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UNDER COLD CONDITIONS

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Interim Report

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PROGRESS REPORT FOR NOAA PROJECT

by

H. J. Morel-Seytoux

December 1974

In this report equations are developed which describe the evolution of the contents and of the temperature of a vertical column of soil. The equations which are derived are not the most general and in particular they only describe water movement in the liquid phase under freezing conditions. From a hydrologic point of view, the thawing period may be of even greater interest than that of freezing and the pertinent equations will be investigated later. In addition moisture movement in vapor form will have to be investigated also.

Even without considering thawing and vapor movement the problem is quite complex, at least at first look. Mathematical modeling is futile if the physical laws governing the processes are not known or misrepresented. For this reason the physical understanding of water movement is soil under freezing will be reviewed.

A. The Physical Phenomena

Experience (and experiments) have shown that as a soil cools below freezing point not only does the (local) water in the "frozen" zone turn to ice but in addition foreign water which was not originally in the frozen zone moves into the frozen zone from the surrounding "unfrozen" soil and then freezes. When this flux of foreign water is important relative to the native water, heaving may take place as a result of ice lens formation (Jumikis, 1956). Heaving has been of concern to construction engineers (highway and foundation engineers) for a long time. For this reason many experiments have been carried to determine the effect of soil porosity (Jumikis, 1969) soil type (Aguirre-Puentes and LeFur, 1967), freezing rate (Penner, 1972), depth to water table (McGaw, 1972) on heaving and rate of heave. One first conclusion is in order: associated with the water movement is a soil deformation. Whereas the soil deformation may not significantly affect the soil temperature distribution, on the contrary it will affect the moisture distribution significantly (or let us say it might). This moisture distribution in space at time of snowmelt may significantly affect the soil capacity to infiltrate water. It appears imperative to include the phenomenon of heave in the description of the system.

What mechanism causes the attraction of water into the frozen zone remained a puzzle for a long time (Jumikis, 1954; Jumikis, 1960; Jumikis,1963; Aguirre-Puente et al. 1971). For a long time an explanation was found in terms of a critical radius of pores, the soil being viewed as a bundle of capillary tubes. This tendency is still alive (Miller,1973; Aguirre-Puente et al. 1972; Aguirre-Puente and Adouni, 1973) not longer to derive quantitative solutions but to secure qualitative answers. Little by little the idea germinated that there exists a capillary pressure between ice and water in a soil just as there is one between air and water (Miller,1963) and that these soil characteristics might possibly be deduced from one another (Koopmans and Miller, 1966). It it not altogether surprising that the idea came from a soil scientist (Miller) rather than from the engineers, probably aware of the capillary rise of water in a capillary tube but not familiar with the concept of a "moisture retention" curve (Baver et al. 1972).

A clear relation between capillary pressure and temperature was introduced by Everett (1961), not surprisingly an expert thermodynamicist, in a contribution termed "definitive" by Miller (Miller, 1963, p. 193). Unfortunately in spite of these accomplishments in understanding the phenomena it is not possible to deduce the ice-water capillary pressure curve from the air-water one except for a soil-soil contact of particles, (Koopmans and Miller, 1966, p. 680) or a pure SLS soil (i.e. a soil with contact between particles always involving an in-between liquid film). In addition the hydraulic characteristics of water in the frozen zone (i.e. the relative permeability to water as a function of water content) are not at all known quantitatively. Based on actual measurements of the ice-water capillary pressure curve (Koopmans and Miller, 1966, p. 683) it is apparent that a temperature drop of only 0.2° C will leave only a small residual unfrozen water content of probably very low mobility. Most probably this drop of temperature of 0.2° C will occur over a relatively thin zone (maybe one inch) and possibly this zone of a drastic reduction from a high water content to a low water content may be treated as a front or at least as a zone of constant properties, with relatively good accuracy. In other words at the ice front the water velocity (i.e. flux per unit bulk area, a Darcy velocity) has a given value. An inch or so further in the frozen zone this velocity is essentially zero. In between the water velocity distribution is not known, but the ignorance of the exact shape of the curve linking two points where the values are known is probably not important as long as the interval is small.

B. Modeling

It is rather surprising that so few attempts at mathematical description and solution of the problem have been carried and only so within the last few years (Kennedy and Lielmezs, 1973; Harlan, 1973; Guymon and Luthin, 1974). The lack of existence of measured properties of the soil, needed for the mathematical description of the system, is probably one of the reasons. We suspect that another cause is the plain difficulty of the problem and the relative scarcity of people trained in the combined areas of thermodynamics, heat transfer, multi-phase flow in porous media and soil mechanics. Petroleum engineers may come closest to meeting these qualifications (except soil mechanics). Combined fluid and heat transfer with phase change has been described in the petroleum literature for some time (Marx and Langenheim, 1959; Ramey, 1959) but many of the solutions developed apply to relatively high speed flow when the major fluid drive is external (injection under high pressure) rather than the natural capillary drive (Fayers, 1962; Morel-Seytoux, 1973).

The assumptions made by the few investigators of water movement under freezing conditions are quite drastic (for example see the long list in Kennedy and Lielmezs, 1973, p. 395). In spite of all these assumptions, Kennedy and Leilmezs did not solve a single problem! They only wrote down the equations to be solved. The fact that the paper was published is an indication of how little work has been done on the subject up to now. The other investigators, most notably Harlan but also Guymon and Luthin, solved the equations for some realistic problems. Harlan solved a problem involving a sequence of freezing, thawing and infiltration. Harlan relaxed several assumptions mentioned in the Kennedy and Lielmezs paper but still assumes that the porous medium is rigid (Harlan, 1973, p. 1316). This same assumption is made by Guymon and Luthin (1974, p. 996). It seems that for these authors the problems of concern are of a type for which important ice lens formation is not to be expected. On the contrary here it is felt that the ice lens formation is of paramount importance.

C. Proposed Equations (no heaving)

First let us define more precisely our terminology. A frozen zone is a zone in which the temperature is everywhere below the (flat interface) freezing point i.e., 0° C. An unfrozen zone (or soil) is a zone where the temperature is above 0° C. In a frozen zone both ice and water coexist. The water content of (liquid) water is denoted θ, of meaning: volumetric fraction of the bulk volume. Similarly an ice content θ_i is defined and occasionally an air content, θ_a. The original porosity of the soil is denoted ϕ_o. In the unfrozen soil only water and air coexist in the unsaturated part of the unfrozen zone.

The ice front is the plane of 0° C temperature. The heaving front is the plane above which no active heaving takes place any longer. Figure 1 illustrates a typical water, ice and air content distribution. The z-coordinate, oriented positive downward, is a fixed coordinate (i.e., not moving with the soil). Figure 1 shows the existence of several zones from top to bottom: (1) a frozen zone, (2) a heaving (therefore also frozen) zone, (3) an (unfrozen) unsaturated zone and (4) a saturated zone. Our approach is to describe each zone separately, then to describe the conditions to be satisfied at the interface of each zone. Because it is the simplest zone to describe we first discuss the unsaturated-saturated zone.

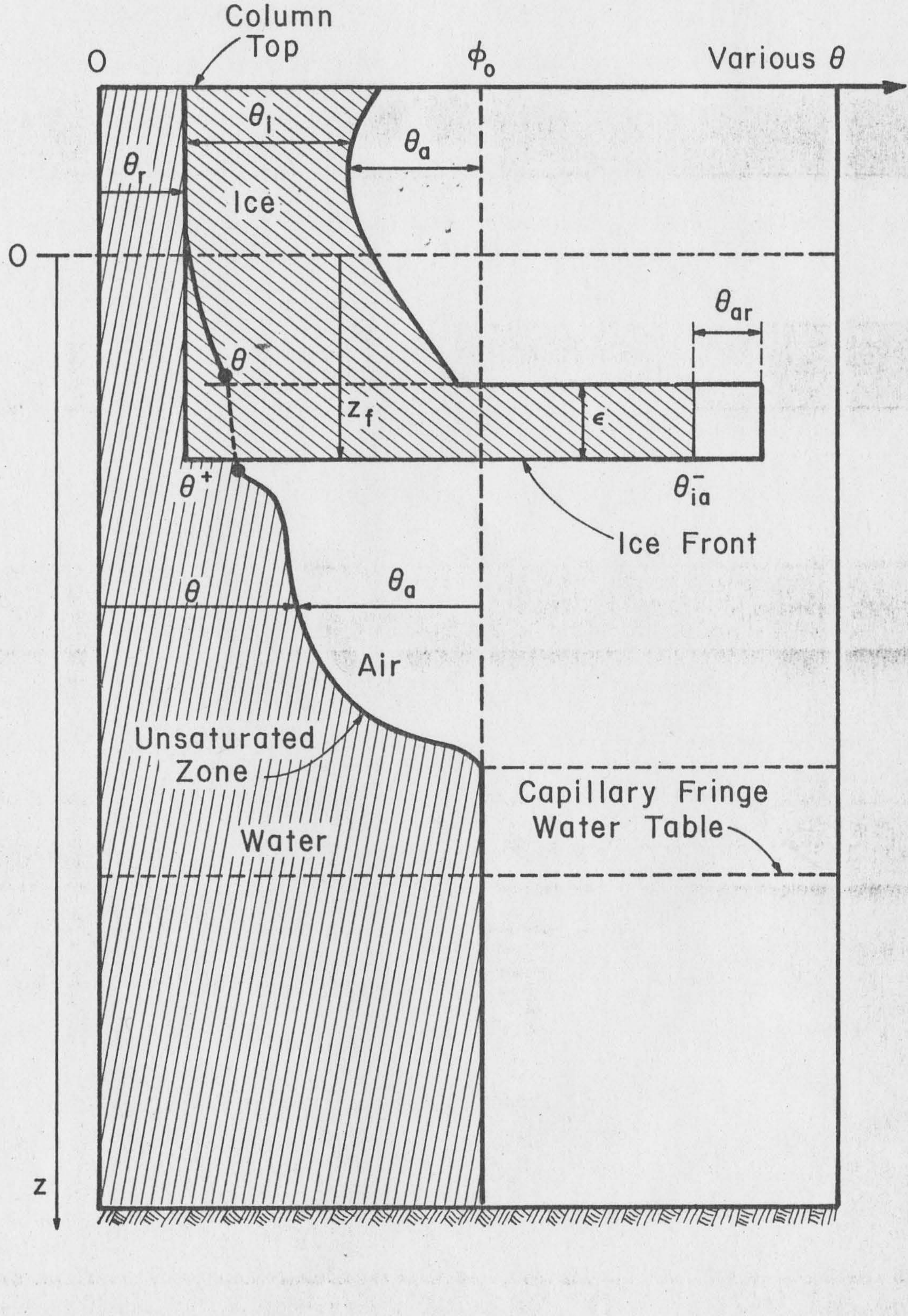

Column Top
0
ϕ_o
Various θ
θ_1
θ_a
θ_r
Ice
0
θ_{ar}
θ^-
z_f
ϵ
θ^+
θ^-_{ia}
Ice Front
θ
θ_a
Air
Unsaturated Zone
Capillary Fringe
Water Table
Water
z

Figure 1

1. The "Unsaturated" Zone

This zone is limited at the top by the ice-front and at the bottom ($z = D$) by a plane deep enough in the aquifer to represent a no-flow and constant temperature boundary. In the zone defined by $z_f \leq z \leq D$ the present fluids are water and air. Movement of both fluids satisfy Darcy's law so that:

$$v_w = -k \frac{k_{rw}}{\mu_w} \frac{\partial p_w}{\partial z} + k \frac{k_{rw}}{\mu_w} \rho_w g \tag{1}$$

$$v_a = -k \frac{k_{ra}}{\mu_a} \frac{\partial p_a}{\partial z} + k \frac{k_{ra}}{\mu_a} \rho_a g \tag{2}$$

with the usual notations (Morel-Seytoux, 1973a) namely: k is intrinsic permeability (L^2, cm^2), k_{rw} is relative permeability (i.e., relative to permeability at natural saturation), μ_w is water viscosity ($M\bar{L}^{-1}T^{-1}$, poises), p_w is water pressure ($ML^{-1}T^{-2}$, usually expressed in cm of water and denoted h_w, thus $p_w = \rho_w^* g h_w$ where ρ_w^* is specific mass of water at standard temperature), ρ_w is water specific mass at temperature T, g is acceleration of gravity (981 cm/sec^2) and the subscript a refers to air. Total velocity is defined as:

$$v = v_w + v_a \tag{3}$$

Mass conservation yields the equations:

$$\frac{\partial(\rho_w \theta)}{\partial t} + \frac{\partial(\rho_w v_w)}{\partial z} = 0 \tag{4}$$

and

$$\frac{\partial(\rho_a \theta_a)}{\partial t} + \frac{\partial(\rho_a v_a)}{\partial z} = 0 \tag{5}$$

Water is essentially incompressible and the variation of ρ_w with temperature in the range 0-25° C is not significant. Thus as a good approximation Eq. (4) reduces to:

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_w}{\partial z} = 0 \tag{6}$$

Because the air pressure deviations from atmospheric pressure are quite small air compressibility effects can be neglected and similarly Eq. (5) reduces to:

$$\frac{\partial \theta_a}{\partial t} + \frac{\partial v_a}{\partial z} = 0 \tag{7}$$

Below the ice-front the porous medium can be considered as nondeformable. Consequently adding Eq. (6) and (7) one obtains:

$$\frac{\partial(\theta+\theta_a)}{\partial t} + \frac{\partial(v_w+v_a)}{\partial z} = 0 \tag{8}$$

or

$$\frac{\partial \phi_o}{\partial t} + \frac{\partial v}{\partial z} = 0 \tag{9}$$

or

$$\frac{\partial v}{\partial z} = 0 \tag{10}$$

that is the important result that the total velocity v in the unfrozen zone does not vary with depth. However, it may vary with time.

First consider the case when v is not zero. Then it is convenient to define the "fractional flow function" F_w as the ratio of v_w/v. Using Eqs. (1) and (2) we obtain for F_w the expression:

$$F_w = f_w \{1 + k\frac{k_{ra}}{\mu_a} \rho_w g \frac{1}{v}(1 + \frac{\partial h_c}{\partial z})\} \tag{11}$$

or defining

$$G_w = k\frac{k_{ra}(\theta)\rho_w g}{\mu_a} f_w(\theta) \tag{12}$$

and $$E_w = -G_w(\theta)\frac{dh_c}{d\theta} \tag{13}$$

we have $$F_w = f_w + \frac{G_w}{v} - \frac{E}{v}\frac{\partial\theta}{\partial z} \tag{14}$$

Substitution of Eq. (14) in Eq. (6) yields the water content equation:

$$\frac{\partial\theta}{\partial t} + vf_w'\frac{\partial\theta}{\partial z} + G_w'\frac{\partial\theta}{\partial z} - \frac{\partial}{\partial z}(E\frac{\partial\theta}{\partial z}) = 0 \tag{15}$$

Equation (15) involves two unknowns θ and v . Another equation is obtained by calculating v directly from Eqs. (1) and (2):

$$v = -\lambda_w\frac{\partial p_w}{\partial z} - \lambda_a\frac{\partial p_a}{\partial z} + \lambda_w\rho_w g + \partial_a\rho_a g \tag{16}$$

or $$\frac{v}{\Lambda} = -f_w\frac{\partial p_w}{\partial z} - f_a\frac{\partial p_a}{\partial z} + f_w\rho_w g + f_a\rho_a g \tag{17}$$

or $$\frac{v}{\Lambda} = -f_w(\frac{\partial p_a}{\partial z} - \frac{\partial p_c}{\partial z}) - (1-f_w)\frac{\partial p_a}{\partial z} + f_w\rho_w g + f_a\rho_a g \tag{18}$$

or $$\frac{v}{\Lambda} = f_w\frac{\partial p_c}{\partial z} - \frac{\partial p_a}{\partial z} + f_w\rho_w g + f_a\rho_a g \tag{19}$$

Integrating Eq. (19) between two positions z_1 and z_2 yields

$$v = \frac{\int_1^2 f_w dp_c + p_{a1} - p_{a2} + \rho_w g\int_1^2 f_w dz}{\int_1^2 \frac{dz}{\Lambda}} \tag{20}$$

where Λ is the total mobility (i.e., $k\frac{k_{rw}}{\mu_w} + k\frac{k_{ra}}{\mu_a}$).

In the case $v = 0$ Eq. 14 is not defined. However the limit of $v_w = vF_w$ when $v \to 0$ is defined namely:

$$v_w = G_w - E_w \frac{\partial \theta}{\partial z}$$

and Eq. (15) applies, setting $v = 0$.

The "natural" limits for Eq. (20) are the bottom and the ice-front. In this case Eq. (20) takes the form:

$$v = \frac{\rho_w g \int_{z_f}^{D} f_w dh_c + \rho_w g \int_{z_f}^{D} f_w dz + (p_{af} - p_{ab})}{\int_{z_f}^{D} \frac{dz}{\Lambda}} \tag{22}$$

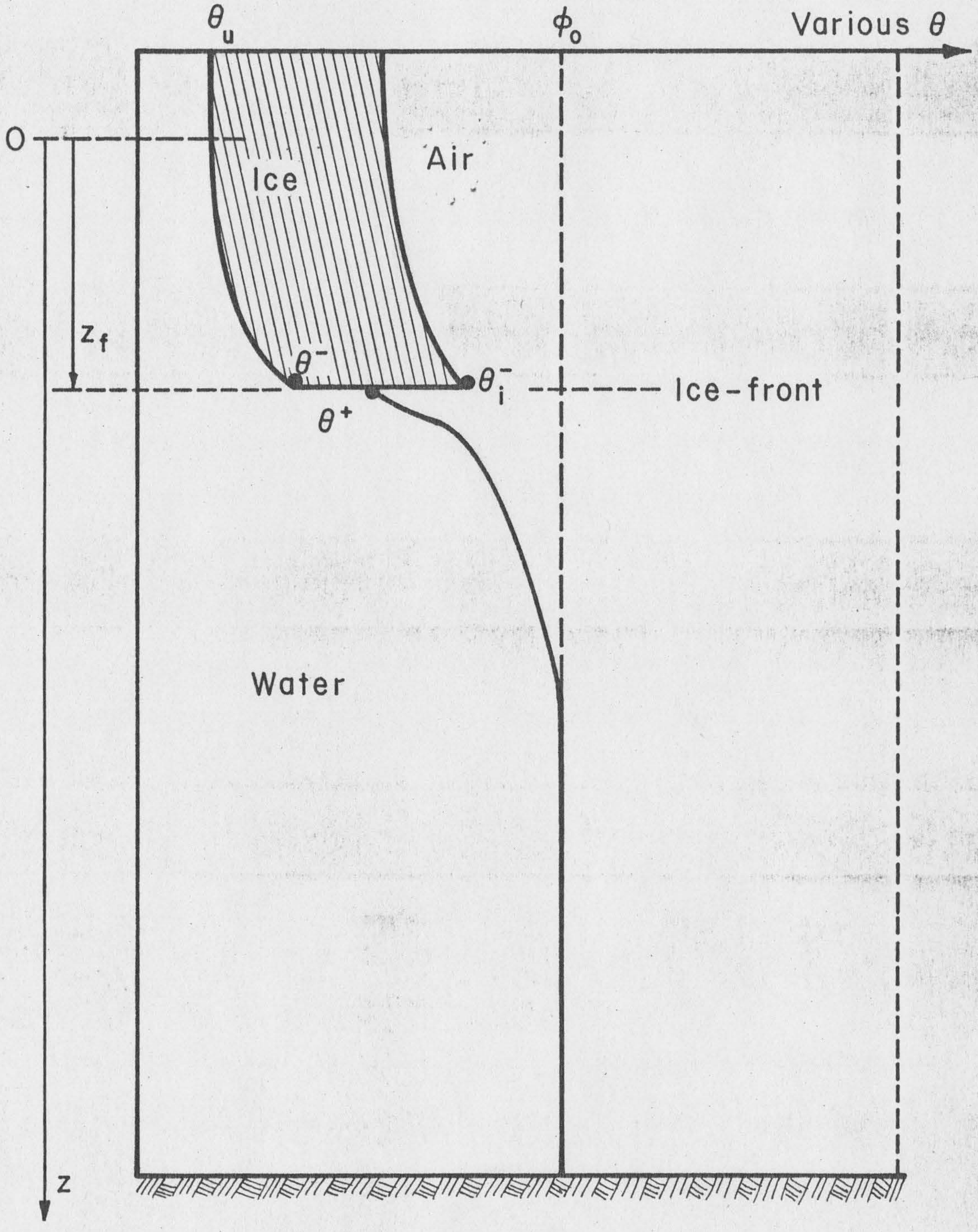
Equation (22) can be rewritten:

$$v = \frac{\rho_w g \left[- \int_{o}^{h_c(\theta^+)} f_w dh_c + \int_{z_f}^{D} f_w dz \right] + (p_{af} - p_{ab})}{\int_{z_f}^{D} \frac{dz}{\Lambda}} \tag{23}$$

where p_{af} is the air pressure at the ice-front and p_{ab} is the air pressure at bottom, which can be taken as atmospheric.

2. <u>The Frozen Zone</u>

We first study the case when no heaving occurs. A typical profile is shown on Figure 2. Three phases may coexist in this zone. For water and air Darcy's law applies, namely:



θ_u
ϕ_o
Various θ
O
Ice
Air
z_f
θ^-
θ_i^-
Ice-front
θ^+
Water
z

Figure 2

$$v_w = -k \frac{k_{rw}}{\mu_w} \frac{\partial p_w}{\partial z} + k \frac{k_{rw}}{\mu_w} \rho_w g \tag{24}$$

$$v_a = -k \frac{k_{ra}}{\mu_a} \frac{\partial p_a}{\partial z} + k \frac{k_{ra}}{\mu_a} \rho_a g \tag{25}$$

Though the notation above does not indicate it, it must be understood that k_{rw} and k_{ra} are not necessarily the same functions of θ and θ_a as in the unfrozen zone. However due to the complete lack of data and in view of the encouraging results of Koopmans and Miller on similarity of capillary pressure curves, we shall not make a notational distinction between the curves for the water-air system and for the water-air-ice system.

We might be justified to write an equation of Darcy's form for ice movement. We speculate however, that progression of ice is not so much a matter of motion as of accretion. Nevertheless very formally we write:

$$v_i = -k \frac{k_{ri}}{\mu_i} \frac{\partial p_i}{\partial z} + k \frac{k_{ri}}{\mu_i} \rho_i g \tag{26}$$

the viscosity of ice being extremely high compared to that of water. For most purposes we can assume that $v_i = 0$.

In the frozen zone the water content is controlled very strongly by temperature and at equilibrium completely. However, when there is no equilibrium the water content at a given temperature is probably not immediately equal to the equilibrium value. From the capillary pressure for the ice-water system we have a relation:

$$h_{ci} = h_{ci}(\theta) \quad \text{or} \quad h_{ci} = h_{ci}(T) \tag{27}$$

and thus $\theta = \theta^*(T)$ (28)

where the * indicates that it is the moisture content at equilibrium for a given temperature. The mass conservation of water in liquid form can be written as:

$$\frac{\partial \theta}{\partial t} + \frac{\partial v_w}{\partial z} = -\alpha(\theta - \theta^*) \tag{29}$$

that for ice (neglecting ice motion):

$$\frac{\partial \theta_i}{\partial t} = \alpha\ (\theta - \theta^*)\frac{\rho_w}{\rho_i} \tag{30}$$

and that for air:

$$\frac{\partial \theta_a}{\partial t} + \frac{\partial v_a}{\partial z} = 0 \tag{31}$$

Defining again a total velocity:

$$v = v_w + v_a + v_i \tag{32}$$

(and again $v_i \simeq 0$)

Adding Eqs. (29), (30) and (31) and since (without heaving):

$$\phi_o = \theta_i + \theta + \theta_a \tag{33}$$

we obtain the result:

$$\frac{\partial v}{\partial z} = \alpha\ (\theta - \theta^*)\left(\frac{\rho_w - \rho_i}{\rho_i}\right) \tag{34}$$

In the frozen zone the total velocity depends on z. From Eqs. (24) and (25) we obtain an equation identical to Eq. (19). Integrating Eq. (19) from top of the column to the ice front we obtain:

$$\bar{v} = \frac{\rho_w g \left[- \int_{h_{ci}(\theta^-)}^{h_{ci}(\theta_a)} f_w dh_{ci} + \int_o^{z_f} f_w dz \right] + p_A - p_{af}}{\int_o^{z_f} \frac{dz}{\Lambda}} \tag{35}$$

where θ_u is the water content at top of the column and θ^- is the water content on the upstream side of the ice-front. The bar above v indicates that it is a space-average value for v. P_A is atmospheric pressure.

3. The Interface Between the Frozen and Unfrozen Zone

Let V_f be the velocity of propagation of the ice-front (0° C plane). We can assume that V_f is known (controlled by the heat transfer equations to be described later). There remain three unknowns defined at the front, to be calculated: θ^+, θ^- and θ_i^- (θ_i^+ is zero; there is no ice above 0° C).

At the ice front several processes take place; a volume of water (per unit time) $V_f\ (\theta^+ - \theta^-)$ is frozen, a volume of water $-V_w^+$ is received from below and partially frozen, and a volume of air $V_f(\theta_i^- + \theta^- - \theta^+)$ is displaced ahead of the front. Mass conservation of water at the front requires that:

$$V_f \theta_i^- = V_f(\theta^+ - \theta^-) \frac{\rho_w}{\rho_i} - (v_w^+ - v_w^-) \frac{\rho_w}{\rho_i} \tag{36}$$

where v_w^- is the water flux penetrating past the front into the frozen zone.

In the unfrozen zone $\frac{\partial v}{\partial t} = 0$ and since $v = 0$ at the bottom, consequently $v = 0$ at the ice front, or in other words:

$$v_w^+ = -v_a^+ \tag{37}$$

The influx of air into the unfrozen zone results from the displacement of air by the penetrating ice front and from a flux of air V_a^- from the frozen zone. Consequently:

$$v_w^+ = -V_f(\theta_i^- + \theta^- - \theta^+) - v_a^- \tag{38}$$

Substitution in Eq. (36) yields:

$$V_f\theta_i^- = \frac{\rho_w}{\rho_i} V_f \; [\theta^+ - \theta^- + \theta_i^- + \theta^- - \theta^+] + v^- \frac{\rho_w}{\rho_i} \tag{39}$$

or $$v^- = -V_f \; \theta_i^- \; (\frac{\rho_w - \rho_i}{\rho_w}) \tag{40}$$

At the interface the water (liquid) is a continuous phase and consequently the equality of water pressure on the two sides of the interface must be satisfied:

$$h_c(\theta^+) = h_{ci}(\theta^-) + h_{af} - h_i \tag{41}$$

where h_{af} is air pressure at ice front and h_i is ice pressure (expressed as water heights).

The condition that the water contents θ^+ and θ^- must travel at the same velocity as the front yields:

$$v_w^+ = G_n(\theta^+) - E(\theta^+)(\frac{\partial\theta}{\partial z})_{\theta^+} \tag{42}$$

and $$V_f = G'(\theta^+) - E'(\theta^+)(\frac{\partial\theta}{\partial z})^+ \tag{43}$$

neglecting curvature of profile in the neighborhood of ice front or using Eq. (38):

$$\frac{-V_f(\theta_i^- + \theta^- - \theta^+) - v_a^- - G^+}{V_f - G'(\theta^+)} = \frac{E(\theta^+)}{E'(\theta^+)} \tag{44}$$

Similarly on the upstream side of the ice front we must have:

$$v_w = v^- f_w^- + G^- - E^- \left(\frac{\partial\theta}{\partial z}\right)^- \tag{45}$$

$$V_f = v^- f'(\theta^-) + G'(\theta^-) - E'(\theta^-) \left(\frac{\partial\theta}{\partial z}\right)^- \tag{46}$$

Since $v_w^- + v_a^- = v^-$ we obtain:

$$\frac{v^-(1-f_w^-) - v_a^- - G^-}{V_f - v^- f'(\theta^-) - G'(\theta^+)} = \frac{E(\theta^-)}{E'(\theta^-)} \tag{47}$$

Equations (40), (41), (44) and (47) provide four equations for the unknowns v^-, θ_i^-, θ^+, θ^-, h_{af} and v_a^-; h_{af} can be calculated from Eq. (23) since $v \equiv 0$ in the unfrozen zone, thus:

$$h_{af} = \int_o^{h_c(\theta^+)} f_w dh_c - \int_{z_f}^{D} f_w dz \tag{48}$$

Given h_{af}, $\overline{v}$ can be calculated from Eq. (35) and then corrected from Eq. (34) to give v^-. Thus we have enough equations for the unknowns. Using Eqs. (44) and (47) to eliminate v_a^-, this new equation and Eqs. (41) and (40) provide three equations for the three basic unknowns at the ice front: θ_i^-, θ^- and θ^+ if V_f, v^- and h_{af} can be calculated otherwise.

4. Heat Equations

In the unfrozen zone the heat conservation equation is: (neglecting transport in air phase)

$$\frac{\partial}{\partial t} \left[\rho_w \theta C_w + \rho_s(1-\phi)C_s\right] T$$

$$+ \frac{\partial}{\partial z} (\rho_w v_w C_w T) - \frac{\partial}{\partial z} \left\{\left[\kappa_w \theta + \kappa_s(1-\phi_o)\right]\frac{\partial T}{\partial z}\right\} = 0 \tag{49}$$

where C_w is heat capacity for water, κ_w the thermal conductivity in Fourier's law of heat, T is temperature and s is the subscript for soil.

In the frozen zone the heat equation is:

$$\frac{\partial}{\partial t}\{[\rho_i C_i \theta_i + \rho_w C_w \theta + \rho_s (1-\phi_o) C_s] T\}$$

$$- \frac{\partial}{\partial z}\{[\kappa_i \theta_i + \kappa_w \theta + \kappa_s (1-\phi_o)] \frac{\partial T}{\partial z}\}$$

$$+ \frac{\partial}{\partial z}(\rho_w v_w C_w T) - L\rho_i \frac{\partial \theta_i}{\partial t} = 0 \qquad (50)$$

where L is the latent heat of fusion.

At the interface the conservation of heat requires that

$$-[\kappa_i \theta_i^- + \kappa_w \theta^- + \kappa_s (1-\phi_o)](\frac{\partial T}{\partial z})^- + \rho_w C_w v_w^- T$$

$$= -[\kappa_w \theta^+ + \kappa_s (1-\phi_o)](\frac{\partial T}{\partial z})^+ + \rho_w C_w v_w^+ T$$

$$-LV_f \rho_i \theta_i^- \qquad (51)$$

which expressing T in °C reduces to:

$$[\kappa_i \theta_i^- + \kappa_w \theta^- + \kappa_s (1-\phi_o)](\frac{\partial T}{\partial z})^- = [\kappa_w \theta^+ + \kappa_s (1-\phi_o)](\frac{\partial T}{\partial z})^+$$

$$+ LV_f \rho_i \theta_i^- \qquad (52)$$

The boundary conditions are that at top of column temperature is T_u and at bottom it is T_b.

5. Solution Procedures

Let the overall moisture profile (and temperature profile) be as shown on Figure 2 at time t_o. One wants to find the new profiles at time $t, (t_o + \Delta t)$. Based on the old profiles one calculates V_f from Eq. (52) or rather one solves for Eq. (49) given T_b which yields $(\frac{\partial T}{\partial z})^+$ and Eq. (50) given T_u which yields $(\frac{\partial T}{\partial z})^-$. Then Eq. (52) yields V_f. (The method of solution, i.e., finite-difference, finite element, Galerkin, method of characteristics, hybrid approaches etc., for the partial differential equation will be studied later.) From Eq. (35) by numerical integration of the t_o-profile one obtains $\bar{v}$.

Knowing $\bar{v}$ and V_f one can solve simultaneously the non-linear system of three equations for θ^+, θ^- and θ_i^-. One then "moves" the temperature and content profiles over a time increment Δt to obtain the new profiles at time t. First the temperature profile is moved. Then $\theta^*(z)$ is known since θ^* is a function of T and $T(z)$ is known. Then one moves the content profiles.

In the unsaturated zone the velocity of propagation of a given θ value is simply:

$$(\frac{dz}{dt})_\theta = \text{slope of } v_w(\theta) \text{ curve}$$

or
$$(\frac{dz}{dt})_\theta = \frac{d}{d\theta}(G - E\frac{\partial\theta}{\partial z}) \tag{53}$$

using Eq. (21).

In the frozen zone one first applies the correction of Eq. (34) to calculate v knowing $\bar{v}$. (This correction may be quite small and not necessary since θ will be close to θ^* and $\rho_w - \rho_i / \rho_i$ is of the order of 0.1). The velocity of propagation of a given θ_o is

$$(\frac{dz}{dt})_{\theta_o} = v \frac{d}{d\theta}(F) \tag{54}$$

where F is given by Eq. (14), but as θ_o moves it changes its value and upon arrival it has a value:

$$\theta = \theta^* + (\theta_o - \theta^*)e^{-\alpha(t-t_o)} \tag{55}$$

One is now ready to repeat the procedures for a new time increment.

6. Remarks

In the experiments of Dirksen and Miller (see their Figure 1, p. 169) the average temperature gradient across the column has a value of the order of 1.3° C per cm. Based on the results of Koopmans and Miller (1966, p. 683) it would appear that residual water content with very low relative permeability would be attained within a temperature drop of the order of 0.3° C (i.e., a distance of about 2 mm) behind the freezing front. If that were true there should be no ice-content increase with time at positions a few millimeters behing the freezing front. In fact, however, there was a significant ice increase 2 cm behind the freezing front. Unfortunately no relative permeability data are available in the Dirksen and Miller paper. Either the soil had still high relative permeability at very low content so that liquid water could percolate downward and feed the ice well below the freezing front or much higher water content than one would expect from the equilibrium capillary pressure curves existed at low temperatures. For this reason a kinetic first-order reaction for the icing was postulated in our model, namely Eq. (30). The larger α the faster the liquid water content reaches its equilibrium value. Additional evidence is provided by the lack of correspondence mentioned by Guymon and Luthin (1974, p. 1000) when using the Clapeyron equation to determine the equilibrium ice content and using the Nakano-Brown empirical relationship for the Barrow soils.

D. Proposed Equations (heaving)

From a soil mechanics point of view heaving will start when the ice pressure exceeds the overburden pressure since then the effective stress on the soil becomes negative. From a practical standpoint heaving starts when the ice and water content exceed 90 percent of the initial porosity or more precisely when:

$$\theta + \theta_i \geq \phi_o - \theta_{ar} \tag{56}$$

where θ_{ar} is residual (trapped) air content (Dirksen and Miller, 1966, p. 170; McGaw, 1972, p. 51).

Thus as θ_i^- is calculated as a function of time (and so is θ^-) one checks whether or not

$$\theta^- + \theta_i^- \geq \phi_o - \theta_{ar} \tag{57}$$

The problem is different from the no-heave case in that a zone of soil of thickness ε changes porosity. This zone is limited from below by the ice-front and from above by the heaving front. It goes without saying that essentially nothing is known about the variations of $h_{ci}(\theta)$ with changes in porosity due to ice lensing. A priori it would seem that the same equations that apply for no-heaving would still apply if the change of ϕ with time can be made explicit.

Equation (33) for example takes the new form:

$$\phi = \theta_i + \theta + \theta_a \tag{58}$$

and formally Eq. (34) becomes:

$$\frac{\partial v}{\partial z} = \alpha(\theta-\theta^*)\left(\frac{\rho_w-\rho_i}{\rho_i}\right) - \frac{\partial \phi}{\partial t} \tag{59}$$

However θ^* is a different function of T when the soil deforms than when the soil had its original ϕ_o porosity (the capillary pressure curve changes).