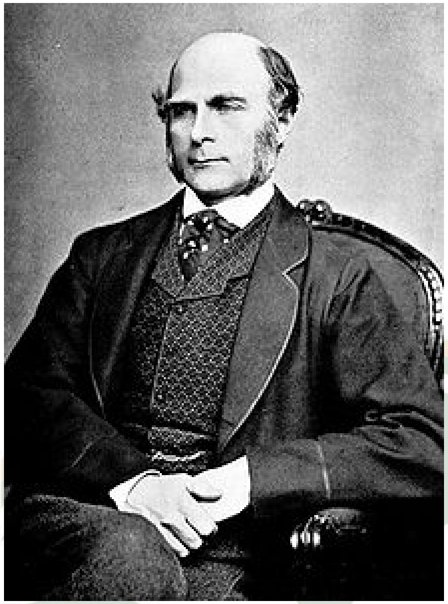
Financial Risk Management

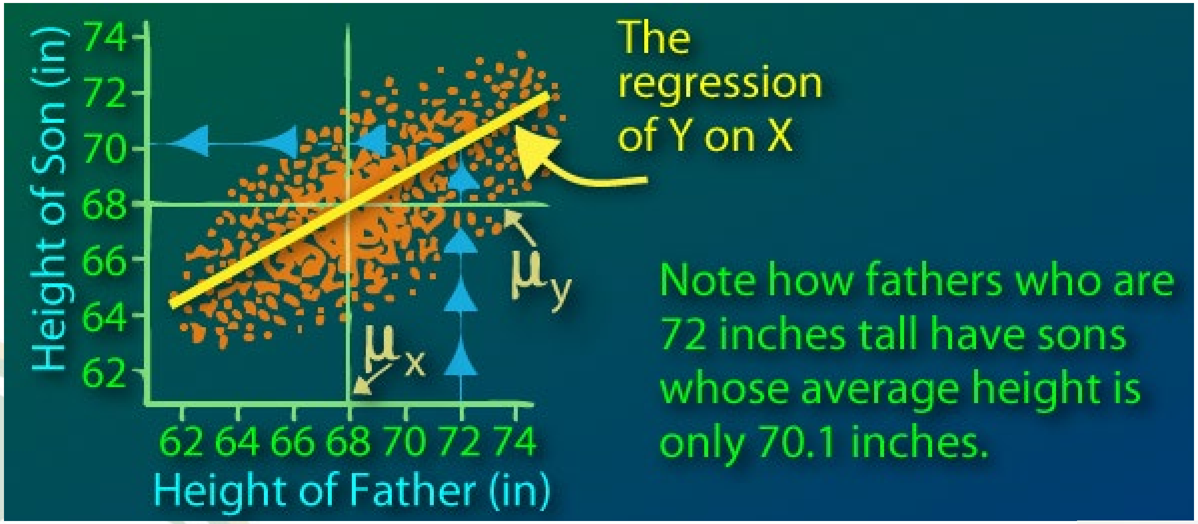
MSBA in Financial Risk Management

Mean Reversion Process

Regression Toward the Mean



Sir Francis Galton
(1822-1911)



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between two far extremes,
It usually swings back naturally
to values in between.

Regression Toward the Mean

If something varies normally between two far extremes,
It usually swings back naturally to values in between.
From sport to crime and illnesses, we see this common theme,
An effect we call, statistically, '*regression to the mean*'.

When you are playing cards and holding up a queen,
You know that next a lower card is likely to be seen,
Because there are so many cards much lower than a queen.
It's simple probability, it's '*regression to the mean*'.

Regression Toward the Mean

If something varies normally between two far extremes,
It usually swings back naturally to values in between.
From sport to crime and illnesses, we see this common theme,
An effect we call, statistically, '*regression to the mean*'.

When Brucie plays his cards right and he's holding up a queen,
You know that next a lower card is likely to be seen,
Because there are so many cards much lower than a queen.
It's simple probability, it's '*regression to the mean*'.

Random fluctuations of performances in sports,
Befuddle sports professionals, who use gimmicks of all sorts,
Crystals, magnets, copper bracelets they esteem,
But improvements in achievement are '*regression to the mean*'.

A man with awful backache, that sometimes gets much worse,
May turn to herbal remedies and swear his pain's reversed.
Perhaps it has, but not because the herbalist intervened,
The pain will ease quite simply through '*regression to the mean*'.

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Regression Toward the Mean

Evidence-based treatments that doctors should assign,
Use tests that will be randomized, controlled and double-blind,
Stopping self-deception before those test-results are seen,
It stops them being fooled by '*regression to the mean*'.

Big international drug firms are rightly criticized,
For concealing contrary evidence and other sorts of lies,
So how come homeopathy and herbalists have been.
Permitted to bamboozle us with '*regression to the mean*'.

While scientific medicine cann't cure all ills,
It does do so much better than those homeopathic sugar pills,
That a child should die through want of proper treatment is obscene,
When wishful-thinking parents fall for '*regression to the mean*'.

When much that was untreatable is curable, it's tragic,
That patients turn from medicine to phony cure-all magic.
The alternative to thinking is delusionary dreams,
Alas, from death there will be no '*regression to the mean*'.

Mean Reversion Process

- As we have seen previously, the variable tends to achieve values very different from its initial value in the GBM.
- Although this can be realistic to model the value of the majority of securities, there is a group of securities that don't behave that way.
- It is believed that many securities like oil, copper, agricultural products and other comodities have their price correlated with its marginal cost of production, while they may suffer random variations in the short term.
- To the extent that the price varies, the producters will increase production to benefit from the high prices and reduce them to avoid losses when the prices are low. This will force prices to revert to their long term equilibrium value.

Mean Reversion Process

- There are many models for mean reverting proceses. One of the most simple is the Ornstein-Uhlenbeck model, which has the following mathematical expression:

$$dP_t = \lambda\left(\overline{P} - P_t\right)dt + \sigma dz$$

where

λ = reversion speed

$\overline{P}$ = the long term mean to which V tends to revert

- The speed of reversion indicates how quickly the variable reverts to its long term equilibrium value.

Intuition

- Change in next price is a function of long run average and current price level

$$\Delta P = f\left(\overline{P}, P_t\right) + \varepsilon$$

- How about

$$\Delta P = \lambda\left(\overline{P} - P_t\right) + \varepsilon$$

$$\overline{P} > P_t \Rightarrow \lambda\left(\overline{P} - P_t\right) > 0 \Rightarrow \Delta P \text{ likely} > 0$$

$$\overline{P} < P_t \Rightarrow \lambda\left(\overline{P} - P_t\right) < 0 \Rightarrow \Delta P \text{ likely} < 0$$

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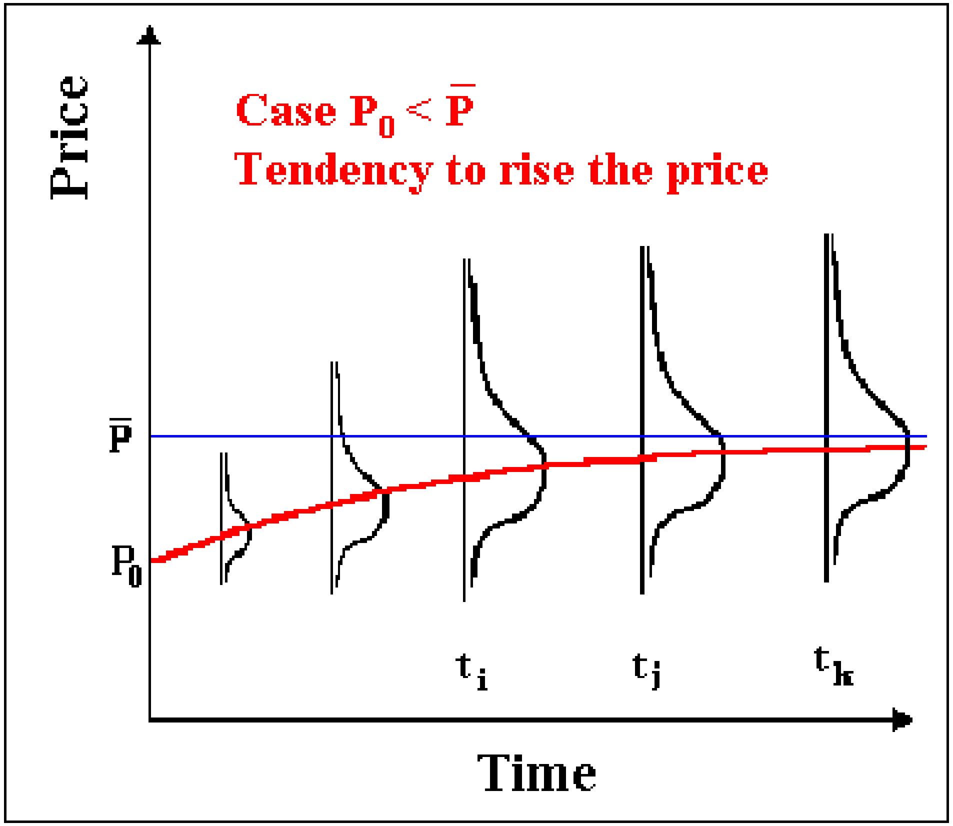
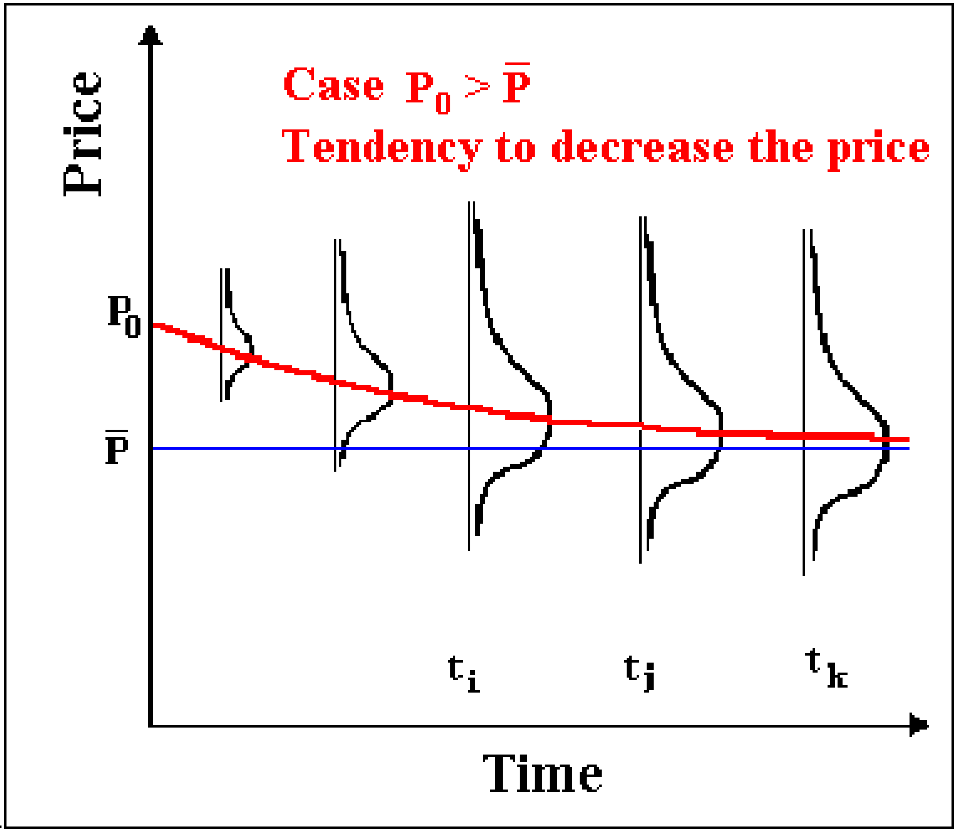
Mean-Reverting Process





Does it Right?

$$\mathrm{E}[P_{t+1}] = P_t \quad ?$$

$$\mathrm{E}\left[\lambda \overline{P} + (1-\lambda)P_t + \varepsilon\right] = P_t$$

$$\Rightarrow \lambda \overline{P} + (1-\lambda)E(P_t) = P_t$$

$$\Rightarrow \lambda \overline{P} + (1-\lambda)\overline{P} = P_t$$

$$\Rightarrow \overline{P} = P_t$$

$$\Rightarrow \mathrm{E}[P_{t+1}] = P_t \;\Rightarrow\; P_t = \overline{P}$$

What are we approximating?

- Mean Reverting (Ornstein–Uhlenbeck) process
  - continuous time

$$dP_t = \boxed{\lambda\left(\overline{P} - P_t\right)} dt + \sigma dz$$

One Factor Models

$$\Delta P = \boxed{\lambda\left(\overline{P} - P_t\right)} + \varepsilon$$

- An AR1 process is the discrete-time analogue

Rearrange

$$dP_t = \lambda\left(\overline{P} - P_t\right)dt + \sigma dz$$

$$\Delta P = \lambda\left(\overline{P} - P_t\right) + \varepsilon$$

$$P_{t+1} - P_t = \lambda\left(\overline{P} - P_t\right) + \varepsilon$$

$$P_{t+1} - P_t = \lambda\overline{P} - \lambda P_t + \varepsilon$$

$$P_{t+1} = \underbrace{\lambda\overline{P}}_{\beta_0} + \underbrace{\left(1 - \lambda\right)}_{\beta_1}P_t + \varepsilon$$

Auto Regressive (1)
Looks back one period

It's a regression

What is the long run mean?

$$P_{t+1} = \underbrace{\lambda \overline{P}}_{\beta_0} + \underbrace{(1-\lambda)}_{\beta_1} P_t + \varepsilon$$

$$\beta_1 = (1-\lambda) \Rightarrow \lambda = 1 - \beta_1$$

$$\beta_0 = \lambda \overline{P} \Rightarrow \overline{P} = \frac{\beta_0}{\lambda} = \boxed{\frac{\beta_0}{1-\beta_1}}$$

Implied Long Run Mean

Price Uncertainty

- The process for the price uncertainty is:

$$dP_t = \lambda\left(\overline{P} - P_t\right)dt + \sigma dz$$

- The discrete simulation model is:

$$P_{t+1} = P_t + \lambda\left(\overline{P} - P_t\right)\Delta t + \sigma\sqrt{\Delta t}\, NORMSINV\left(RAND()\right)$$

Example

Year	Price(t+1)	Price(t)
1955	$ 0.44	
1956	$ 0.41	$ 0.44
1957	$ 0.27	$ 0.41
1958	$ 0.25	$ 0.27
1959	$ 0.30	$ 0.25
1960	$ 0.31	$ 0.30
1961	$ 0.29	$ 0.31
1962	$ 0.29	$ 0.29
1963	$ 0.29	$ 0.29
1964	$ 0.44	$ 0.29
1965	$ 0.59	$ 0.44
1966	$ 0.69	$ 0.59
1967	$ 0.51	$ 0.69
1968	$ 0.56	$ 0.51
1969	$ 0.66	$ 0.56

	Coefficients	*P-value*
Intercept	0.0603	0.480
Price(t)	0.8901	0.001

$$P_{t+1} = 0.0603 + 0.8901 P_t + \varepsilon$$

$$\overline{P} = \frac{\beta_0}{1-\beta_1} = \frac{0.0603}{1-0.8901} = \$0.548$$

$$\lambda = 1-\beta_1 = 1-0.8901 = 0.1099$$

$$\Delta P = \lambda\left(\overline{P} - P_t\right) + \varepsilon$$

Slow to Adjust

Example

	Price(t+1)	Price(t)
1955	$ 0.44	
1956	$ 0.41	$ 0.44
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1966	$ 0.69	$ 0.59
1967	$ 0.51	$ 0.69
1968	$ 0.56	$ 0.51
1969	$ 0.66	$ 0.56

Regression Statistics	
Multiple R	79.71%
R Square	63.53%
Adjusted R Square	60.49%
Standard Error	0.09813
Observations	14

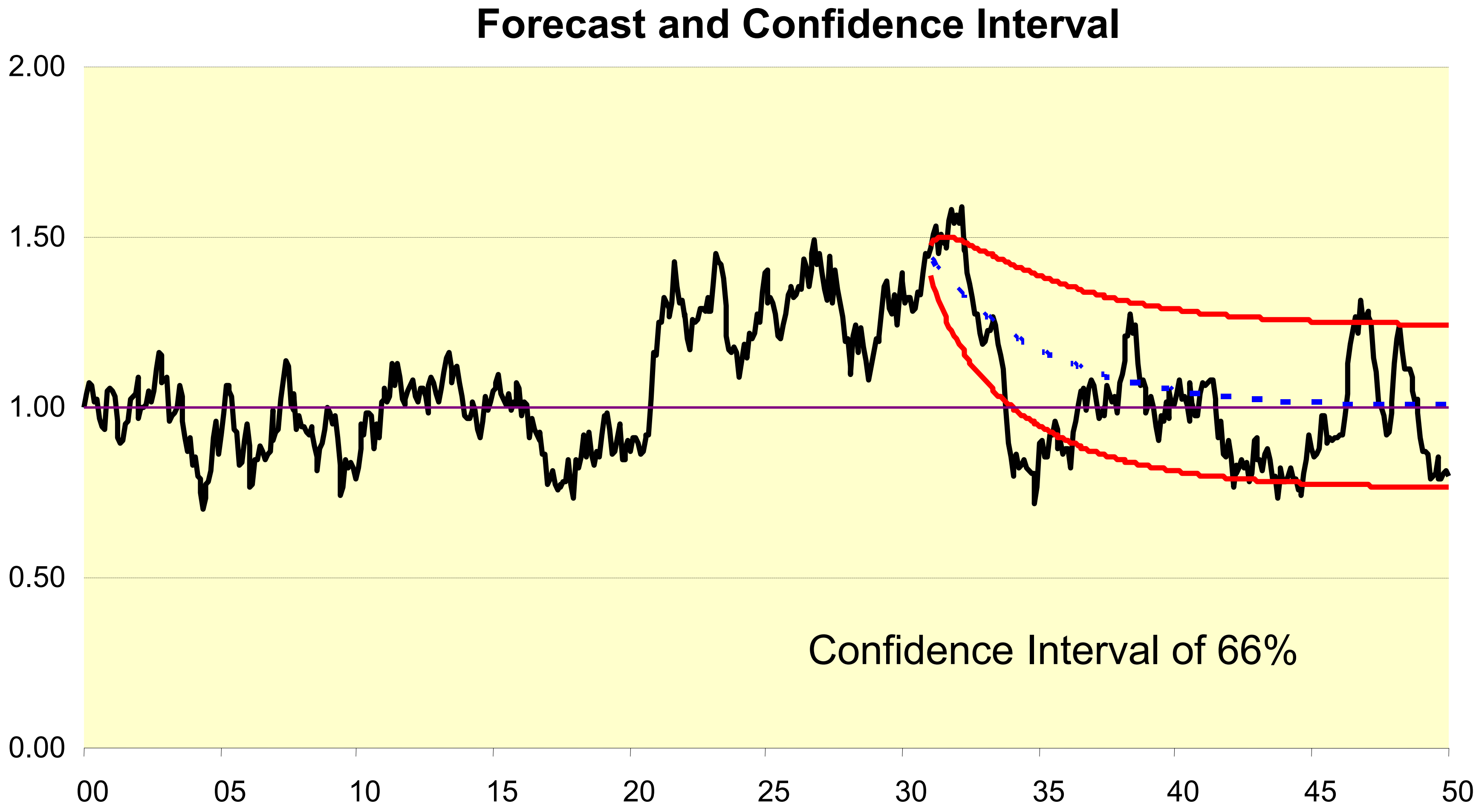
$$P_{t+1} = 0.0603 + 0.8901 P_t + \varepsilon$$

$$\varepsilon \sim N(0, 0.09813)$$

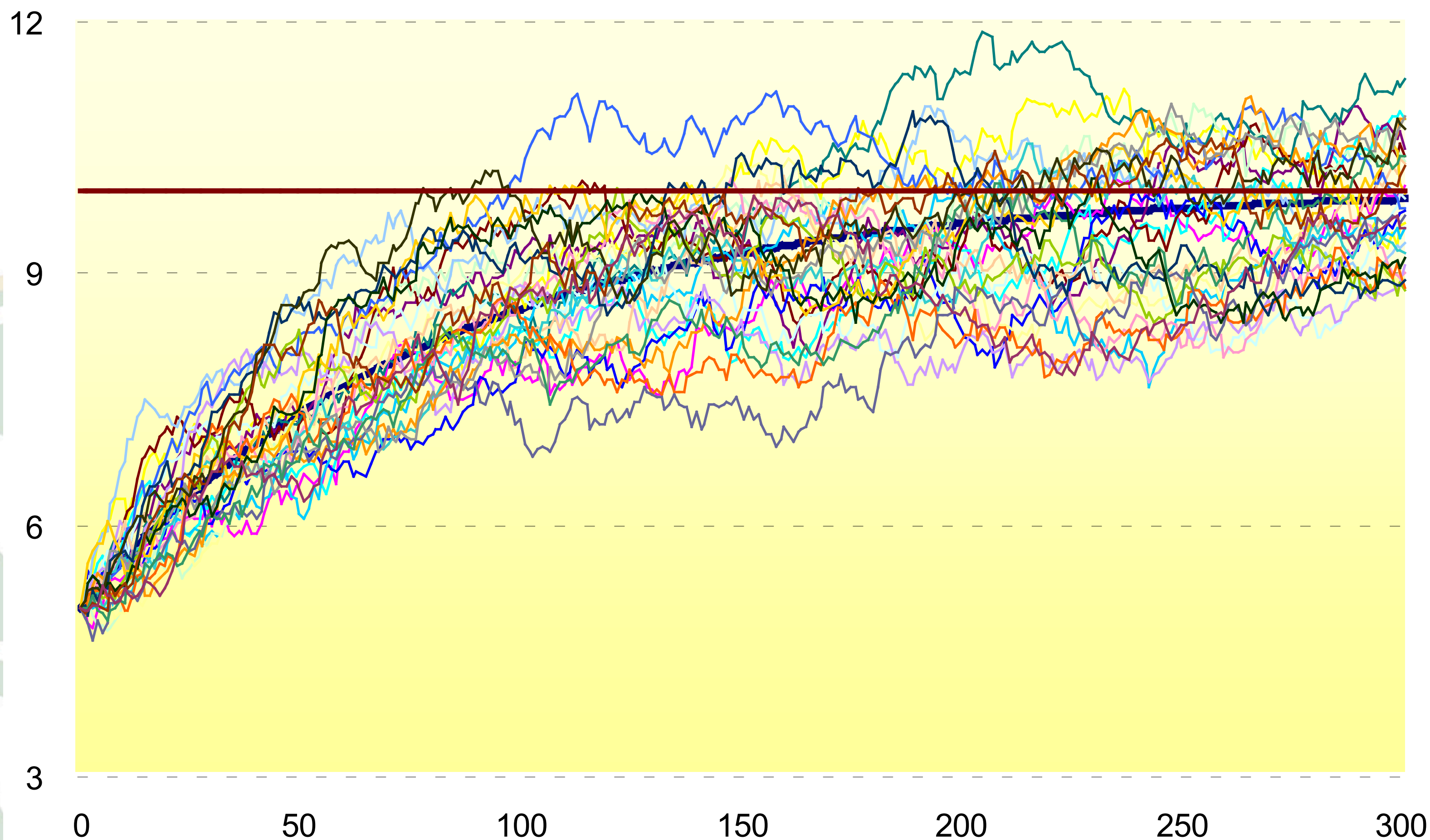
$$P_{t+1} = \lambda \overline{P} + (1-\lambda) P_t + \varepsilon$$

$$\lambda = 0.1099$$

Process of Reversing to the Mean

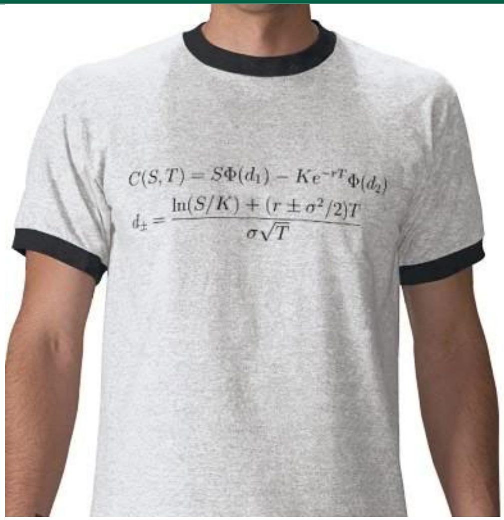


Simulations of a Mean Reverting Process



Final Comments

- The ABM, GBM models and the Mean Reverting process are also known as "models of diffusion," where the value of the variable changes in small increments each time.
- Processes where the value of the variable changes suddenly are named "jump" models.
- The ABM is more utilized for physics processes, while the GBM is widely utilized to model prices of financial securities and real securities. This will be the principal process that we'll use in this course.
- Mean reverting process are utilized to model interest rates and prices of commodities.



Black-Sholes Model

The Formula that Shook the World

Fischer Black
(1938-1995)

Myron Scholes
(1941-)

Robert K. Merton
(1910-2003)

Black and Scholes Formula

$$c = S_0\, e^{-\delta T} N(d_1) - K\, e^{-rT} N(d_2)$$

$$\text{where} \quad d_1 = \frac{\ln(S_0 / K) + (r - \delta + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - \delta - \sigma^2 / 2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

N(.) is the cumulative normal distribution function

- Assumption:
  - The distribution of the underlying security value is lognormal. The stock price follows a very specific type of random process called geometric Brownian motion.
  - Applicable only to European options
  - Continuous trading is possible.

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

Derivation of the Black-Scholes Differential Equation

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z$$

We set up a portfolio consisting of

$$-1: \text{ derivative}$$

$$+\frac{\partial f}{\partial S}: \text{ shares}$$

This gets rid of the dependence on Δz.

Derivation of the Black-Scholes Differential Equation

The value of the portfolio, Π, is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The return on the portfolio must be the risk - free rate. Hence

$$\Delta \Pi = r \, \Pi \Delta t$$

$$- \Delta f + \frac{\partial f}{\partial S} \Delta S = r \left(- f + \frac{\partial f}{\partial S} S \right) \Delta t$$

Derivation of the Black-Scholes Differential Equation

$$- \Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(- f + \frac{\partial f}{\partial S} S \right)\Delta t$$

$$\Delta S = \mu S\,\Delta t + \sigma S\,\Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z$$

We substitute for Δf and ΔS in this equation

$$-[\left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S\,\Delta z] + \frac{\partial f}{\partial S}[\mu S\,\Delta t + \sigma S\,\Delta z] = r\left(- f + \frac{\partial f}{\partial S} S \right)\Delta t$$

$$-[\left(\frac{\partial f}{\partial t} + ½ \frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t] = r\left(- f + \frac{\partial f}{\partial S} S \right)\Delta t$$

to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation

Attack the equation directly….

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

Risk-Neutral Valuation

- The variable m does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Price of a put option

- To compute the price of a put option, you should first calculate C_0 using the Black-Scholes formula.
- Then, use the *put-call parity*:

$$c + Ke^{-rT} = p + S_0 e^{-\delta T}$$

$$p = c + Ke^{-rT} - S_0 e^{-\delta T}$$

$$p = S_0 e^{-\delta T} N(d_1) - K e^{-rT} N(d_2) + K e^{-rT} - S_0 e^{-\delta T}$$

$$p = S_0 e^{-\delta T} (N(d_1) - 1) - K e^{-rT} (N(d_2) - 1)$$

$$p = K e^{-rT} (1 - N(d_2)) - S_0 e^{-\delta T} (1 - N(d_1))$$

$$p = K e^{-rT} N(-d_2) - S_0 e^{-\delta T} N(-d_1)$$

Determinants of call option values

$$c = S_0\, e^{-\delta T} N(d_1) - K\, e^{-rT} N(d_2)$$

$$p = K\, e^{-rT}\ N(-d_2) - S_0\, e^{-\delta T} N(-d_1)$$

If this variable increase	The value of a call option	The value of a put option
Stock price	Increases	Decreases
Exercise price	Decreases	Increases
Volatility	Increases	Increases
Time to expiration	?(E)/Increases(A)	?(E)/Increases(A)
Interest rate	Increases	Decreases
Dividend payout	Decreases	Increases

Example: Pricing a call option

$S_0 = 100,\ X = 95,\ r = 10\%$ per year

T = 3 months (=0.25 year), σ=50% per year

$$d_1 = \frac{\ln\left(\frac{100}{95}\right) + \left(0.10 - 0 + \frac{0.50^2}{2}\right)(0.25)}{0.50\sqrt{0.25}} = 0.43$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.43 - (0.50)\left(\sqrt{0.25}\right) = 0.18$$

where $N(d_1) = N(0.43) = 0.6664$, $\quad N(d_2) = N(0.18) = 0.5714$

$$C_0 = S_0 e^{-\delta T} N(d_1) - Xe^{-rT} N(d_2)$$

$$= (100 \times 1 \times 0.6664) - \left(95 \times e^{-0.10 \times 0.25} \times 0.5714\right)$$

$$= \$13.70$$

Black-Sholes Excel Function

Black-Scholes Calculator

Parameters	
Stock Price	100
Strike Price	120
Expiry Time	1
Risk-Free Rate	6%
Dividend Yield	1%
Volatility	20%

	Call	Put
Excel	3.215	17.222

Intermediate Calculations			
d1	d2	N(d1)	N(d2)
-0.562	-0.762	0.287	0.223

Example

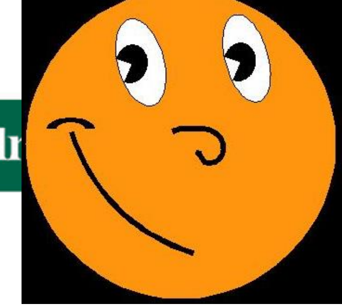
- This formula can be calculated utilizing tables or various softwares available in the market.
- Ex: A European option to buy stock has exercise price of $120 and expires in a year. The actual value of the stock is $100, the volatility is 35% and the risk free discount rate is 10%. What is the value of the option today?
- Using the Black-Sholes formula:

$$\left.\begin{array}{l} S = \$100 \\ X = \$120 \\ \sigma = 35\% \\ r = 10\% \\ T = 1 \end{array}\right\}$$

Call option price = 10.59

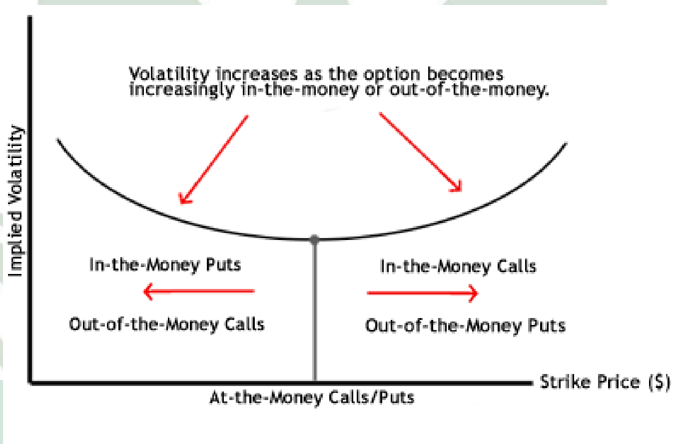
Implied Volatility

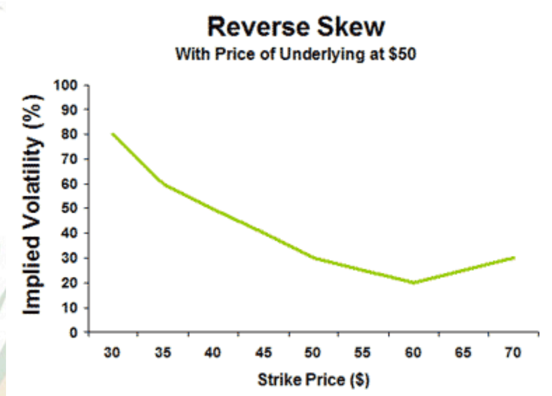
- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

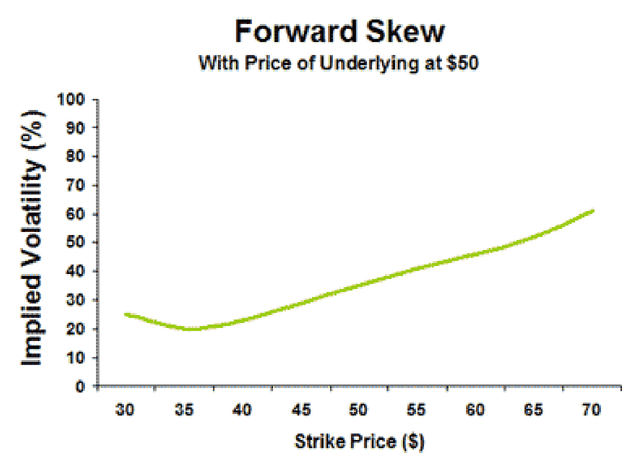


Volatility Smile (Smirk)

- A volatility smile shows the variation of the implied volatility with the strike price
- The volatility smile is the same whether calculated from European call options or European put options. (This follows from put-call parity.)



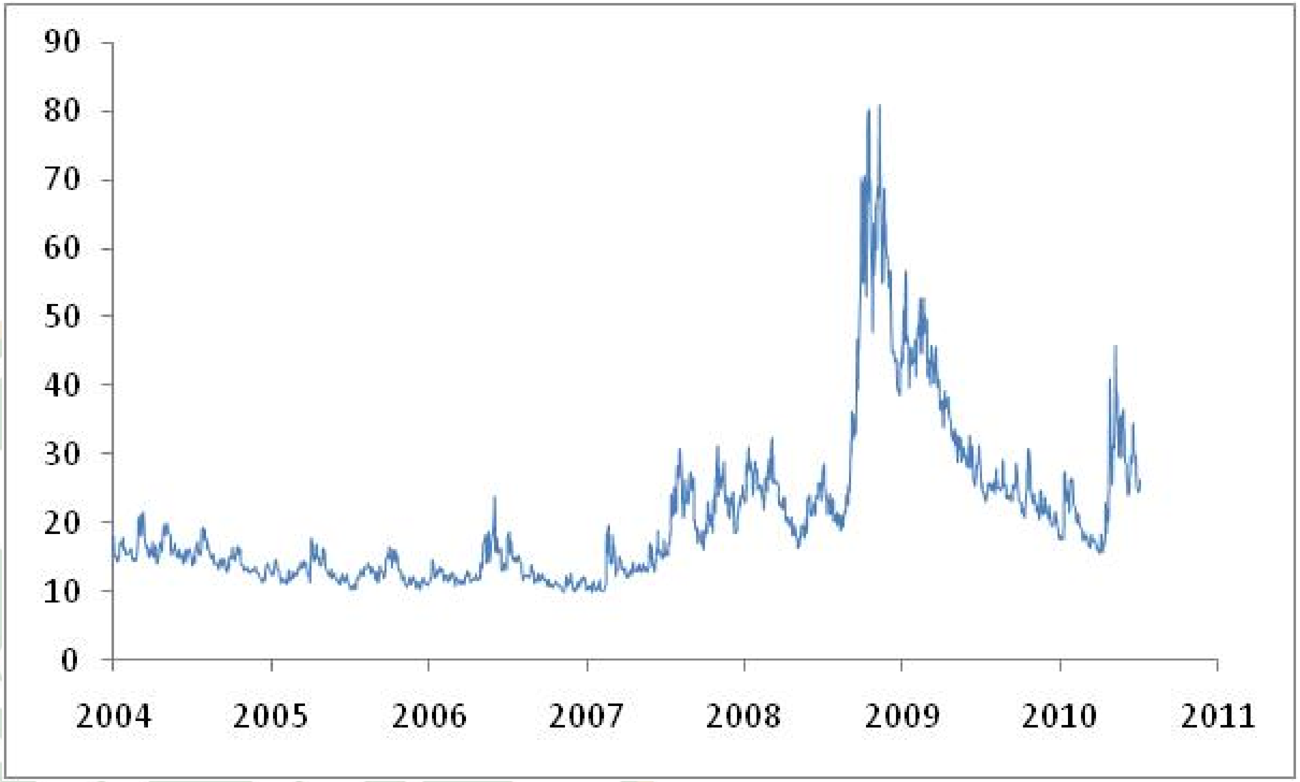




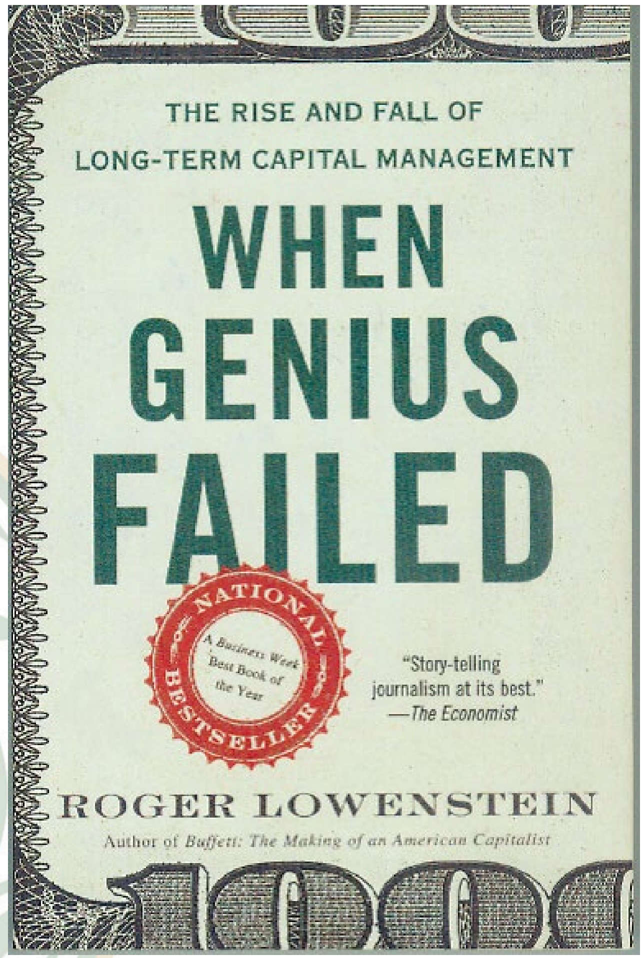
Volatility Term Structure

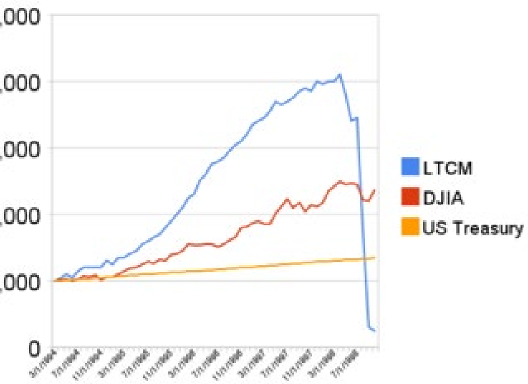
- In addition to calculating a volatility smile, traders also calculate a volatility term structure
- This shows the variation of implied volatility with the time to maturity of the option

The VIX Index of S&P 500 Implied Volatility



Lesson from LTCM





Option Pricing with Skew and Kurtosis

- The Black-Scholes option pricing model has several well-known deficiencies. Perhaps most significantly, Black-Scholes assumes that prices are log-normally distributed.

- In reality, however, investors see more extreme behavior than predicted by the normal distribution; in fact, extreme events occur 10 times more often than the normal distribution would have you assume.

Option Pricing with Skew and Kurtosis

- Corrado & Su (1996) extended the standard Black-Scholes scheme for option pricing by capturing the effect of skew and kurtosis. Their novel approach expanded the normal density function with a Gram-Charlier approach. This resulted in a pricing formula that was equal to the standard Black-Scholes equations plus terms that capture excess skew and kurtosis.
- The prices predicted by the Corrado & Su (1996) equations are equal to those predicted by Black-Scholes for a skew of 0 and kurtosis of 3.

Option Pricing with Skew and Kurtosis

$$d = \frac{ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

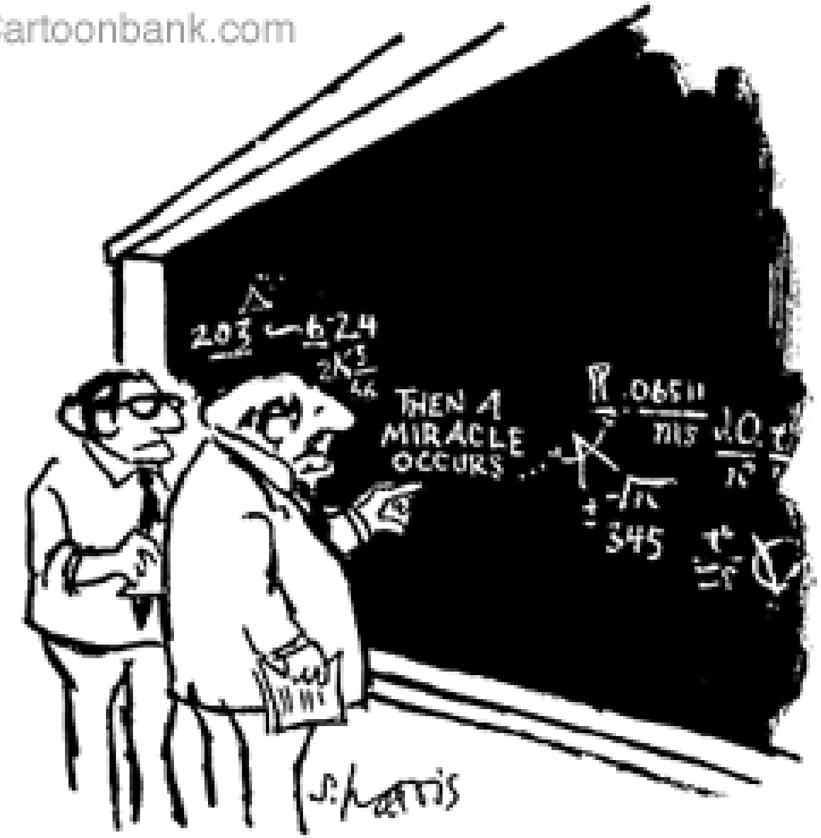
$$Q_3 = \frac{1}{3!} S_0\, \sigma \sqrt{T} \left(\left(2\, \sigma \sqrt{T} - d\right) n(d) + \sigma^2\, T\, N(d) \right)$$

$$Q_4 = \frac{1}{4!}\, S_0\, \sigma \sqrt{T} \left(\left(d^2 - 1 - 3\, \sigma \sqrt{T} \left(d - \sigma \sqrt{T}\right)\right) n(d) + \sigma^3\, T^{\frac{3}{2}}\, N(d) \right)$$

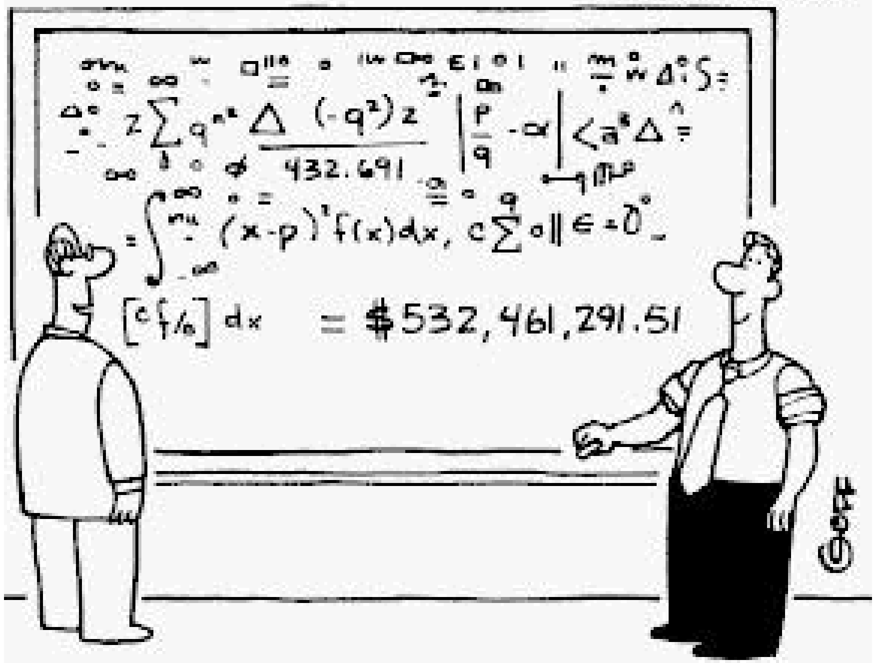
$$C_{BS} = S_0\, N(d) - K\, e^{-r\,t} N\left(d - \sigma \sqrt{t}\right)$$

$$C_{GC} = C_{BS} + \mu_3\, Q_3 + (\mu_4 - 3) Q_4$$

Breakthrough of The Miracle



"I think you should be more explicit here in step two."



"Now that's what I call a breakthrough!"