Investment Timing under Regime Switching

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We investigate the optimal investment timing strategy in a real option framework. Depending on the state of the economy, whose changes are modeled by a Markov chain, the investment cost can take one of two values. The optimal investment timing decision is determined by finding the free boundary of a perpetual American option. Three investment timing policies, based on different assumptions of investors' information sets, are determined and compared. In the full information case, a significantly earlier optimal exercising time is indicated. We show that an optimal timing policy suggested by the conventional real option model might ruin the investment opportunities.

Keywords: Regime Switching, Real Option, Investment Timing.

1. Introduction

The investment timing problem is usually solved by the standard real option technique which is addressed by papers including Titman (1985), Brennan and Schwartz (1985), McDonald and Siegel (1986), and Dixit (1989). The books by Dixit and Pindyck (1994), and Trigeorgis (1996) offer a classical treatment. Schwartz and Trigeorgis (2004) collect classical readings and recent contributions in real options and investment under uncertainty, as the title of the book indicates. Grenadier (2000) studies firms' optimal option exercise policies in a continuous time Nash Cournot Equilibrium. The option value under strategic competition deteriorates quickly.

Grenadier and Wang (2005) derive an optimal contract that best aligns the incentives of owners and managers. They assume that the project generates two sources of value, one of which can be observed only by the manager. This source of value can take two different values and the manager can increase the possibility of the realization of the high value by exerting effort. However, in reality, managers have much limited ability to increase the probability of the high value.

We argue that it might be macroeconomic variables, rather than the manager's effort, which triggers the change of the parameters used in the model. We model the state of the economy by a Markov chain with a finite state space. Markov regime switching models have been used in electrical engineering since the 1960s. Hamilton (1989) proposed the a regime switching model to study postwar US real GNP associated with business cycle. See Hamilton (2005) for a recent survey. The modelling of asset price processes with regime shifts has also been adopted by others. Veronesi (1999) derives a rational expectations equilibrium model of asset values in the present of regime shifts. David and Veronesi (2000), and Buffington and Elliott (2002) values the contingent claims on an underlying with uncertain parameters.

This paper is intended to elaborate regime switching and optimal investment timing in a real option framework. The paper differs from the existing literature in a significant way. In this paper we first consider an irreversible investment timing decision by adding a hidden Markov process to model the state of the economy in continuous time. The cost of the investment is driven by the Markov chain. Therefore, the investment cost is either K_1 or K_2 depending on whether the economy is in a 'low cost' or 'high cost' state. K_1 and K_2 can be considered as strike prices of a perpetual American option. This is reasonable as people do observe business cycles of macroeconomic variables that determine the investment cost of the project. By introducing this specific structure of stochastic investment cost, the paper presents a different optimal exercising policy for the firm. We have shown that it is optimal for the firm to exercise much earlier than otherwise suggested by a standard real option model. Moreover, a range of exercising trigger prices is determined. It has been shown that the value of the growth option that the firm faces is also higher than the one derived by a conventional real option model.

There are a number of papers dealing with stochastic investment costs for real option valuation, or equivalently stochastic strike prices for the pricing of a financial contingent claim. Fischer (1978) and Margrabe (1978) derive explicit solutions to an option which gives the holder the right to exchange one risky asset to another. McDonald and Siegel (1986) studies a similar irreversible investment timing policy when the value of the project and the required investment cost follow two correlated geometric Brownian motions. This type of problem could be solved by using either of above processes as numeraire. Like the optimal exercising interval derived in our paper, their approach shows the optimal trigger value of the project could take values from zero to infinity, though the critical ratio of the value and the cost is uniquely determined. However, McDonald and Siegel (1986) points out that for the analysis it is necessary to assume Value F and V are geometric Brownian motions. However, this assumption is reasonable for the project value V, but may be less so for the investment cost F. McDonald and Siegel (1986) also suggests a mean reverting process for prices which are not present values. Since Markov switching models can model business cycles easily, they could be a better candidate for modeling non-present value prices.

Pindyck (1993) isolates the uncertainty embedded in the investment cost and examines a firm's irreversible investment decision when the cost is exposed to both technical and input risks. The project value in his model is constant. Blenman and Clark (2005) derives closed-form solutions to European options with stochastic strike prices, which preserve constant elasticity of the strikes with respect to the price. They applied the technique to value an IPO decision of a privately owned firm.

The modeling of stochastic investment costs driven by geometric Brownian motions has, at least, two drawbacks. First, the investment cost could go to everywhere in the interval of zero to infinity at the next second. This is obviously not a realistic assumption of the uncertainty costs. Second, as pointed out by McDonald and Siegel (1986), the prices of physical assets have to converge to their equilibrium levels, but not grow exponentially as suggested by a geometric Brownian motion.

The main contribution of our paper is to overcome the limitations discussed above. We assume, instead, that the investment costs follow a Markov chain with a finite state space, though for convenience of exposition we restrict our analysis to a two state Markov chain. This model confines the uncertainty of the investment cost to a controllable range instead of diffusively moving from zero to infinity. Moreover, by calibrating the parameters in our model, that is the elements in the Q-matrix, it is possible to have our stochastic investment cost return to its equilibrium level in the long run. Hence, our assumption of regime switching investment cost is less misspecified than ones with constant or stochastic costs driven by Brownian motions.

The paper is organized as follows. In section 2, real option models are developed to investigate the investment timing problem. Three cases with different information availability are considered. Two thresholds for exercising the projects with full information are derived. In section 3, the actual payoffs of the firm are derived and discussed in the presence of a hidden Markov chain, and the advantages of the full information model is presented by a numerical example. Section 4 concludes the paper.

2. Optimal Investment Models

Consider a firm which has an investment opportunity. If the firm invests in the project, it will produce a product. We suppose the entire value of the project at time t, S_t, follows a geometric Brownian motion,

$$dS_t = \mu S_t dt + \sigma S_t dw_t, \tag{2.1}$$

where μ is the instantaneous percentage change, or drift term, S_t, σ is the standard deviation per unit of time, or the volatility of the dynamics, and w_t is a standard Brownian motion.

In the classical real option models, people model a possible investment project as a perpetual American call option with a fixed strike price. However, for some projects this is not realistic. For instance, in a very good economy environment, the fixed cost for building a factory is substantially higher than in a bad economy since labor costs are much higher in a good economy and thus a high demand environment. Some equipment is also more expensive in a good economy than in a bad one. Therefore, the economy regime affects the investment cost at least for some projects. To model reality, we suppose the economy has two states, a 'high cost' and a 'low cost' state, in which the cost of investment is different.

We then consider the project as a perpetual American call option. It is optimal for the firm to launch the project if certain values of the project pass some thresholds. In a standard real option framework, the strike price is constant. However, in our model, we suppose the project has two strike prices, or different investment costs, depending on the states of the economy, or the states of the company.

To introduce our model formally, we make the following assumptions.

Assumption 2.1. *We consider a real option model on a filtered probability space* $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{Q}\right)$ *where* $\mathcal{F}_t$ *is to be determined for different cases. The capital market is free of arbitrage and complete.*

Assumption 2.2. *In the market there exists a riskless asset that is essentially a bank account. Let the continuously compounded bank rate be* r*. Then if* B_t *denotes the value in the bank account at time* t *of* \$1 *at time* 0,

$$B_t = e^{rt}. \tag{2.2}$$

We suppose that $\mu < r$ *in order to obtain a finite value of the project.*

The firm will launch the project when the expected value of the project is maximized. The optimal time to invest is then an extension of the usual perpetual American option problem to one where the strike is either K_1 or K_2, depending on the state of the economy. Without loss of generality, we suppose $K_1 < K_2$. The investment timing problem now provides a solution of when to exercise the option.

2.1. *Model Setting*

Recall that under the risk neutral probability

$$\frac{dS_t}{S_t} = \mu dt + \sigma dw_t. \tag{2.3}$$

We define an instantaneous dividend yield, or convenience yield, as

$$\delta = r - \mu.$$

Therefore, we have

$$\mu = r - \delta. \tag{2.4}$$

Substituting (2.4) into (2.3) , we have the following familiar dynamics:

$$\frac{dS_t}{S_t} = (r - \delta)\, dt + \sigma dw_t, \tag{2.5}$$

Now model the states of the world by a continuous time Markov chain $X = \{X_t, t \geq 0\}$. We suppose the economy has only two states: 'low cost' and 'high cost'. The state space of the chain X can be taken to be the two unit vectors $e_1 = (1,0)'$, $e_2 = (0,1)'$ in $\mathbb{R}^2$. The value of the project is then the payoff of a perpetual American call option, so we shall discuss the value of a perpetual call on S for which the strike price can take one of the two values K_1 or K_2. The strike price at time t is determined by the Markov chain $X = \{X_t, t \geq 0\}$ which takes two values corresponding to K_1 or K_2. If K denotes the vector (K_1, K_2) and $\langle , \rangle$ is the scalar product in $\mathbb{R}^2$ then the strike price in effect for the call at time t is

$$K_t = \langle K, X_t \rangle .$$

The strike price K_t reflects the state of the economy at time t. That is, in a low cost state of economy, the strike price is K_1, and K_2 otherwise.

Assumption 2.3. *Suppose the rate matrix of the chain X is $A = (a_{ij})$, $1 \leq i, j \leq 2$. Here A is a $Q-$matrix and $a_{ij} > 0$ for $i \neq j$, $a_{1j} + a_{2j} = 0$ for $j = 1, 2$ so $a_{11} < 0$ and $a_{22} < 0$.*

Write $a_{11} = -a_1$, and $a_{22} = -a_2$ so $A = \begin{pmatrix} -a_1 & a_2 \\ a_1 & -a_2 \end{pmatrix}$. The semimartingale representation of the Markov chain (see Elliott (1993)) is

$$X_t = X_0 + \int_0^t AX_s ds + M_t, \tag{2.6}$$

where $M = \{M_t, t \geq 0\}$ is an $\mathbb{R}^2$ valued Martingale process.

Assumption 2.4. *We assume the independence of the Brownian motion and the Markov chain. Accordingly the internal filtrations generated by respective Brownian motion, $\mathcal{F}_t^w = \sigma(w_s, 0 \leq s \leq t)$ and Markov chain $\mathcal{F}_t^X = \sigma(X_s, 0 \leq s \leq t)$ are independent.*

Definition 2.1. The full information set, $\mathcal{F}_t^{w,X}$, at time t is defined as

$$\mathcal{F}_t^{w,X} = \sigma(w_s, X_s, 0 \leq s \leq t). \tag{2.7}$$

It is optimal for the firm to start the project at time τ such that the Net Present Value of the firm, D, is maximized,

$$D = \sup_{\tau \geq 0} E\left[e^{-r\tau}(S_\tau - K_\tau) | \mathcal{F}_0\right]. \tag{2.8}$$

Here $\mathcal{F}$ is determined case by case. In the subsequent subsections, we study, case by case, the optimal strategies that the firm might take, given different available information to the decision maker.

2.2. *The Benchmark Case*

We shall decide the optimal investment policy for the firm under the regime switching framework. For comparision, we first consider a simple case with a rational investment cost based on the state of the economy when the project is initiated. Therefore, the manger faces an optimal exercise problem with a constant investment cost identical to the rational expectation of the investment cost at time zero.

The rational investment cost should be the expectation of $\langle K, X_t \rangle$. We first estimate the investment costs $\bar{K}_1$ and $\bar{K}_2$ by computing the expectation of $\langle K, X_t \rangle$ based on observing the original state of the chain. Thus, we shall first calculate the expectation of the Markov chain at time t, i.e. $\mathbb{E}[X_t]$.

We know X_t is a Markov chain with $Q-$Matrix A, thus, X_t has a semimartingale representation:

$$dX_t = AX_t dt + dM_t.$$

Solving the above SDE by multiplying the integrating factor, e^{At}, and assuming the initial value of $(X_t)_{t\geq 0}$ is a known value X_0, we have,

$$X_t = e^{At}\left(X_0 + \int_0^t e^{-Au} dM_u\right), \tag{2.9}$$

So

$$\mathbb{E}[X_t] = e^{At} X_0. \tag{2.10}$$

Here the stochastic integral with respect to dM_t disappears because it is a martingale. The following Lemma 1 gives the value of e^{At} :

Lemma 2.1.

$$e^{At} = \frac{1}{c}\begin{pmatrix} a_2 & a_2 \\ a_1 & a_1 \end{pmatrix} + \frac{e^{-ct}}{c} A. \tag{2.11}$$

where, $c = a_1 + a_2$.

Then the investment costs are:

(1) If the initial value of $X_0 = e_1$,, i.e. the economy at time zero is at a low cost state

$$\bar{K}_1 = \left(a_2 + a_1 e^{-ct}\right)\frac{K_1}{c} + \left(1 - e^{-ct}\right)\frac{a_1 K_2}{c}. \tag{2.12}$$

(2) If the initial value of $X_0 = e_2$,, i.e. the economy starts at a high cost state

$$\bar{K}_2 = \left(1 - e^{-ct}\right)\frac{a_2 K_1}{c} + \left(a_1 + a_2 e^{-ct}\right)\frac{K_2}{c}. \tag{2.13}$$

Note that we also obtain the probabilities of K_t being K_1 or K_2 conditional on the initial state of the chain. To be more precise, define these probabilities as follows:

Definition 2.2.

$$\begin{aligned}
q_{11}^t &= \mathbb{P}\left(K_t = K_1 | X_0 = e_1\right) = \mathbb{P}\left(X_t = e_1 | X_0 = e_1\right), \\
q_{12}^t &= \mathbb{P}\left(K_t = K_2 | X_0 = e_1\right) = \mathbb{P}\left(X_t = e_2 | X_0 = e_1\right), \\
q_{21}^t &= \mathbb{P}\left(K_t = K_1 | X_0 = e_2\right) = \mathbb{P}\left(X_t = e_1 | X_0 = e_2\right), \\
q_{22}^t &= \mathbb{P}\left(K_t = K_2 | X_0 = e_2\right) = \mathbb{P}\left(X_t = e_2 | X_0 = e_2\right).
\end{aligned}$$

Proposition 2.1. *The conditional probabilities defined in Definition 2.2 can be calculated as follows.*

$$\begin{aligned}
q_{11}^t &= \frac{a_2 + a_1 e^{-ct}}{c}, \quad q_{12}^t = \frac{a_1 - a_1 e^{-ct}}{c}, \\
q_{21}^t &= \frac{a_2 - a_2 e^{-ct}}{c}, \quad q_{22}^t = \frac{a_1 + a_2 e^{-ct}}{c}.
\end{aligned}$$

Proof. Note that in the proof below all expectations, as well as probabilities, are taken conditional on X_0.

$$\begin{aligned}\bar{K}_1 &= \mathbb{E}[K_t] = \mathbb{E}[\langle K, X_t\rangle] \\ &= \langle K, \mathbb{E}[X_t]\rangle = \langle K, e_1\mathbb{P}(X_t = e_1) + e_2\mathbb{P}(X_t = e_2)\rangle \\ &= \left\langle K, (q_1 1^t, q_1 2^t)\right\rangle = K_1 q_1 1^t + K_2 q_1 2^t.\end{aligned}$$

Analogously, we have

$$\bar{K}_2 = K_1 q_{21}^t + K_2 q_{22}^t.$$

The proof is then completed by comparing the terms on the right hand side of equations here and of equations (2.12) and (2.13). □

Note that unlike a standard perpetual American option which has a time homogeneous environment, here the problem is more complicated because conditional probabilities and, thus, strike prices are time-dependent. It might be difficult to argue that the trigger prices are still constants. Since this benchmark case is not the main focus of the paper, we shall make the following simplifying assumption.

Assumption 2.5. *The firm's rational expectation of the investment cost is some value $\bar{K} \in [K_1, K_2]$.*

The problem now becomes like a standard real option one. We wish to compute the investment timing, or critical values, indicating when to start the project and the consequent present value of the project. Applying the standard approach as in Dixit and Pindyck (1994) , we obtain

Proposition 2.2. *The values of the project, and the critical values $\bar{S}_i^*$ are: If the economy is at the cost state i, $i = 1, 2$, at time zero (when the project is initiated), that is, $X_0 = e_i$, $i = 1, 2$, the present value of the option at time zero and the critical value $\bar{S}_i^*$ are*

$$\bar{D}_i\left(S_0, \bar{K}_1\right) = \begin{cases} \left(\frac{S_0}{\bar{S}_i^*}\right)^{\gamma_1}\left(\bar{S}_i^* - \bar{K}_i\right), & \text{for } S_0 < \bar{S}_i^*, \\ S_0 - \bar{K}_i. & \text{for } S_0 \geq \bar{S}_i^*. \end{cases}$$

where

$$\bar{S}_i^* = \frac{\gamma_1}{\gamma_1 - 1}\bar{K}_i,$$

and

$$\gamma_1 = \frac{1}{\sigma^2}\left[-\left(r - \delta - \frac{\sigma^2}{2}\right) + \sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right] > 1. \tag{2.14}$$

In the simplifying case, the common critical value is $\bar{S}^* = \bar{S}_1^* = \bar{S}_2^*$.

In this case the investment costs are rationally chosen and the existence of the regime changes are known. We call this case the benchmark and we shall show that the optimal investment policy improves on the benchmark case. That is, using the full information does increase the value of the project for the firm. We also notice that the optimal timing policies for the two different initial states of economy coincide because the long run expectation of the investment costs are identical.

2.3. *Standard Real-options Model*

The second case is one in which the manager does not know the timing of the regime changes, but at time zero, the manager does observe the state of the economy, that is she knows X_0. The filtrations, or Information sets, of both are $\mathcal{F}_t^w$. If the economy is in the 'low cost' state, she chooses the investment cost K_1 and maintain that during the whole life of the project; otherwise, she chooses K_2. This is then a standard real-option situation where the manager can decide either to invest in the project or wait. The objective of the manager is to maximize the value of the project. The strike prices of the option are K_i , $i = 1$, 2. Similarly to the previous case, applying the standard approach as in Dixit and Pindyck (1994) , we have:

Proposition 2.3. *If the economy is in state i, $i = 1, 2$, at time zero, the present value of the project and the critical value $\hat{S}_i^*$ are:*

$$\hat{D}_i\left(S_0, K_i\right) = \begin{cases} \left(\frac{S_0}{\hat{S}_i^*}\right)^{\gamma_1}\left(\hat{S}_i^* - K_i\right), & \text{for } S_0 < \hat{S}_i^*, \\ S_0 - K_i. & \text{for } S_0 \geq \hat{S}_i^*. \end{cases} \tag{2.15}$$

where

$$\hat{S}_i^* = \frac{\gamma_1}{\gamma_1 - 1} K_1. \tag{2.16}$$

Here γ_1 is defined in Equation (2.14).

Here the values of the project should be less than those in the benchmark case since the decision is not rational. Although the economy has regime changes, the manager does not observe the changes.

When compared to the benchmark case, the standard real options model implies an investment strategy which is too early when the low cost state economy occurs at time zero, and which is too late when the economy starts in the high cost state.

2.4. *Full Information Case*

In this situation, the manager observes the existence of the regime changes, and will invest depending on the state of the world. Her information set is $\mathcal{F}_t^{w,X}$. The investment timing problem now becomes a perpetual American option with two strike prices, K_1 and K_2. The investment timing depends on the payoff of the project.

Recall that we suppose $K_1 \leq K_2$; the case when $K_1 \geq K_2$ is just the opposite of the following discussion. Analogously to the case when $K_1 = K_2$, we suppose there are two critical prices S_1^* and S_2^*. When $X_t = e_i, i = 1, 2$, it is better to hold the option if $S_t \leq S_i^*$ and to exercise if $S_t > S_i^*$. Again the value process has the dynamics

$$\frac{dS_t}{S_t} = (r - \delta)\,dt + \sigma dw_t. \tag{2.17}$$

As usual, if $D(S, X)$ denotes the value of the option, then

$$D(S, X) = \sup_{\tau \geq 0} \mathbb{E}\left[e^{-r\tau}(S_\tau - K_\tau)\,|S_0, X_0\right]. \tag{2.18}$$

Here, X_0 can take either the value e_1 or the value e_2.

Write

$$D_1(S) = D(S, e_1), \text{ and } D_2(S) = D(S, e_2), \tag{2.19}$$

then, $D_t(S, X) = \langle (D_1(S), D_2(S)), X_t \rangle$. As in Buffington and Elliott (2002), it can be shown that for $0 < S \leq S_1^*$, $D_t(S, X)$ satisfies the Black-Scholes equation:

$$-rD + (r - \delta) S \frac{\partial D}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 D}{\partial S^2} + \langle D, AX \rangle = 0. \tag{2.20}$$

This gives the following two equations when $X = e_1$ or $X = e_2$, and $0 < S \leq S_1^*$,

$$-rD_1 + (r - \delta) S \frac{\partial D_1}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_1}{\partial S^2} - a_1 D_1 + a_1 D_2 = 0, \tag{2.21}$$

$$-rD_2 + (r - \delta) S \frac{\partial D_2}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_2}{\partial S^2} + a_2 D_1 - a_2 D_2 = 0. \tag{2.22}$$

In this section we shall determine critical prices S_1^* and S_2^* which provide the optimal investment policy. That is, if the economy is in the 'low cost' state, invest if $S_t > S_1^*$; if the economy is in the 'high cost' state, invest if $S_t > S_2^*$. We shall also derive the values of the project if it is launched.

Proposition 2.4. *For $0 < S \leq S_1^*$, the manager does not invest, and under the two regimes the project has the values*

$$D_1 = \alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}, \tag{2.23}$$
$$D_2 = \lambda_1 \alpha_1 S^{\gamma_1} + \lambda_2 \alpha_2 S^{\gamma_2}.$$

where

$$\gamma_1 = \frac{\left(\sigma^2 + 2\delta - 2r + 2\sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}\right)}{2\sigma^2}, \tag{2.24}$$

$$\gamma_2 = \frac{\left(\sigma^2 + 2\delta - 2r + 2\sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2(r + a_1 + a_2)}\right)}{2\sigma^2}, \tag{2.25}$$

$$\lambda_1 = -\frac{1}{a_1}\left(\frac{\sigma^2}{2}\gamma_1(\gamma_1 - 1) + (r - \delta)\gamma_1 - a_1 - r\right), \tag{2.26}$$

$$\lambda_2 = -\frac{1}{a_1}\left(\frac{\sigma^2}{2}\gamma_2(\gamma_2 - 1) + (r - \delta)\gamma_2 - a_1 - r\right). \tag{2.27}$$

For $S_1^* < S \leq S_2^*$*, the manager should launch the project if* $X = e_1$*, and wait if* $X = e_2$*. The the values of the project in these cases are*

$$D_1 = S - K_1, \tag{2.28}$$

$$D_2 = \omega_1 S^{\gamma_3} + \omega_2 S^{\gamma_4} + F(S). \tag{2.29}$$

Here,

$$F(S) = \frac{a_2 S}{\delta + a_2} - \frac{a_2 K_1}{r + a_2}, \tag{2.30}$$

and

$$\gamma_3, \gamma_4 = \frac{\left(\sigma^2 + 2\delta - 2r \pm 2\sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 (r + a_2)}\right)}{2\sigma^2}. \tag{2.31}$$

For $S_2^* \leq S$*, the manager launches the project in both states of the economy, and the values of the project is given by:*

$$D_1 = S - K_1,$$

and

$$D_2 = S - K_2.$$

Proof. See Appendix A. □

Proposition 2.5. *The critical values* S_1^**, and* S_2^* *for launching the project are determined by the following two equations, which must be solved numerically:*

$$\begin{aligned} &\frac{\lambda_1(\gamma_1 - \gamma_3)}{\gamma_2 - \gamma_1}(\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2(\gamma_2 - \gamma_3)}{\gamma_1 - \gamma_2}(\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= -\left(\frac{S_1^*}{S_2^*}\right)^{\gamma_4}\left[\frac{\delta S_2^*(\gamma_3 - 1)}{\delta + a_2} + \frac{a_2\gamma_3 K_1 - a_2\gamma_3 K_2 - r\gamma_3 K_2}{r + a_2}\right] \\ &\quad + \frac{a_2 S_1^*(1 - \gamma_3)}{\delta + a_2} + \frac{a_2\gamma_3 K_1}{r + a_2}, \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} &\frac{\lambda_1(\gamma_1 - \gamma_4)}{\gamma_2 - \gamma_1}(\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2(\gamma_2 - \gamma_4)}{\gamma_1 - \gamma_2}(\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= -\left(\frac{S_1^*}{S_2^*}\right)^{\gamma_3}\left[\frac{\delta S_2^*(\gamma_4 - 1)}{\delta + a_2} + \frac{a_2\gamma_4 K_1 - a_2\gamma_4 K_2 - r\gamma_4 K_2}{r + a_2}\right] \\ &\quad + \frac{a_2 S_1^*(1 - \gamma_4)}{\delta + a_2} + \frac{a_2\gamma_4 K_1}{r + a_2}. \end{aligned} \tag{2.33}$$

Proof. See Appendix B. □

The tractability is unfortunately lost due to the highly non-linear system giving S_1^* and S_2^*. We resort to numerical analysis to compare the payoffs of different cases in a subsequent section.

The recent advances in Asset Pricing theory use a firm's real investment activities to explain the conditional dynamics in expected asset returns. We are also interested in the risk implications of a firm's growth opportunity given that investment cost is stochastic. To focus on the added uncertainty of investment cost, we confine our analysis on the case that a firm has no assets in place and has only one growth opportunity. There is no fixed operating cost, either. Following Carlson, Fisher, and Giammarino (2004), we calculate the firm's systematic risk sensitivity, η, as follows,

$$\eta = \frac{\partial D_i}{\partial S}\frac{S}{D_i}. \tag{2.34}$$

Without losing of generality, we compute the beta when the economy starts at e_1. Differentiating equation (2.23) with respect to S and using the definition of η by equation (2.34), we have

$$\begin{aligned}\eta &= \frac{\alpha_1\gamma_1 S^{\gamma_1} + \alpha_2\gamma_2 S^{\gamma_2}}{\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}} \\ &= 1 + \frac{\alpha_1(\gamma_1 - 1)S^{\gamma_1} + \alpha_2(\gamma_2 - 1)S^{\gamma_2}}{\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}} \\ &> 1 + (\gamma_1 - 1) \\ &= \gamma_1,\end{aligned}$$

because $\gamma_2 > \gamma_1 > 1$. We interpret the derived η as follows. Firstly, our result is identical to the one derived in Carlson, Fisher, and Giammarino (2004) after recognizing $V_i^G = V_i$ (because value of asset in place is zero) and $V_i^F = 0$[a]. Secondly, our model suggests a higher risk premium of a growth stock because of the additional economy wide shock of the investment cost. Thirdly, our η nests the β in Carlson, Fisher, and Giammarino (2004) by letting $a_1 = a_2 = 0$ (hence $\gamma_1 = \gamma_2$). Obviously, assuming $a_1 = a_2 = 0$, we go back to an economy with constant investment cost.

3. Hidden Losses

In this section, we investigate hidden losses due to not knowing the hidden Markov chain for a set of base parameters. In other words, the 'optimal' option values derived above for the benchmark and standard Real Option cases are different from the expected actual payoffs that the firm receives. This is because the investment costs are in fact stochastic but are assumed to be constant in these two cases.

3.1. *Expected Actual Payoff in the Benchmark Case*

In the previous section, we have computed the present values of the project subject to different assumptions. In the benchmark case, we simplified the problem by assuming the strike price to be the firm's rational expectation of the investment

[a]For definititons of V_i^G, V_i, and, V_i^f, see Carlson, Fisher, and Giammarino (2004).

cost. In the real world, when the project is launched, even the manager does not observe the state of the economy, the investment cost will still be K_i depending on the state of the economy at that time. Thus, the actual payoff will be different from the values in Proposition 2.2. Therefore, to compare the value of the project in the three cases we compute the expected actual payoff of the project.

Recall that in the benchmark case the optimal timing policy is independent of the initial state of the economy. Thus we have:

Proposition 3.1. *The expected actual payoff to the firm in the Benchmark Case is strictly greater than the one derived by their 'optimal' timing policy derived in Proposition 2.2. We denote the actual payoff by* $\bar{P}$. *Explicitly we have*

$$\bar{P} = \bar{D}_1\left(S_0, \bar{K}\right) + \left(\frac{a_1 \Delta K}{c}\right)\left(\frac{S_0}{\bar{S}_1^*}\right)^{\gamma_2} > \bar{D}_1\left(S_0, \bar{K}\right), \tag{3.1}$$

where $\Delta K = K_2 - K_1 > 0$ *and* γ_2 *is defined in Proposition 2.4.*

Proof. In the following, all expectations are taken under $\mathbb{Q}$ and conditional on $\mathcal{F}_0$.

$$\begin{aligned}
\bar{P}_1 = \bar{P}_2 &= \mathbb{E}\left[e^{-r\bar{\tau}}\left(\bar{S}_1^* - K_{\bar{\tau}}\right)\right] \\
&= \mathbb{E}\left[e^{-r\bar{\tau}}\left(\bar{S}_1^* - K_1 q_{11}^{\bar{\tau}} - K_2 q_{22}^{\bar{\tau}}\right)\right] \\
&= \mathbb{E}\left[e^{-r\bar{\tau}}\left(\bar{S}_1^* - K_1\frac{a_2}{c} - K_2\frac{a_1}{c} + e^{-c\bar{\tau}}\frac{a_1}{c}\left(K_2 - K_1\right)\right)\right] \\
&= \mathbb{E}\left[e^{-r\bar{\tau}}\right]\left(\bar{S}_1^* - \bar{K}_1\right) + \mathbb{E}\left[e^{-(r+c)\bar{\tau}}\frac{a_1}{c}\left(K_2 - K_1\right)\right] \\
&= \bar{D}_1\left(S_0, \bar{K}_1\right) + \left(\frac{a_1\Delta K}{c}\right)\left(\frac{S_0}{\bar{S}_1^*}\right)^{\gamma_2}.
\end{aligned}$$

The equality in the second line follows from the definition of $K_{\bar{\tau}}$ with $\vartheta = r + c$. Setting $\kappa = \gamma_2$ gives the last equality. □

Proposition 3.1 can be interpreted as saying that the expected actual payoff of the firm is higher than the option value we have derived in Proposition 2.2. Therefore, the exercise strategy in Proposition 2.2 is optimal only with respect to the information set $\mathcal{F}_t^w$. That is, if $\Delta K = 0$ and/or $a_1 = 0$, the difference vanishes and, in fact, we have a hidden trivial single state Markov chain.

3.2. *Expected Actual Payoffs in a Standard Real Option case*

Similarly to Proposition 3.1, we have the following analogous results.

Proposition 3.2. *If the economy starts in the 'low cost' state, the expected actual payoff of the firm is strictly less than the option value we derived in Proposition 2.3. Otherwise, the expected actual payoff is strictly greater. Denote the expected payoffs* *by $\hat{P}_1$ and $\hat{P}_2$, respectively. Then,*

$$\hat{P}_1 = \hat{D}_1\left(S_0, K_1\right) + \left(\frac{a_1 \Delta K}{c}\right)\left[\left(\frac{S_0}{\hat{S}_1^*}\right)^{\gamma_2} - \left(\frac{S_0}{\hat{S}_1^*}\right)^{\gamma_1}\right] < \hat{D}_1\left(S_0, K_1\right), \tag{3.2}$$

$$\hat{P}_2 = \hat{D}_2\left(S_0, K_2\right) + \left(\frac{a_2 \Delta K}{c}\right)\left[\left(\frac{S_0}{\hat{S}_2^*}\right)^{\gamma_1} - \left(\frac{S_0}{\hat{S}_2^*}\right)^{\gamma_2}\right] > \hat{D}_2\left(S_0, K_2\right). \tag{3.3}$$

Proof. Similar to the proof of Proposition 3.1. □

Again, Proposition 3.2 shows that the expected payoffs to the firm differ from the option value determined 'optimally' in Proposition 2.3. Although we do observe the deviation of the expected payoff from the option value, we have not said anything about which model is superior. Intuitively, one expects the full information model would improve on the other two models which consider only partially revealed information. To this end, we resort to a numerical example.

3.3. *A Numerical Example*

Suppose we have a possible project which fits into the previous framework. We shall numerically show how the optimal policy works and why knowing the full information about the project creates value for the firm.

We assume the Q-matrix takes the value:

$$A = \begin{pmatrix} -0.5 & 0.3 \\ 0.5 & -0.3 \end{pmatrix}.$$

That is we set $a_1 = 0.5$, and $a_2 = 0.3$. The other parameter values are taken to be $r = 3\%$, $\delta = 2.5\%$, $\sigma = 10\%$, $S_0 = 100$, $K_1 = 105$, and $K_2 = 110$. Thus, applying Proposition 2.2 to Proposition 3.2, we have the following results:

1. For the Benchmark case, $\bar{K}_1 = \bar{K}_2 = 108.125$, $\gamma_1 = 2.45$, the present value of the project, $\bar{D}_1\left(S_0, \bar{K}\right) = 17.0399$, and the critical values $\bar{S}^* = 182.72$. The actual expected payoff is $\bar{P}_1 = 17.0412$ which is very close to the present value.
2. For the standard real option case: If the economy is in the 'low cost' state at time zero, the present value of the project, the critical value, and the actual expected payoff are $\hat{D}_1\left(S_0, K_1\right) = 17.78$, $\hat{S}_1^* = 177.44$ and $\hat{P}_1 = 17.0148$, respectively. If the economy is in the 'high cost' state at time zero, the values are: $\hat{D}_2\left(S_0, K_2\right) = 16.62$, $\hat{S}_2^* = 188.89$, and $\hat{P}_2 = 17.0305$ respectively.
3. For the full information case, we have: $S_1^* = 169.49$, $S_2^* = 220.71$,and $D_1 = 17.4076$, $D_2 = 17.4056$ which are the values of the project depending on the state of the economy. Notice both of these two values are greater than the values provided by the benchmark case and the standard real option case. Therefore, when S_t reaches 169.49, and the economy is in the 'low cost' state, the manager should launch the project according to the full information case optimal investment policy. If S_t reaches 220.7, and during the period of time when S_t is in the

interval [169.49, 220.71) the economy stays in the 'high cost' state, the manager launches the project immediately, and the state of the economy does not matter anymore.

Comparing the results from the three cases, we have the following remarks:

- If the economy is in the 'low cost' state at time zero, the standard real option approach suggests an earlier exercise than the Benchmark case since $\hat{S}_1^*(177.44) < \bar{S}^*(182.72)$. If the economy is in the 'high cost' state at time zero, the model implies a later exercise than the Benchmark case, since $\hat{S}_2^* > \bar{S}^*$.
- The standard Real Option approach decreases the value of the project since both $\hat{P}_1$, and $\hat{P}_2$ are smaller than $\bar{P}_1$. This is also expected because in this case investment costs used to determine the optimal timing policy are far way from the actual costs incurred when the project is launched. The firm is penalized by adopting a misspecified exercising strategy.
- The full information case optimal strategy implies a lowest trigger value $S_1^* = 169.49$ with the highest present value, of the project's payoff to the firm, 17.4076. The values of the project dominates the values from the other two cases. This is remarkable, since the result indicates an earlier exercise and an higher payoff than both the benchmark and the standard real option case. On the other hand, we also notice that $S_2^* = 220.71$, but it is unlikely to happen that the project is going to be launched at this threshold. As we mentioned above, this can only happened if the economy is in the 'high cost' state before the first trigger value $S_1^* = 169.49$, and it maintains in the state until the value process S_t reaches the second trigger value $S_2^* = 220.71$. For the continuous Markov chain X_t, the probability of having no transition in some time s is

$$\mathbb{P}\left[X\left(t+s\right)=i|X(t)=i\right]=e^{-sq_i},$$

and the exponential function approaches zero quickly as s gets big. We are considering a perpetual option, so this probability is almost surely close to zero. Therefore, the full information case optimal strategy really implies an earlier exercise and increase the value of the project.
- The full information case suggests a range of exercising values, that is, the project has more flexibility than the previous two cases. The standard real option case provides a lower exercising value for the 'high cost' state and generates a lower payoff for the firm which means this case decreases some value of the project by exercising it in a not really 'optimal' time.

This simple example says that if there are different capital costs in different situation of the economy, which is common in the real world, the regime switching approach is a better method of investigating investment timing problems. Although the standard real option is broadly used, it does often ruin investment opportunities.

4. Conclusions

This paper investigates the investment timing problem in a regime switching real option framework which extends the standard real option approach. By assuming the economy has two different cost states, a 'low cost' state and a 'high cost' state, we consider the investment timing decision in a regime switching framework. We derive closed form investment policies for the manager which maximize the value of the project. The numerical example shows that the investment policy provided by our framework is better than the investment strategies suggested by the usual standard real option approach. The optimal investment policy implies a lower threshold, or trigger value, to launch the project. This increases the possibility of investing in the project and creates value for the firm. A wide range of trigger prices, rather than a single exercise threshold, of the project is also proposed.

Some possible extensions of the framework would be: firstly, the model can be generalized to more than two states, secondly, the investment costs may be fluctuating, or time dependent because of advances in technology, thirdly, the model may include more than one project. These might provide future research directions.

Appendix A. Proof of Proposition 2.4.

Proof. When $S_1^* < S \leq S_2^*$ it is optimal to invest if $X = e_1$ so then the value of the project is

$$D_1 = S - K_1,$$

and if $X = e_2$ from (2.22),

$$-rD_2 + (r-\delta)\, S\frac{\partial D_2}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_2}{\partial S^2} + a_2\,(S - K_1) - a_2 D_2 = 0. \tag{A.1}$$

When $S > S_2^*$ it is optimal to exercise whether $X = e_1$ or $X = e_2$ so

$$D_1 = S - K_1,$$
$$\text{and } D_2 = S - K_2.$$

When $0 < S \leq S_1^*$ we look for solutions of (2.21) and (2.22) in the form

$$D_1 = D_1\,(S) = \alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}, \tag{A.2}$$
$$D_2 = D_2\,(S) = \beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2}. \tag{A.3}$$

Now $(A.2)$ is a solution of (2.21) if:

$$\frac{\sigma^2}{2}\left(\alpha_1\gamma_1\,(\gamma_1 - 1)\, S^{\gamma_1} + \alpha_2\gamma_2\,(\gamma_2 - 1)\, S^{\gamma_2}\right) + (r-\delta)\left(\alpha_1\gamma_1 S^{\gamma_1} + \alpha_2\gamma_2 S^{\gamma_2}\right)$$
$$-\,(r + a_1)\left(\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}\right) + a_1\left(\beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2}\right) = 0, \tag{A.4}$$

and $(A.3)$ is a solution of (2.22) if:

$$\frac{\sigma^2}{2}\left(\beta_1\gamma_1\,(\gamma_1 - 1)\, S^{\gamma_1} + \beta_2\gamma_2\,(\gamma_2 - 1)\, S^{\gamma_2}\right) + (r-\delta)\left(\beta_1\gamma_1 S^{\gamma_1} + \beta_2\gamma_2 S^{\gamma_2}\right)$$
$$-\,(r + a_1)\left(\beta_1 S^{\gamma_1} + \beta_2 S^{\gamma_2}\right) + a_2\left(\alpha_1 S^{\gamma_1} + \alpha_2 S^{\gamma_2}\right) = 0. \tag{A.5}$$

Equating coefficient of S^{γ_1} and S^{γ_2} in $(A.4)$ and $(A.5)$ we must have:

$$\frac{\sigma^2}{2}\alpha_1\gamma_1(\gamma_1-1)+(r-\delta)\alpha_1\gamma_1-a_1\alpha_1+a_1\beta_1-\alpha_1 r=0, \tag{A.6}$$

$$\frac{\sigma^2}{2}\alpha_2\gamma_2(\gamma_2-1)+(r-\delta)\alpha_2\gamma_2-a_1\alpha_2+a_1\beta_2-\alpha_2 r=0, \tag{A.7}$$

$$\frac{\sigma^2}{2}\beta_1\gamma_1(\gamma_1-1)+(r-\delta)\beta_1\gamma_1+a_2\alpha_1-a_2\beta_1-\beta_1 r=0, \tag{A.8}$$

$$\frac{\sigma^2}{2}\beta_2\gamma_2(\gamma_2-1)+(r-\delta)\beta_2\gamma_2+a_2\alpha_2-a_2\beta_2-\beta_2 r=0, \tag{A.9}$$

From $(A.6)$

$$-a_1\frac{\beta_1}{\alpha_1}=\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+(r-\delta)\gamma_1-a_1-r. \tag{A.10}$$

From $(A.8)$

$$-a_2\frac{\alpha_1}{\beta_1}=\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+(r-\delta)\gamma_1-a_2-r. \tag{A.11}$$

Therefore, γ_1 is a solution of the fourth order equation

$$a_1a_2=\left(\frac{\sigma^2}{2}\gamma(\gamma-1)+(r-\delta)\gamma-a_1-r\right)\left(\frac{\sigma^2}{2}\gamma(\gamma-1)+(r-\delta)\gamma-a_2-r\right). \tag{A.12}$$

Similarly from $(A.7)$ and $(A.9)$ γ_2 is a solution of the same equation. This factors as

$$\left(\frac{\sigma^2}{2}\gamma(\gamma-1)+(r-\delta)\gamma-r\right)\left(\frac{\sigma^2}{2}\gamma(\gamma-1)+(r-\delta)\gamma-r-a_1-a_2\right)=0. \tag{A.13}$$

The call option must have a finite value at $S=0$ so only the positive roots of $(A.13)$ are of interest. We see these are

$$\gamma_1=\frac{\left(\sigma^2+2\delta-2r+2\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2r\sigma^2}\right)}{2\sigma^2}, \tag{A.14}$$

$$\gamma_2=\frac{\left(\sigma^2+2\delta-2r+2\sqrt{\left(r-\delta-\frac{\sigma^2}{2}\right)^2+2\sigma^2(r+a_1+a_2)}\right)}{2\sigma^2}. \tag{A.15}$$

Write

$$\lambda_1=\frac{\beta_1}{\alpha_1}=-\frac{1}{a_1}\left(\frac{\sigma^2}{2}\gamma_1(\gamma_1-1)+(r-\delta)\gamma_1-a_1-r\right)\quad\text{from (A.6)}, \tag{A.16}$$

and

$$\lambda_2=\frac{\beta_2}{\alpha_2}=-\frac{1}{a_1}\left(\frac{\sigma^2}{2}\gamma_2(\gamma_2-1)+(r-\delta)\gamma_2-a_1-r\right)\quad\text{from (A.7)}. \tag{A.17}$$

Then $\beta_1=\lambda_1\alpha_1$ and $\beta_2=\lambda_2\alpha_2$ where λ_1 and λ_2 are known.

From $(A.1)$ we have seen that when $S_1^* < S \leq S_2^*$,

$$-rD_2 + (r-\delta) S \frac{\partial D_2}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 D_2}{\partial S^2} + a_2 (S - K_1) - a_2 D_2 = 0. \tag{A.18}$$

A particular solution of $(A.18)$ is

$$F(S) = \frac{a_2 S}{\delta + a_2} - \frac{a_2 K_1}{r + a_2}. \tag{A.19}$$

The general solution of $(A.18)$ is

$$D_2(S) = \omega_1 S^{\gamma_3} + \omega_2 S^{\gamma_4} + F(S), \tag{A.20}$$

where γ_3 and γ_4 are the roots of the quadratic equation

$$\frac{\sigma^2}{2}\gamma^2 + \left(r - \delta - \frac{\sigma^2}{2}\right)\gamma - a_2 - r = 0. \tag{A.21}$$

That is

$$\gamma_3, \gamma_4 = \frac{\left(\sigma^2 + 2\delta - 2r \pm 2\sqrt{\left(r - \delta - \frac{\sigma^2}{2}\right)^2 + 2\sigma^2 (r + a_2)}\right)}{2\sigma^2}. \tag{A.22} \qquad \square$$

Appendix B. Proof of Proposition 2.5.

Proof. The remaining quantities to be determined are the coefficients $\alpha_1, \alpha_2, \omega_1, \omega_2$ and the critical prices S_1^*, S_2^*. These will be found using continuity and smooth fit of the option values D_1 and D_2 at the critical prices S_1^*, and S_2^*.

Consider D_1 at the boundary S_1^*. To ensure D_1 and its derivative are continuous at S_1^*:

$$\alpha_1 (S_1^*)^{\gamma_1} + \alpha_2 (S_1^*)^{\gamma_2} = S_1^* - K_1, \tag{B.1}$$

$$\alpha_1 \gamma_1 (S_1^*)^{\gamma_1} + \alpha_2 \gamma_2 (S_1^*)^{\gamma_2} = S_1^*. \tag{B.2}$$

These give

$$\alpha_1 (S_1^*)^{\gamma_1} = (\gamma_2 - \gamma_1)^{-1} [\gamma_2 S_1^* - S_1^* - \gamma_2 K_1], \tag{B.3}$$

$$\alpha_2 (S_1^*)^{\gamma_2} = (\gamma_1 - \gamma_2)^{-1} [\gamma_1 S_1^* - S_1^* - \gamma_1 K_1]. \tag{B.4}$$

For $S \leq S_1^*$, $D_2 = \lambda_1 \alpha_1 S^{\gamma_1} + \lambda_2 \alpha_2 S^{\gamma_2}$ and for $S_1^* < S \leq S_2^*$, $D_2 = \omega_1 S^{\gamma_3} + \omega_2 S^{\gamma_4} + F(S)$. To ensure D_2 and its derivative are continuous at S_1^* :

$$\lambda_1 \alpha_1 (S_1^*)^{\gamma_1} + \lambda_2 \alpha_2 (S_1^*)^{\gamma_2} = \omega_1 (S_1^*)^{\gamma_3} + \omega_2 (S_1^*)^{\gamma_4} + F(S_1^*), \tag{B.5}$$

$$\lambda_1 \alpha_1 \gamma_1 (S_1^*)^{\gamma_1} + \lambda_2 \alpha_2 \gamma_2 (S_1^*)^{\gamma_2} = \omega_1 \gamma_3 (S_1^*)^{\gamma_3} + \omega_2 \gamma_4 (S_1^*)^{\gamma_4} + \frac{a_2 S_1^*}{\delta + a_2}. \tag{B.6}$$

For $S \leq S_2^*$, $D_2 = S - K_2$. To ensure D_2 and its derivative are continuous at S_2^* :

$$\omega_1 (S_2^*)^{\gamma_3} + \omega_2 (S_2^*)^{\gamma_4} + F(S_2^*) = S_2^* - K_2, \tag{B.7}$$

$$\omega_1 \gamma_3 (S_2^*)^{\gamma_3} + \omega_2 \gamma_4 (S_2^*)^{\gamma_4} = \frac{\delta S_2^*}{\delta + a_2}. \tag{B.8}$$

These give

$$\omega_1 (S_2^*)^{\gamma_3} = \frac{1}{\gamma_4 - \gamma_3} \left[\frac{\delta S_2^* (\gamma_4 - 1)}{\delta + a_2} + \frac{a_2\gamma_4 K_1 - a_2\gamma_4 K_2 - r\gamma_4 K_2}{r + a_2} \right], \tag{B.9}$$

$$\omega_2 (S_2^*)^{\gamma_4} = \frac{1}{\gamma_3 - \gamma_4} \left[\frac{\delta S_2^* (\gamma_3 - 1)}{\delta + a_2} + \frac{a_2\gamma_3 K_1 - a_2\gamma_3 K_2 - r\gamma_3 K_2}{r + a_2} \right]. \tag{B.10}$$

Substituting $(B.3)$, $(B.4)$ and $(B.9)$, $(B.10)$ into $(B.5)$, $(B.6)$ gives

$$\begin{aligned} &\frac{\lambda_1}{\gamma_2 - \gamma_1} (\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2}{\gamma_1 - \gamma_2} (\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= \left(\frac{S_1^*}{S_2^*}\right)^{\gamma_3} \frac{1}{\gamma_4 - \gamma_3} \left[\frac{\delta S_2^* (\gamma_4 - 1)}{\delta + a_2} + \frac{a_2\gamma_4 K_1 - a_2\gamma_4 K_2 - r\gamma_4 K_2}{r + a_2} \right] \\ &\quad + \left(\frac{S_1^*}{S_2^*}\right)^{\gamma_4} \frac{1}{\gamma_3 - \gamma_4} \left[\frac{\delta S_2^* (\gamma_3 - 1)}{\delta + a_2} + \frac{a_2\gamma_3 K_1 - a_2\gamma_3 K_2 - r\gamma_3 K_2}{r + a_2} \right] \\ &\quad + \frac{a_2 S_1^*}{\delta + a_2} - \frac{a_2 K_1}{r + a_2}, \end{aligned} \tag{B.11}$$

$$\begin{aligned} &\frac{\lambda_1\gamma_1}{\gamma_2 - \gamma_1} (\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2\gamma_2}{\gamma_1 - \gamma_2} (\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= \left(\frac{S_1^*}{S_2^*}\right)^{\gamma_3} \frac{\gamma_3}{\gamma_4 - \gamma_3} \left[\frac{\delta S_2^* (\gamma_4 - 1)}{\delta + a_2} + \frac{a_2\gamma_4 K_1 - a_2\gamma_4 K_2 - r\gamma_4 K_2}{r + a_2} \right] \\ &\quad + \left(\frac{S_1^*}{S_2^*}\right)^{\gamma_4} \frac{\gamma_4}{\gamma_3 - \gamma_4} \left[\frac{\delta S_2^* (\gamma_3 - 1)}{\delta + a_2} + \frac{a_2\gamma_3 K_1 - a_2\gamma_3 K_2 - r\gamma_3 K_2}{r + a_2} \right] \\ &\quad + \frac{a_2 S_1^*}{\delta + a_2}. \end{aligned} \tag{B.12}$$

Rearranging $(B.11)$ and $(B.12)$ we obtain the following two equations for S_1^*, and S_2^* :

$$\begin{aligned} &\frac{\lambda_1(\gamma_1 - \gamma_3)}{\gamma_2 - \gamma_1} (\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2(\gamma_2 - \gamma_3)}{\gamma_1 - \gamma_2} (\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= -\left(\frac{S_1^*}{S_2^*}\right)^{\gamma_4} \left[\frac{\delta S_2^* (\gamma_3 - 1)}{\delta + a_2} + \frac{a_2\gamma_3 K_1 - a_2\gamma_3 K_2 - r\gamma_3 K_2}{r + a_2} \right] \\ &+ \frac{a_2 S_1^* (1 - \gamma_3)}{\delta + a_2} + \frac{a_2\gamma_3 K_1}{r + a_2}, \end{aligned} \tag{B.13}$$

$$\begin{aligned} &\text{and } \frac{\lambda_1(\gamma_1 - \gamma_4)}{\gamma_2 - \gamma_1} (\gamma_2 S_1^* - \gamma_2 K_1 - S_1^*) + \frac{\lambda_2(\gamma_2 - \gamma_4)}{\gamma_1 - \gamma_2} (\gamma_1 S_1^* - \gamma_1 K_1 - S_1^*) \\ &= -\left(\frac{S_1^*}{S_2^*}\right)^{\gamma_3} \left[\frac{\delta S_2^* (\gamma_4 - 1)}{\delta + a_2} + \frac{a_2\gamma_4 K_1 - a_2\gamma_4 K_2 - r\gamma_4 K_2}{r + a_2} \right] \\ &\quad + \frac{a_2 S_1^* (1 - \gamma_4)}{\delta + a_2} + \frac{a_2\gamma_4 K_1}{r + a_2}. \end{aligned} \tag{B.14}$$

S_1^*, and S_2^* are determined numerically from $(B.13)$ and $(B.14)$. Then α_1, α_2, and ω_1, ω_2 can be found from $(B.3)$, $(B.4)$ and $(B.9)$, $(B.10)$ respectively. □

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