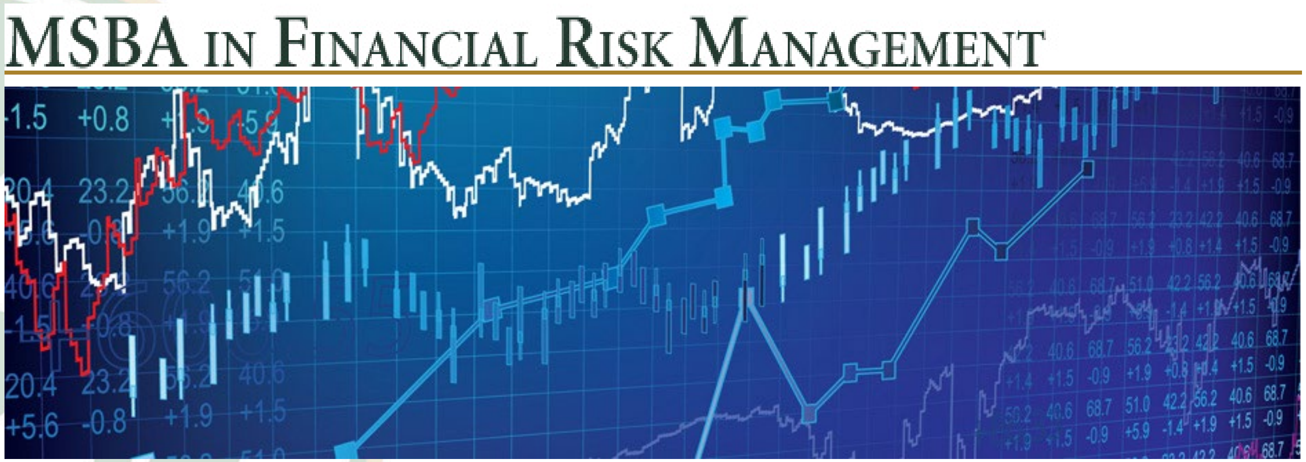
Colorado State University

Finance & Real Estate | Personal and Professional Business Explorations in Finance and Real Estate

Financial Risk Management



Tianyang Wang FIN 670

Modeling Stochastic Process

Colorado State University

Random Walk (Drunkard's Walk)

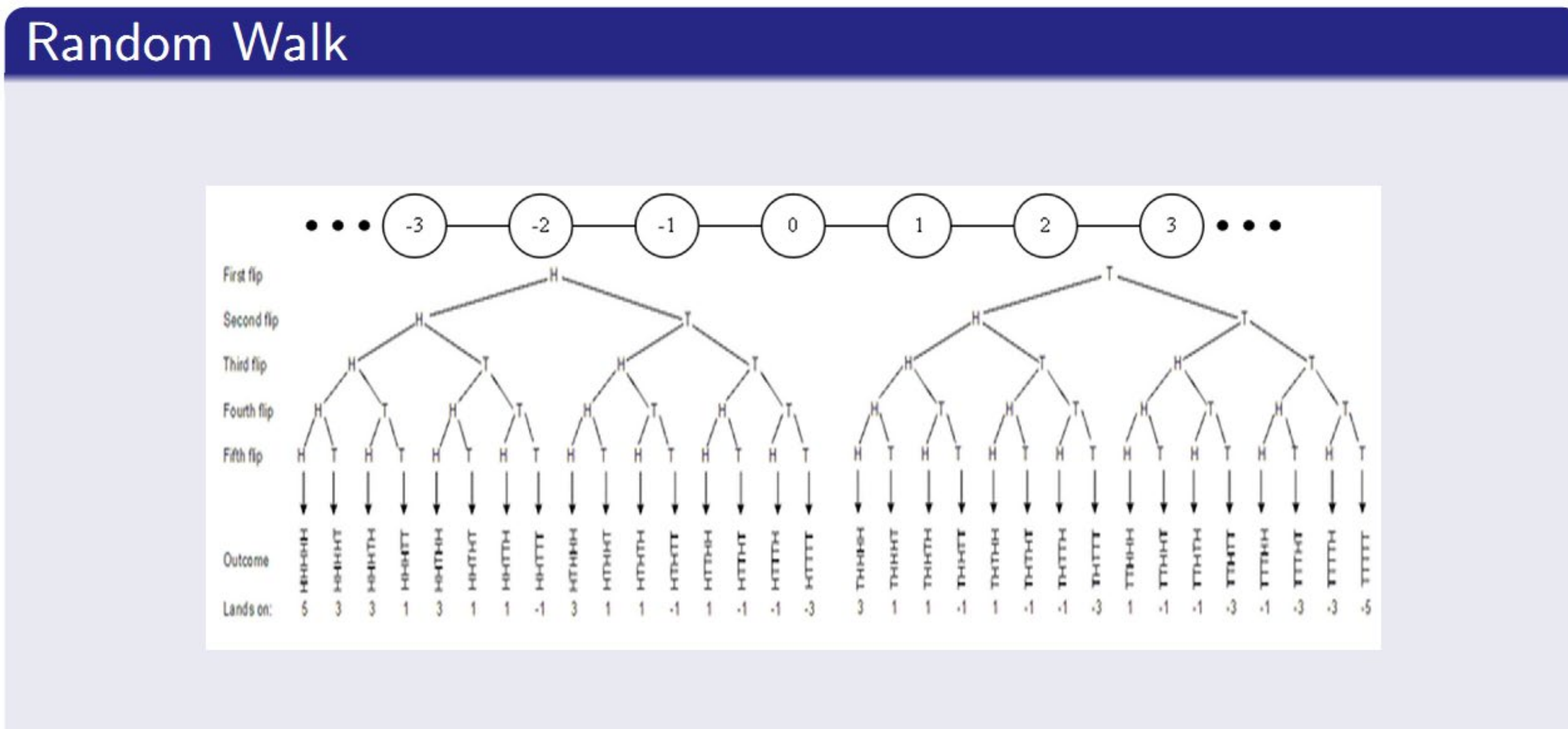
Drunken Walk on Street

A random walk is one of the most basic stochastic processes.
The simplest random walk is a path constructed according to the following rules:
1. There is a starting point.

2. The distance from one point in the path to the next is a constant.
3. The direction from one point in the path to the next is chosen at random, and no direction is more probable than another.

Random Walk Example

- Say you have a coin. If it lands on heads, you move one to the right on the number line. If it lands on tails, you move one to the left. So after five flips, you have the possibility of landing on 1, -1, 3, -3, 5, or -5.



Random Walk Example

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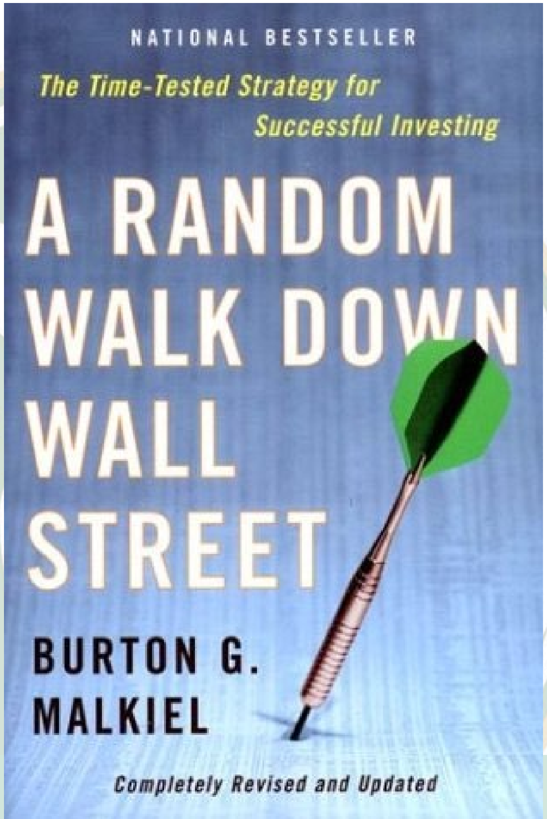
Pascal Triangle

n	-5	-4	-3	-2	-1	0	1	2	3	4	5
$f_0(n)$						1					
$2f_1(n)$					1		1				
$2^2f_2(n)$				1		2		1			
$2^3f_3(n)$			1		3		3		1		
$2^4f_4(n)$		1		4		6		4		1	
$2^5f_5(n)$	1		5		10		10		5		1

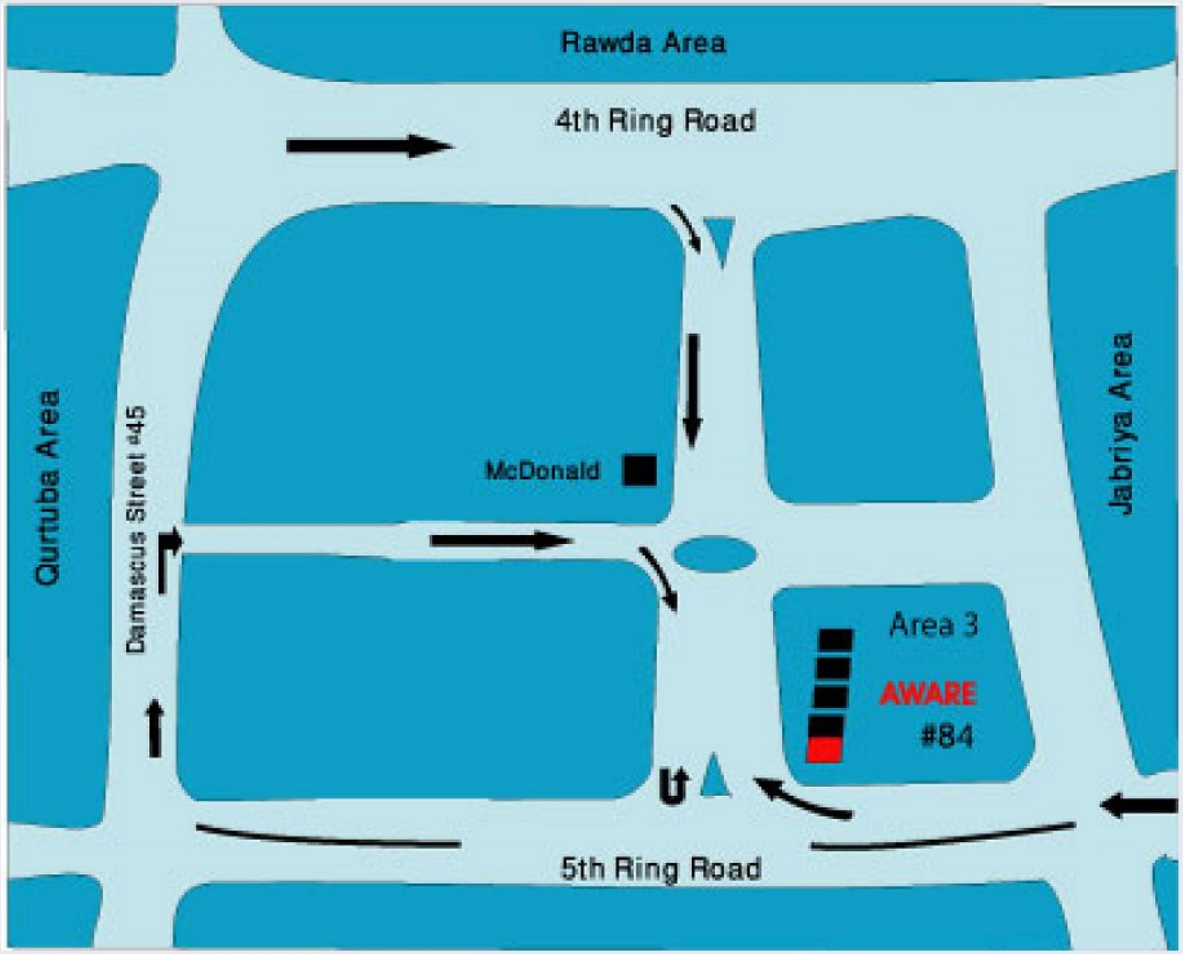
The Recurrence Problem

- Suppose we draw a line some distance from the origin of the walk. How many times will the random walk cross the line?
- The *recurrence* problem or the *gambler's ruin* problem.



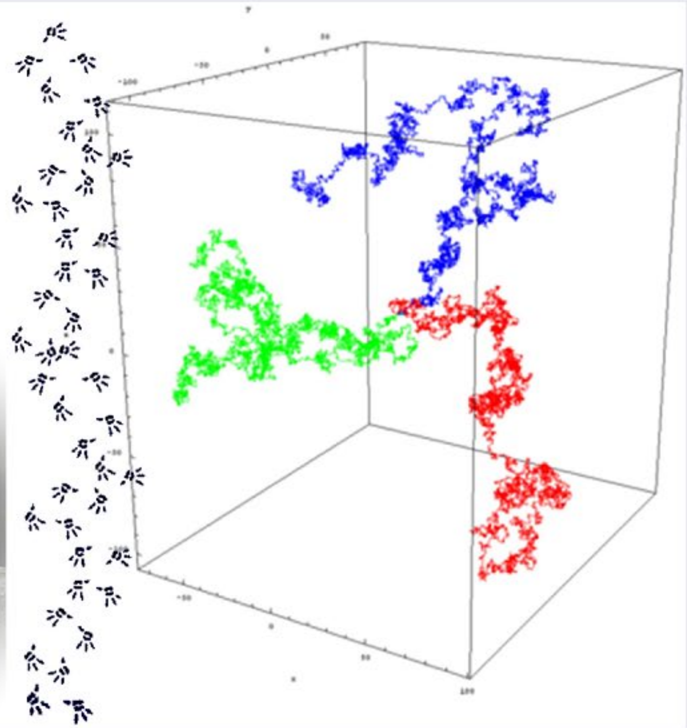
Two Dimension Random Walk

Drunken Walk in Neighborhood



Higher Dimension Random Walk

Drunk Bird

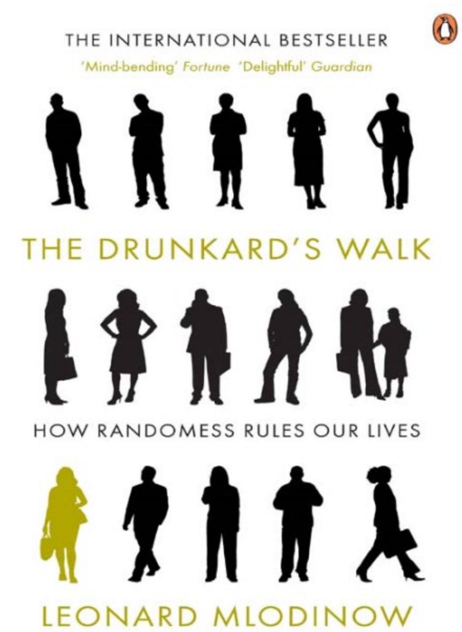


More intriguing thoughts on Random Walk

A drunk man will find his way home, but a drunk bird may get lost forever.

---- Shizuo Kakutani

- What is the intuition behind this?
- What if the city has tunnels and bridge?
- What if there are two drunkards on the road?
- What if there are two drunkards in the city?
- What if there are two drunk birds in the sky?



The Black Swan

- One of the important points is that there's a huge difference between "unlikely" and "impossible."

- Over enough trials, every unlikely result is likely to occur. And in a single trial, anything can happen.
- If you roll two dice together, the most likely result is that they will total 7, but that only happens 1/6 of the time. It is twice as likely that you will roll one of the "unlikely" results of 2, 3, 4, 10, 11, or 12, because their *combined* chance is 1/3.

- Unlikely; Impossible
- Likely; Probably; Possibly; Plausibly; Almost Surely

Stochastic Process

- A variable whose value changes randomly through time is said to follow a stochastic process.

- Most models of asset price behavior for pricing derivatives are fomulated in a continuous time framework by assuming a stochastic differential equation describing the stochastic process followed by the asset price.

- Continuous time process can be approximated through discrete process, which are simpler to model and more flexible for modeling decision-making.

- We will study the principal models of continuous process, and subsequently, their corresponding discrete models.

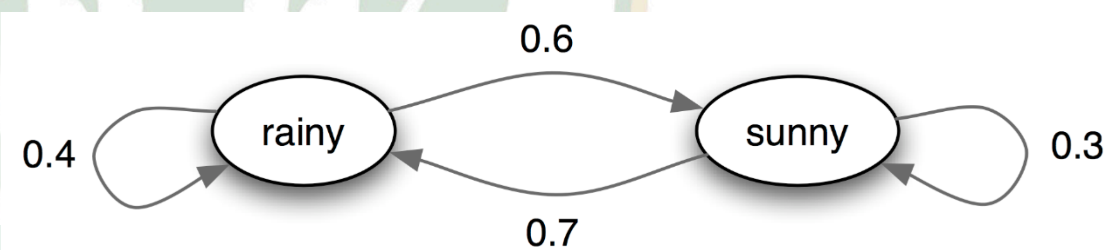
Stochastic Process

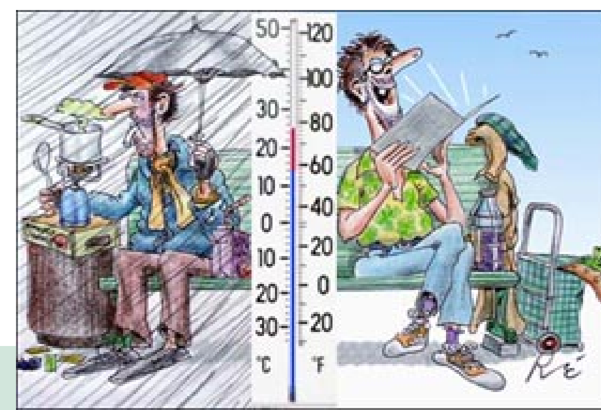
- Stochastic processes were initially used in physics to describe the motion of particles. They can be classified in the following categories:
  - **Continuous Time Process:** The variable can change its value at any moment in time.
  - **Discrete Time Process:** The variable can only change its value during fixed intervals.
  - **Continuous Variable:** The variable can assume any value within a determined interval.
  - **Discrete Variable:** The variable can assume only a few discrete values.
  - **Stationary Process:** The mean and variance are constant over time.
  - **Non-stationary Process:** The expected value of the random variable can grow without limit and its variance increases over time.

Andrey Markov
(1856-1922)

Markov Processes

- A Markov Process is a Stochastic process where only the present value of the variable is relevant to predict the future evolution of the process.
  - This means that historic values or even the path through which the variable arrived at its present value are irrelevant in determining its future value.
- Markov processes can be drawn as a *transition diagram*.





Andrey Markov
(1856-1922)

Markov Processes

- It is often assumed that the price of securities, like stock and comodities, follow a Markov process.

- Given this premise, we assume that the current price of a stock reflects all the historical information as well as expectations about its future price.

- Using this model, it would be impossible to predict the future value of a stock based on historical price infomation. The market is efficient (weak form efficiency).

"A blindfolded monkey throwing darts at a newspaper's financial pages could select a portfolio that would do just as well as one carefully selected by experts."

Random Walk

- Random Walk is one of the most basic stochastic processes.
- A random walk is a Markov process in discrete time that has independent increments in the form of:

$$S_{t+1} = S_t + \varepsilon_t$$

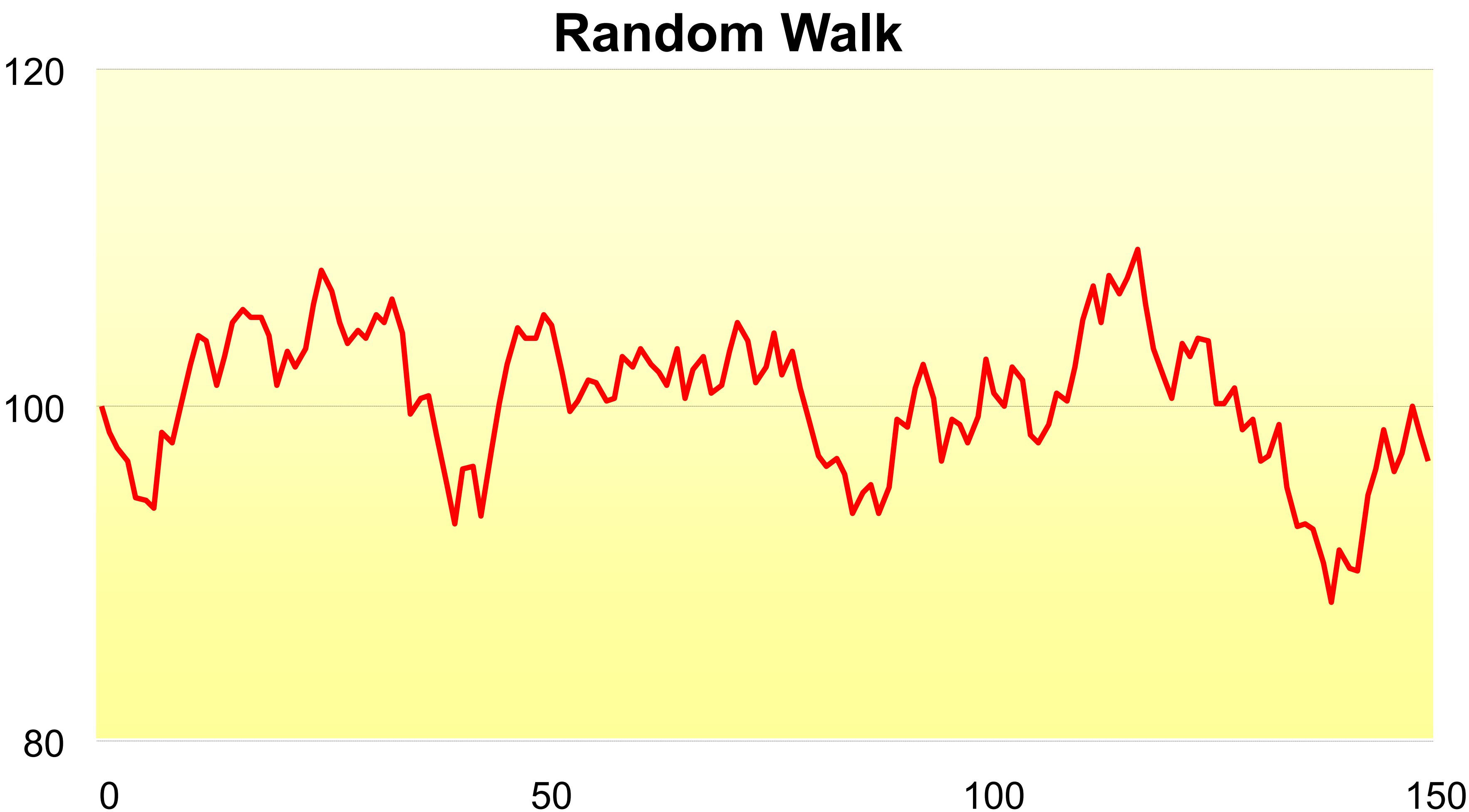
where S_{t+1} is the value of the variable at $t+1$

S_t is the value of the variable at t

ε_t is a random variable with probability

$P(\varepsilon_t = 1) = P(\varepsilon_t = -1) = 0.5$

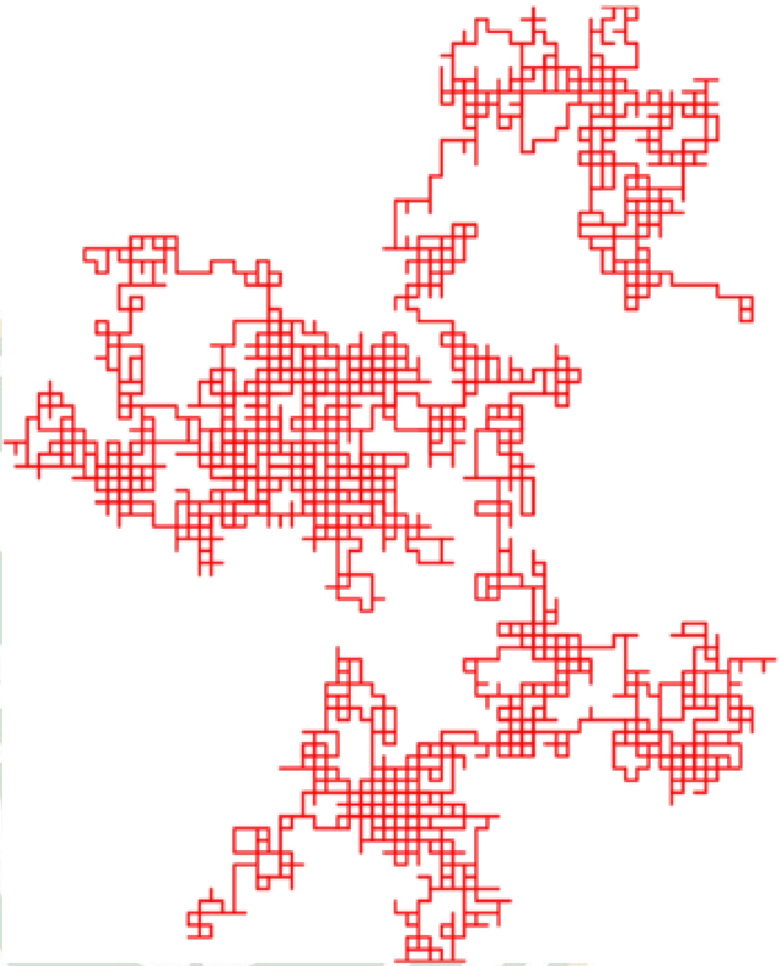
Random Walk

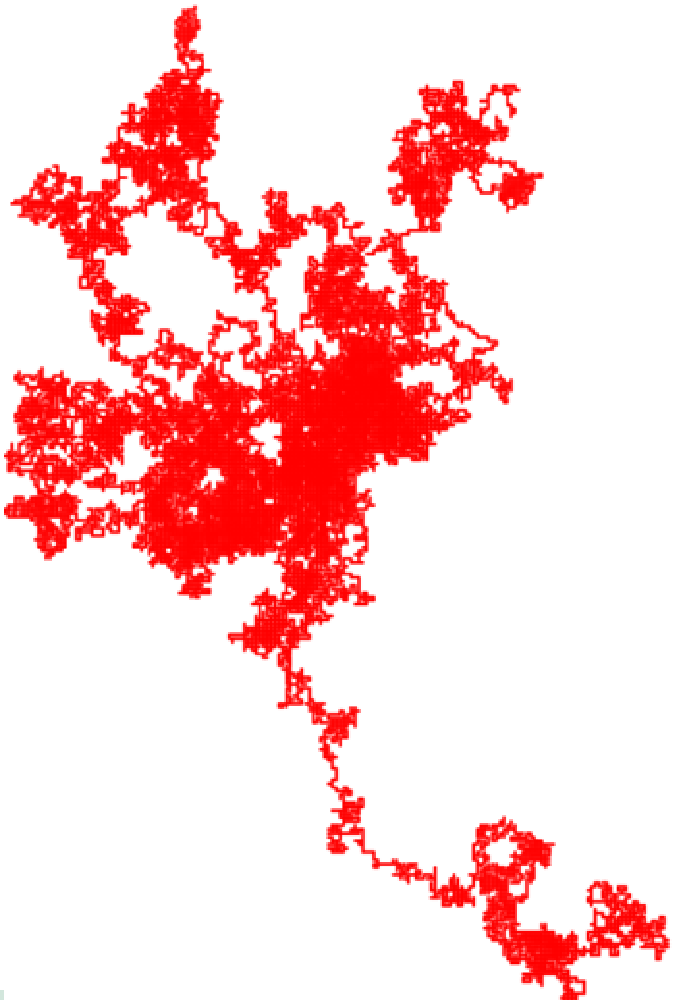


Random Walk with Drift

- Random Walks can include a growth term, or drift, that represents a long term growth.
- $S_{t+1} = \mu t + S_t + \varepsilon_t$
- Without the drift term, the best estimate of the next value of the variable S_{t+1} is the present value, if the term of error is normally distributed with a mean of zero.
- With the drift term, or growth, the future values of the variable tend to grow in a proportional manner to the rate of growth.

From Random Walk to Wiener Process





Wiener Process

- A Wiener process is a particular type of Markov process
  - Named after mathematician Norbert Wiener, although also referred to as a Brownian Motion
  - Changes in the process are normally distributed with a variance that increases with time.
- The Wiener process has the form:

$$x_{t+1} = x_t + dz \qquad where \qquad dz = \varepsilon\sqrt{dt} \quad and \quad \varepsilon \approx N(0,1)$$

- This means that:
  - $E[dz]=0$ (expected change is zero)
  - $Var[dz]=dt$ (variance is proportional to the time increment)

Generalized Wiener Process

- If we add a growth ("drift") term to the Wiener process we obtain the following mathematic representation:

$$dx = a\,dt + b\,dz$$

- The evolution of this process is a combination of two parts:
  - A drift term, with a rate *a*
  - A variance term, with standard deviation *b*

Arithmetic Brownian Motion

Arithmetic Brownian Motion (ABM)

- If we add a long term growth to the Wiener process we obtain a Arithmetic Brownian Motion (ABM), that has the following mathematic representation:

$$S_{t+1} = S_t + \mu dt + \sigma dz$$

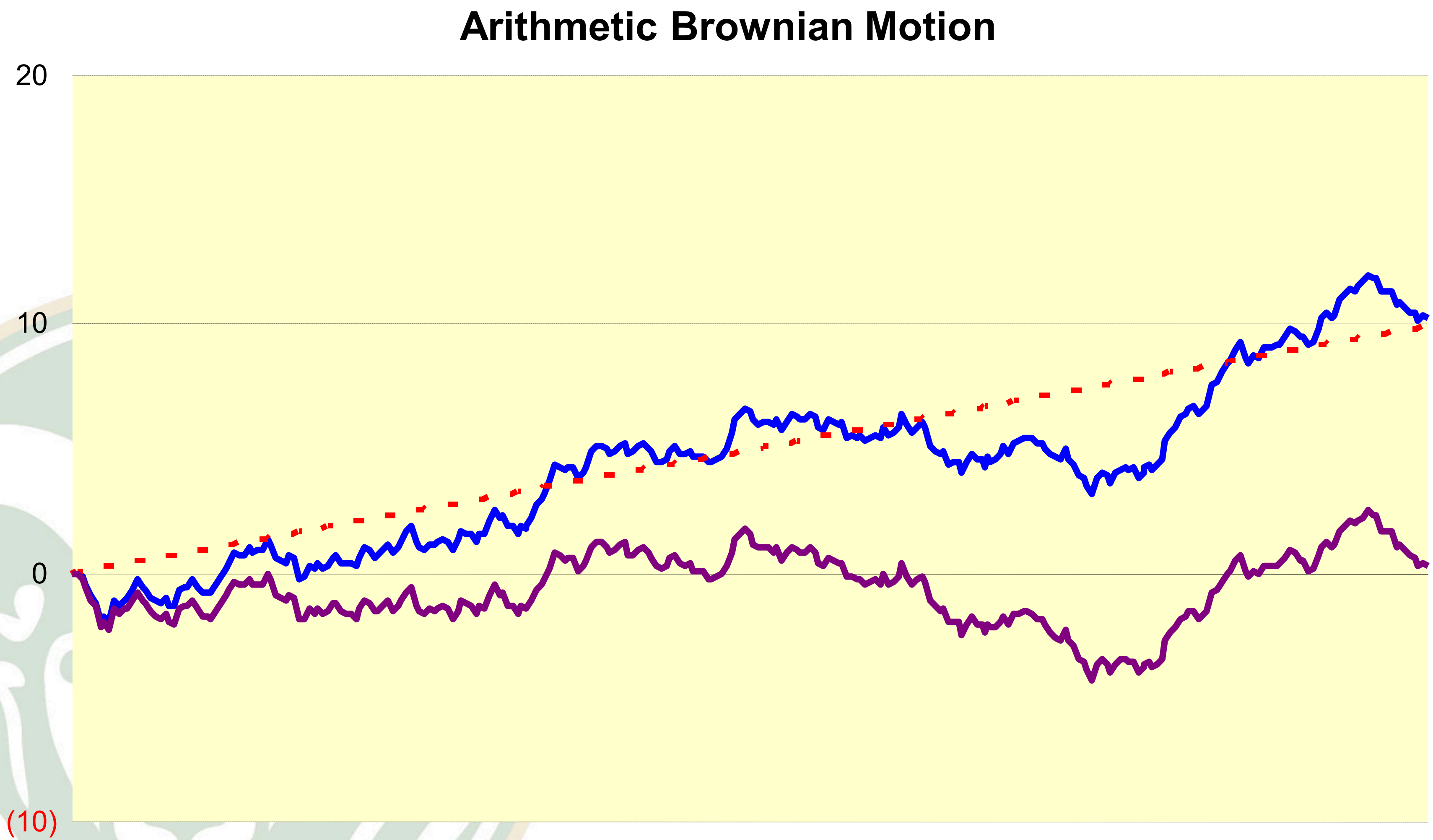
$$dS = \mu dt + \sigma dz \qquad dS \approx N(\mu dt, \sigma^2 dt)$$

- The evolution of a ABM is a combination of two parts:
  - A linear growth, with a rate μ
  - A random growth with a normal distribution and a standard deviation σ

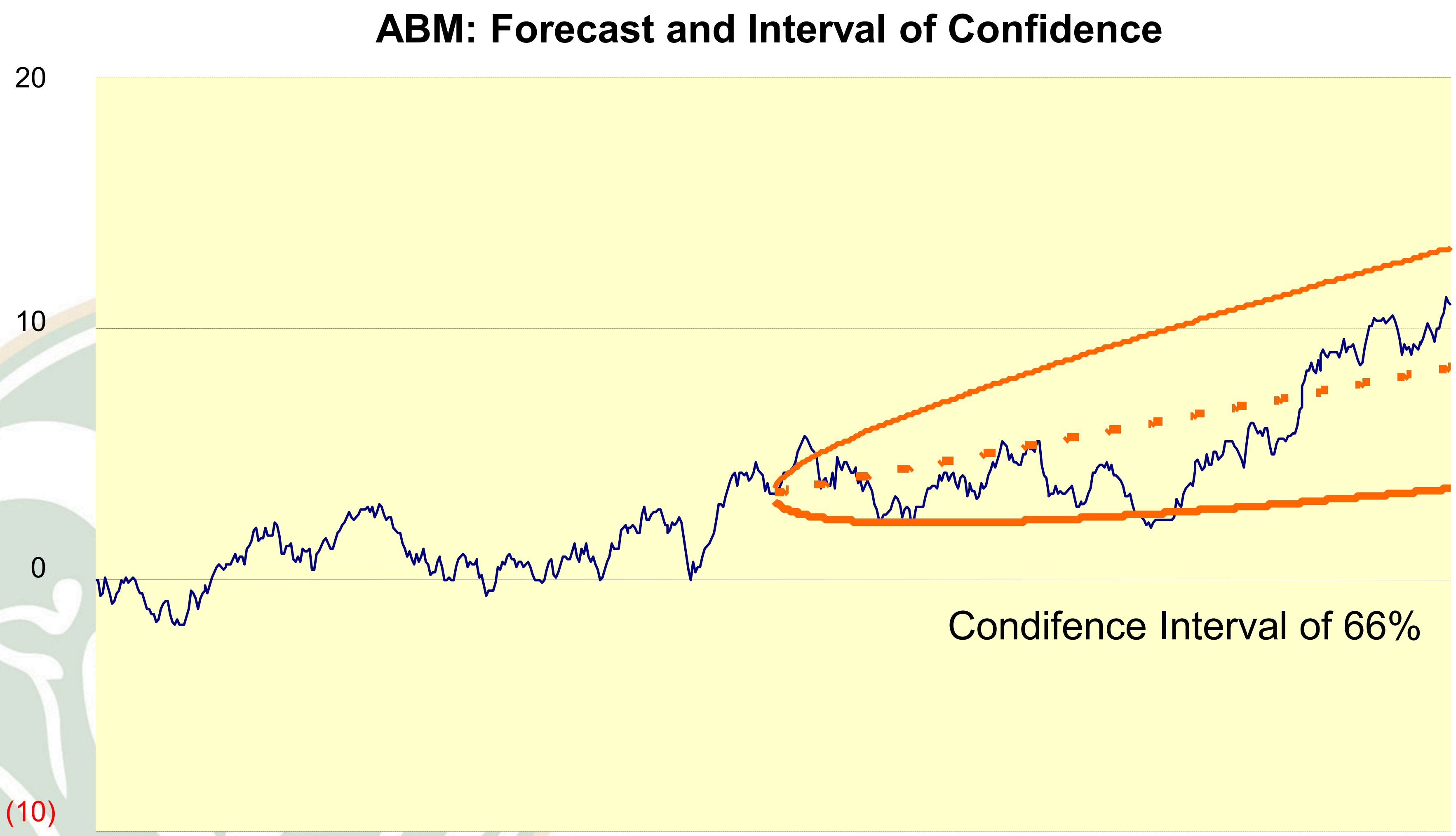
- The focus of the ABM is in the change in the value of the variable, instead of the value of the variable itself.

- By being a Random Walk, the ABM also has a normal distribution.
- ABM Process.xls

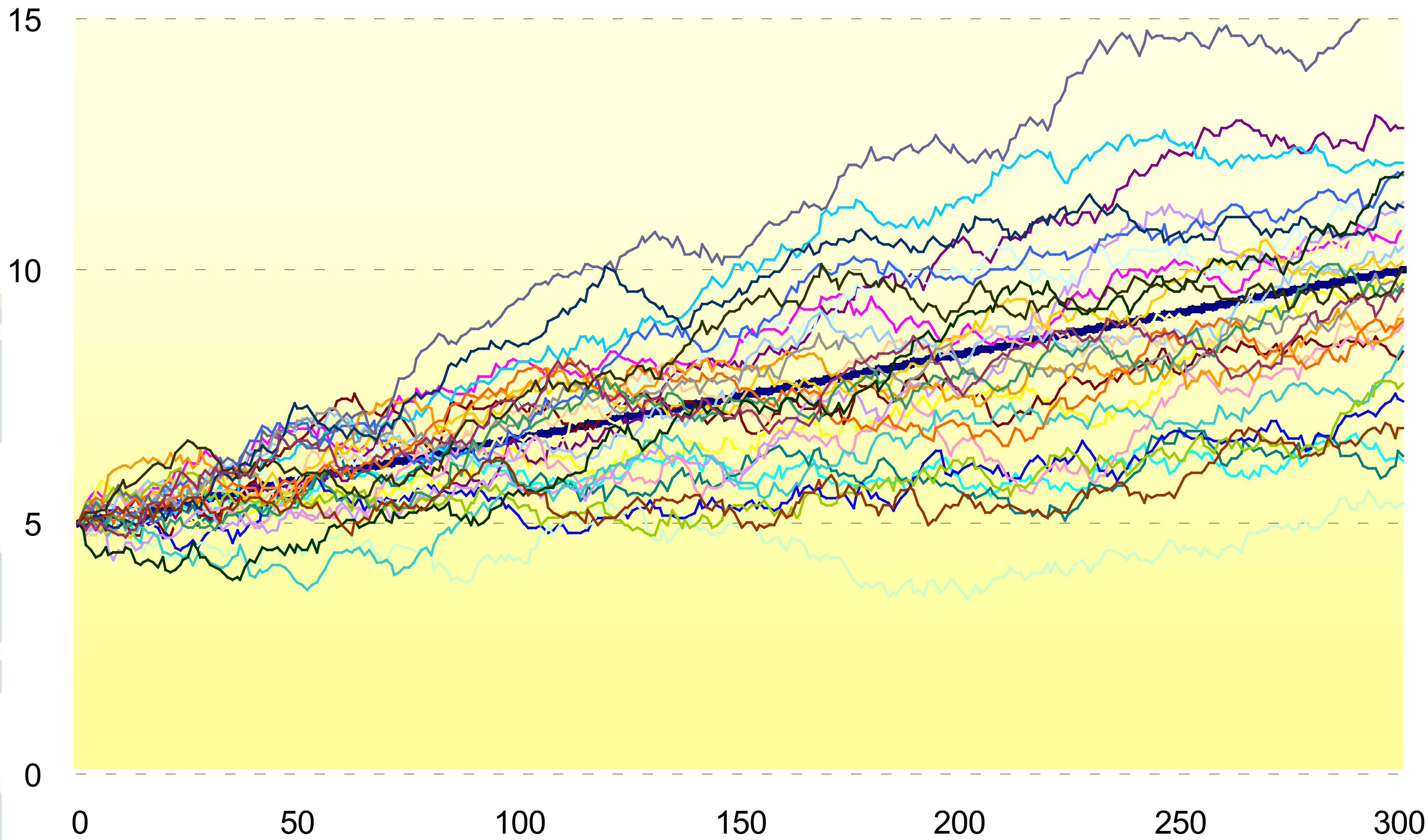
Arithmetic Brownian Motion



Arithmetic Brownian Motion



Arithmetic Brownian Motion



ABM Model Limitations

- The ABM is also known as an additive model because the variable grows by a constant value every period.
- However, modeling securities with ABM presents some problems:
  - Since the random term is a normally distributed variable, the value of the variable can occasionally become negative, which cannot happen with the price of securities.
  - For a stock that doesn't pay dividends, the rate of return of the stock decreases with time as the value of the stock increases for ABM. We know, however, that investers require a constant rate of return, independent of the price of the stock.
  - In ABM the standard deviation is constant throughout time, while to better model securities the standard deviation should be proportional to the value of the security.
- Because of these reasons ABM is not the most appropriate process to model the prices of stock or securities in general.

Geometric Brownian Motion

Geometric Brownian Motion (GBM)

- Dividing through by S, we get a Geometric Brownian Motion:

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- Recall form differential calculus that $\frac{dx}{x} = d \ln x$
- Here, we have the derivative of the log of value $\frac{dS}{S} = d(\ln S)$
- It can be shown using stochastic calculus that the process for log of value is: $d(\ln S) = \nu dt + \sigma dz$

where $\nu = \mu - \frac{1}{2}\sigma^2$

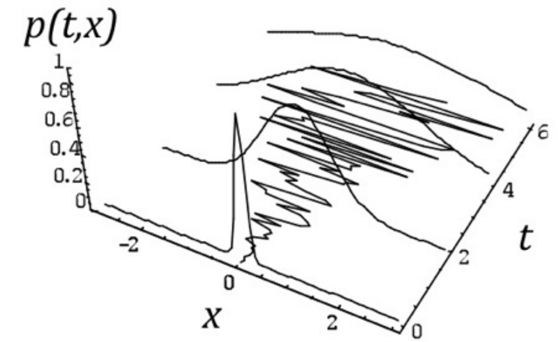
Kiyoshi Itô
(1915-2008)

Ito’s Lemma

- If we know the stochastic process followed by x, Itô’s lemma tells us the stochastic process followed by some function $F(x, t)$

- Since a derivative is a function of the price of the underlying asset and time, Itô’s lemma plays an important part in the analysis of derivatives

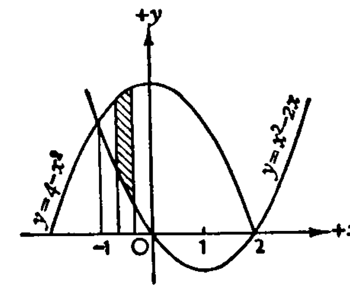
Ito's Lemma



- We define an Ito process as:

$$dx = a(x,t)dt + b(x,t)dz$$

- The mathematician Ito discovered that for $F(x,t)$



$$dF = \left(\frac{\partial F}{\partial x} a(x,t) + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial x^2} b^2(x,t) \right) dt + \frac{\partial F}{\partial x} b(x,t) dz$$

- This is Ito's Lemma. The lemma states that if $dS = \mu S dt + \sigma S dz$

then

$$dF = \left(\frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial F}{\partial S} \sigma S dz$$

Stochastic version of Taylor's expansion

$$f(x) = f(x_0) + \frac{1}{1} f'(x_0)(x - x_0) + \frac{1}{1\times 2} f''(x_0)(x - x_0)^2 + \text{higher order terms}$$

$$dF = \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 + \frac{\partial^2 F}{\partial S \partial t} dSdt + \frac{1}{2}\frac{\partial^2 F}{\partial t^2}(dt)^2 + \text{higher order terms}$$

$$(dz)^2 = dt \qquad (dt)^2 = 0 \qquad dzdt = 0$$

$$\Rightarrow dF = \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2 + \frac{\partial^2 F}{\partial S \partial t} dSdt$$

Stochastic version of Taylor's expansion

$$\Rightarrow dF = \frac{\partial F}{\partial S} dS + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (dS)^2 + \frac{\partial^2 F}{\partial S \partial t} dS dt$$

$$\Rightarrow dF = \frac{\partial F}{\partial S} (\mu S dt + \sigma S dz) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (\mu S dt + \sigma S dz)^2 + \frac{\partial^2 F}{\partial S \partial t} (\mu S dt + \sigma S dz) dt$$

$$\Rightarrow dF = \frac{\partial F}{\partial S} (\mu S dt + \sigma S dz) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (\mu^2 S^2 (dt)^2 + \sigma^2 S^2 (dz)^2) + \frac{\partial^2 F}{\partial S \partial t} (\mu S (dt)^2 + \sigma S dz dt)$$

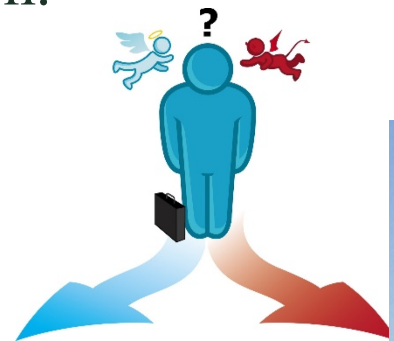
$$(dz)^2 = dt \qquad (dt)^2 = 0 \qquad dz dt = 0$$

$$\Rightarrow dF = \frac{\partial F}{\partial S} (\mu S dt + \sigma S dz) + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} (\mu^2 S^2 \times 0 + \sigma^2 S^2 dt) + \frac{\partial^2 F}{\partial S \partial t} (\mu S \times 0 + \sigma S \times 0)$$

$$\Rightarrow dF = \left(\frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial F}{\partial S} \sigma S dz$$

Ito's Lemma

- What if $F(S,t) = \ln S$?
- We use Ito's Lemma to answer the question.
- Given: $dS = \mu S dt + \sigma S dz$



- The process of *dF* will be:

$$dF = \left(\frac{\partial F}{\partial S} \mu S + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial F}{\partial S} \sigma S dz$$

where $\frac{\partial F}{\partial S} = \frac{1}{S}$ $\quad \frac{\partial F}{\partial t} = 0$ $\quad \frac{\partial^2 F}{\partial S^2} = -\frac{1}{S^2}$

Ito's Lemma

- Substituting we obtain

$$dF = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dz$$

- Using $v = \mu - \frac{1}{2}\sigma^2$ we obtain the process for the returns,

  that has a natural distribution

$$d(\ln S) = vdt + \sigma dz$$

- Note that the lognormal distribution can be completely defined with just a few parameters, the mean and the corresponding normal distribution variance, beyond its initial value.

Simulating Distributions

- Once the desired distribution is chosen, it is necessary to generate random values from this distribution.

- There are diverse ways to do this:
  - To develop a software program to perform the simulation.
  - Use the distribution functions of Excel in connection with the RAND function.
  - Use the functions of specialized programs like Crystall Ball.

- In this course, we will utilize the software Crystal Ball to carry out the necessary simulations.

Simulating a GBM

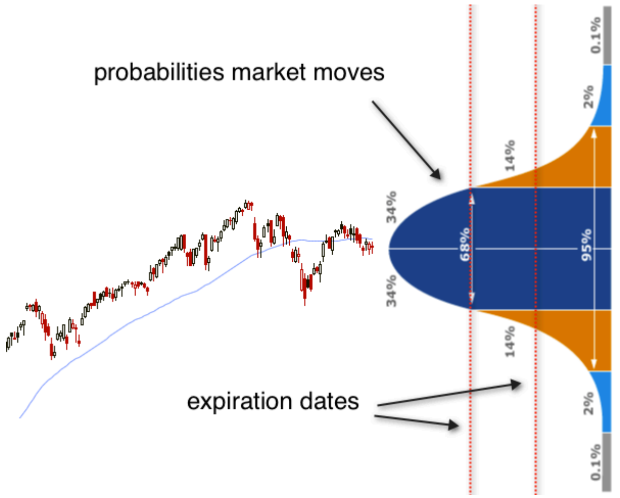
GBM

- If the underlying asset follows GBM, we have $dS = \mu S dt + \sigma S dz$
- To simulate the path followed by S we use a discrete model:
- This can be modeled in Excel as:

$$S_{t+1} - S_t = \mu S_t \Delta t + \sigma S_t \varepsilon \sqrt{\Delta t} \qquad \varepsilon \sim N(0,1)$$

- With Excel the representation is:

$$S_{t+1} - S_t = \mu S_t \Delta t + \sigma S_t \text{Normsinv}(\text{Rand}()) \sqrt{\Delta t}$$



Simulating a GBM

- We can also simulate Ln(S) as follows:

$$\ln S_{t+1} - \ln S_t = \left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\ln S_t - \ln S_0 \sim N\left[\left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

or

or,
$$d(\ln S) = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma dz$$

$$\ln S_t \sim N\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)t, \sigma^2 t\right]$$

- To simulate the paths in an Excel worksheet we have:

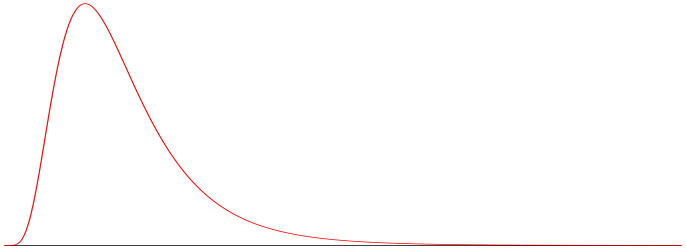
$$S_{t+1} = S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

or,
$$S_{t+1} = S_t e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\, NORMSINV(RAND())}$$

GBM Parameters

- S_t: $$E\left[\tilde{S}_t\right] = S_0 e^{\tilde{\mu} t}$$

$$\operatorname{var}\left[\tilde{S}_t\right] = S_0^2 e^{2\tilde{\mu} t}\left(e^{\sigma^2 t} - 1\right)$$



- Ln S_t:

$$E\left[\ln\frac{\tilde{S}_t}{S_0}\right] = E\left[\tilde{v}\right] = vt$$

$$\operatorname{var}\left[\ln\frac{\tilde{S}_t}{S_0}\right] = \operatorname{var}\left[\tilde{v}\right] = \sigma^2 t$$

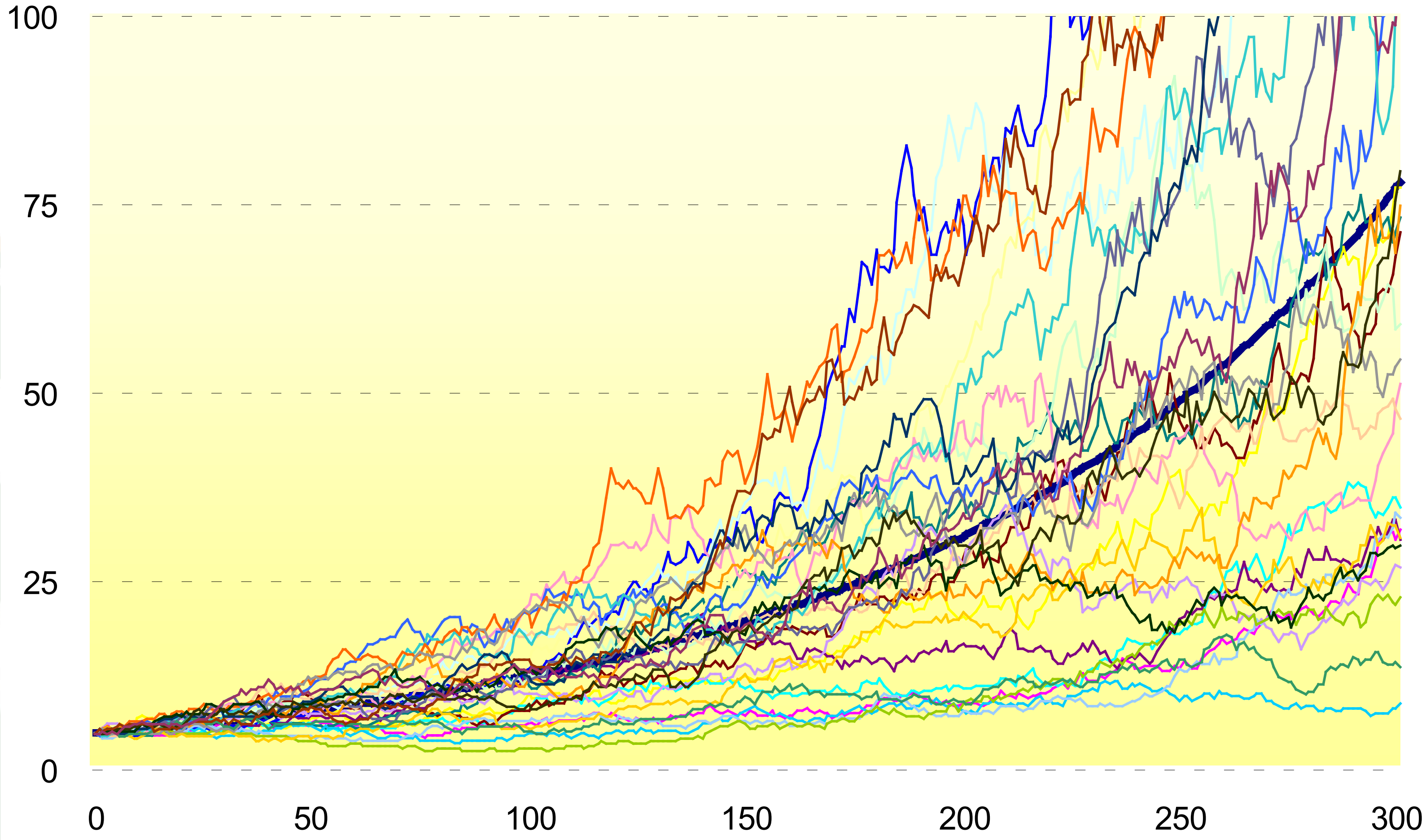
$$v = \mu - \frac{\sigma^2}{2}$$

μ and $\mu - \sigma^2/2$

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is $\mu - s^2/2$ not μ
- This is because $\ln[E(S_T / S_0)]$ and $E[\ln(S_T / S_0)]$ are not the same

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ
- The geometric mean of 12.4% is analogous to $\mu - \sigma^2/2$

Simulation Paths of GBM's Realization



Simulation Paths of GBM's Realization