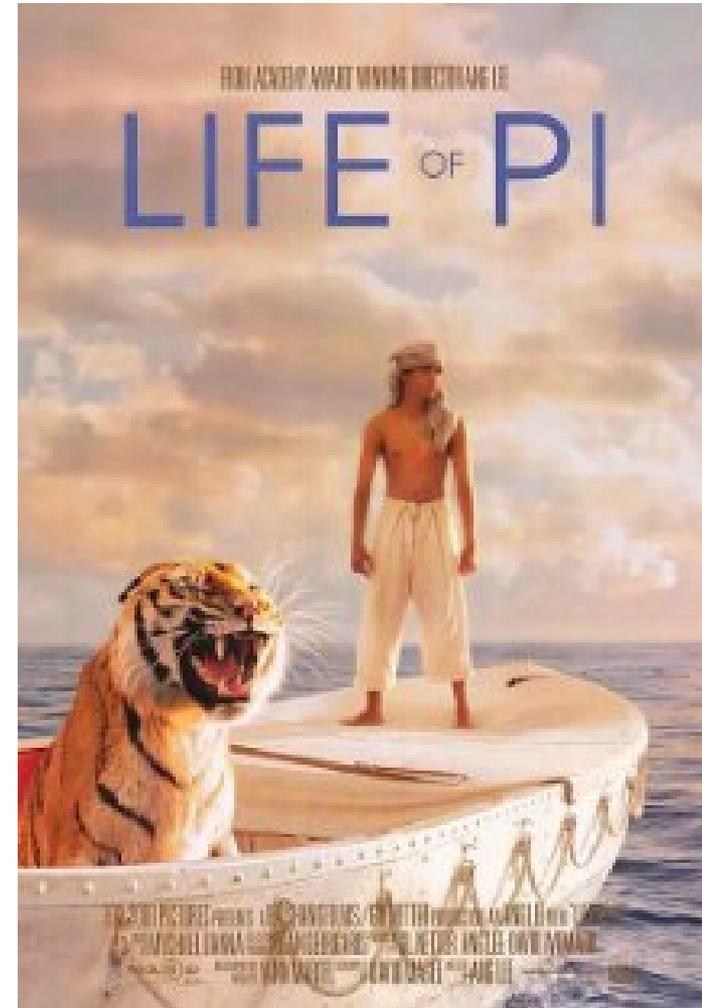
- The technique was named for Monte Carlo, Monaco, where the primary casino attractions are games of chance such as roulette wheels, dice, and slot machines.



Monte Carlo Example: Estimating π



3.1415926
 π

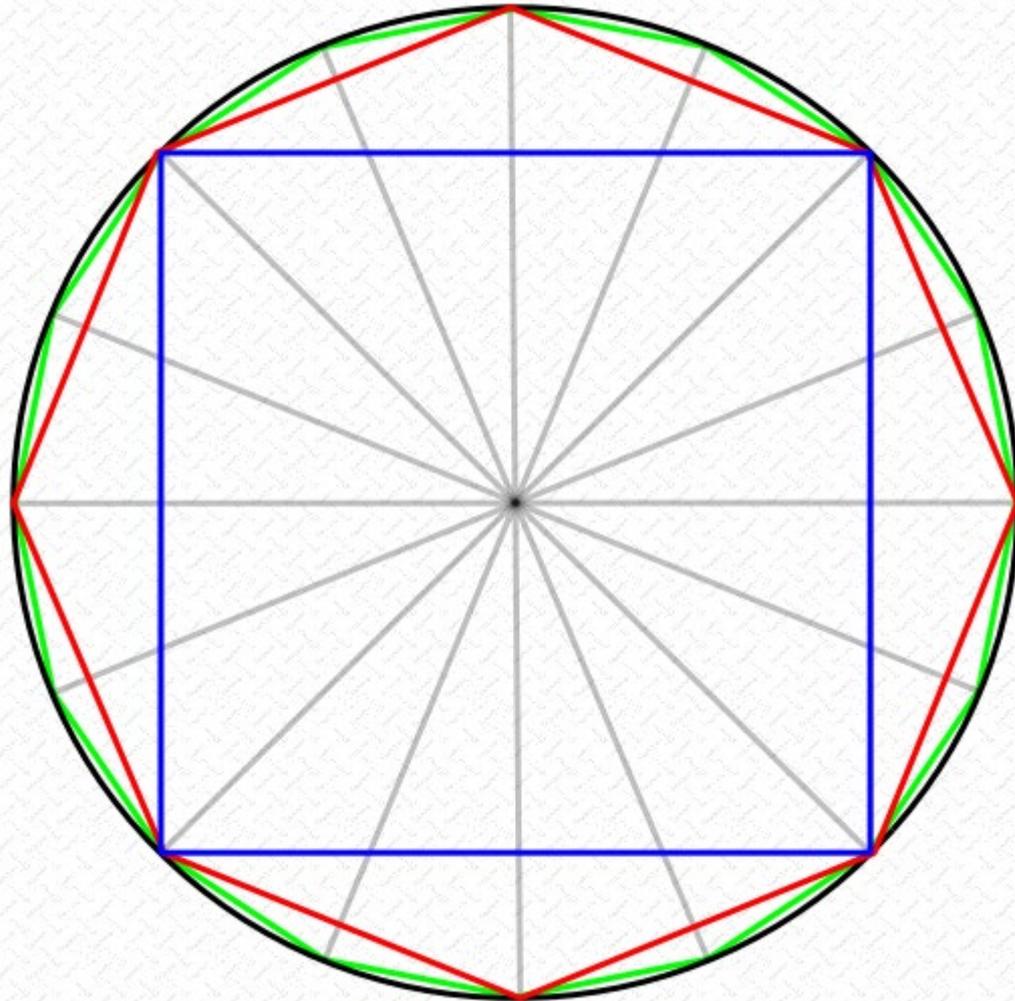


Monte Carlo Example: Estimating π

$$C = 2\pi r$$

- You can measure Pi by constructing a physical wheel and rolling it out.
- When a circle's diameter is 1 unit, its circumference is π units.
- - but you won't get more than a digit or two of accuracy.

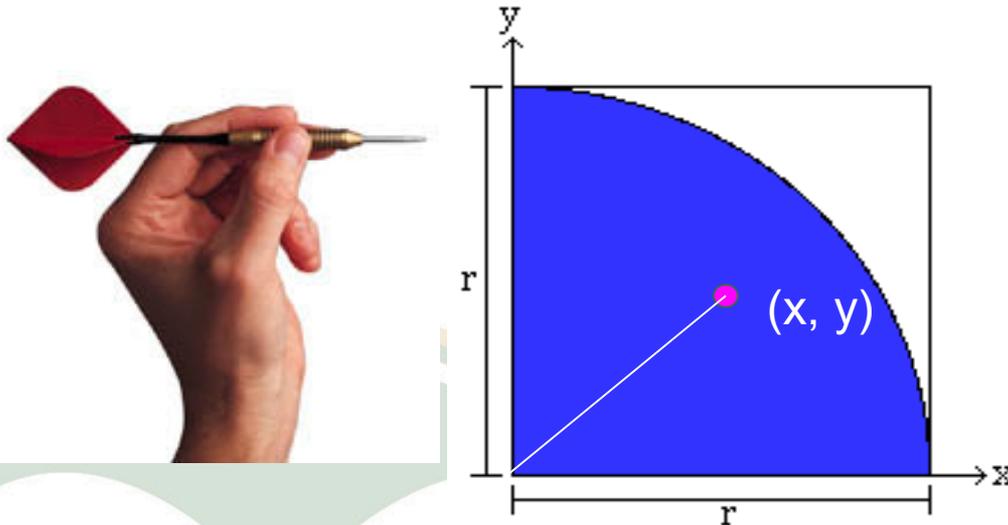
Monte Carlo Example: Estimating π



$$A = \pi r^2$$

You can measure Pi by bracketing a circle with polygons. This is easy with polygons with small numbers of sides, but it gets harder as you add sides.

Monte Carlo Example: Estimating π



$$\frac{\# \text{ darts hitting shaded area}}{\# \text{ darts hitting inside square}} = \frac{\frac{1}{4} \pi r^2}{r^2} = \frac{1}{4} \pi$$

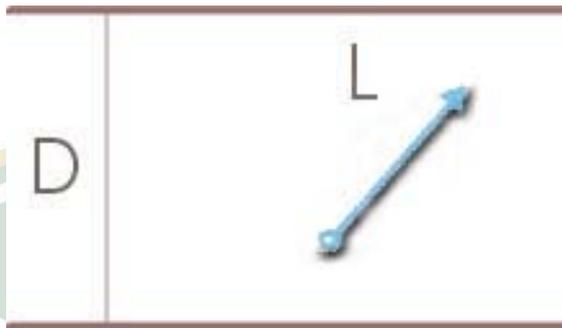
or

$$\pi = 4 \frac{\# \text{ darts hitting shaded area}}{\# \text{ darts hitting inside square}}$$

- If you are a very poor dart player, it is easy to imagine throwing darts randomly at the above figure, and it should be apparent that of the total number of darts that hit within the square, the number of darts that hit the shaded part (circle quadrant) is proportional to the area of that part.

Monte Carlo Example: Estimating π

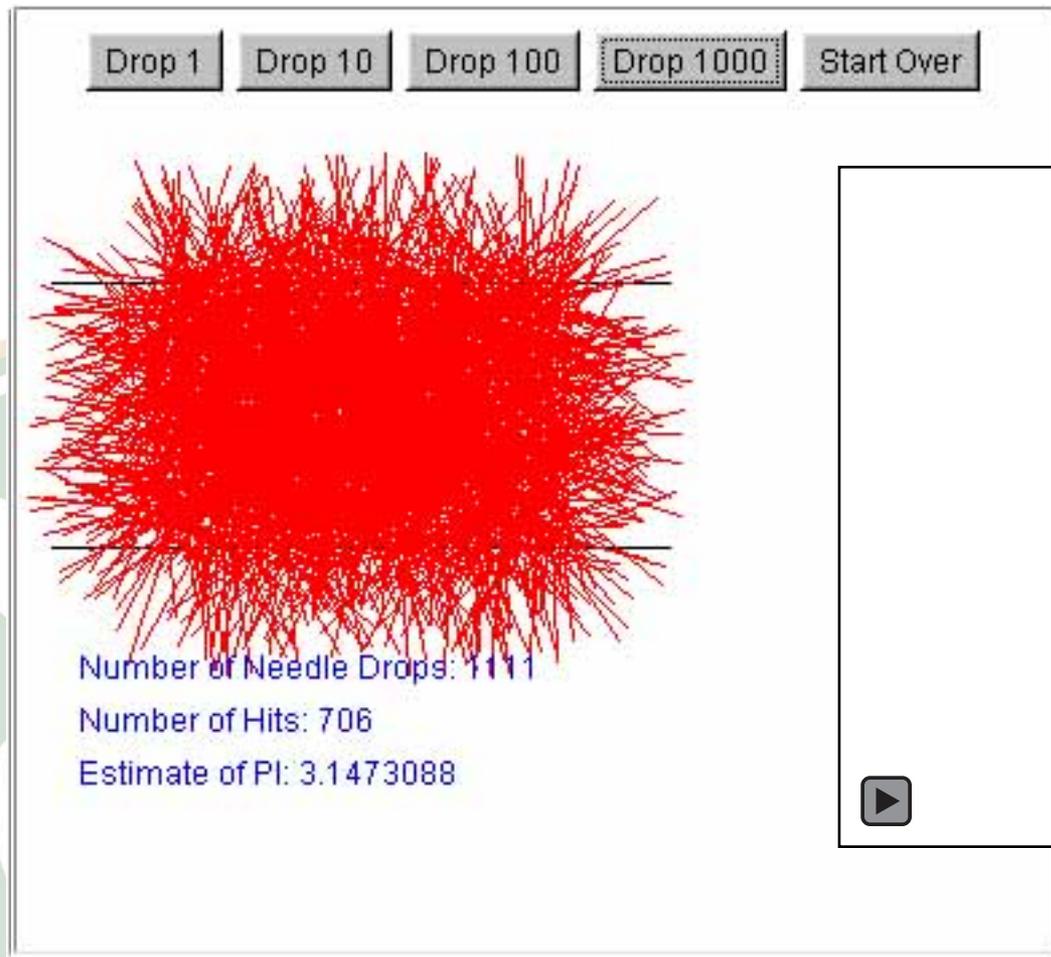
- Buffon's Needle Experiment:
- Drop a needle of length L at random on grid of parallel lines of spacing D .



Georges Louis Leclerc Comte de Buffon
(1707-1788)

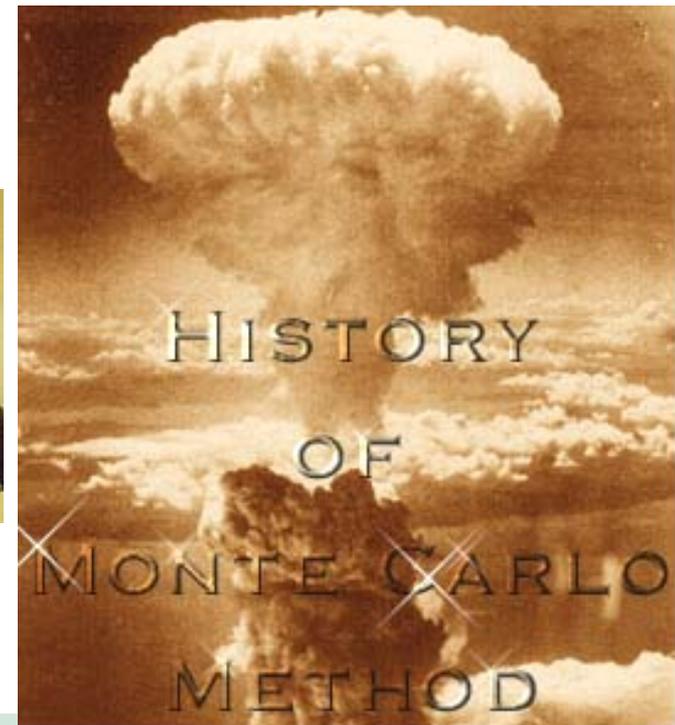
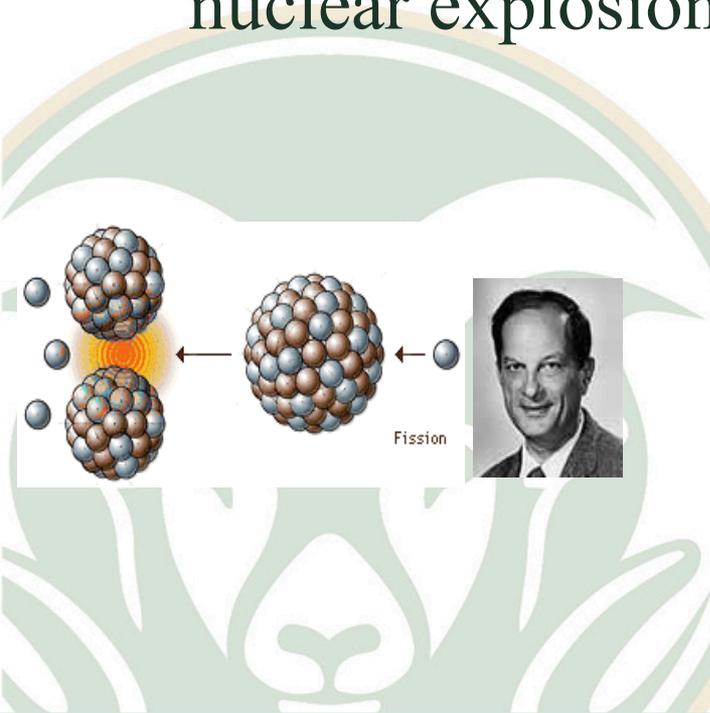
- For L less than or equal D we obtain
- $P(\text{needle intersects the grid}) = \frac{L}{D} \cdot \frac{2}{\pi}$
- If we drop the needle N times and count R intersections we obtain
- $P = \frac{L}{D} \cdot \frac{2}{\pi} = \frac{R}{N}$ \rightarrow $\pi = 2 \cdot \frac{L}{D} \cdot \frac{N}{R}$

Monte Carlo Example: Estimating π



What is Monte Carlo Simulation?

- In the late 1940s, scientists at the Manhattan Project at Los Alamos National Laboratory used Monte Carlo Simulation method to predict the range of possible nuclear explosion results.



Simulation

- Models that imitate real-life
- Same basic idea
 - Decision variables
 - Inputs (certain or uncertain?)
 - Outputs
- Key distinction
 - Use random numbers to drive results
- The advantage: It shows an entire distribution of results, not simply a bottom-line result

Simulation

- You Can Simulate Anything!
 - Playing poker
 - Investing in the stock market
 - Running a grocery store
- Cheap and fast experimentation: Like a lab
 - What if we use 4 checkout registers instead of 3 in the grocery store?
 - Simulation is cheaper than actually buying the equipment and see what happens.

Simulation: When NPV meet Reality

- An example: Calculating the NPV of the profits from a new modal car
- Lots of uncertainties, such as?
- We can build a spreadsheet model with our best guesses and report **a single NPV**
 - incomplete and probably misleading
- Or we can enter probability distributions for the uncertain quantities and see how the NPV varies as these quantities vary → **a distribution of NPVs**

Simulation

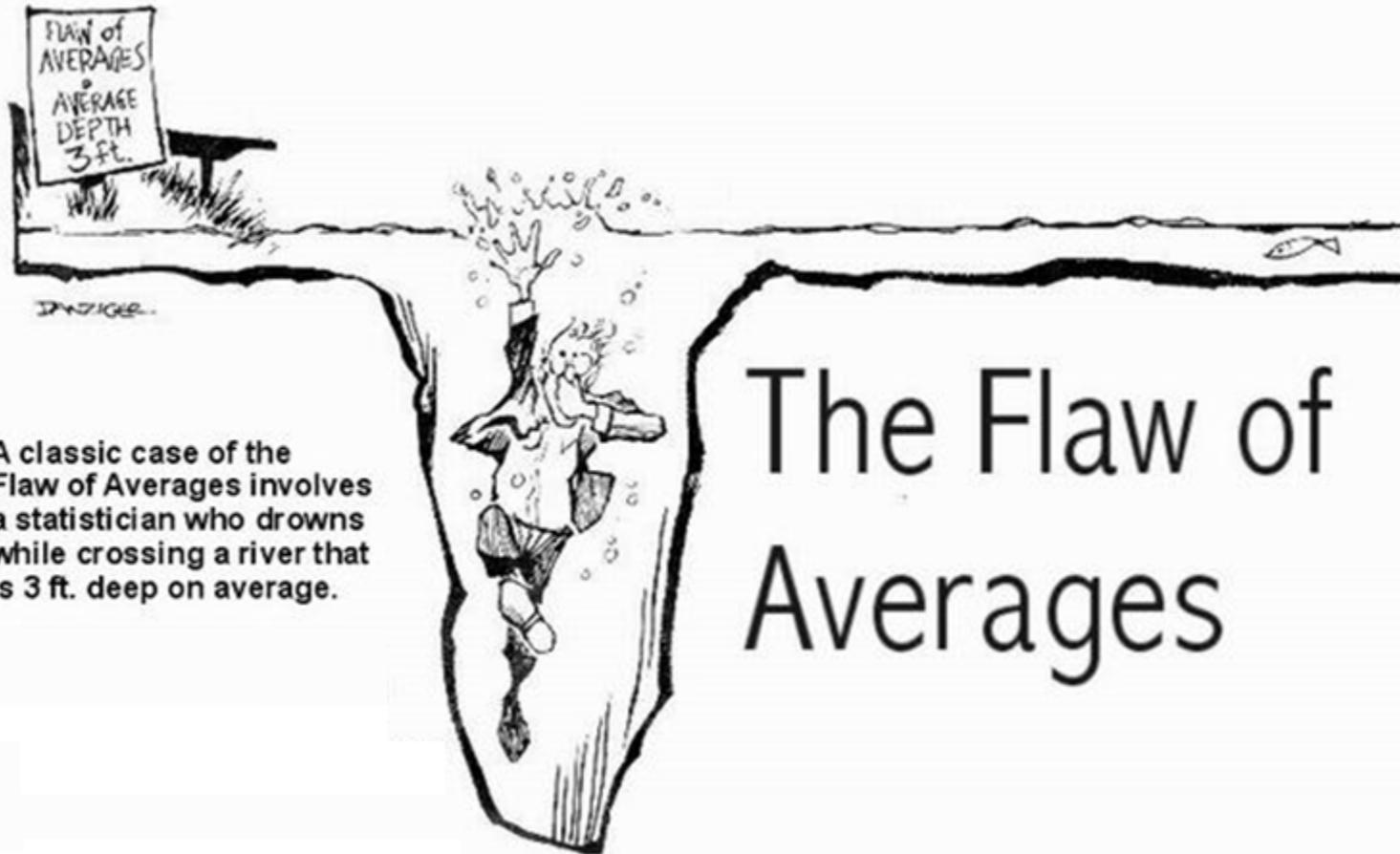
You can do risk analysis

- Find the worst and best case scenarios, the variance of payoffs etc.
- Answer questions like:
 - Which project is the riskiest?
 - What is the probability that the firm will not break-even?
 - What is the probability that the investment will yield at least a 20 % return?

Drawback of Simulation

- Monte Carlo method is often referred to as the “method of last resort”, as it is apt to consume large computing resources;
- Characteristics:
 - consuming vast computing resources
 - have historically had to be executed upon the fastest computers available at the time
 - and employ the most advanced algorithms
 - implemented with substantial programming acumen.

Cartoon



A classic case of the Flaw of Averages involves a statistician who drowns while crossing a river that is 3 ft. deep on average.

The Flaw of Averages



Generating Random Numbers

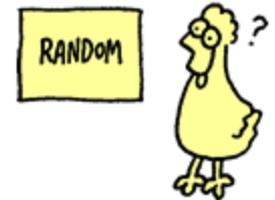
What is really about Randomness?

- **Does Anything Really Happen at Random?**
- According to Stanford statistics professor [Persi Diaconis](#), we shouldn't, or at least we should take great care in doing so.
- Diaconis looks closely at the physical and mathematical details behind such things as coin tosses, roulette wheels, dice-throwing and card-shuffling. He discovers that even the smallest changes in initial conditions (type of coin, force of spin, wind conditions, number of shuffles) can have big consequences.
- Coins spun on their edges on a tabletop do not come up 50/50.
- Coins spun in the air and allowed to land on the ground will exhibit less randomness than those caught in the hand.
- It takes at least how many riffle shuffles of a deck of cards to approximate randomness.



Pseudo-random numbers

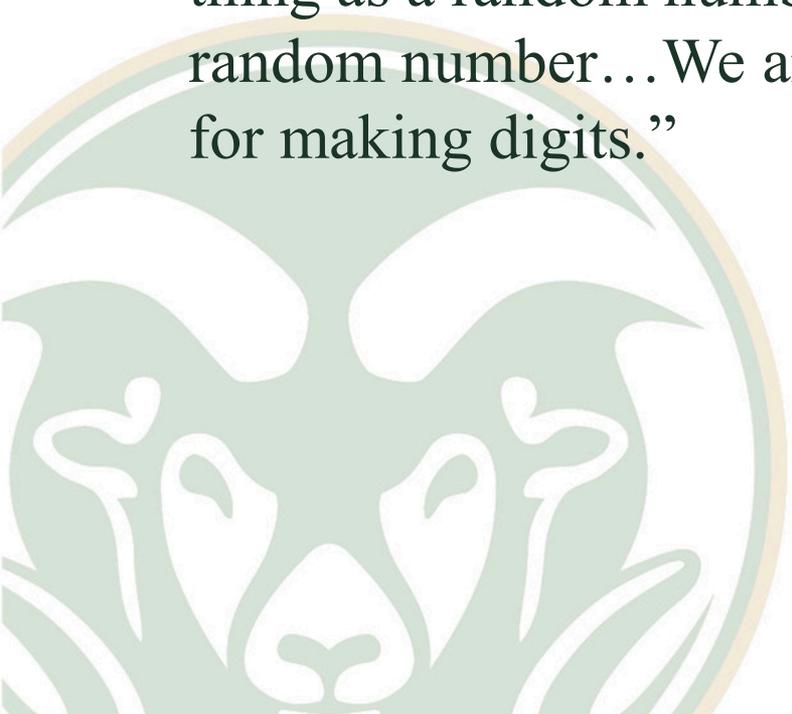
- **Almost random numbers!**
- Actually, they are “computer-generated random numbers.”
- Not truly random because there is an inherent pattern in any sequence of pseudo-random numbers.
- Lot of mathematics goes into building pseudo-random number generators (algorithms) so that the output is sufficiently “random”.
- Most of these algorithms generate $U(0,1)$ pseudo-random numbers.



Pseudo-random numbers

- **John von Neumann** (1951):

“Any one who considers arithmetical methods of producing random digits is, of course, in a state of sin....There is no such thing as a random number – there are only methods to produce random number... We are dealing with mere ‘cooking recipes’ for making digits.”



Random number generators

Properties of a “good” generator:

1. The number should appear to be distributed uniformly on $[0,1]$ and should not exhibit any correlation.
2. Fast and avoid need for lot of storage.
3. Able to reproduce a given stream of random numbers exactly. Why?
4. Provision for producing several separate streams of random numbers.



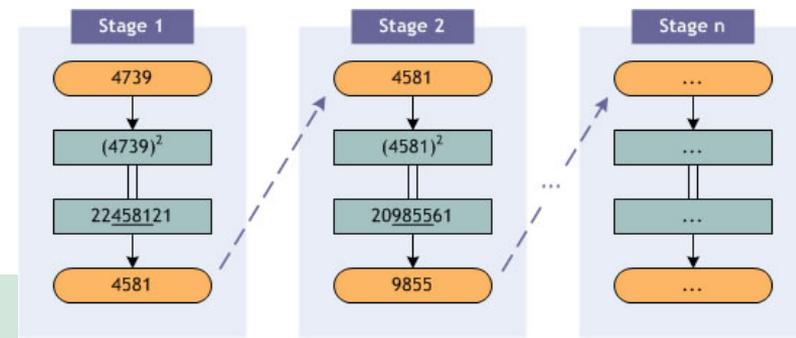
Middle Square Generator

The Middle-Square method –developed by von Neumann for use in high-speed calculations with the first true computer (ENIAC) – is seriously flawed. But it provides a simple means to examine pseudorandom numbers. The algorithm that creates these numbers is pretty simple and straightforward:

1. Begin with an n -digit seed number.
2. Square it to obtain a $2n$ -digit number, adding a leading zero if necessary.
3. Take the middle n digits as the next random number.
4. Repeat.



John von Neumann
(1903-1957)



Linear Congruential Generators (LCG)

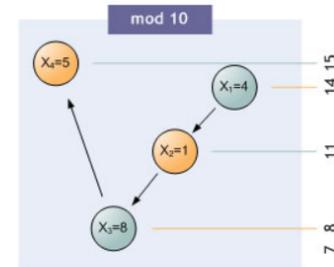
- Let X_0 be the seed value (initial value of the sequence).
- Then the next values are generated by:

$$X_n = (aX_{n-1} + c) \pmod{m}$$

- a = the multiplier; c = increment; and m = modulus.
- Obviously, X_n can take values: $0, 1, \dots, m-1$.
- The $U(0,1)$ pseudo-random number is generated by X_n/m .
- Values of a , m , and c are chosen such that LCG satisfies the properties of “good” generator.
- The Mersenne Twister: $2^{19937}-1$

Let $X_0 = 7$ and integers $a = 1$, $b = 7$, and $c = 10$.

- $X_1 = (1 \cdot 7 + 7) \pmod{10} = 4$
- $X_2 = (1 \cdot 4 + 7) \pmod{10} = 1$
- $X_3 = (1 \cdot 1 + 7) \pmod{10} = 8$
- $X_4 = (1 \cdot 8 + 7) \pmod{10} = 5$
- $X_5 = (1 \cdot 5 + 7) \pmod{10} = 2$
- $X_6 = (1 \cdot 2 + 7) \pmod{10} = 9$
- $X_7 = (1 \cdot 8 + 7) \pmod{10} = 5$



- $2^2-1 = 3$
- $2^3-1 = 7$
- $2^5-1 = 31$
- $2^7-1 = 127$
- $2^{13}-1 = 8,191$
- $2^{17}-1 = 131,071$
- $2^{19}-1 = 524,287$

Marin Mersenne
(1588 - 1648)



Linear Congruential Generators (LCG)

Example: $a = 13$, $c=0$, $m = 2^6=64$, and $X_0 = 1, 2, 3, 4$.

Period Determination Using Various seeds

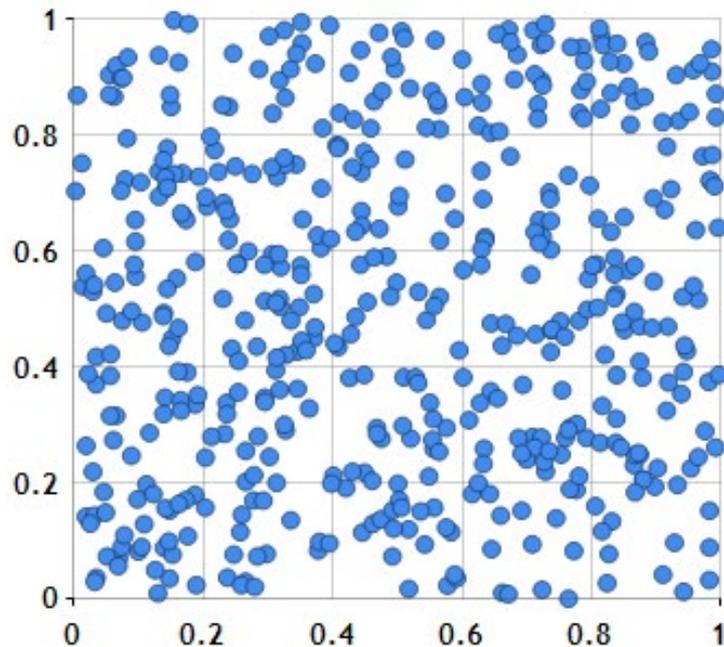
i	X_i	X_i	X_i	X_i
0	1	2	3	4
1	13	26	39	52
2	41	18	59	36
3	21	42	63	20
4	17	34	51	4
5	29	58	23	
6	57	50	43	
7	37	10	47	
8	33	2	35	
9	45		7	
10	9		27	
11	53		31	
12	49		19	
13	61		55	
14	25		11	
15	5		15	
16	1		3	

$$X_n = (aX_{n-1} + c) \pmod{m}.$$

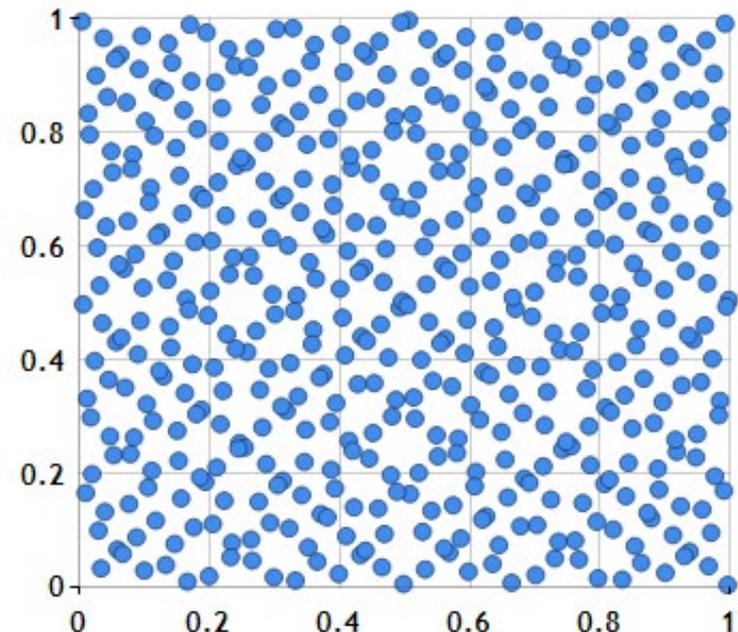


Pseudorandom vs Quasirandom

Pseudorandom



Quasirandom



Testing Random Number

Runs test

- Directly tests for independence.
- We examine the sequence of values for unbroken subsequence of maximal length within which the U_i 's increase/decrease monotonically.
- Such a sequence is called a *run*.
- Concerns:
 - Number of runs
 - Length of runs

Testing Random Number

Runs test

If N is the number of numbers in a sequence, the maximum number of runs is $N-1$, and the minimum number of runs is one.

If “ a ” is the total number of runs in a sequence, For $N > 20$, the distribution of “ a ” approximated by a normal distribution, $N(\mu_a, \sigma_a^2)$

$$\mu_a = \frac{(2N - 1)}{3}$$

$$\sigma_a^2 = \frac{(16N - 29)}{90}$$

Testing Random Number

Based on runs up and runs down, determine whether the following sequence of 40 numbers is such that the hypothesis of independence can be rejected where $\alpha = 0.05$.

0.41	0.68	0.89	0.94	0.74	0.91	0.55	0.62	0.36	0.27
0.19	0.72	0.75	0.08	0.54	0.02	0.01	0.36	0.16	0.28
0.18	0.01	0.95	0.69	0.18	0.47	0.23	0.32	0.82	0.53
0.31	0.42	0.73	0.04	0.83	0.45	0.13	0.57	0.63	0.29

The sequence of runs up and down is as follows:

+++ - + - + - - - + + - + - - + - + - - + - - + - + + - - + + - + - - + + -

Testing Random Number

- There are 26 runs in this sequence.
- With $N=40$ and $a=26$,

$$\mu_a = \frac{(2 \times 40 - 1)}{3} = 26.33 \quad \sigma_a^2 = \frac{(16 \times 40 - 29)}{90} = 6.79$$

- Then,
- $Z = (26 - 26.33) / \text{sqrt}(6.79) = -0.13$
 - Now, the critical value is 1.96, so the independence of the numbers cannot be rejected on the basis of this test.

Inverse transformation method

- We wish to generate a random variate X that is continuous and has a distribution function that is continuous and strictly increasing when $0 < F(x) < 1$.
- Let F^{-1} denote the inverse of the function F .
- Then the inverse transformation algorithm is:
 1. Generate $U \sim U(0, 1)$
 2. Return $X = F^{-1}(U)$.
- To show that the returned value X has the desired distribution F , we must show $\Pr\{X \leq x\} = F(x)$.

$$\begin{aligned}\Pr(F^{-1}(U) \leq x) &= \Pr(F(F^{-1}(U)) \leq F(x)) \\ &= \Pr(U \leq F(x)) \\ &= F(x)\end{aligned}$$

Generating Random Variables

Uniform (a, b)

Using inverse-transform method

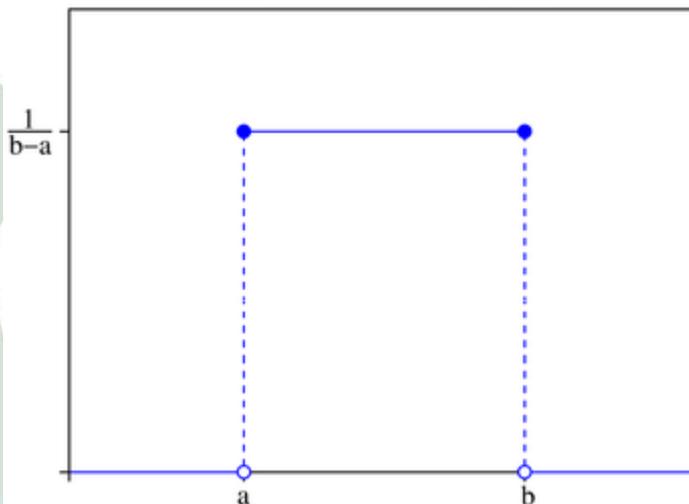
1. Generate $U \sim U(0, 1)$.
2. Return $X = a + (b - a)U$.

$$f(x) = \frac{1}{b - a}$$

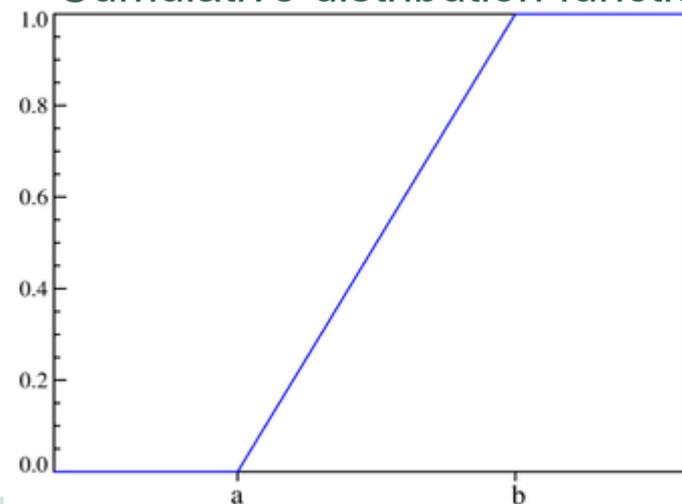
$$F(x) = \frac{x - a}{b - a}$$

$$X = F^{-1}(u) = a + (b - a)u$$

Probability density function



Cumulative distribution function



Generating Random Variables

Exponential (β)

Once again, using inverse transform method

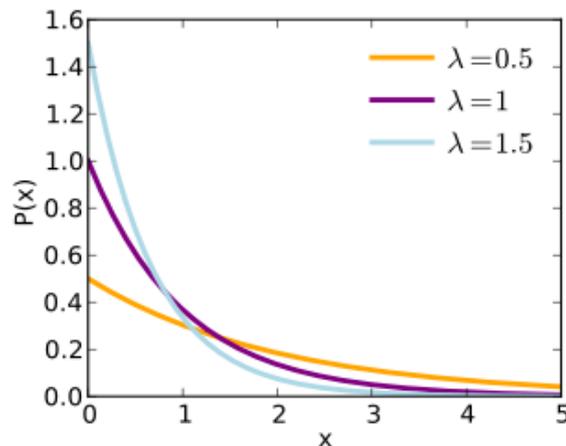
1. Generate $U \sim U(0,1)$.
2. Return $X = -1/\lambda * \ln(U)$.

$$f(x) = \lambda e^{-\lambda x}$$

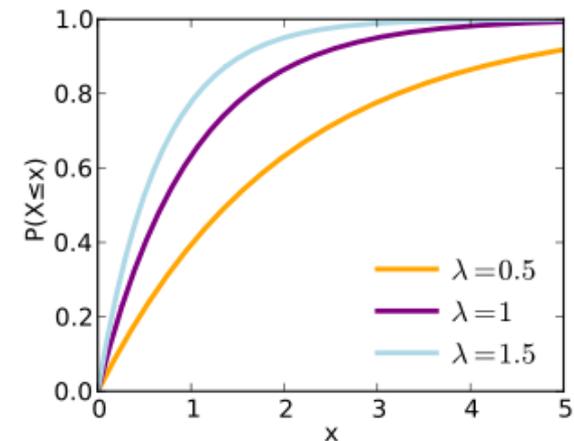
$$F(x) = 1 - e^{-\lambda x}$$

$$X = F^{-1}(u) = -\frac{\ln(u)}{\lambda}$$

Probability density function



Cumulative distribution function



Generating Normal Variable

Normal – $N(0, 1)$

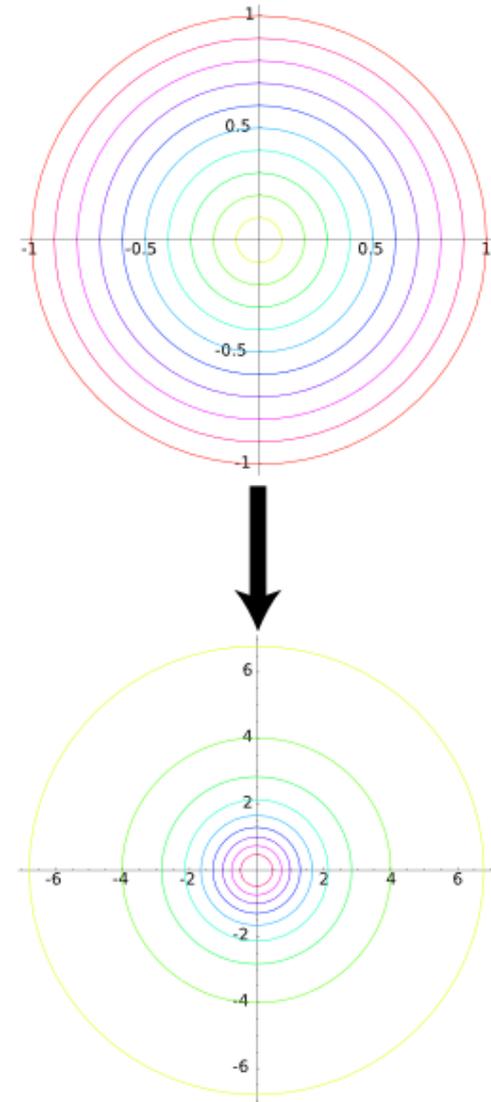
- The Box-Muller Transformation

1. Generate U_1, U_2 as IID $U(0, 1)$.

2. Set $X_1 = \sqrt{-2 \ln(U_1)} \cos(2\pi U_2)$

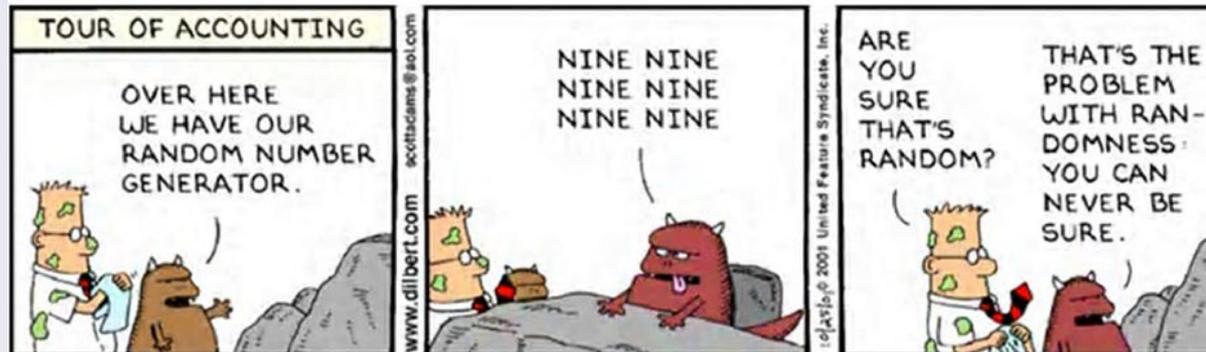
$$X_2 = \sqrt{-2 \ln(U_1)} \sin(2\pi U_2)$$

$$X_1, X_2 \sim N(0, 1).$$



Cartoon

Randomness





Monte Carlo Simulation with Excel

Monte Carlo Simulation with Excel

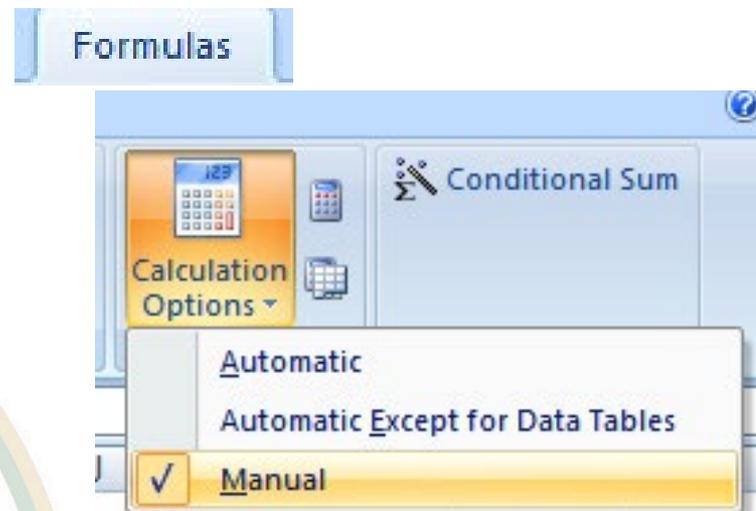
- Use pseudo-random numbers in place of fixed inputs in a spreadsheet
- Draw a random number many times
- Perform a **statistical analysis** of the results

In Excel:

- RAND()
 - Pseudo-random number between (0,1)
- Each time the spreadsheet recalculates (e.g. whenever you calculate a new formula), new random numbers appear → new output values

Side Note: Working with Random #s

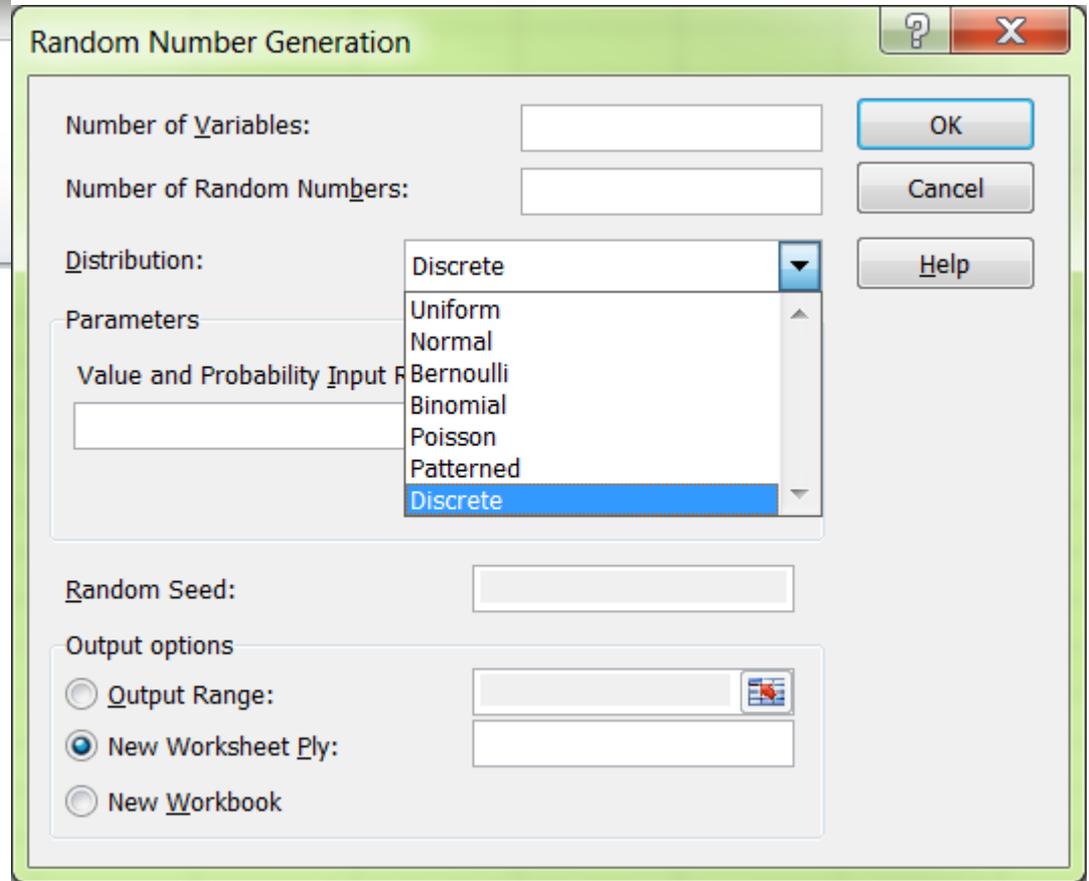
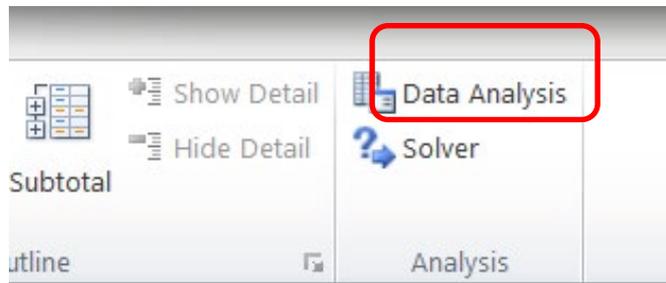
- Recalculation can be slow and annoying



F9 to recalculate

- Or, Paste Special Values to replace random #s

Monte Carlo Simulation with Excel

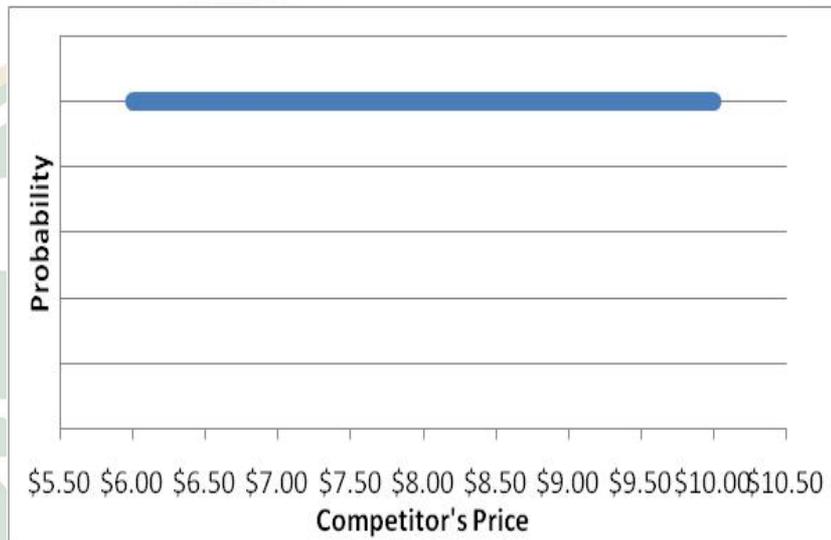


Some Common Distributions

- Uniform
 - Equally likely to be between two points
- Discrete
 - Few outcomes, probability and value
- Normal
- Triangular
 - Min, Mode, Max

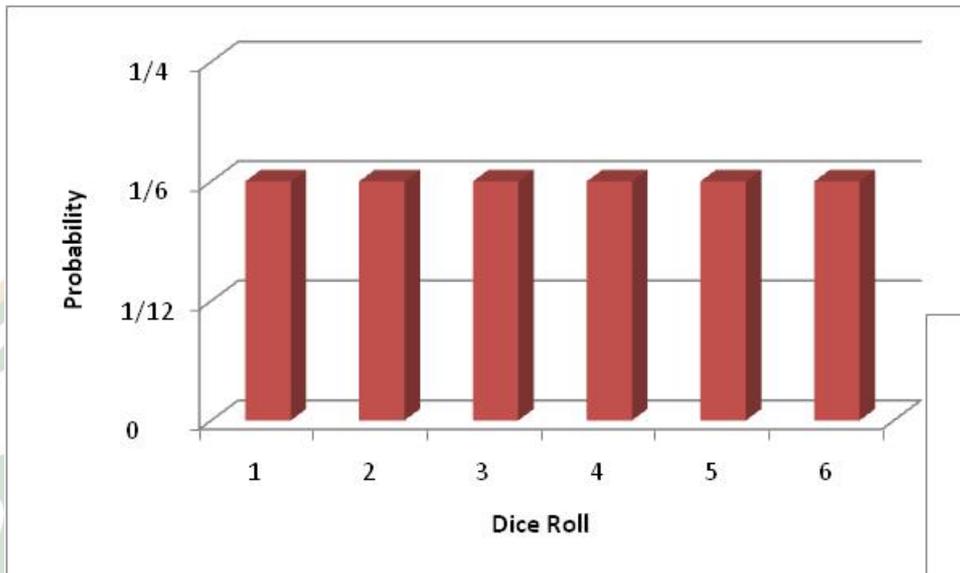
Uniform Distribution

- The value will be between these two numbers and all possibilities are equally likely
 - Competitor will charge a price between \$6 and \$10

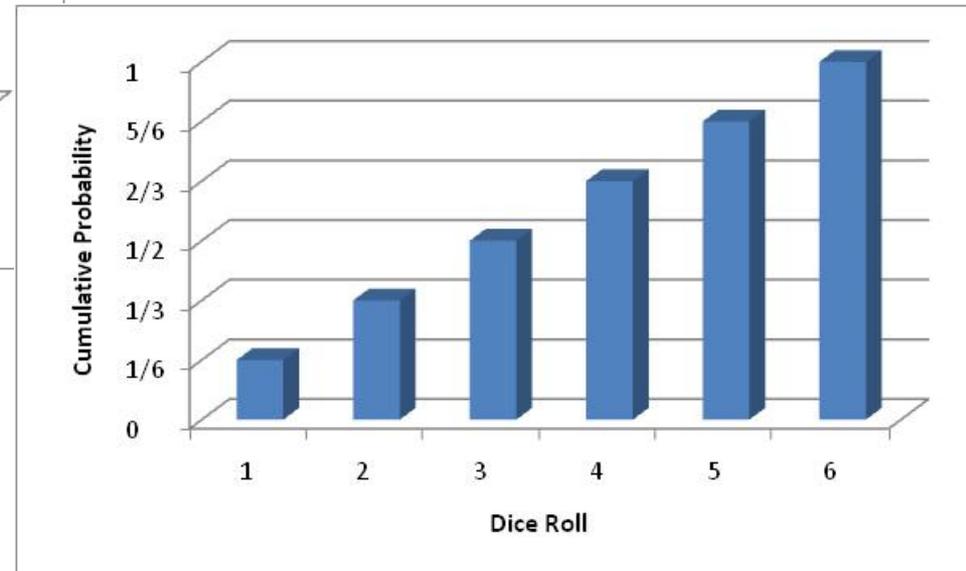


=6 + RAND()*(10-6) with Excel

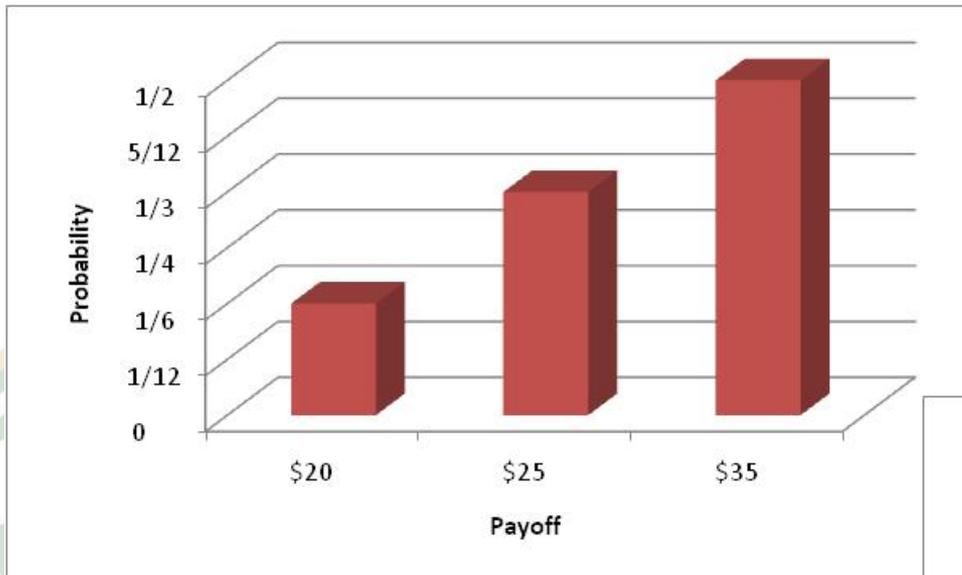
Discrete Distribution



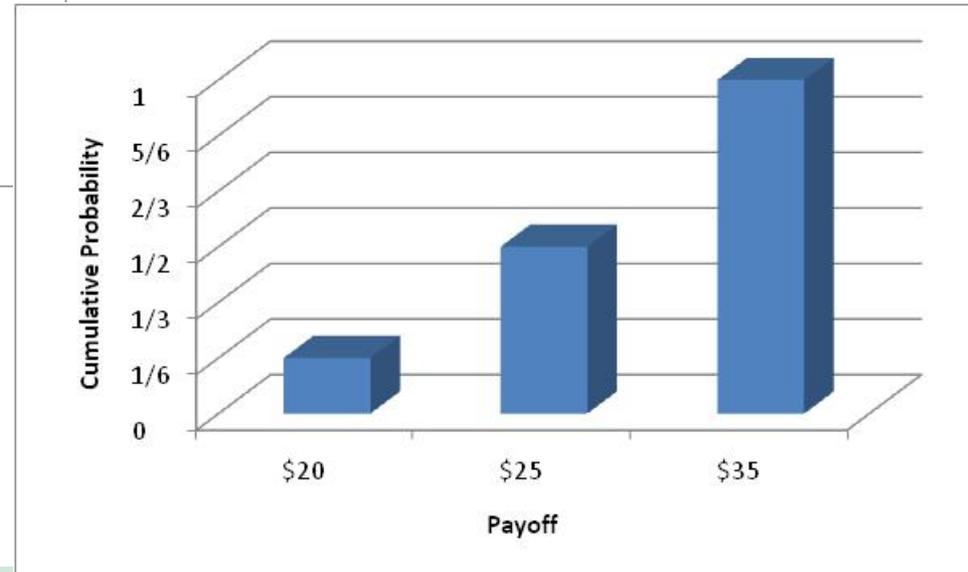
Dice Roll	Probability	Cumulative Probability
1	1/6	1/6
2	1/6	1/3
3	1/6	1/2
4	1/6	2/3
5	1/6	5/6
6	1/6	1



Can be General



Payoff	Probability	Cumulative Probability
\$20	1/6	1/6
\$25	1/3	1/2
\$35	1/2	1



Discrete Distribution

- Consider the following demand distribution

Demand	Probability
100	0.30
150	0.20
200	0.30
250	0.15
300	0.05

- You can use the RAND() function to simulate:
 - If $\text{RAND}() \leq 0.3 \rightarrow \text{Demand} = 100$
 - If $0.3 < \text{RAND}() \leq 0.5 \rightarrow \text{Demand} = 150$
 - If $0.5 < \text{RAND}() \leq 0.8 \rightarrow \text{Demand} = 200$
 - If $0.8 < \text{RAND}() \leq 0.95 \rightarrow \text{Demand} = 250$
 - If $0.95 < \text{RAND}() \leq 1 \rightarrow \text{Demand} = 300$

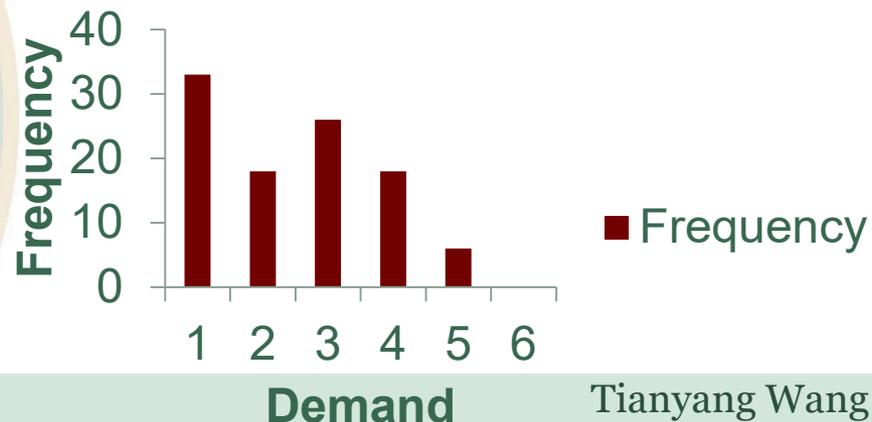
Discrete Distribution

- Consider the following demand distribution

G3 fx =VLOOKUP(F3,\$A\$3:\$C\$7,2)

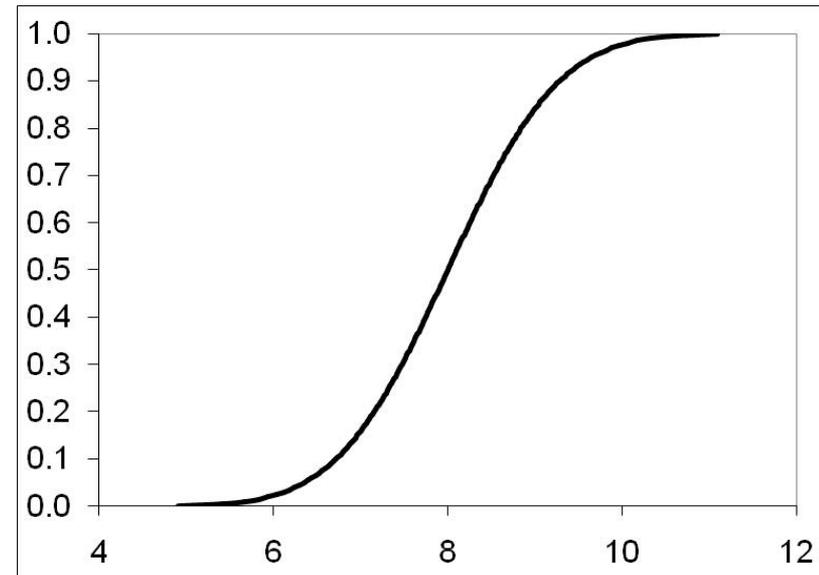
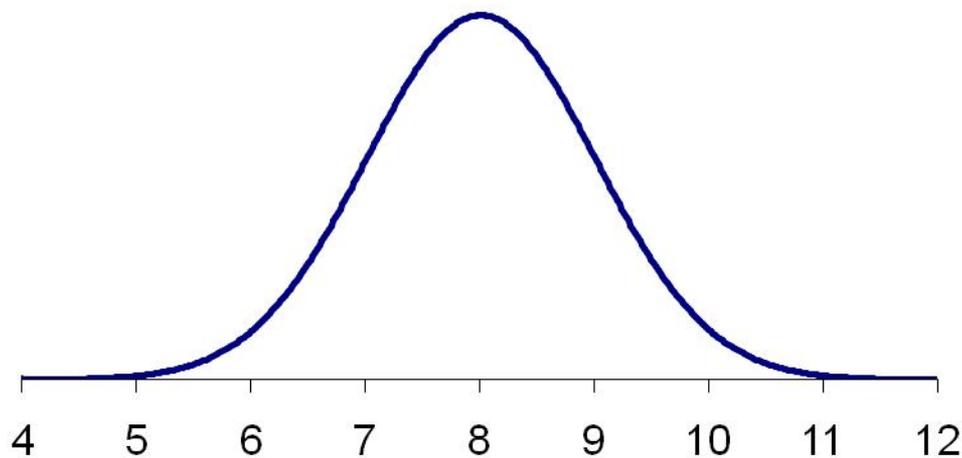
	A	B	C	D	E	F	G
1		Demand distribution			Simulation		
2	Cum. Probability	Demand	Probability		Replication	Random #	Demand
3	0.00	100	0.30		1	0.871642	250
4	0.30	150	0.20		2	0.229104	100
5	0.50	200	0.30		3	0.565745	200
6	0.80	250	0.15		4	0.841623	250
7	0.95	300	0.05		5	0.663662	200
8					6	0.437191	150

Histogram



Normal Distribution

- The price charged by our competitor will have mean of \$8 and a standard deviation of 1



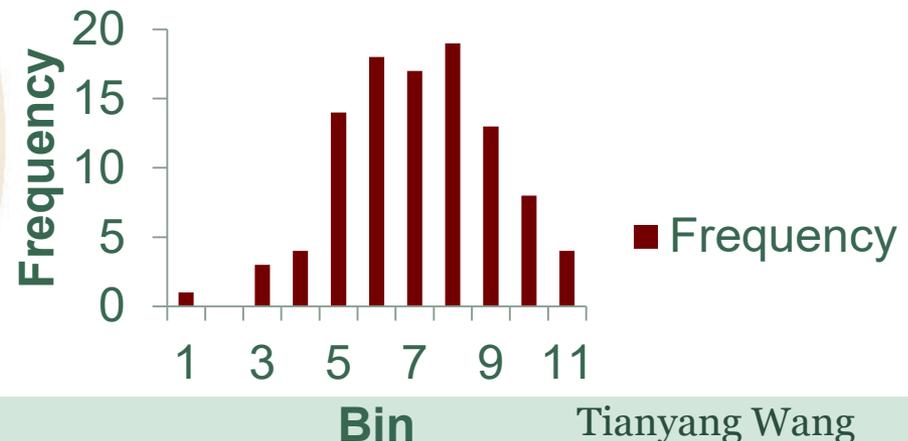
=NORMINV(RAND(),Mean,Stdev) with Excel

Normal Distribution

- Consider the following Price distribution

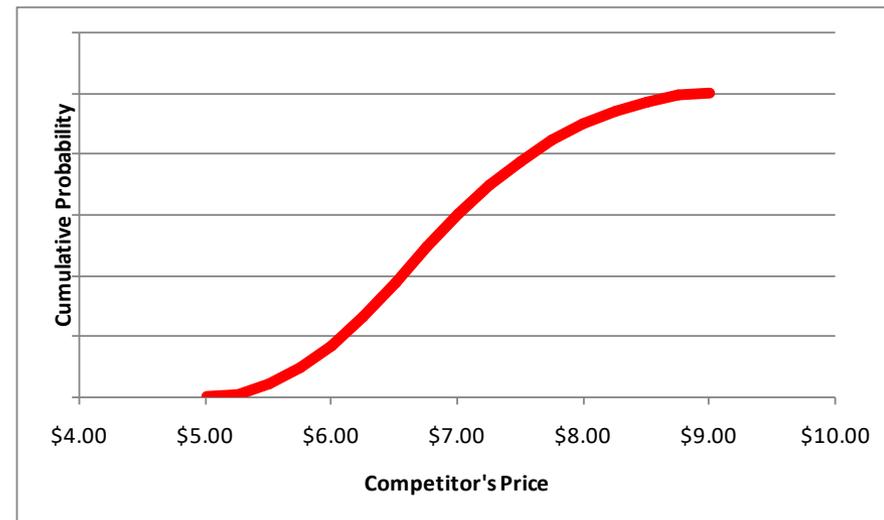
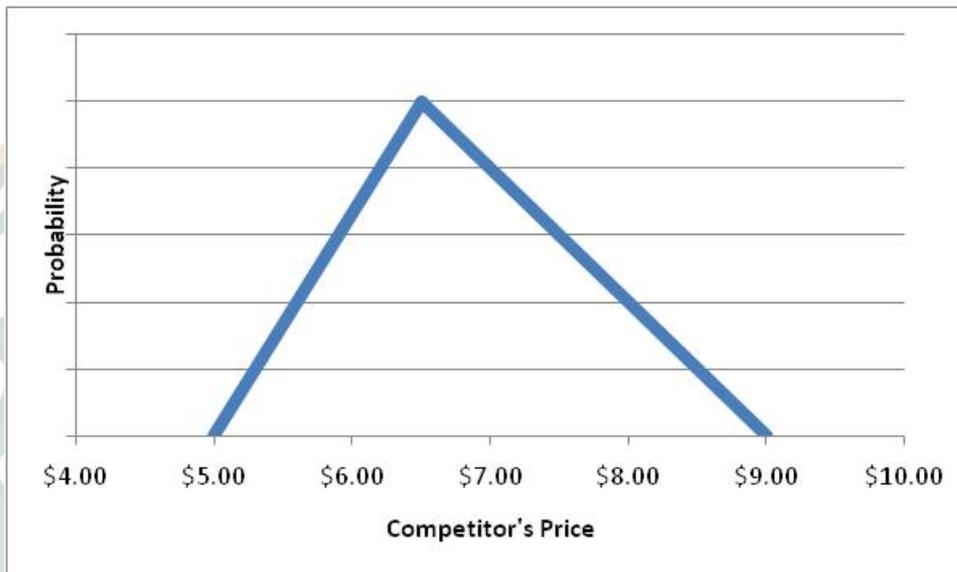
		fx		=NORMINV(F3,\$B\$2,\$B\$3)			
G3							
	A	B	C	D	E	F	G
1		Normal Demand			Simulation		
2	Mean	8			Replication	Random #	Demand
3	Standard Deviation	1			1	0.906903	9.321924
4					2	0.859087	9.076226
5					3	0.525009	8.06273
6					4	0.324294	7.544275

Histogram



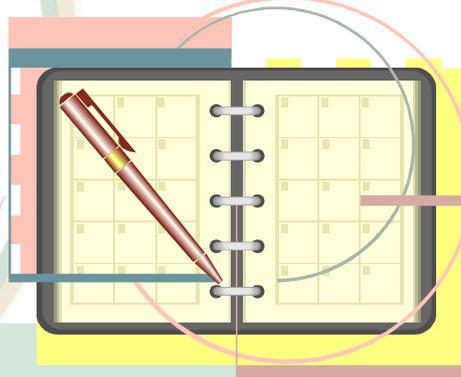
Triangular Distribution

- The price our competitor will charge is between \$5 and \$9 and is most likely \$6.50



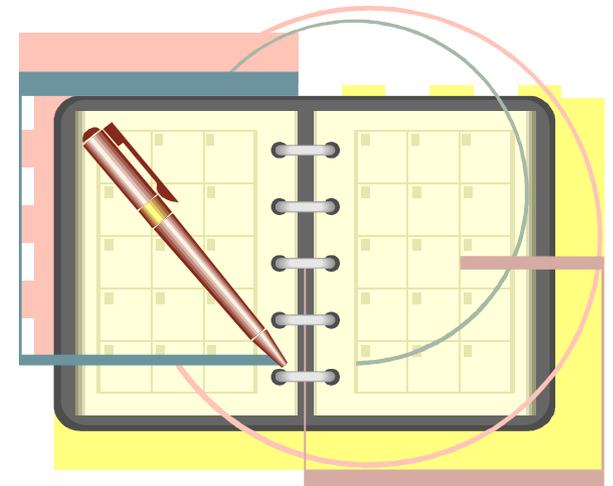
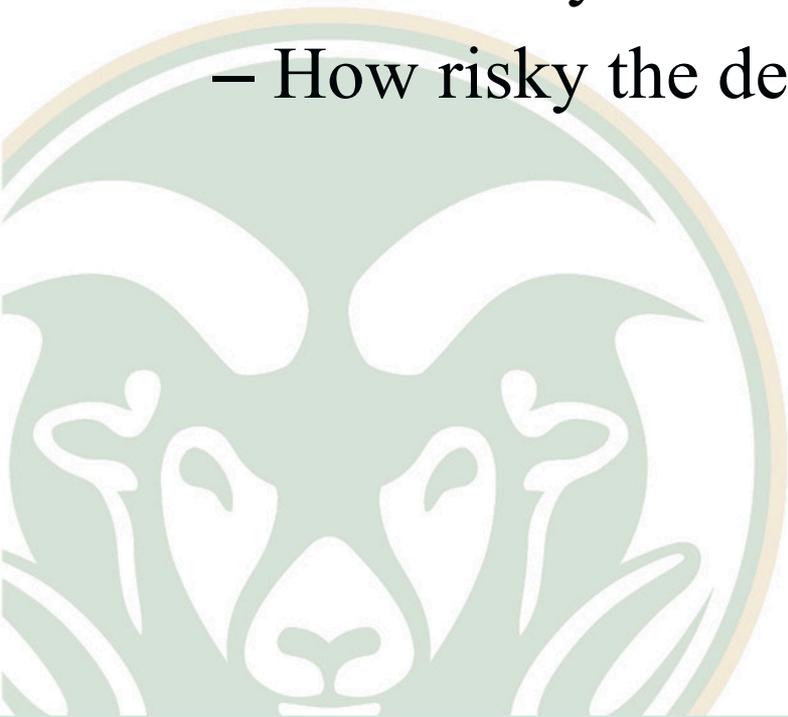
Walton Bookstore: Calendar Order

- Problem description
 - Each calendar costs \$7.50; sells for \$10.00
 - Calendar is a **perishable** product
 - Unsold calendars returned to publisher for \$2.50
 - Demand is uncertain
 - Probability distribution
 - Order quantity must be decided **before** knowing demand



Walton Bookstore: Calendar Order

- Managerial questions:
 - What happens if ordering 200?
 - How many calendars to order?
 - How risky the decision is? Best/worst case?



Walton Bookstore: Calendar Order

- What should be the objective ?
- What is (are) the decision variable(s) ?
- Other than the decision variable(s), what else is needed to find the payoff (profit) ?

Set up the Basic Model

- Model setup
 - Objective – Expected profit
 - Decision variable – Order quantity (trial value: 200)
 - Input Probability Distribution – *We will first use a discrete distribution, then a triangular distribution*
 - Statistical Summary: Min, max, average profit and the standard deviation
- Run simulation (1000 iterations)

Use Excel's Built-in Functions

- Walton believes that the number of calendars it can sell by February 1 follows this probability distribution

Probability Distribution of Demand	
Demand	Probability
100	0.30
150	0.20
200	0.30
250	0.15
300	0.05

- In reality, it is clear that other values of demand are possible

Use Excel's Built-in Functions

- Generate random demand values following this discrete distribution → use RAND() and VLOOKUP() functions as we did before.
- Calculate the profit for each demand value.
- Then calculate the statistics like the average profit, min profit, max profit and the standard deviation of the profit using the Excel functions AVERAGE(), MIN(), MAX() and STDEV().

Use Excel's Built-in Functions

- The setup looks like this:

Simulation of Walton's bookstore						
Cost data		Demand distribution				
Unit cost	\$7.50		Demand	Probability		
Unit price	\$10.00	0	100	0.30		
Unit refund	\$2.50	0.30	150	0.20		
		0.50	200	0.30		
Decision variable		0.80	250	0.15		
Order quantity	200	0.95	300	0.05		
Summary measures for simulation below						
Average profit	\$183.50	95% confidence interval for expected profit				
Stdev of profit	\$327.87	Lower limit	\$163.18			
Minimum profit	-\$250.00	Upper limit	\$203.82			
Maximum profit	\$500.00					
Simulation						
Replication	Random #	Demand	Revenue	Cost	Refund	Profit
1	0.8716424	250	\$2,000	\$1,500	\$0	\$500
2	0.2291037	100	\$1,000	\$1,500	\$250	-\$250
3	0.5657451	200	\$2,000	\$1,500	\$0	\$500
4	0.8416225	250	\$2,000	\$1,500	\$0	\$500

Observations

- If we order 200 calendars
 - On average we earn \$183.50
 - Risk: the standard deviation of the profit is \$327.87!
 - 31.60% of the time the profit is below 0!
- With computer simulation each time it is run the answers will be slightly different. → Use a confidence interval.

Observations

- Confidence interval is given by:

$$\bar{X} \pm 1.96 \frac{s}{\sqrt{n}}$$

Average Profit

Standard deviation obtained from the simulation

n=# of iterations

Gives a 95% confidence interval

- In this case, we are 95% confident that the profit is between \$163.18 and \$203.82, when we order 200.

$$\$183.50 \pm 1.96 \frac{327.87}{\sqrt{100}} = (\$163.18, \$203.82)$$

Observations

- Further managerial questions:
 - Is 200 a good decision?
 - Can we do better than that?
- Do a data table of average profit versus order quantity with different order quantities.

Order quantity	AvgProfit
	\$183.50
100	\$250.00
120	\$252.60
140	\$255.20
160	\$241.90
180	\$212.70
200	\$183.50
220	\$113.05
240	\$42.60
260	(\$39.48)
280	-133.175
300	-226.875

Choosing Probability Distributions

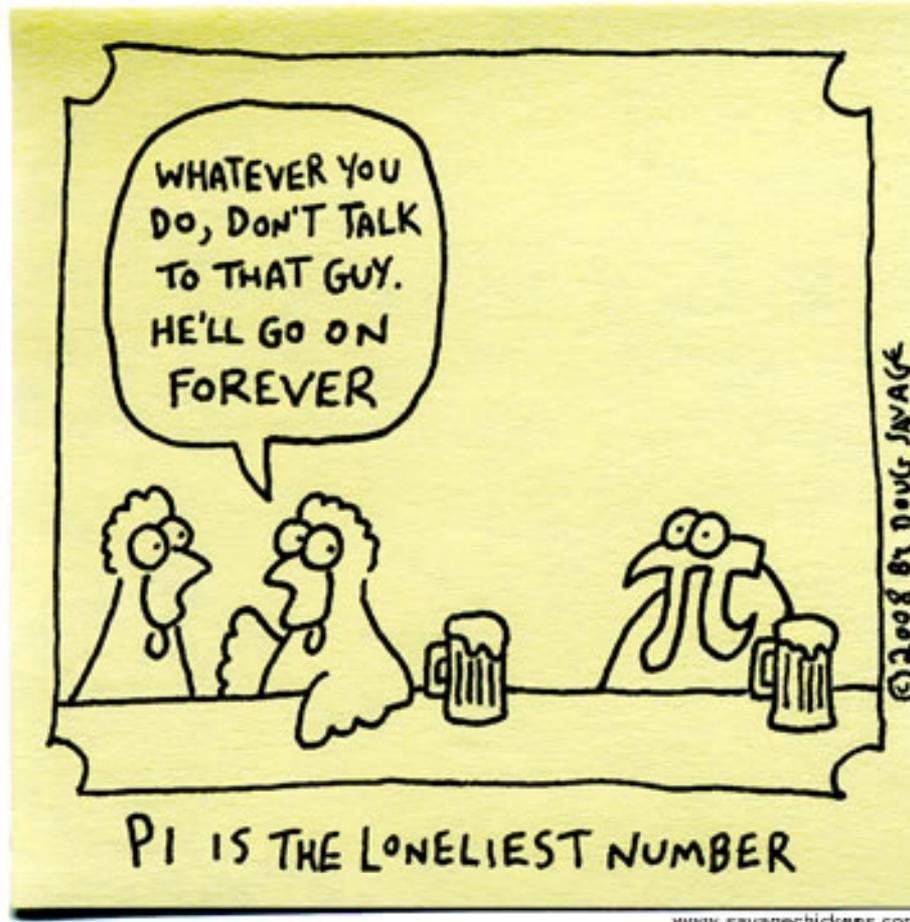
- Bootstrapping
- Parameter estimation
- Best fit
- Subjective guess



Cartoon

Savage Chickens

by Doug Savage



www.savagechickens.com