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A MATHEMATICAL STUDY OF PRECIPITATION PHENOMENA

by

Petar Todorovic
Associate Professor

Colorado State University

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1. SOME BASIC NOTATION

We begin by presenting some definitions and notations that are basic in this study. This will be given in abbreviated form with a minimum of explanatory discussion.

Symbol	Meaning
1^o ε, ($\notin$)	is (is not) an element of
2^o $\subset$ ($\supset$)	is contained in, contains
3^o $\Rightarrow$	implies
4^o $\{a,b,c,...\}$	the set consisting of the elements a,b,c...
5^o $A \cup B$	the union of sets A and B
6^o $A \cap B$	the intersection of sets A and B
7^o $\bigcup_\nu A_\nu$	the union of sets A_1, A_2, A_3,...
8^o $\bigcap_\nu A_\nu$	the intersections of sets A_1, A_2, A_3,...
9^o A^c	the complement of A
10^o θ	the empty set (impossible event)
11^o Ω	the space of elementary event
12^o ω, ($\omega \varepsilon \Omega$)	an elementary event
13^o $\forall$	for all (for every)
14^o $\mathcal{A}$	σ-field
15^o $\mathcal{B}$	borel field
16^o $\|x\|$	absolute value of x
17^o $[x]$	the greatest integer not greater than x
18^o P	probability measure
19^o $\exists$	there exist (s)

Symbol	Meaning
20^o ξ_t	the rainfall intensity at some instant of time t
21^o X_t	total amount of precipitation in some interval of time (t_o,t)
22^o η_t	number of storm periods in (t_o,t)
23^o t_o	moment of time when observations begin
24^o $\tau_\nu - t_o$	the elapsed time up to the end of ν-th storm period
25^o X_ν	the total amount of precipitation during exactly ν storm periods
26^o Z_ν	total amount of precipitation during ν-th storm period
27^o $E_\nu^{t_o,t}$	the event that exactly ν complete storm periods will occur in (t_o,t)
28^o $G_\nu^{x_o,x}$	the events that total amount of precipitation during ν storms will be less or equal $(x-x_o)$
29^o $\int_{t_o}^{t} \lambda_1(s)\, ds$	the average number of storms in (t_o,t)
30^o $A_\nu(t)$	the distribution function of τ_ν
31^o $a_\nu(t)$	the density function of τ_ν
32^o $F_\nu(x)$	the distribution function of X_ν
33^o $f_\nu(x)$	the density function of X_ν
34^o $B_\nu(z)$	the distribution function of Z_ν
35^o $b_\nu(z)$	the density function of Z_ν
36^o $\Gamma\ (\nu)$	Gamma function

Symbol	Meaning
37° $\sup(x,y)$	$= x$ if $x > y$ or y if $y > x$
38° $\inf(x,y)$	$= x$ if $x < y$ or y if $y < x$
39° $\sup\limits_{1\leq\nu\leq n} x_\nu$	$= x_j$ if $x_j > x_\nu$ $\forall\ \nu = 1,2\ldots n$
40° $\inf\limits_{1\leq\nu\leq n} x_\nu$	$= x_i$ if $x_i < x_\nu$ $\forall\ \nu = 1,2\ldots n$
41° $\underline{Z}(n) = \inf\limits_{1\leq\nu\leq n} Z_\nu$	is the smallest storm among $Z_1,\ldots Z_n$
42° $\overline{Z}(n) = \sup\limits_{1\leq\nu\leq n} Z_\nu$	is the largest storm among $Z_1,\ldots Z_n$
43° $Q_n(z)$	is the distribution function of $\underline{Z}(n)$
44° $q_n(z)$	is the density function of $\underline{Z}(n)$
45° $H_n(z)$	is the distribution function of $\overline{Z}(n)$
46° $h_n(z)$	is the density function of $\overline{Z}(n)$
47° $F_t(x)$	is the distribution function of X_t for all $t \geq t_o$
48° $f_t(x)$	is the density function of X_t
49° $F_{1t}(x)$	is the lower approximation of $F_t(x)$
50° $F_{2t}(x)$	is the upper approximation of $F_t(x)$
51° $f_{1t}(x)$	is equal to $\dfrac{dF_{1t}(x)}{dt}$
52° $f_{2t}(x)$	is equal to $\dfrac{dF_{2t}(x)}{dt}$

Symbol	Meaning
53° $E\ (X)$	is the mathematical expectation of the random variable X
54° $\{ \xi_t ;\ t \geqslant t_o \}$	represents a stochastic process or a family of random variables
55° T_{ν}	is the length of the ν-th storm period

2. PREFACE

The main purpose of this work is to provide a method for mathematical treatment and analysis of some aspects of rainfall phenomena which have shown themselves to be important in many of the water resource problems. Those characteristics of precipitation such as the frequency of storm periods, total amount of precipitation during one or more storm periods, the elapsed time up to the end of ν-th storm period where $\nu = 1,2,\ldots$, etc., represent objectives of this study.

In the further exposition, an attempt has been made to establish a theoretical base for an investigation of these problems. Because precipitation is a random phenomenon depending on time, the theory of stochastic processes represents one of the most appropriate mathematical tools for the interpretation and analysis of the phenomena considered.

Toward this end, a particular stochastic process has been considered whose sample functions provide the total amount of precipitation in the given interval of time (t_o,t). These functions are nondecreasing functions of time.

In this study, an analysis of the precipitation phenomenon has been performed by means of the stochastic process considered. Some effort was made to give the phenomenological interpretation and discussion of every important mathematical result. It is hoped that this will make the study easier for reading.

Finally, I wish to express my gratitude to Professor V. Yevjevich who gave me the opportunity to work on this problem. The criticism given by him and by Professor H. Morel-Seytoux, was always prized even when not accepted.

I would especially like to acknowledge Professor J. W. N. Fead and Professor D. B. Simons for their discreet support during the work on this study. I am also indebted to Don Collins for editing the manuscript. Thanks are due Mrs. Arlene Nelson for her patience in typing the first version of this study and to Mrs. Ann Brown for the final typing.

April 1968 P. Todorovic

Fort Collins

3. SUMMARY

The purpose of this study is to provide a method for mathematical treatment and analysis of some aspects of rainfall phenomena which have shown themselves to be important in many water resource problems. As the theoretical base for such an analysis, a particular stochastic process, $X_t = \pi(t,\omega)$, has been considered.

Consider a hydrograph of a rainfall gaging station (Fig. 1) and

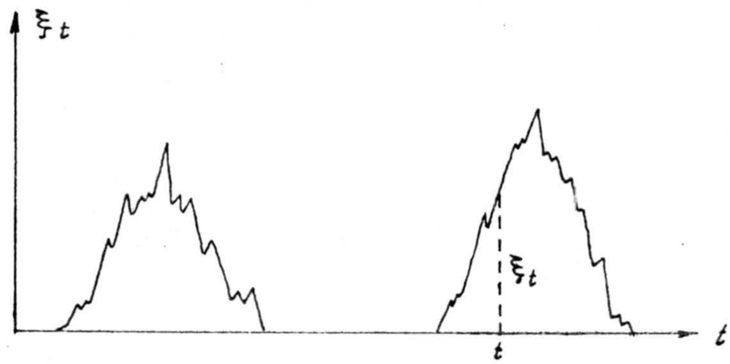

Fig. 1 Graphical representation of a rainfall hydrograph

denote by ξ_t the rainfall intensity at some instant of time t. If t_o stands for the moment of time when observation begins, the total amount of precipitation X_t up to time $t > t_o$ is equal to the following integral

$$X_t = x_o + \int_{t_o}^{t} \xi_s \, ds \qquad (3.1)$$

where x_o is the total amount of precipitation up to time t_o.

Since ξ_t is a random variable for any $t \geqslant t_o$, it is apparent that

$$\{ \xi_t ; t \geqslant t_o \}$$

represents a stochastic process; since $\xi_t \geqslant 0$ for every $t \geqslant t_o$

the sample function of the process $X_t = \pi(t,\omega)$ are nondecreasing t functions (Fig. 2)

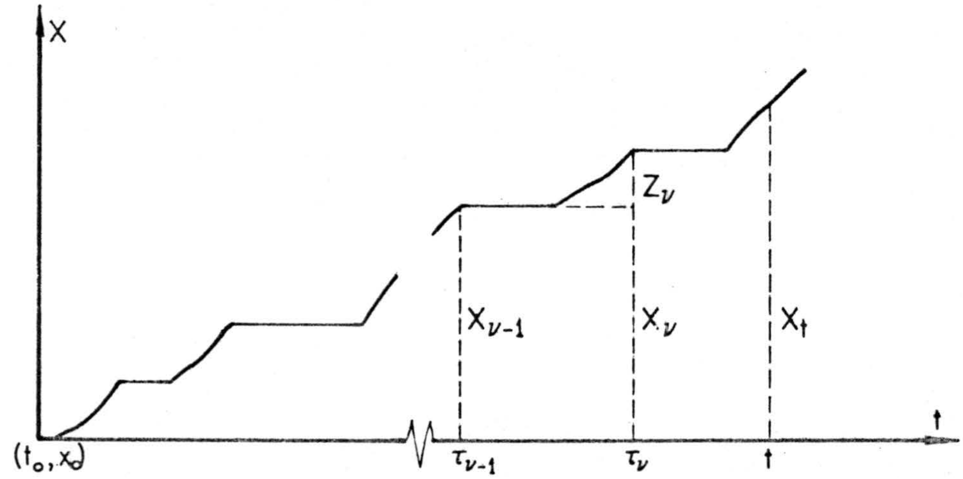

Fig. 2 Graphical representation of a sample function of the process $\pi(t,\omega)$

and provide total amount of precipitation during the interval of time (t_o, t). The following characteristics of the precipitation phenomenon are objectives of this study:

η_t	the number of storm periods in (t_o, t),
$\tau_\nu - t_o$	the elapsed time up to the end of ν-th storm period,
X_ν	the total amount of precipitation during exactly ν storm periods,
Z_ν	the total amount of precipitation during ν-th storm,
X_t	the total amount of precipitation during the interval of time (t_o, t).

It is apparent that τ_ν , X_ν , Z_ν are random variables for all $\nu = 1,2,\ldots,$ and η_t and X_t for all $t > t_o$. For the purpose of applications it is indispensable to possess distribution functions of these random variables. Toward this end, a series of theorem has been proved providing analytical expressions for these distributions.

For this purpose two classes of random events have been considered E_t and G_x, where

$$E_t = \{ E_\nu^{t_o,t} \quad ; \quad \nu = 0,1,2,\ldots \} \tag{3.2}$$

$$G_x = \{ G_\nu^{x_o,x} \quad ; \quad \nu = 0,1,2,\ldots \} \tag{3.3}$$

where $E_\nu^{t_o,t}$ represents the event that exactly ν storm periods will occur in the interval of time (t_o,t) and $G_\nu^{x_o,x}$ represents the event that the total amount of precipitation during ν storm periods will be less or equal to $(x-x_o)$ and for $(\nu+1)$ storms will exceed this value.

Probability $P(E_\nu^{t_o,t})$ satisfies the following differential equations

$$\begin{aligned} \frac{\partial P(E_\nu^{t_o,t})}{\partial t} &= \lambda_1(t,\nu-1)\, P(E_{\nu-1}^{t_o,t}) - \lambda_1(t)\, P(E_\nu^{t_o,t}) \\ \frac{\partial P(E_o^{t_o,t})}{\partial t} &= \lambda_1(t,0)\, P(E_o^{t_o,t}) \quad . \end{aligned} \tag{3.4}$$

Under the assumption that

$$\lambda_1(t,\nu) = \lambda_1(t) \tag{3.5}$$

for every $\nu = 0,1,2,\ldots$ we have

$$P(E_\nu^{t_o,t}) = e^{-\int_{t_o}^{t} \lambda_1(s)\, ds} \; \frac{(\int_{t_o}^{t} \lambda_1(s)\, ds)^\nu}{\nu!} \quad . \tag{3.6}$$

The function

$$\Lambda_1(t_o,t) = \int_{t_o}^{t} \lambda_1(s)\, ds \tag{3.7}$$

represents the average number of complete storm periods in (t_o,t).

According to definition $\lambda_1(t) \geqslant 0$ for all $t \geqslant t_o$ and represents the instantaneous intensity of storm periods. With regard to the seasonal variations it is realistic to expect that $\lambda_1(t)$ is a periodic function of time (Fig. 3).

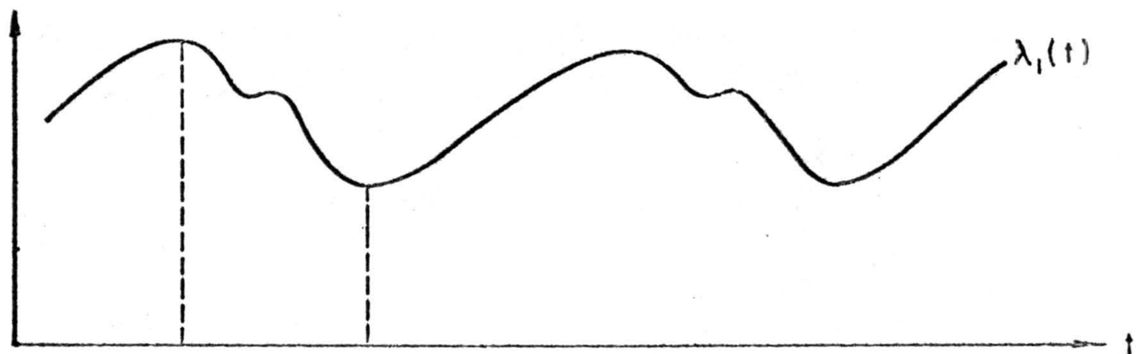

Fig. 3 Graphical presentation of $\lambda_1(t)$

The distirbution function $A_\nu(t)$ of random variable τ_ν is equal to

$$A_\nu(t) = \sum_{j=\nu}^{\infty} P\,(E_j^{t_o,t}) \qquad (3.8)$$

or using (3.6) we have

$$A_\nu(t) = e^{-\int_{t_o}^{t} \lambda_1(s)\,ds} \sum_{j=\nu}^{\infty} \frac{(\int_{t_o}^{t} \lambda_1(s)\,ds)^j}{j!} \qquad (3.9)$$

Corresponding density function $a_\nu(t)$ is equal to

$$a_\nu(t) = \frac{\lambda_1(t)}{\Gamma(\nu)}\, e^{-\int_{t_o}^{t} \lambda_1(s)\,ds} \left(\int_{t_o}^{t} \lambda_1(s)\,ds \right)^{\nu-1} \qquad (3.10)$$

for all $\nu = 1,2,\ldots$.

Probability $P(G_\nu^{x_o,x})$ satisfies the following differential equations

$$\frac{\partial P(G_\nu^{x_o,x})}{\partial x} = \lambda_2(x,\nu-1)\ P(G_{\nu-1}^{x_o,x}) - \lambda_2(x,\nu)\ P(G_\nu^{x_o,x}) \ . \tag{3.11}$$

Under assumption that

$$\lambda_2(x,\nu) = \lambda_2(x) \tag{3.12}$$

for every $\nu = 0,1,2,\ldots$ we have

$$P(G_\nu^{x_o,x}) = e^{-\int_{x_o}^{x} \lambda_2(s)\,ds} \frac{\left(\int_{x_o}^{x} \lambda_1(s)\,ds\right)^{\nu-1}}{\nu!} \ . \tag{3.13}$$

Consider now random variable X_ν which represents the total amount of precipitation during exactly ν storm periods. Let $F_\nu(x)$ be its distribution function then for every $\nu = 1,2,\ldots$ we have

$$F_\nu(x) = \sum_{j=\nu}^{\infty} P(G_j^{x_o,x}) \ . \tag{3.14}$$

By virtue of (6.13), $F_\nu(x)$ becomes

$$F_\nu(x) = e^{-\int_{x_o}^{x} \lambda_2(s)\,ds} \sum_{j=\nu}^{\infty} \frac{\left(\int_{x_o}^{x} \lambda_2(s)\,ds\right)^{j}}{j!} \tag{3.15}$$

and corresponding density function $f_\nu(x)$ is equal to

$$f_\nu(x) = \frac{\lambda_2(x)}{\Gamma(\nu)}\, e^{-\int_{x_o}^{x} \lambda_2(s)\,ds} \left(\int_{x_o}^{x} \lambda_2(s)\,ds\right)^{\nu-1} \ . \tag{3.16}$$

If the functions $\lambda_1(t)$ and $\lambda_2(x)$ are constant ones, i.e.,

$$\lambda_1(t) = \lambda_1 = \text{const.} \qquad \lambda_2(x) = \lambda_2 = \text{const.}$$

assuming $t_o = 0$ and $x_o = 0$, (3.10) and (3.16) become the Gamma density functions

$$a_\nu(t) = \frac{\lambda_1^\nu}{\Gamma(\nu)} e^{-\lambda_1 t} t^{\nu-1} \tag{3.17}$$

$$f_\nu(x) = \frac{\lambda_2^\nu}{\Gamma(\nu)} e^{-\lambda_2 x} x^{\nu-1} \quad . \tag{3.18}$$

Suppose that $Z_1, Z_2, \ldots Z_\nu, \ldots$ are independent random variables and let $b_\nu(z)$ stand for the density function of Z_ν , then it could be obtained as a solution of the following integral equation

$$f_\nu(x_o+u) = \int_o^u f_{\nu-1} [(x_o+u)(1-\frac{z}{u})] \, b_\nu(z) \, dz \quad . \tag{3.19}$$

If $\lambda_2(x) = \lambda_2 = \text{const.}$ the solution of (3.19) is equal to

$$f_\nu(z) = \lambda_2 e^{-\lambda_2 t} \quad . \tag{3.20}$$

The following important problem is the problem of extreme storms. If n storms $Z_1, Z_2, \ldots Z_n$ are expected in the interval of time (t_o, t) then among them one is minimum $\underline{Z}(n)$ and another one is maximum $\overline{Z}(n)$, i.e.,

$$\underline{Z}(n) = \inf_{1 \leq \nu \leq n} Z_\nu \qquad \overline{Z}(n) = \sup_{1 \leq \nu \leq n} Z_\nu \quad . \tag{3.21}$$

The distribution function $Q_n(z)$ of the minimum storm $\underline{Z}(n)$ has the following form

$$Q_n(z) = 1 - \prod_{\nu=1}^{n} [1 - B_\nu(z)] \tag{3.22}$$

where

$$B_\nu(z) = P \{ Z_\nu \leq z \}$$

and corresponding density function $q_n(z)$ is equal to

$$q_n(z) = \sum_{\nu=1}^{n} b_\nu(z) \prod_{k \neq \nu}^{n} [1 - B_k(z)] \quad . \tag{3.23}$$

Let $H_n(z)$ denote the distribution function of the random variable $\overline{Z}(n)$ and $h_n(z)$ corresponding density function, then

$$H_n(z) = \prod_{\nu=1}^{n} B_\nu(z) \tag{3.24}$$

$$h_n(z) = \sum_{\nu=1}^{n} b_\nu(z) \prod_{k \neq \nu}^{n} B_k(z) \quad .$$

If the parameter λ_2 = const. then by virtue of (3.20) we have

$$q_n(z) = \lambda_2 \cdot n\, e^{-\lambda_2 nz} \tag{3.25}$$

$$h_n(z) = \lambda_2 \cdot n\, e^{-\lambda_2 nz} (1-e^{-\lambda_2 z})^{n-1} \quad . \tag{3.26}$$

The corresponding mathematical expectations are

$$E(\underline{Z}_n) = \frac{1}{\lambda_2 n} \tag{3.27}$$

$$E(\overline{Z}_n) = \frac{n}{\lambda_2} \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+k)^2} \binom{n-1}{k} \quad . \tag{3.28}$$

Finally, consider the stochastic process

$$\{ X_t ; t > t_o \}$$

where X_t represents the total amount of precipitation during time $(t-t_o)$. Let $F_t(x)$ stand for the distribution function of X_t then

$$F_t(x) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_o,t}) F_\nu^*(x,t) \tag{3.29}$$

where

$$F_\nu^*(x,t) = P\{X_t \leqslant x \mid E_\nu^{t_o,t}\} \quad . \tag{3.30}$$

Since the method for calculation (3.30) is not known, an attempt has been made to obtain some information about this function. It has been shown that the following inequality is valid

$$\sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} \mid E_\nu^{t_o,t}) \leqslant F_\nu^*(x,t) \leqslant \sum_{j=\nu}^{\infty} P(G_j^{x_o,x} \mid E_\nu^{t_o,t}) \tag{3.31}$$

hence we have

$$F_{1t}(x) = \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_\nu^{t_o,t} \cap G_j^{x_o,x}) \tag{3.32}$$

$$F_{2t}(x) = \sum_{\nu=0}^{\infty} \sum_{j=\nu}^{\infty} P(E_\nu^{t_o,t} \cap G_j^{x_o,x}) \tag{3.33}$$

where

$$0 \leqslant F_{1t}(x) \leqslant F_t(x) \leqslant F_{2t}(x) \leqslant 1 \tag{3.34}$$

(see Fig. 4).

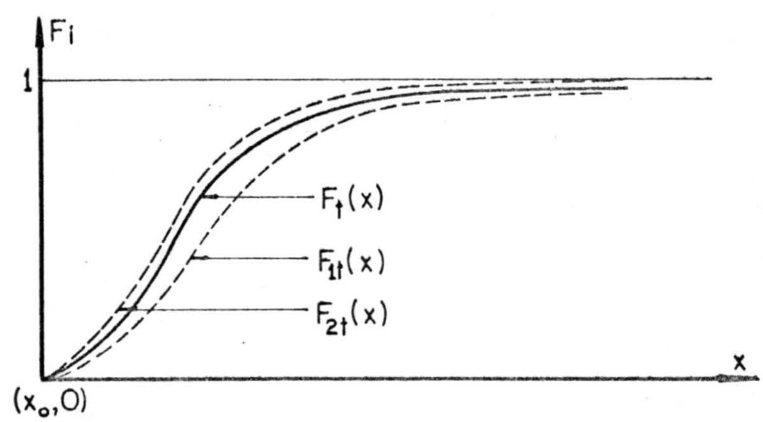

Fig. 4 Graphical representation of the distribution function $F_t(x)$ and its approximations

The approximations $F_{jt}(x)$ could be obtained as solutions of the following partial differential equation

$$\frac{\partial^2 F_{1t}(x)}{\partial x \partial t} + \psi_1(x,t) \frac{\partial F_{1t}(x)}{\partial x} + \psi_2(x,t) \frac{\partial F_{1t}(x)}{\partial t} = 0 \qquad (3.35)$$

$$\frac{\partial^2 F_{2t}(x)}{\partial x \partial t} + \overline{\psi}_1(x,t) \frac{\partial F_{2t}(x)}{\partial x} + \overline{\psi}_2(x,t) \frac{\partial F_{2t}(x)}{\partial t} = 0 \qquad (3.36)$$

where

$$\psi_1 \equiv \lambda_1(x,t) - \frac{\partial \ln \lambda_2(x,t)}{\partial x} \qquad \psi_2 \equiv \lambda_2(x,t)$$

$$\overline{\psi}_1 \equiv \lambda_1(x,t) \qquad \overline{\psi}_2 \equiv \lambda_2(x,t) - \frac{\partial \ln \lambda_2(x,t)}{\partial t} \quad .$$

Chapter I

1. INTRODUCTION

The analysis of hydrologic time series and other sequences by appropriate mathematical models, which describe the patterns and sequence of a river flow and precipitation, is one of the objectives of modern hydrologic investigations.

Among the known mathematical approaches that have been used for this purpose, one can distinguish two basically different concepts, deterministic and probabilistic ones. In the following exposition the two examples will be outlined, which point out distinction between these two philosophical aspects.

It is known that a hydrologic (or generally speaking a physical) phenomenon is subject to laws which govern its evolution. A physical phenomenon can be assumed as a deterministic one if, on the basis of the present state, the future states are determined (are sure outcomes). For instance, the newton laws of motions are deterministic in the sense that the given present state of a moving particle uniquely determines its future position.

The laws which govern the rainfall phenomenon evolutions are stochastic in the sense that on the basis of the present state only probabilities of the future outcomes are determined. For example, if η_t denotes the number of storm periods during the interval of time (o,t), (the present state), the number of storm periods $\eta_{t,T}$ in the interval of time (t,T) can never be predicted with certainty for any $T > t$, i.e., $\eta_{t,T}$ is a random variable (defined over some space

$\Omega = \{\omega\}$ of elementary events ω). Since $\eta_{t,T}$ is a discrete random variable for every $T > t$ ($\eta_{t,T} = 0, 1, 2, \ldots$) only the probabilities of the future states

$$P\{\eta_{t,T} = \nu\} = P_\nu(t,T) \qquad \nu = 0, 1, 2, \ldots$$

are determined.

The present exposition follows in principle probabilistic ideas as one of the most modern theoretical approaches to the problem of analysis and predicting the future characteristics of the hydrologic and metereologic phenomena such as the river flow, precipitation, etc., which are of the greatest importance in many of the water resource problems.

To be precise, the precipitation phenomenon which is the subject of this study will be considered from the aspect of the stochastic process theory. A stochastic process is a mathematical abstraction of an empirical process whose development is governed by probabilistic laws. Most of the hydrologic and metereologic phenomena are of this kind.

With respect to the complexity of the precipitation phenomenon (a random phenomenon which depends on time t) we have separately studied its "dynamic" and "quantitative" characteristics. This separation needs some explanation. Under the "dynamic" properties of the phenomenon considered one understands those features which give information concerning the frequency of storm periods in the given interval of time (t_0, t), duration of a storm period, the elapsed time up to the end of ν-th storm period where $\nu = 1, 2, \ldots$, etc.

As quantitative characteristics of the rainfall phenomenon, one understands the total amount of precipitation during one storm period or during ν storm periods or the amount of precipitation for the time $(t - t_0)$, etc.

Finally, for the purpose of the practical application, it is of interest to consider these properties combined. For example, information that the average number of storm periods is ν in the interval of time (t_0, t) is insufficient if we know nothing about the average amount of precipitation during a storm period. Likewise, information that the average amount of precipitation during one storm is $\bar{x}$ does not mean very much if it is not known how many storms could be expected in the considered interval of time.

The second and third chapters of this study are devoted to investigation of the "dynamic" and "quantitative" properties of rainfall phenomena respectively. As far as the fourth chapter is concerned, it represents an attempt to investigate a problem such as obtaining the distribution function $F_t(x)$ of the random variable X_t for every $t > t_0$, where X_t represents the total amount of precipitation up to time t , or obtaining the distribution function of random variable T_x , where T_x represents the time necessary for that amount of precipitation to be equal x , etc.

Toward this end, a particular stochastic process of nondecreasing sample functions, denoted by

$$X_t = \pi\ (t, \omega) \quad ,$$

has been introduced. The most of the theoretical results that have been considered in this study and connected with this process are of an original nature and appear here for the first time.

2. OBJECTIVES

The general purpose of this paper is to present a mathematical study of precipitation phenomenon (not entering into its physical nature, although some conclusions concerning the physics of the phenomenon can be drawn indirectly from the results obtained) based on probability theory and stochastic processes. In fact, a family of stochastic processes will be established (derived from the stochastic process of non-decreasing sample functions $\pi(t,\omega)$) and used for an interpretation of some aspects of rainfall phenomenon.

This paper should represent the initial steps in establishing a general mathematical theory which makes it possible to predict the future behavior of rainfall. It is expected that these results will help to better understand the precipitation phenomenon and that this understanding will make possible better predictions of future characteristics. It is also hoped that this study will discover new problems and open the way to new investigations and discussions.

In this paper, studying of the rainfall will be restricted to the consideration of that portion of the total amount of precipitation which reached the ground and has been measured or recorded at the existing network of raingage stations.

In order to achieve these goals, let us denote by ξ_t the rainfall intensity over a small part ΔS (of an area S), at some moment of time t . ξ_t is non-negative and in addition is dependent not only on time t , but on the position of ΔS within S , i.e.,

$$\xi_t = \xi(t,x,y) \tag{1.2.1}$$

where (x,y) is a point which belongs to S (see Fig. 5).

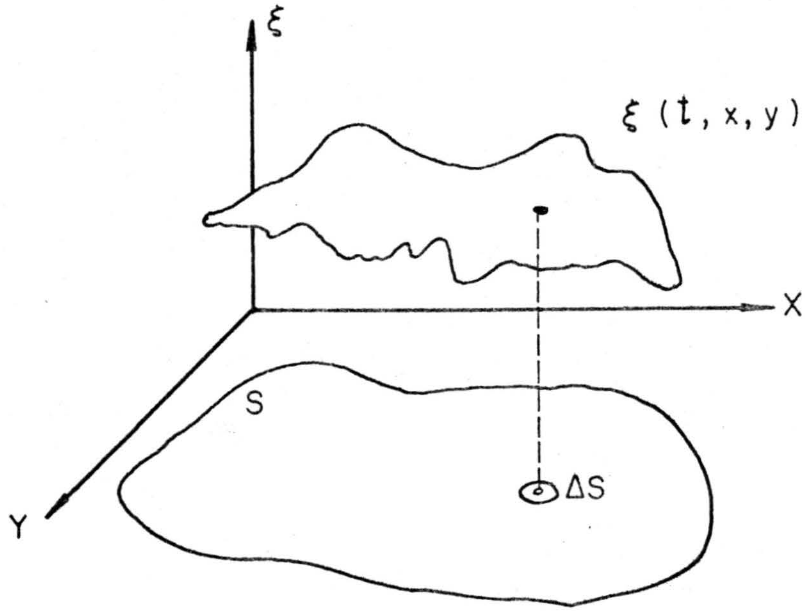

Fig. 5 Graphical representation of intensity of precipitation over an area S at the moment of time t

It is assumed that precipitation intensity is uniform over ΔS at any moment of time, i.e., if the area ΔS is sufficiently small, the following is valid: for any two points $(x_1,y_1) \varepsilon\Delta S$ and $(x_2,y_2) \varepsilon\Delta S$ we have

$$\xi(t,x_1,y_1) \approx \xi(t,x_2,y_2) \quad .$$

This small part ΔS could be, for instance, a precipitation station in the area S .

In addition, the function $\xi(t,x,y)$ is a random variable for every t , x and y . Therefore, ξ_t is a continuous parameter stochastic process. Since the rainfall phenomenon will be considered over a small part ΔS , obviously $\xi_t = \xi(t,x,y)$ becomes a stochastic process with parameter t only. Consequently we have

$$\{ \xi_t : t \geqslant t_o \}$$

as an objective of further investigation.

3. APPROACH

The last several years have seen an extraordinary increase of interest in the problems of planning of water resource projects and in the problems of increasing natural water supplies. For the first of these two problems (or the group of problems) a method for predicting future characteristics of the water supplies is indispensible. As far as the second problem is concerned, it is important to develop sound mathematical methods for evaluating weather modification attainments applicable to a variety of natural conditions.

In order to achieve these goals, it is necessary to establish a quantitative (mathematical) theory of some aspects of weather phenomena. It is realistic to expect that the theory must be probabilistic in nature, but since weather phenomena occur randomly and depend on time t, it is apparent that the theory of stochastic processes will play the basic role.

We are far from concluding that the results represent a complete theory (mathematical) of the rainfall phenomenon. In fact, this paper represents the initial steps in establishing methods which should make it possible to predict the future behavior of precipitation and to estimate development of practical weather modification techniques. In addition, the aim of this theory is not only a simple evaluation of weather modification attainment but also to point out better application of weather modification techniques such that optimization of the seeding procedure may be examined.

In this study, we have resolved to take as a subject of the investigation a precipitation station, i.e., all changes in the weather phenomena related to precipitation that could be recorded in the raingage station being considered. We are not going to talk about advantages of such an approach, but it would seem to be a very natural one.

Let t_o denote the instant of time when observation of the rainfall phenomenon begins and let ξ_t be the rainfall intensity at some moment of time t. Obviously, ξ_t is a random variable for all $t \geqslant t_o$. Therefore, we have a family of random variables

$$\{ \xi_t ; t \geqslant t_o \}$$

or a stochastic process. With respect to the nature of precipitation phenomenon, it is apparent that for every $t \geqslant t_o$, $\xi_t \geqslant 0$.

An intermittent hydrograph at a rainfall gaging station has the shape of the curve in Fig. 6 and represents a sample function of the stochastic process, ξ_t. Obviously then, the total amount of precipitation during some interval of time (t_o,t) recorded at the gaging instrument is equal to the following integral:

$$\int_{t_o}^{t} \xi_s \, ds$$

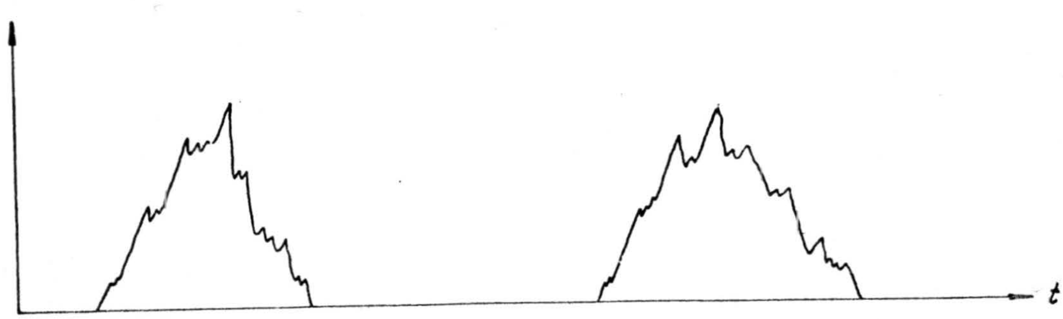

Fig. 6 Graphical representation of a rainfall hyetograph

Since ξ_s is a random variable for every $s \geqslant t_o$, it follows that the integral considered is a random variable for every $t > t_o$. In the present study, we are not going to investigate properties of the process ξ_t directly. In fact, we are going to deal with the stochastic process X_t which represents a cumulative process, i.e.,

$$X_t = x_o + \int_{t_o}^{t} \xi_s \, ds \quad ,$$

where x_o represents the total amount of precipitation up to the instant of time t_o . Since X_t is a random variable for every $t > t_o$, we have a new family of random variables:

$$\{ X_t ; t > t_o \}$$

or a continuous parameter stochastic process, which will be denoted by

$$X_t = \pi \ (t,\omega) \quad .$$

Since $\xi_s \geqslant 0$ for every $s \geqslant t_o$, it is apparent that any sample function of $\pi(t,\omega)$ is a nondecreasing t function. In other words, for every $t \geqslant t_o$ and Δt , the following inequality is valid

$$\pi(t,\omega) \leqslant \pi(t + \Delta t,\omega) \qquad \forall \ \Delta t > 0 \quad .$$

The following exposition is devoted to the problems of interpretation and investigations of the rainfall phenomena by the stochastic process $X_t = \pi(t,\omega)$. One will see that studying of some important characteristics of precipitation phenomenon can be reduced to studying corresponding properties of the stochastic process. Therefore, the more we know about the process $X_t = \pi(t,\omega)$, the more we know about the rainfall phenomenon. It is hoped that this study represents a contribution to investigations of this problem.

Chapter II

1. THE FUNDAMENTAL CONSIDERATIONS

1°. In this section will be discussed how and why the precipitation phenomenon should be studied from the aspect of the theory of stochastic processes. For the sake of clarity, the exposition of the first section will start with nonmathematical description of the quantitative properties of the rainfall phenomenon not entering into its physical nature.

Let us first explain why this phenomenon should be considered as a stochastic process or why probabilistic approach is more realistic than deterministic. Toward this end, consider the rainfall hydrograph at a rainfall gaging station; it is known that it has the shape of the curve in Fig. 6. An ordinate ξ_t of this curve at some instant of time t represents rainfall intensity at the moment t. If t_o denotes the moment of time when observation of the rainfall phenomenon begins and ξ_{t_o} is the rainfall intensity at this moment, then with respect to its nature it is not possible to predict with certainty, the value of the variable ξ_t at any moment of time t after t_o. In other words, on the basis of the present state the future outcomes cannot be predicted with certainty, i.e., ξ_t is a <u>random variable</u> for any $t \geqslant t_o$ or for any $t \varepsilon (t_o, \infty)$. For example, if an arbitrary sequence $t_1, t_2, \ldots, t_\nu, \ldots$ from (t_o, ∞) is selected such that $t_\nu < t_{\nu+1}$ for $\nu = 1, 2, \ldots,$ then corresponding rainfall intensities

$$\xi_{t_1}, \xi_{t_2}, \ldots \xi_{t_\nu}, \ldots$$

are obviously random variables.

Since it is valid for any sequence of moments of time from (t_o,∞) , obviously we have a family or random variables (one variable for each t) which is denoted in the following manner:

$$\{ \xi_t ; \ t \geqslant t_o \} \tag{2.1.1}$$

This family or random variables represents a stochastic process of a continuous parameter (see Doob (5) p. 46). A particular hydrograph represents a sample function of this process (see Fig. 7), where T_ν denotes the length of the ν-th storm period (starting from the instant of time t_o) and τ_ν is its upper bound. Obviously T_ν and τ_ν are random variables for every $\nu = 1,2,\ldots$. Therefore, we have two more families of random variables

$$\{ T_\nu ; \quad \nu = 1,2,\ldots \} \quad \text{and} \quad \{ \tau_\nu ; \quad \nu = 1,2,\ldots \}$$

or the two stochastic processes of the discrete parameter.

In the following exposition we shall not consider the stochastic process (2.1.1), but we will deal with integral of the function ξ_t of the following form

$$X_t = x_o + \int_{t_o}^{t} \xi_s \, ds \tag{2.1.2}$$

where x_o is a constant. It is obvious that the integral considered represents the total amount of precipitation during the interval of time (t_o,t) , so that it is a random variable for every $t > t_o$. Therefore X_t is a random variable for every $t > t_o$, and the following

$$\{ X_t ; \ t > t_o \} \tag{2.1.3}$$

represents a continuous parameter stochastic process, which will be

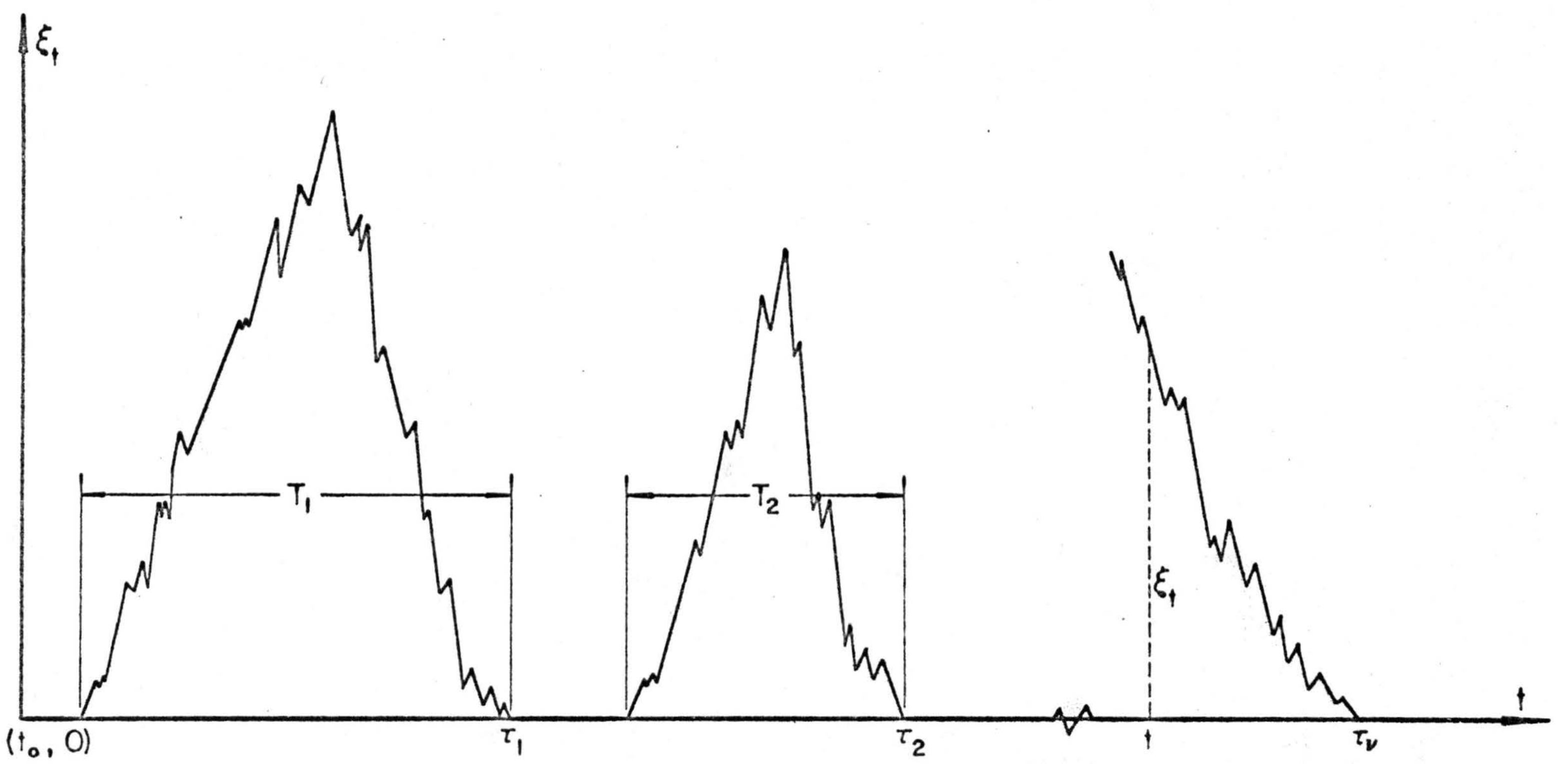

Fig. 7 A sample function of the stochastic process $\{\xi_t;\ t > t_o\}$

denoted by

$$X_t = \pi(t,\omega) \quad .$$

Since for every $s \geqslant t_o$, $\xi_s \geqslant 0$, it is obvious that for every $\Delta t > 0$ the following inequality is valid:

$$X_t \leqslant X_{t+\Delta t} \qquad \forall t \geqslant t_o \ , \quad \Delta t > 0 \quad .$$

Therefore, sample functions of the process (2.1.3) are nondecreasing t functions (see Fig. 8).

2^o. Probability background - The purpose of this part of the paper is to replace preliminary intuitive notions with a sound mathematical base for further investigations. Toward this end, assume that the following system is given:

$$(\Omega' = \{ \omega' \} \ , \ \mathcal{A} \ , \ P_o)$$

where

a) $\Omega' = \{\omega'\}$ is the space (or sure event) of elementary events ω' , and domain of definition of random variables ξ_t , for every $t \varepsilon T^*$, such that (see Doob (5) p. 10)

$$\omega' : (x_t ;\ t \varepsilon T^*) \tag{2.1.4}$$

where $x_t \geqslant 0$ is any real number. In other words, Ω' is the space of sample functions of $t \varepsilon T^*$, or, from another point of view, the coordinate space, whose dimensionality is the cardinal number of the set T^* . The value of a t function at the point $t = s$ defines an ω' function ξ_s if we set

$$\xi_s (\omega') = x_s \quad . \tag{2.1.5}$$

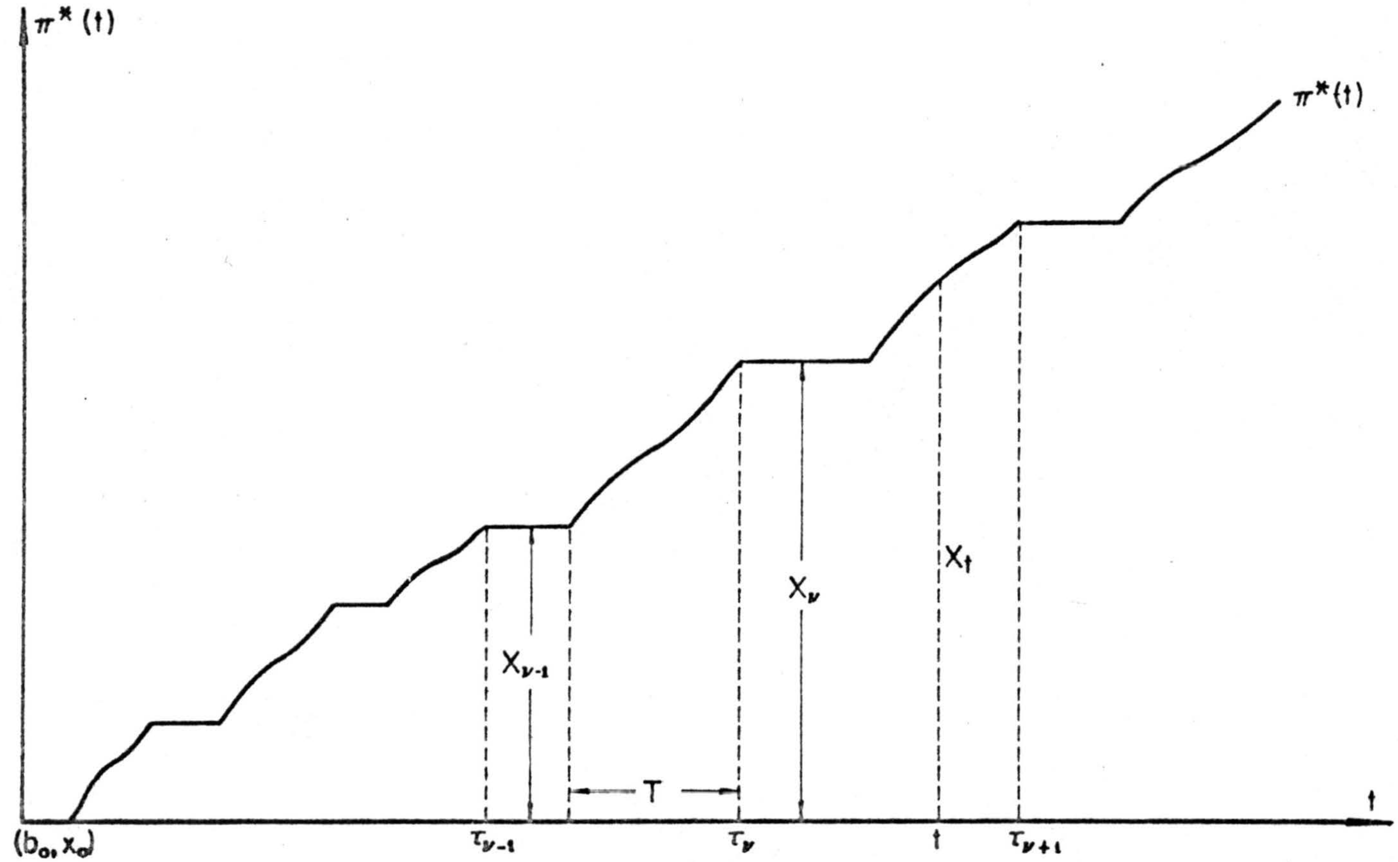

Fig. 8 A sample function of the stochastic process $\pi(t,\omega)$

b) The class $\mathcal{A}$ is a smallest σ-field (or Borel field) generated by the class of Ω' subsets of the following form

$$\{ \omega'; \xi_t \; \varepsilon I \} \quad t \varepsilon T^*$$

where I is an interval of the real line.

c) Finally, one assumes that the probability measure P_o, with the class of sets $\mathcal{A}$ as the domain of definition, is complete and perfect, following Gnedenko and Kolmogorov (see (8) §§ 2, 3).

For the purpose of the further investigations, it would be necessary to assume ξ_t is a separable stochastic process, i.e., if $\overline{T}$ denotes an everywhere dense set in (t_o,T), then the stochastic process ξ_t is called $\overline{T}$-separable or separable with respect to $\overline{T}$, if there exists an event A (i.e., a subset of Ω' such that $A \varepsilon \mathcal{A}$) having probability zero, i.e., $P_o(A) = 0$ such that

$$A \cup \{\omega'; \xi_t \; \varepsilon F \; \forall t \varepsilon I^*\} \supseteq \{\omega'; \xi_t \; \varepsilon F \; \forall t \varepsilon \; \overline{T} \cap I^*\} \qquad (2.1.6)$$

where F is a closed set and I^* an open interval from T^*. The set on the right side of the relation (2.1.6) is a measurable one and contains the following set:

$$\{ \omega'; \xi_t \; \varepsilon F \;\; \forall t \varepsilon I^* \}$$

which then, under separability hypothesis, is also a measurable one (see Doob (5) p. 51 or Skorohod (26) p. 6).

On the basis of the separability hypothesis it follows that ω set

$$\{ \omega'; \xi_t = 0 \; \forall t \varepsilon \Delta t \} \qquad (2.1.7)$$

where $\Delta t \subset (t_o,T)$ is also measurable. In the further investigations, we shall suppose that probability of this set is not equal zero for any

$\Delta t \subset (t_o,T)$, i.e.,

$$P_o \{ \omega'; \; \xi_t = 0 , \; \forall t \varepsilon \Delta t \} \neq 0 \quad . \tag{2.1.8}$$

Finally, we assume that the stochastic process is measurable and integrable (Doob (5), p. 60). The last hypothesis is of great importance in the study of the stochastic process $\pi(t,\omega)$. It will be seen that on the basis of this hypothesis it is possible to prove that the last process is a separable one as well.

3^o. Comment - For better understanding of the notions and hypothesis of the previous exposition (Probability Background), let us try to give the phenomenologic interpretation of these symbols and assumptions. In other words, let us express all these notions and definitions in hydrologic terms.

First of all, the evolution of the rainfall phenomenon is considered in the time interval $T^* = (t_o,T)$ where $T < \infty$ so that instead of (2.1.1) we have the following:

$$\{ \xi_t ; \; t \varepsilon (t_o,T) \} \quad . \tag{2.1.9}$$

If we start to consider the rainfall phenomenon at the moment of time t_o , then it is not possible to predict the shape of the hydrograph in the time interval (t_o,T) , since there exists an infinite number of outcomes (hydrograph curve) which could be realized. The set of all these curves is Ω' , and ω' is any of these curves. In other words, the space Ω' is the set of all sample functions of the stochastic process (2.1.9).

The class $\mathcal{A}$ consists of the sets of sample functions which are particular (measurable) subsets of Ω' . For example, the set

$$\{ \omega'; \; \xi_t \leqslant x \}$$

belongs to the class $\mathcal{A}$ and consists of all sample functions of the process (2.1.9) with the property that the ordinates at the point t, $(t_o \leqslant t \leqslant T)$ are less than x. The sets belonging to the class are called (random) events; the space Ω' is an event, i.e. $\Omega' \varepsilon \mathcal{A}$.

The probability measure P_o is a function (so-called set function) defined over the class $\mathcal{A}$ in the sense that to every event $A \varepsilon \mathcal{A}$ corresponds a number $P_o(A)$, the probability of this event. This function is completely additive, i.e., if $B_1, B_2, \ldots$ is an accountably infinite set of mutually disjointed events, then

$$P_o \left(\bigcup_{i=1}^{\infty} B_i \right) = \sum_{i=1}^{\infty} P_o (B_i) \quad .$$

Finally, let us give the meaning of separability hypothesis; according to definition, if we have an denumerable (countable) family of events, say

$$A_1, A_2, \ldots A_\nu, \ldots, \quad A_\nu \in \mathcal{A}, \quad \forall \nu = 1, 2, \ldots$$

then, according to definition of the class $\mathcal{A}$, the union and intersection of this event is an event as well, i.e.,

$$\bigcup_{i=1}^{\infty} A_i \;\varepsilon\; \mathcal{A} \qquad \bigcap_{i=1}^{\infty} A_i \;\varepsilon\; \mathcal{A} \quad .$$

The situation is more complicated, however, if one deals with non-denumerable family of events, for instance if $A_t \varepsilon \mathcal{A}$ for every $t \varepsilon I^*$, then

$$\bigcap_{t \varepsilon I^*} A_t$$

is not necessarily an event; i.e., generally speaking,

$$\bigcap_{t \varepsilon I^*} A_t \notin \mathcal{A}$$

and therefore

$$P_o\,(\bigcap_{t\varepsilon I^*} A_t\,)$$

has no sense. In our case, the set (2.1.7) can be represented as an nondenumerable intersection of events, and therefore it is not necessarily an event; i.e.,

$$\{\,\omega';\ \xi_t = 0 \ \ \forall t\varepsilon\Delta t\,\} = \bigcap_{t\varepsilon\Delta t}\{\,\omega';\ \xi_t = 0\,\}\quad . \tag{2.1.10}$$

On the basis of (2.1.6), we have the following:

$$A \cup \bigcap_{t\varepsilon\Delta t}\{\,\omega';\ \xi_t = 0\,\} \supseteq \bigcap_{t\varepsilon\Delta t\cap\overline{T}}\{\,\omega';\ \xi_t = 0\,\}\ .$$

Since the second set on the left side is a subset of the set on the right side (which is measurable; i.e., an event) of the last relation, it follows that (2.1.10) is also measurable under separability hypothesis.

2. AN ANALYSIS OF THE PROCESS $\pi(t,\omega)$

1^o. On the basis of definition (2.1.2), it is obvious that the stochastic process

$$X_t = \pi(t,\omega),\ \omega\varepsilon\Omega\ ,\quad t\varepsilon T^* = (t_o,T) \tag{2.2.1}$$

represents a cumulative process, namely if at the instant of time t_o when observation begins the total amount of precipitation was x_o, then X_t denotes the total amount of precipitation up to time t (see Fig. 8). The difference

$$\pi\ (t,\omega) - x_o$$

represents the total amount of precipitation recorded at a rainfall gaging station during the time interval (t_o,t).

For the purpose of establishing an appropriate theoretical description of the quantitative aspect of the rainfall phenomenon which will give us a complete as possible analysis of precipitation, it is more convenient to study the properties of stochastic process (2.2.1) than (2.1.1) or (2.1.9). Therefore, investigation of characteristics of (2.2.1) will be the main objective of the further exposition.

To achieve a full analysis of the rainfall phenomenon, it is necessary to study the various aspects of the stochastic process considered. This leads to a new series of stochastic processes, derived from the process $\pi(t,\omega)$.

Consider η_t , the number of the full storm periods in an interval of time (t_o,t) , where $t\varepsilon(t_o,T)$. Obviously, since $\eta_t = 0, 1, 2,...$ is a random variable for every t from (t_o,T) , we have a new continuous parameter stochastic process:

$$\{ \eta_t ;\ t\varepsilon T^* \} \quad .$$

The upper bound of the ν-th storm period has been denoted by τ_ν (see Fig. 7 or 8); information about τ_ν for every $\nu = 1, 2,...$ is of remarkable phenomenological interest. Since τ_ν is a random variable for all ν , we have a discrete parameter stochastic process

$$\{ \tau_\nu ;\ \nu = 1, 2,... \} \quad . \tag{2.2.2}$$

The other two variables are of great importance for analysis of the precipitation phenomenon; the first X_ν represents the total amount of precipitation during exactly ν storm periods (see Fig. 8) and Z_ν , where

$$Z_\nu = X_\nu - X_{\nu-1} \quad ,$$

which represents the total amount of precipitation during ν-th storm period only. Both X_ν and Z_ν are random variables for every $\nu = 1, 2, \ldots$ so that two more discrete parameter stochastic processes will be considered:

$$\{ X_\nu ; \nu = 1, 2, \ldots \} \text{ and } \{ Z_\nu ; \nu = 1, 2, \ldots \} \quad . \quad (2.2.3)$$

Finally, duration T_ν of the ν-th storm period has particular phénomenological interest for the rainfall phenomenon study. Since T_ν is a stochastic variable for every $\nu = 1, 2, \ldots$ we have another stochastic process

$$\{ T_\nu ; \nu = 1, 2, \ldots \} \qquad (2.2.4)$$

which is of interest in the following investigations.

2^o. Some definitions - For the purpose of further study, the two particular classes of measurable sets (events) whose elements are sample functions of the stochastic process $\pi(t,\omega)$ will be defined. It will be shown that events such as

$$\{ \omega ; \tau_\nu \leqslant t \} , \{ X_\nu \leqslant x \} , \forall \nu = 1, 2, \ldots$$

could be expressed over unions and intersections of sets from these classes. Since we are able to calculate probability of these sets, the probability of the previous event can be obtained as well.

Let $E_\nu^{t_o,t}$ represents the set of all sample functions of the stochastic process $\pi(t,\omega)$ having exactly ν points τ_j in the interval of time (t_o,t) , or, in other words, the set of all sample functions for which the following is valid: $\tau_\nu \leqslant t < t_{\nu+1}$, i.e.,

$$E_\nu^{t_o,t} = \{ \omega ; \tau_\nu \leqslant t < \tau_{\nu+1} \} \qquad (2.2.5)$$

(see Fig. 8). On the basis of definition of these sets of sample functions, obviously the following class

$$E_t = \{ E_\nu^{t_o,t} \;;\; \nu = 0, 1, 2, \ldots \} \tag{2.2.6}$$

for fixed t is a countable one. Elements of this class are disjointed sets, i.e., for every $i \neq j$, the following is valid

$$E_i^{t_o,t} \cap E_j^{t_o,t} = \theta \quad \text{and} \quad \bigcup_{\nu=0}^{\infty} E_\nu^{t_o,t} = \Omega \tag{2.2.7}$$

where symbol θ denotes an empty set.

Phenomenologically speaking, the set $E_\nu^{t_o,t}$ represents the event that exactly ν storm periods will occur in the interval of time (t_o,t).

Let us define another class of sets of sample functions. Let $G_\nu^{x_o,x}$ be the set of sample functions of the process $\pi(t,\omega)$, which have exactly ν points $\pi^*(\tau_j)$ in the interval of time (x_o,x), i.e.,

$$G_\nu^{x_o,x} = \{ \omega;\ \pi^*(\tau_\nu) \leqslant x < \pi^*(\tau_{\nu+1}) \} \tag{2.2.8}$$

(see Fig. 4). By definition, the following relations are valid

$$\forall i \neq j \quad G_i^{x_o,x} \cap G_j^{x_o,x} = \theta \qquad \bigcup_{\nu=0}^{\infty} G_\nu^{x_o,x} = \Omega \quad . \tag{2.2.9}$$

The phenomenological interpretation of the set $G_\nu^{x_o,x}$ of sample functions is evident; it represents the event that the total amount of precipitation during exactly ν storm periods will be less than or equal to $(x - x_o)$ and for $(\nu + 1)$ storms it will be larger than $(x - x_o)$.

3^o. Probability background - Relation (2.1.2) represents a transformation with $\Omega' = \{ \omega' \}$ as the domain of definition. The set of all values of this transformation will be denoted by $\Omega = \{ \omega \}$. In order to avoid purely technical difficulties, we shall assume that the following system is given:

$$(\Omega = \{ \omega \} , \mathcal{B} , P)$$

where Ω is the space of elementary events ω, $\mathcal{B}$ is the smallest σ-field generated by the class of Ω subsets of the following form:

$$\{ \omega; X_t \varepsilon I \}, (E_i^{t_o,t} \cap G_j^{t_o,t}) \quad i, j = 0, 1, 2, \ldots$$

where I is an interval of the real line. Elementary event ω should be realized as a sample function of the process $\pi(t,\omega)$. P is a probability measure defined over class $\mathcal{B}$.

Since $\xi_t \geqslant 0$ for every $(t,\omega) \varepsilon (T^* x\Omega)$, (where $T^* x\Omega$ represents the Cartesian product of the sets T^* and Ω, (see Halmosh (12) p. 137) obviously the sample functions of the process $X_t = \pi(t,\omega)$ are nondecreasing t functions, i.e., for any $\omega\varepsilon\Omega$

$$X_t = \pi(t,\omega) \leqslant X_{t+\Delta t} = \pi(t+\Delta t,\omega) \quad \forall \Delta t > 0 \quad .$$

Finally, let us examine the question of separability of stochastic process $X_t = \pi(t,\omega)$. First, if it is supposed that almost all sample functions of the process

$$\{ \xi_t ; t\varepsilon T^* \}$$

are continuous functions then $\pi(t,\omega)$ is a separable stochastic process. If ξ_t is stochastically continuous, then $\pi(t,\omega)$ is separable. Let us now prove the following theorem:

Theorem 1.

If almost all sample functions of the measurable stochastic process

$$\{ \xi_t ; t \varepsilon T^* \}$$

are integrable, then stochastic process

$$\{ X_t ; t \varepsilon T^* \}$$

is separable.

Proof.

Consider two instants of time t_1 and t_2 and assume $t_1 < t_2$. On the basis of monotony of sample functions of $\pi(t,\omega)$, we have $X_{t_2} - X_{t_1} \geqslant 0$. Therefore, for every $\varepsilon > 0$, probability of the following event

$$\{ \omega ; X_{t_2} - X_{t_1} \geqslant \varepsilon \} = \{ \omega ; \int_{t_1}^{t_2} \xi_s \, ds \geqslant \varepsilon \}$$

obviously tends to zero, if $t_2 - t_1 \to 0$, since

$$\int_{t_1}^{t_2} \xi_s \, ds \to 0$$

i.e.,

$$\lim_{t_2 - t_1 \to 0} P \{ \omega ; X_{t_2} - X_{t_1} \geqslant \varepsilon \} = 0 \quad .$$

Therefore, the stochastic process $\pi(t,\omega)$ is a stochastically continuous one so that by virtue of the foregoing (see Skorohod (27), p. 209), it is separable.

3. STOCHASTIC PROCESSES η_t and τ_ν

1^o. It has been mentioned in the previous section that η_t, where $t \varepsilon T^*$, denotes the number of the complete storm periods occurring in the interval of time (t_o, t). Since η_t is a random variable for any t from T^*, we have a continuous parameter stochastic process

$$\{ \eta_t ; t \varepsilon T^* \} \tag{2.3.1}$$

where $\eta_t = 0, 1, 2, \ldots$.

In order to estimate the average number of storm periods in some interval (t_o, t), which is obviously a t function, and other characteristics of the variable η_t it is necessary to calculate the following probabilities:

$$P \{ \eta_t = \nu \} = P_\nu(t)$$

for every $t \varepsilon T^*$ and $\nu = 0, 1, 2, \ldots$.

According to definition of the events $E_\nu^{t_o, t}$ (see 2.2.5), it follows that

$$P_\nu(t) = P(E_\nu^{t_o, t}) \tag{2.3.2}$$

and on the basis of (2.2.7) it follows that

$$\sum_{\nu=0}^{\infty} P_\nu (t) = 1 \quad .$$

The corresponding distribution function $F(x|t)$ of the random variable η_t has the form

$$F(x|t) = P \{ \omega ; \eta_t \leqslant x \} = \sum_{\nu=0}^{[x]} P_\nu(t) \tag{2.3.3}$$

where the symbol $[x]$ denotes the greatest integer not greater than x .

On the basis of (2.3.2) the average number of storm periods $E(n_t)$ in the interval of time (t_o,t) is equal to

$$E(n_t) = \sum_{\nu=1}^{\infty} \nu \, P(E_\nu^{t_o,t}) \quad . \tag{2.3.4}$$

For an effective calculation of the probabilities $P_\nu(t)$ and mathematical expectation $E(n_t)$, it is necessary to possess the probabilities $P(E_\nu^{t_o,t})$ for any $t \varepsilon T^*$ and $\nu = 0, 1, 2,\ldots$. In the next section, it will be shown how this probability can be obtained under very general assumptions about the phenomenon considered.

The next characteristic which will be studied is the upper bound τ_ν of the ν-th storm period. Study of this characteristic has a particular phenomenologic interest. Since τ_ν is a random variable for every $\nu = 1, 2,\ldots$ we have a family of random variables or a continuous parameter stochastic process

$$\{ \tau_\nu ; \quad \nu = 1, 2, \ldots \} \quad .$$

For practical application, it is necessary to possess information about distribution function $A_\nu(t)$ of random variable τ_ν for every $\nu = 1, 2,\ldots$. The following theorem gives the relationship between $A_\nu(t)$ and the probabilities (2.3.2).

Theorem 2.

For every $\nu = 1, 2,\ldots$ and $t \geqslant t_o$, we have

$$A_\nu(t) = \sum_{k=\nu}^{\infty} P(E_k^{t_o,t}) \quad . \tag{2.3.5}$$

The proof of this theorem is very simple (see appendix to this section). Phenomenologically speaking, the relation (2.3.5) means that τ_ν will be less than t if at least ν storm period will occur in (t_o,t).

Let $a_\nu(t)$ represent the corresponding density function of the distribution function (2.3.5), i.e.,

$$a_\nu(t) = \frac{\partial A_\nu(t)}{\partial t} \quad .$$

Then the following theorem can be proven:

<u>Theorem 3.</u>

Assume that the following conditions are satisfied:

a) $$\lim_{\Delta t\to 0} \frac{\sum_{\tau=2}^{\infty} P(E_\tau^{t,t+\Delta t})}{\Delta t} = 0 \quad \forall t \geqslant t_o$$

b) $$\lim_{\Delta t\to 0} \frac{P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t})}{\Delta x} = \lambda_1(t) \quad \forall t \geqslant t_o \quad ,$$

then for every $\nu = 1, 2, \ldots$

$$a_\nu(t) = \lambda_1(t) \quad P(E_{\nu-1}^{t_o,t}) \quad . \tag{2.3.6}$$

The proof of this theorem will be given later.

Let us now discuss the conditions a) and b) of the theorem and try to give their pure phenomenological interpretation. Toward this end, consider first the condition a) and its physical meaning. First of all, the following sum

$$\sum_{\tau=2}^{\infty} P(E_\tau^{t,t+\Delta t}) \tag{2.3.7}$$

expresses the probability that in the interval of time $(t,t+\Delta t)$ at least the two events τ_ν will appear (i.e., will occur and, of a rainfall and the next storm period, will belong wholly to the same time interval).

Condition a) means that the sum (2.3.7) is an infinitesimal of a higher order than Δt, when $\Delta t \to 0$. With respect to the nature of the precipitation phenomenon, this condition is very realistic.

Consider now the condition b) of the theorem and its phenomenological interpretation. On the basis of definition of the events $E_\nu^{t_o,t}$, obviously

$$P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t}) \tag{2.3.8}$$

represents the conditional probability that the upper bound of the ν-th storm period will belong to the interval of time $(t,t+\Delta t)$, under the condition that exactly $(\nu-1)$ storms occurred in the interval of time (t_o,t).

Since the conditional probability (2.3.8) depends on t, Δt and ν, it represents a function which in the most general case depends on these variables, i.e.,

$$P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t}) = \lambda_1(t,\Delta t,\nu-1) \quad . \tag{2.3.9}$$

If it is assumed that:

1. Probability that a termination instant will lie between the two instants of time, t and $t+\Delta t$ do not depend on the number of storm periods up to time t.

2. For very small Δt, λ_1 is a linear function with respect to Δt, then the following is valid

$$P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t}) = P(E_1^{t,t+\Delta t}) = \lambda_1(t,\Delta t)$$

and

$$\lambda_1(t,\Delta t) = \lambda_1(t)\ \Delta t$$

then condition b) of the previous theorem is satisfied.

Certainly the assumption (2.3.9) that conditional probability (2.3.8) is a function of t , Δt and ν is most general. As far as the other two hypotheses are concerned, there is no doubt that the second,

$$\lambda_1(t,\Delta t,\nu-1) = \lambda_1(t,\nu-1)\ \Delta t$$

is realistic, but the first, that

$$\lambda_1(t,\Delta t,\nu-1) = \lambda_1(t,\Delta t)$$

is discussible and needs experimental testing. In fact, a very realistic hypothesis is to suppose that the relation

$$P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t}) = \lambda_1(t,\nu-1)\ \Delta t \tag{2.3.10}$$

is valid.

Note:

Since the random variable τ_ν for $\nu = 1, 2, \ldots$ can assume any value from the time interval (t_o,∞) , it is supposed that $T^* = (t_o,\infty)$. In the following exposition it will be of particular interest to consider no interval $T^* = (t_o,\infty)$ but an interval of time $T^* = (t_o,T)$ where $T < \infty$. In other words, let us consider the problem of calculation of probability of the random event

$$\{\ \omega;\ \tau_\nu \leqslant t\ \}\quad \nu = 1, 2, \ldots$$

under the condition that at least ν storm periods have occurred in (t_o,T) , i.e.,

$$P\ \{\ \omega;\ \tau_\nu \leqslant t,\ \tau_\nu \varepsilon\ (t_o,T)\ \}\quad .$$

Let us denote this conditional probability by $F_\nu(t|T^*)$, i.e.,

$$F_\nu(t|T^*) = P\ \{\ \omega;\ \tau_\nu < t|\tau_\nu \varepsilon T^*\ \}\quad .$$

Then the following theorem is valid.

Theorem 4.

For every $t\varepsilon(t_o,T)$ and $\nu = 1, 2, \ldots$

$$F_\nu(t|T^*) = \frac{\sum_{i=\nu}^{\infty} P(E_i^{t_o,t})}{\sum_{i=\nu}^{\infty} (E_i^{t_o,T})} \qquad (2.3.11)$$

If $f_\nu(t\ T^*)$ denotes the corresponding density function

$$f_\nu(t|T^*) = \frac{\partial F_\nu(t|T^*)}{\partial t}$$

then

$$f_\nu(t|T^*) = \frac{\lambda_1(t) P(E_{\nu-1}^{t_o,t})}{\sum_{i=\nu}^{\infty} P(E_i^{t_o,T})} \qquad (2.3.12)$$

APPENDIX

In this part of the paper the proofs of the previous theorems will be given.

Proof of Theorem 2.

First we have

$$\{\omega;\ \tau_\nu \leq t\} = \bigcup_{i=\nu}^{\infty} \{\omega;\ \tau_i \leq t < \tau_{i+1}\}$$

so that on the basis of definition of the events $E_\nu^{t_o,t}$, the following is valid

$$\{\omega; \tau_\nu \leq t\} = \bigcup_{i=\nu}^{\infty} E_i^{t_o,t}$$

Finally, by virtue of the first of relation (2.2.7) we have

$$P\{\omega; \tau_\nu \leq t\} = P\left(\bigcup_{i=\nu}^{\infty} E_i^{t_o,t}\right) = \sum_{i=\nu}^{\infty} P(E_i^{t_o,t})$$

and the assertion follows.

Proof of Theorem 3.

It is obvious that a) and b) represent sufficient conditions for existence of derivatives of the function $A_\nu(t)$. In order to prove the theorem, consider the distribution function (2.3.5). It is not difficult to see that the following is valid:

$$A_\nu(t+\Delta t) = \sum_{i=\nu}^{\infty} P(E_i^{t,t+\Delta t})$$

Since

$$E_i^{t_o,t+\Delta t} = \bigcup_{\tau=0}^{i} (E_{i-\tau}^{t_o,t} \cap E_\tau^{t,t+\Delta t})$$

then the following is valid:

$$A_\nu(t+\Delta t) = \sum_{i=\nu}^{\infty} \sum_{\tau=0}^{i} P(E_{i-\tau}^{t_o,t} \cap E_\tau^{t,t+\Delta t}) = \sum_{i=\nu}^{\infty} P(E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) +$$

$$+ \sum_{i=\nu}^{\infty} P(E_{i-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + \sum_{i=\nu}^{\infty} \sum_{\tau=2}^{i} P(E_{i-\tau}^{t_o,t} \cap E_\tau^{t,t+\Delta t})$$

On the basis of the following inequality,

$$\sum_{i=\nu}^{\infty} \sum_{\tau=2}^{i} P(E_{i-\tau}^{t_o,t} \cap E_\tau^{t,t+\Delta t}) = \sum_{i=\nu}^{\infty} \{P(E_{i-2}^{t_o,t} \cap E_2^{t,t+\Delta t}) + \dots$$

$$+ P(E_o^{t_o,t} \cap E_i^{t,t+\Delta t})\} \le \sum_{\tau=2}^{\infty} P(E_\tau^{t,t+\Delta t})$$

Therefore, on the basis of condition a) of the theorem, we have

$$A_\nu(t+\Delta t) = \sum_{i=\nu}^{\infty} P(E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) +$$

$$+ \sum_{i=\nu}^{\infty} P(E_{i-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

and consequently,

$$\frac{\partial A_\nu(t)}{\partial t}\,\Delta t = \sum_{i=\nu}^{\infty} \{P(E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) - P(E_i^{t_o,t})\} +$$

$$+ \sum_{i=\nu}^{\infty} P(E_{i-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

On the basis of the following equality,

$$\text{if } A \subseteq B \text{ , then } P(B-A) = P(B) - P(A) \text{ ,} \tag{2.3.13}$$

we have

$$\frac{\partial A_\nu(t)}{\partial t}\,\Delta t = - \sum_{i=\nu}^{\infty} P(E_i^{t_o,t} - E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) +$$

$$+ \sum_{i=\nu}^{\infty} P(E_{i-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

so that by virtue of the set relation

$$A-B = A \cap B^c \tag{2.3.14}$$

where B^c means the complement of the set B, the following equality is valid:

$$P(E_i^{t_o,t} - E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) = P(E_i^{t_o,t} \cap (E_i^{t_o,t} \cap E_o^{t,t+\Delta t})^c) \text{ .}$$

By virtue of de Morgan's laws (see (1) p. 17 or (13), p. 10), it follows that

$$P(E_i^{t_o,t} - E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) = P(E_i^{t_o,t} \cap (E_o^{t,t+\Delta t})^c)$$

and since

$$(E_o^{t,t+\Delta t})^c = \bigcup_{\tau=1}^{\infty} E_\tau^{t,t+\Delta t} \tag{2.3.15}$$

the following relation holds:

$$P(E_i^{t_o,t} - E_i^{t_o,t} \cap E_o^{t,t+\Delta t}) = P(E_i^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) \quad .$$

Therefore, on the basis of these results the following is obtained:

$$\frac{\partial A_\nu(t)}{\partial t} \Delta t = \sum_{i=\nu}^{\infty} P(E_i^{t_o,t} \cap E_1^{t,t+\Delta t}) +$$

$$+ \sum_{i=\nu}^{\infty} P(E_{i-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

and by virtue of condition b) of the theorem the assertion follows.

Proof of Theorem 4.

The proof of this theorem is very simple; indeed on the basis of the following obvious relation

$$\{\omega;\ \tau_\nu \leq t,\ \tau_\nu \leq T\} = \{\omega;\ \tau_\nu \leq t\}$$

and Theorem 2, we have

$$P\{\omega; \tau_\nu \leq t, \tau_\nu \leq T\} = \sum_{i=\nu}^{\infty} P(E_i^{t_o,t})$$

or

$$P\{\omega; \tau_\nu \leq t \mid \tau_\nu \leq T\} = \frac{\sum_{i=\nu}^{\infty} P(E_i^{t_o,t})}{P\{\omega; \tau_\nu \leq T\}}$$

from which the proof of the theorem follows.

4. CALCULATION PROBABILITIES OF THE EVENTS $E_\nu^{t_o,t}$

1^o. We have seen from the previous exposition that several very important characteristics of the stochastic process $\pi(t,\omega)$ are closely related to the probabilities of the events $P(E_\nu^{t_o,t})$. More precisely, the distribution functions of the two stochastic processes may be expressed as sums of these probabilities.

Therefore, an effective obtaining of one-dimensional distribution functions of the processes η_t and τ_ν, depends on our capability to calculate probabilities $P(E_\nu^{t_o,t})$ for every $\nu = 0, 1, 2, \ldots$ and $t \geqslant t_o$. On the basis of conditions a) and b) of Theorem 3, this calculation can be done. To accomplish this objective it is necessary to prove the following theorem:

Theorem 5.

Assuming that conditions a) and b) of Theorem 3 are satisfied, then the probabilities $P(E_\nu^{t_o,t})$ for every $\nu = 0, 1, 2, \ldots$ are solutions of the following system of differential equations:

$$\frac{\partial P(E_\nu^{t_o,t})}{\partial t} = -\lambda_1(t)[P(E_\nu^{t_o,t}) - P(E_{\nu-1}^{t_o,t})] \qquad (2.4.1)$$

To obtain a solution of the system (2.4.1) we will use the method of generating functions which has been applied by Khintchin (see (17), p. 18 and p. 23), under the assumption that the following condition is satisfied:

$$\forall t \geqslant t_o \quad P(E_\nu^{t_o,t}) \equiv 0 \quad \text{if} \quad \nu < 0 \quad . \qquad (2.4.2)$$

The general solution of the system (2.4.1) has the following form:

$$P(E^{t_o,t}) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \frac{(\int_{t_o}^{t} \lambda_1(s)ds)^\nu}{\nu!} \qquad (2.4.3)$$

The expression (2.4.3) represents a solution to the system of equations under conditions a) and b) of Theorem 3. If it is assumed that condition a) of Theorem 3 is satisfied and b) is modified in the following manner

$$\lim_{\Delta t \to o} \frac{P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t})}{\Delta t} = \lambda_1(t,\nu-1)$$

then the system (2.4.1) becomes

$$\frac{\partial P(E_\nu^{t_o,t})}{\partial t} = \lambda_1(t,\nu-1)\ P(E_{\nu-1}^{t_o,t}) - \lambda_1(t,\nu)P(E_\nu^{t_o,t}) \qquad (2.4.4)$$

The particular important case is if the function λ_1 does not depend on t but on ν only, then (2.4.4) becomes

$$\frac{\partial P(E_\nu^{t_o,t})}{\partial t} = \lambda_1(\nu-1)P(E_{\nu-1}^{t_o,t}) - \lambda_1(\nu)P(E_\nu^{t_o,t}) \qquad (2.4.5)$$

In the following, the proofs of the previous assertions will be given.

2^o. Appendix - Let us prove Theorem 5 and other assertions in the previous exposition.

Proof of Theorem 5.

In order to prove the theorem consider the following relation:

$$P(E_\nu^{t_o,t+\Delta t}) = P\Big[\bigcup_{\tau=o}^{\nu} (E_{\nu-\tau}^{t_o,t} \cap E_\tau^{t,t+\Delta t})\Big] =$$

$$= P(E_\nu^{t_o,t} \cap E_o^{t,t+\Delta t}) + P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

Therefore, we have

$$P(E_\nu^{t_o,t+\Delta t}) - P(E_\nu^{t_o,t}) = -P(E_\nu^{t_o,t} - E_\nu^{t_o,t} \cap E_o^{t,t+\Delta t}) +$$

$$+P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) = -P(E_\nu^{t_o,t} \cap (E_\nu^{t_o,t} \cap E_o^{t,t+\Delta t})^c) +$$

$$+ P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

so that the following is obtained:

$$\frac{\partial P(E_\nu^{t_o,t})}{\partial t}\Delta t = -P(E_\nu^{t_o,t} \cap (E_o^{t,t+\Delta t})^c) + P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

Finally, on the basis of (2.3.15) we have

$$\frac{\partial P(E_\nu^{t_o,t})}{\partial t}\Delta t = -P(E_\nu^{t_o,t} \cap E_1^{t,t+\Delta t}) + P(E_{\nu-1}^{t_o,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

which proves the theorem.

Let us now state the procedure to obtain the solution of the system of differential equations (2.4.1). In order to achieve this goal, consider the function

$$\phi(t,z) = \sum_{\nu=0}^{\infty} P(E^{t_o,t})\, z^\nu \tag{2.4.6}$$

and multiply both left and right sides of the system of equations (2.4.1) by z^ν and take the sum of the both sides from $\nu = 0$ up to $\nu = \infty$. Then, obviously, the following is obtained:

$$\sum_{\nu=o}^{\infty} \frac{\partial P(E_\nu^{t_o,t})}{\partial t} z^\nu = -\lambda_1(t) \sum_{\nu=o}^{\infty} \{P(E_\nu^{t_o,t}) - P(E_{\nu=1}^{t_o,t})\} z^\nu =$$

$$= \lambda_1(t)(z-1) \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t}) z^\nu$$

Therefore, on the basis of (2.4.6) the last relation results in the following form

$$\frac{\partial\phi(t,z)}{\partial t} = \lambda_1(t)(z-1)\phi(t,z)$$

or

$$\frac{\partial \ln\phi(t,z)}{\partial t} = \lambda_1(t)\cdot(z-1) \quad ,$$

wherefrom we have

$$\ln\phi(t,z) - \ln\phi(t_o,z) = (z-1)\int_{t_o}^{t} \lambda_1(s)ds \tag{2.4.7}$$

Since, by definition,

$$P(E_\nu^{t_o,t_o}) = \begin{cases} 0 \text{ if } \nu < 0 \\ \\ 1 \text{ if } \nu = 0 \end{cases}$$

then the following is valid

$$\phi(t_o,z) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_o,t_o})\, z^\nu = 1 \quad ,$$

so that (2.4.7) becomes

$$\ln\phi(t,z) = (z-1)\,\Lambda_1\,(t_o,t) \quad .$$

Therefore, on the basis of (2.4.6) we have

$$\phi(t,z) = e^{(z-1)\Lambda_1(t_o,t)} =$$

$$= e^{z\Lambda_1(t_o,t)} \cdot e^{-\Lambda_1(t_o,t)} = \sum_{\nu=0}^{\infty} e^{-\Lambda_1(t_o,t)} \frac{[\Lambda_1(t_o,t)]^\nu}{\nu!} z^\nu$$

i.e.,

$$\sum_{\nu=0}^{\infty} P(E_\nu^{t_o,t}) z^\nu = \sum_{\nu=0}^{\infty} e^{-\Lambda_1(t_o,t)} \frac{[\Lambda_1(t_o,t)]^\nu}{\nu!} z$$

and finally

$$P(E_\nu^{t_o,t}) = e^{-\Lambda_1(t_o,t)} \frac{[\Lambda_1(t_o,t)]^\nu}{\nu!}$$

which proves the theorem.

The results (2.4.3) represent the most general expression for the probability of the events $E_\nu^{t_o,t}$ $\nu = 0, 1, 2, \ldots$ $t \geqslant t_o$, under conditions a) and b) of Theorem 5. Obviously, these probabilities depend on an unknown function $\lambda_1(t)$, therefore it is important to possess a method for its evaluation.

As will be seen in the following section, in some particular cases important for practical application function $\lambda_1(t)$ can be obtained immediately. In any case, this function or its integral $\Lambda_1(t_o,t)$ must be obtained either from the properties of the rainfall or experimental data, or it has to be given.

5. DISCUSSION AND APPLICATIONS

In the previous section, an analytic expression for the probabilities $P(E_\nu^{t_o,t})$, for every $\nu = 0, 1, 2,\ldots$ and $t \geqslant t_o$, has been obtained. This result is very important since it can be seen from (2.3.3) and (2.3.5) that the one-dimensional distribution functions of the processes τ_ν $\nu = 1, 2,\ldots$ and η_t $t \geqslant t_o$, can be expressed over these probabilities. Only the question of how the function $\lambda_1(t)$ can be effectively obtained remains open.

In order to contribute to the solution of this problem, it is necessary, besides the pure probabilistic definition of the function $\lambda_1(t)$ given by the limit

$$\lim_{\Delta t \to 0} \frac{P(E_1^{t,t+\Delta t} \mid E_{\nu-1}^{t_o,t})}{\Delta t} = \lambda_1(t) \tag{2.5.1}$$

to possess its phenomenological interpretation. Toward this end, consider first the stochastic process η_t. Since by (2.3.2)

$$P_\nu(t) = P(E_\nu^{t_o,t})$$

then on the basis of (2.4.3) the following is valid

$$P_\nu(t) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \frac{\left(\int_{t_o}^{t} \lambda_1(s)ds\right)^\nu}{\nu!}$$

where, as it has been seen, $P_\nu(t)$ represents the probability that exactly ν complete storm periods will occur in the interval of time (t_o,t), for every $t \varepsilon T^*$. The expected (average) number of storm periods during this interval is given by the function

$$E(n_t) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \sum_{\nu=1}^{\infty} \nu \cdot \frac{(\int_{t_o}^{t} \lambda_1(s)ds)^\nu}{\nu!} =$$

$$= e^{-\int_{t_o}^{t} \lambda_1(s)ds} \sum_{\nu=1}^{\infty} \frac{\int_{t_o}^{t} \lambda_1(s)ds)^\nu}{(\nu-1)!} =$$

$$= e^{-\int_{t_o}^{t} \lambda_1(s)ds} (\int_{t_o}^{t} \lambda_1(s)ds) \sum_{\nu=1}^{\infty} \frac{(\int_{t_o}^{t} \lambda_1(s)ds)^\nu}{\nu!} = \int_{t_o}^{t} \lambda_1(s)ds$$

Therefore, the average number of storm periods in the interval of time (t_o,t) is given by the integral

$$E(n_t) = \int_{t_o}^{t} \lambda_1(s)\, ds \quad . \tag{2.5.2}$$

On the basis of this, integrand $\lambda_1(s)$ represents some kind of intensity measure of storms.

Obviously, if the function $\lambda_1(t)$ is larger then the average number of storms is larger. Particularly, if in some subinterval (t_1,t_2) of T^* values of $\lambda_1(t)$ are larger than for instance in (t_o,t_1), a larger number of storms can be expected in (t_1,t_2), even if the following equality is valid

$$(t_1 - t_o) = (t_2 - t_1) \quad .$$

Consider now the distribution function $A_\nu(t)$ of the random variable τ_ν, $\nu = 1, 2, \ldots$. On the basis of (2.3.5) and (2.4.3), the following relation is valid

$$A_\nu(t) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \sum_{j=\nu}^{\infty} \frac{1}{\Gamma(j+1)} \left(\int_{t_o}^{t} \lambda_1(s)ds\right)^j \qquad (2.5.3)$$

Since (2.5.3) could be written in the following manner,

$$A_\nu(t) = e^{-\int_{t_o}^{t} \lambda_1(s)ds} \left(e^{-\int_{t_o}^{t} \lambda_1(s)ds} - \sum_{j=o}^{\nu-1} \frac{1}{\Gamma(j+1)} \left(\int_{t_o}^{t} \lambda_1(s)ds\right)^j\right) =$$

$$= 1 - e^{-\int_{t_o}^{t} \lambda_1(s)ds} \sum_{j=o}^{\nu-1} \frac{1}{\Gamma(j+1)} \left(\int_{t_o}^{t} \lambda_1(s)ds\right)^j$$

we have

$$A_1(t) = 1 - e^{-\int_{t_o}^{t} \lambda_1(s)ds}$$

$$A_2(t) = 1 - e^{-\int_{t_o}^{t} \lambda_1(s)ds} - e^{-\int_{t_o}^{t} \lambda_1(s)ds} \int_{t_o}^{t} \lambda_1(s)ds$$

etc.

By virtue of Theorem 3, the corresponding density function $a_\nu(t)$ of distribution function $A_\nu(t)$ has the following shape:

$$a_\nu(t) = \frac{\lambda_1(t)}{\Gamma(\nu)} e^{-\int_{t_o}^{t} \lambda_1(s)ds} \left(\int_{t_o}^{t} \lambda_1(s)ds\right)^{\nu-1} \qquad (2.5.4)$$

Therefore, for $\nu = 1$ and $\nu = 2$ we have,

$$a_1(t) = \lambda_1(t)\, e^{-\int_{t_o}^{t} \lambda_1(s)ds}$$

$$a_2(t) = \lambda_1(t)\, e^{-\int_{t_o}^{t} \lambda_1(s)ds} \int_{t_o}^{t} \lambda_1(s)ds$$

etc.

The mathematical expectation of the random variable τ_ν for $\nu = 1, 2, \ldots$ is given by the following expression:

$$E(\tau_\nu) = \frac{1}{\Gamma(\nu)} \int_{t_o}^{\infty} t\lambda_1(t) e^{-\int_{t_o}^{t} \lambda_1(s)ds} \left(\int_{t_o}^{t} \lambda_1(s)ds\right)^{\nu-1} dt \quad (2.5.5)$$

Formula (2.5.5) represents the arithmetic mean of the upper bound of the ν-th storm period. In the following, the difference

$$E(\tau_\nu) - t_o \qquad \nu = 1, 2, \ldots$$

represents the minimal average time during which exactly ν storm periods will occur.

It is possible to give another interpretation of the difference (2.5.4); it is the average total time elapsed up to the end of the ν-th storm period.

Let us now return to the problem of evaluation of the function

$$\Lambda_1(t_o,t) = \int_{t_o}^{t} \lambda_1(s)\, ds \quad . \qquad (2.5.6)$$

On the basis of definition $\lambda_1(t) \geqslant 0$, therefore, integral (2.5.6) represents a monotonous nondecreasing t function. Further, since integral (2.5.5) represents the average number of storm periods in the interval of time (t_0, t) , the integrand $\lambda_1(t)$ represents some kind of intensity measure of storm periods.

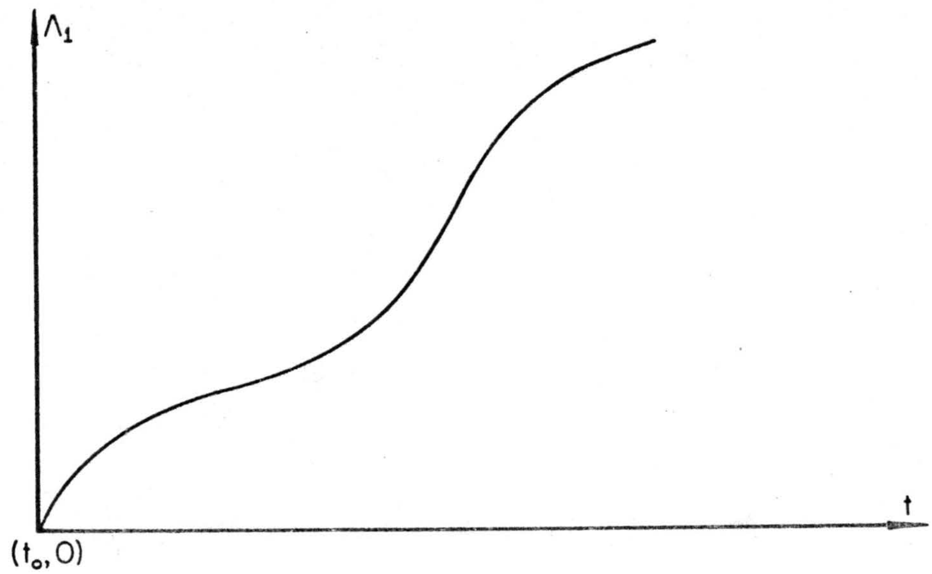

Fig. 9 Graphical presentation of the function $\Lambda_1(t_0, t)$

Let us now consider a particular case where the function $\lambda_1(t)$ can be easily evaluated and which is of a great importance for practical applications. Toward this end, consider relation (2.5.1); as is seen, it expresses the stipulation that the probability of belonging to some interval $(t, t+\Delta t)$ of the termination instant of a storm period does not depend on ν , i.e., on the number of storm periods up to time t , but on t and Δt only. The immediate consequence of this hypothesis is the nonnegative function $\lambda_1(t)$, which represents a kind of measure of storm period intensity.

With respect to the seasonal variation, it is realistic to expect that the function $\lambda_1(t)$ is a periodic function. For a temperate zone, the corresponding period is usually one year; generally speaking for different climatic zones the function $\lambda_1(t)$ has different shapes.

As an example, consider a tropical zone. In this case, it is realistic to assume that the function $\lambda_1(t)$ can assume the two different values $\lambda_{1.1}$ and $\lambda_{1.2}$ only, for the wet and dry season respectively, i.e.,

$$\lambda_1(t) = \begin{cases} \lambda_{1.1} & \text{if } t \text{ belongs to the wet season} \\ \lambda_{1.2} & \text{if } t \text{ belongs to the dry season} \end{cases}$$

(see Fig. 10a)

To justify this assumption, suppose that the instant of time t_o represents the beginning of the wet season; then the following integral

$$\Lambda_1(t_o,t) = \int_{t_o}^{t} \lambda_1(s)ds$$

where t, $(t \geq t_o)$ belongs to the wet season as well, representing the average number of storms in (t_o,t) is a linear t function, i.e., if t increases, then the average number of storms increases as a linear function,

$$\Lambda_1(t_o,t) = at + b \quad .$$

Since, $\Lambda_1(t_o,t_o) = 0$ it follows that $b = 0$, and, by virtue of the following relations

$$\frac{d\Lambda_1}{dt} = \lambda_1(t), \quad \frac{d\Lambda_1}{dt} = a$$

where a = constant, it follows that $\lambda_1(t)$ is a constant as well.

If it is assumed that $t_o = 0$ and T^* represents the wet season, then on the basis of (2.3.2) and (2.4.3) probability of ν storm periods occurring in the interval of time $T^* = (0,T)$ is

$$P_\nu(T) = e^{-\lambda_{11}T} \frac{(\lambda_{11}T)^\nu}{\nu!} \tag{2.5.7}$$

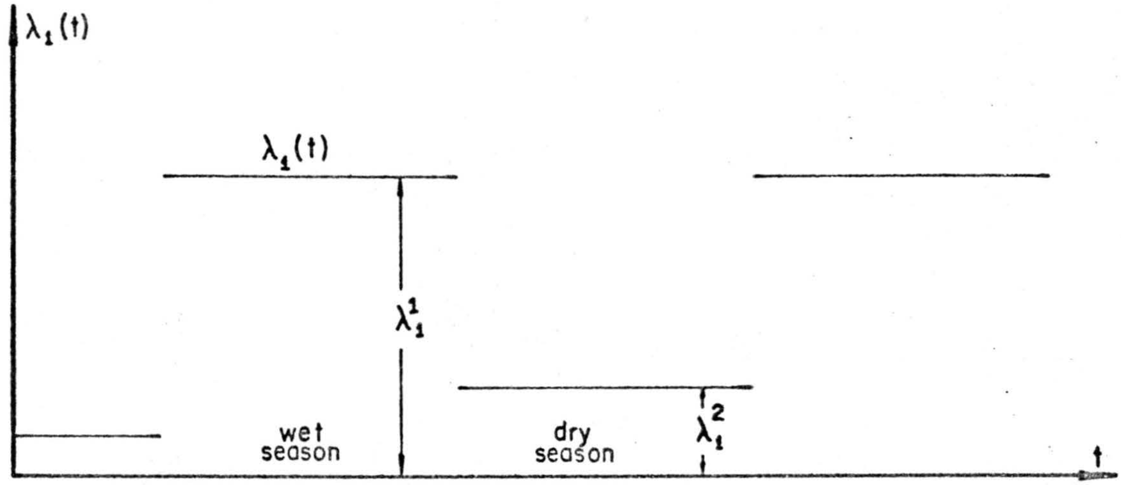

Fig. 10a A graphic illustration of intensity function $\lambda_1(t)$ for a tropical zone

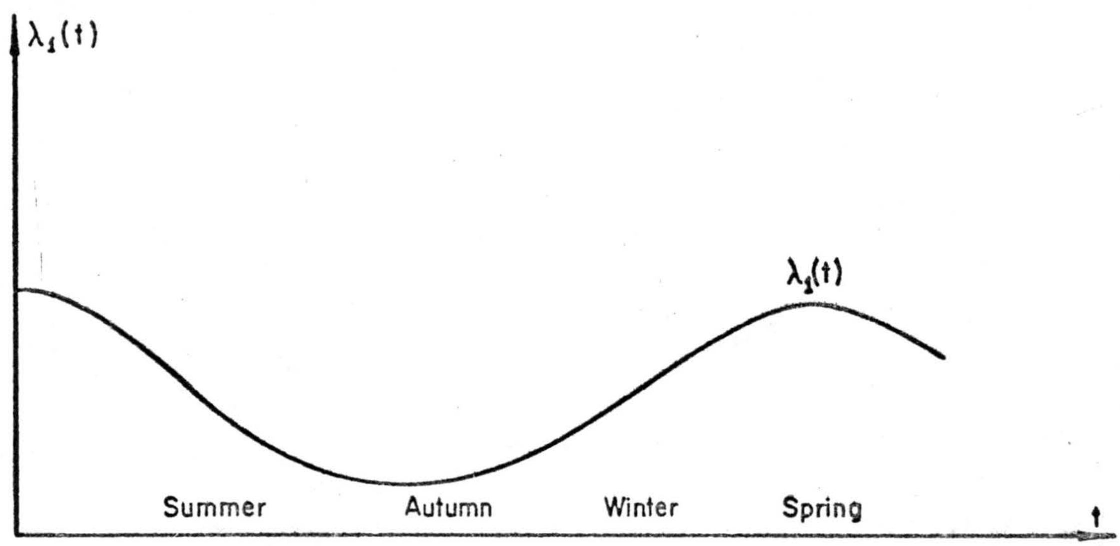

Fig. 10b Graphic illustration of intensity function $\lambda_1(t)$ for a temperate zone

Therefore, the number of storm periods in the wet season is distributed according to Poisson law. The corresponding distribution function $F(x|t)$ given by (2.3.3) is equal to the following expression

$$F(x|t) = e^{-\lambda_{11}\cdot T} \sum_{\nu=0}^{[x]} \frac{(\lambda_{11}T)^{\nu}}{\nu!} \tag{2.5.8}$$

Since the random variable τ_ν for $\nu = 1, 2,\ldots$ can take any value from the interval of time (t_o,∞) , it is assumed that $T = \infty$. Therefore, assuming λ_1 = cons. on the basis of (2.5.4), the following is obtained:

$$A_\nu(t) = \frac{\lambda_1^{\nu}}{\Gamma(\nu)} e^{-\lambda_1 t} \; t^{\nu-1} \tag{2.5.9}$$

where $t_o=0$. Obviously, (2.5.9) represents a Gamma density function with parameter λ_1 . As is known,

$$E(\tau_\nu) = \frac{\nu}{\lambda_1} \qquad D(\tau_\nu) = \frac{\nu}{\lambda_1^2} \; . \tag{2.5.10}$$

Finally, on the basis of (2.4.3), the conditional density function $f_\nu(t|T^*)$ becomes

$$f_\nu(t|T^*) = \frac{\lambda_1(t)e^{\int_t^T \lambda_1(s)ds} \; (\int_{t_o}^{t} \lambda_1(s)ds)^{\nu-1}}{\Gamma(\nu) \sum_{i=\nu}^{\infty} \frac{(\int_{t_o}^{T}\lambda_1(s)ds)^i}{i!}} \tag{2.5.11}$$

If a tropical zone is of interest (assume, for instance, that the interval of time (t_o,T) represents a wet season) then $\lambda_1(t)=\lambda_{11}$, and (2.5.11) becomes

$$f_\nu(t|T^*) = \frac{\lambda_{11}^{\nu}}{\Gamma(\nu)} \frac{(t-t_o)^{\nu-1} e^{\lambda_{11}(T-t)}}{\sum_{i=\nu}^{\infty} \frac{[(T-t_o)\lambda_{11}]^i}{i!}} \tag{2.5.2}$$

or, setting $t_o=0$,

$$f_\nu(t|T^*) = \frac{\lambda_1^{\nu}}{\Gamma(\nu)} \frac{t^{\nu-1} e^{\lambda_{11}(T-t)}}{\sum_{i=\nu}^{\infty} \frac{(\lambda_{11}T)^i}{i!}} .$$

6. PRECIPITATION AND MARKOV CHAIN

1. Up to the present time, the precipitation phenomenon and its most important characteristics have been considered in a given interval of time (t_o,T) independent of and isolated from the previous behavior of the phenomenon considered. For example, does the number of storm periods in a previous interval of time influence the number of storms in the interval (t_o,T)?

According to experience, there are cases where this relationship can be assumed as justified. For example, the number of storms in the springtime influences the number of storms in the summer time. But the question, is there correlation between number of storm periods in two successive years, is discussible, according to the opinion of some hydrologists and metereologists.

The purpose of this section is not to discuss in which cases there exists such a stochastic relationship and in which there is none. The response to such a question can be obtained by studying corresponding data only. In the following, we are going to present some methods by the help of which this problem could be studied.

Consider a sequence of successive time intervals

$$(T_o, T_1),\ (T_1, T_2),\ \ldots\ (T_{\nu-1}, T_\nu),\ \ldots$$

and denote by

$$\eta_1,\qquad \eta_2,\qquad \ldots\ \eta_\nu,\qquad \ldots \tag{2.6.1}$$

the corresponding number of storm periods respectively; then η_ν is a random variable for every $\nu=1,2,\ldots$ such that

$$P\{\omega; \eta_\nu = i_\nu\} = P(E_{i_\nu}^{T_{\nu-1}, T_\nu}) \tag{2.6.2}$$

Of particular interest is the following question: does the knowledge of the number of storm periods in the past and the present time give to us some information concerning the future behavior of the phenomenon considered? In other words, if it is known that

$$\eta_1 = i_1,\quad \eta_2 = i_2,\ \ldots\ \eta_\nu = i_\nu$$

what could be said about future $\eta_{\nu+1}$, i.e., what about the following conditional probability

$$P\{\omega; \eta_{\nu+1} = i_{\nu+1} \mid \eta_1 = i_1,\ \ldots\ \eta_\nu = i_\nu\} \tag{2.6.3}$$

On the basis of

$$\{\omega; \eta_k = i_k\} = E_{i_k}^{T_{k-1}, T_k},$$

conditional probability (2.6.3) becomes

$$P\{\omega; \eta_{\nu+1}=i_{\nu+1} \mid \eta_k=i_k \quad k=\overline{1,\nu}\} = \frac{P\bigcap_{\tau=1}^{\nu+1}(E_{i_\tau}^{T_{\tau-1},T_\tau})}{P\bigcap_{\tau=1}^{\nu}(E_{i_\tau}^{T_{\tau-1},T_\tau})}$$

$$= P(E_{i_{\nu+1}}^{T_\nu,T_{\nu+1}} \mid \bigcap_{\tau=1}^{\nu} E_i^{T_{\tau+1},T_\tau}) \qquad (2.6.4)$$

In the simplest case, if it is assumed that (2.6.4) represents a sequence of independent random variables, then (2.6.3) becomes

$$P\{\omega; \eta_{\nu+1} = i_{\nu+1}\} = P(E_{i_{\nu+1}}^{T_\nu,T_{\nu+1}}) \quad . \qquad (2.6.5)$$

The other possibility is to assume that the future state depends on the present state only, so that (2.6.3) can be written in the following manner:

$$P\{\omega; \eta_{\nu+1}=i_{\nu+1} \mid \eta_\nu=i_\nu\} = P(E_{i_{\nu+1}}^{T_\nu,T_{\nu+1}} \mid E_{i_\nu}^{T_\nu-1,T_\nu}) \quad . (2.6.6)$$

Therefore, under this condition, the sequence (2.6.1) represents a Markov chain with (2.6.6) as transition probabilities. If (2.6.6) is denoted by $P_{i_\nu,i_{\nu+1}}(\nu, \nu+1)$, i.e.,

$$P_{i_\nu,i_{\nu+1}}(\nu,\nu+1) = P(E_{i_{\nu+1}}^{T_\nu,T_{\nu+1}} \mid E_{i_\nu}^{T_\nu-1,T_\nu}) \qquad (2.6.7)$$

then obviously the following is valid:

$$\sum_{i_\nu=0}^{\infty} P_{i_\nu,i_{\nu+1}}(\nu, \nu+1) = 1 \quad \forall i_\nu = 0,1,2,\ldots \quad \nu=0,1,2,\ldots$$

If each time interval (T_{k-1},T_k) , $k=0,1,2,\ldots$ represents a year, then it seems realistic to suppose that transition probabilities are independent on ν , i.e., (2.6.1) represents a Markov chain with stationary transition probabilities. Then (2.6.7) can be written in the following manner:

$$P_{ij} = P(E_j^{T_\nu,T_{\nu+1}} | E_i^{T_{\nu-1},T_\nu}) \quad . \tag{2.6.8}$$

In order to effectively obtain transition probabilities (2.6.8), consider the expression

$$P_{ij}(t) = P(E_j^{T_\nu,t} \mid E_i^{T_{\nu-1},T_\nu}) \tag{2.6.9}$$

where $T_\nu < t \leq T_{\nu+1}$; then, if the condition is satisfied that

$$\lim_{\Delta t \to o} \frac{P(E_1^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu} \cap E_j^{T_\nu,t})}{\Delta t} = \lambda_i(t) \tag{2.6.10}$$

the system (2.6.11) of differential equations is obtained:

$$\frac{\partial P(E_j^{T_\nu,t} \mid E_i^{T_{\nu-1},T_\nu})}{\partial t} = \lambda_i(t)[P(E_{j-1}^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) - P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu})] \quad . \tag{2.6.11}$$

The solution of this system of equations can be obtained in a manner similar to the solution of the system (2.4.1), so applying the same method for solution, the following is obtained:

$$P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) = e^{-\Lambda_i(T_\nu,t)} \frac{[\Lambda_i(T_\nu,t)]^j}{j!} \tag{2.6.12}$$

where

$$\Lambda_i(T_\nu,t) = \int_{T_\nu}^{t} \lambda_i(s)ds \quad . \tag{2.6.13}$$

Let us discuss now the condition (2.6.10) which represents the basic hypothesis under which results (2.6.12) have been obtained. For sufficiently small Δt , (2.6.10) can be written as

$$P(E_1^{t,t+\Delta t} \mid E_i^{T_{\nu-1},T_\nu} \cap E_j^{T_\nu,t}) \approx \lambda_i(t)\Delta t \quad . \tag{2.6.14}$$

Phenomenologically speaking, the last relation expresses the following: the expression on the left side of (2.6.14) represents the conditional probability that a termination instant will lie between t and $t+\Delta t$, under the condition that exactly i storms have occurred in the previous period $(T_{\nu-1},T_\nu)$ and exactly j storms from T_ν up to t . In other words, this is the probability that the end of the $(j+1)$ storm period will occur somewhere in $(t,t+\Delta t)$ under the condition that $\eta_\nu=i$ and $E_j^{T_\nu,t}$.

The right side of (2.6.14) is a function of t , i and Δt only, therefore under condition (2.6.10) or (2.6.14) it follows that the probability that a termination instant will belong to a time interval $(t,t+\Delta t)$ depends on the number of storms i in the previous period $(T_{\nu-1},T_\nu)$, on t but not on the number of j storms in (T_ν,t) .

The more general hypothesis represents the assumption that the function $\lambda_i(t)$ depends on j as well, i.e., of the number of storms in (T,t):

$$\lim_{\Delta t\to 0} \frac{P(E_1^{t,t+\Delta t} \mid E_i^{T_{\nu-1},T_\nu} \cap E_j^{T_\nu,t})}{\Delta t} = \lambda_{ij}(t) \quad . \tag{2.6.15}$$

Under this condition, the system (2.6.11) becomes

$$\frac{\partial P(E_j^{T_\nu,t} \cap E_i^{T_{\nu-1},T})}{\partial t} =$$

$$= \lambda_{i,j-1}(t)P(E_{j-1}^{T_\nu,t} \mid E_i^{T_{\nu-1},T_\nu}) - \lambda_{i,j}(t)P(E_j^{T_\nu,t} \mid E_i^{T_{\nu-1},T_\nu}) \; .$$

(2.6.16)

On the basis of definition of (2.6.9), the following relation obviously is valid

$$\sum_{j=0}^{\infty} P_{ij}(t) = 1 \quad t\varepsilon(T_\nu,T_{\nu+1}) \; .$$

Therefore, on the basis of (2.6.12) mathematical expectation

$$m_i(t) = \sum_{j=1}^{\infty} jP_{ij}(t) =$$

$$= e^{-\Lambda_i(T_\nu,t)} \sum_{j=1}^{\infty} j \frac{\Lambda_i(T_\nu,t)^j}{j!} = \Lambda_i(T_\nu,t)$$

i.e.,

$$m_i(t) = \int_{T_\nu}^{t} \lambda_i(s)ds \; . \tag{2.6.17}$$

In this manner, the phenomenologic interpretation of the integral (2.6.13) has been obtained and represents the average number of storm periods in some interval (T_ν,t) under condition that $\eta_\nu=i$. Therefore, the expected number of storm periods in $(T_\nu,T_{\nu+1})$ under condition that exactly i have occurred in $(T_{\nu-1},T_\nu)$ is equal to $m_i(T_{\nu+1})$, i.e.,

$$m_i(T_{\nu+1}) = E(\eta_{\nu+1} \mid \eta_\nu=i) \; .$$

Obviously, the theory of Markov chains can be applied in the whole to the problem of investigation of properties of the sequence (2.6.1). Since, with respect to the extent of all these problems, such an investigation represents a separate study, we shall restrict consideration to the problems which have been studied in this section.

2. Appendix - Let us prove now that under condition (2.6.10), the function (2.6.9) satisfies the system of differential equations (2.6.11). In order to achieve this goal, consider the following expression:

$$P(E_j^{T_\nu, t+\Delta t} \mid E_i^{T_{\nu-1}, T_\nu}) =$$

$$= \frac{1}{P(E_i^{T_{\nu-1}, T_\nu})} P(E_j^{T_\nu, t+\Delta t} \cap E_i^{T_{\nu-1}, T_\nu}) =$$

$$= \frac{1}{P(E_i^{T_{\nu-1}, T_\nu})} \sum_{\tau=0}^{j} P(E_{j-\tau}^{T_\nu, t} \cap E_\tau^{t, t+\Delta t} \cap E_i^{T_{\nu-1}, T_\nu}) =$$

$$= \frac{1}{P(E_i^{T_{\nu-1}, T_\nu})} \sum_{\tau=0}^{j} P(E_{j-\tau}^{T_\nu, t} \cap E_\tau^{t, t+\Delta t} \cap E_i^{T_{\nu-1}, T_\nu})$$

On the basis of the inequality

$$\sum_{\tau=2}^{j} P(E_{j-\tau}^{T_\nu, t} \cap E_\tau^{t, t+\Delta t} \cap E_i^{T_{\nu-1}, T_\nu}) \leq \sum_{\tau=2}^{j} P(E_\tau^{t, t+\Delta t}) ,$$

it is not difficult to see that for sufficiently small Δt the following relation is valid:

$$P(E_j^{T_\nu,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) = P(E_j^{T_\nu,t} \cap E_o^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) +$$

$$+ P(E_{j-1}^{T_\nu,t} \cap E_1^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) + o(\Delta t) \quad .$$

Therefore, we have

$$P(E_j^{T_\nu,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) - P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) =$$

$$= - P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) + P(E_j^{T_\nu,t} \cap E_o^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) +$$

$$+ P(E_{j-1}^{T_\nu,t} \cap E_1^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) + 0(\Delta t) \quad .$$

Since

$$-P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) + P(E_j^{T_\nu,t} \cap E_i^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) =$$

$$= - P(E_j^{T_\nu,t} - E_j^{T_\nu,t} \cap E_o^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) =$$

$$= - P(E_j^{T_\nu,t} \cap (E_j^{T_\nu,t} \quad E_o^{t,t+\Delta t})^c | E_i^{T_{\nu-1},T_\nu}) =$$

$$= - P(E_j^{T_\nu,t} \cap (E_o^{t,t+\Delta t})^c | E_o^{T_{\nu-1},T_\nu})$$

and

$$(E_o^{t,t+\Delta t})^c = \bigcup_{\tau=1}^{\infty} E_\tau^{t,t+\Delta t} \quad ,$$

the following is obtained:

$$P(E_j^{T_\nu,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) - P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu}) =$$

$$\frac{\partial P(E_j^{T_\nu,t} | E_i^{T_{\nu-1},T_\nu})}{\partial t} \Delta t = - P(E_j^{T_\nu,t} \cap E_1^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) +$$

$$+ P(E_{j-1}^{T_\nu,t} \cap E_1^{t,t+\Delta t} | E_i^{T_{\nu-1},T_\nu}) + 0(\Delta t) \quad . \qquad (2.6.18)$$

On the basis of this relation, the following conclusions can be drawn. If it is assumed that the condition (2.6.10) is valid, then apparently the system of equations (2.6.11) follows immediately. On the other hand, if instead of (2.6.10) we suppose that the condition (2.6.15) is truthful, then the probability

$$P_{ij}(t) = P(E_j^{T_\nu,t} \mid E_i^{T_{\nu-1},T_\nu})$$

is a solution of the system of equations (2.6.16).

Note:

With respect to the nature of the rainfall phenomenon, obviously the relationship considered among the sequence of time interval $(T_{\nu-1},T_\nu)$ where $\nu=1,2,\ldots$, depends on these intervals. In other words, the conditional probability (2.6.4) up to a certain point depends on these intervals.

The hypothesis established that the sequence (2.6.4) represents a Markov chain with stationary transition probabilities (homogenous Markov chains) is very realistic under the assumption that $(T_{\nu-1},T_\nu)$ represents a year for every $\nu=1,2,\ldots$. We arrive at the Markov chain of order k, if given fixed k, for all ν and for all possible values of the variables η_ν $(\nu=1,2,\ldots)$ it is true that

$$P\{\omega;\ \eta_{\nu+1}=i_{\nu+1} \mid \eta_1=i_1,\ \eta_2=i_2,\ \ldots\ \eta_\nu=i_\nu\}=$$

$$P\{\omega;\ \eta_{\nu+1}=i_{\nu+1} \mid \eta_{\nu-\tau+1}=i_{\nu-k+1},\ \ldots\ \eta_\nu=i_\nu\}$$

etc. All these questions, for example, are of the greatest interest in water storage problems and evaluation of weather modification attainments.

However, if it has been taken that $(T_{\nu-1}, T_\nu)$ represents smaller periods of time, then, generally speaking, the hypothesis is not true that the sequence of random variables represents a Markov chain with stationary transition probabilities. Therefore, the conditional probability P_{ij} given by the relation (2.6.8) becomes a function on ν , i.e.,

$$P_{ij}(\nu) = P(E_j^{T_\nu, T_{\nu+1}} \mid E_i^{T_{\nu-1}, T_\nu}) \quad .$$

For example, it could be taken that (T_o, T_1) represents the spring time, (T_1, T_2) summertime, (T_2, T_3) autumn, etc. In this case it looks very realistic to take into consideration the possibility that (2.6.1) may represent a Markov chain of a higher order than one.

All these problems, of the greatest importance for applications in hydrologic investigation, will not be considered in this paper. Their consideration can be the subject of a separate study. In this paper, we shall restrict ourselves to the previously stated problems.

Chapter III

1. Introductory Remarks

The two fundamental characteristics of the stochastic process of nondecreasing sample functions $X_t=\pi(t,\omega)$ have been considered in the previous chapter. The first of these characteristics, η_t , giving us information concerning the frequency of storms, represents the number of full storm periods in some interval of time (t_o,t) . As has been shown, the distribution function $F(x|t)$ of the random variable can be written in the form

$$F(x|t) = \sum_{\nu=o}^{[x]} P(E_\nu^{t_o,t}) \quad \forall t \geq t_o$$

where $E_\nu^{t_o,t}$ represents the random event that exactly ν full storm periods will occur in the interval of time (t_o,t).

Another characteristic is the random variable τ_ν , representing the total elapsed time up to the end of the ν-th storm period or the upper bound of the ν-th storm period (see Fig. 3). As has been seen, the corresponding distribution function $A_\nu(t)$ was

$$A_\nu(t) = \sum_{j=\nu}^{\infty} P(E_j^{t_o,t})$$

etc.

Obviously, the random variables η_t and τ_ν, $\nu=1,2,\ldots$ do not give us information concerning the quantitative aspect of the rainfall phenomenon, i.e., we have no idea about amount of precipitation during these storms. In other words, these variables represent some "dynamic" characteristics of the phenomenon considered.

In this chapter, the quantitative aspect of the precipitation phenomenon will be studied. The random variables, such as the total amount of precipitation during exactly ν storm periods, or during one storm period only, etc., will be investigated.

2. STOCHASTIC PROCESSES 1_x and $\dot{X}_\nu$

1. Suppose that at the instant of time t_o, when the observation of the rainfall phenomenon begins, the total amount of precipitation was x_o, if 1_x denotes the maximum number of complete storm periods such that their total amount of precipitation is smaller or equal to $(x-x_o)$, then obviously

$$P\{\omega;\ 1_x = \nu\} = P(G_\nu^{x_o,x}) \tag{3.2.1}$$

where $G_\nu^{x_o,x}$ is given by (2.2.8). In other words, the random variable 1_x represents the number of storms such that the corresponding total amount of precipitation is less or equal to $(x-x_o)$, while total amounts for 1_x+1 storms exceed $(x-x_o)$. Since 1_x represents the random variable for all $x \geq x_o$, we have a family of random variables or a continuous parameter stochastic process

$$\{1_x;\ x > x_o\} \tag{3.2.2}$$

where $1_x = 0,1,2,\ldots$.

We should be very careful with the probability of the value 0 of the random variable 1_x. Namely, the event

$$\{\omega;\ 1_x = 0\} = G_o^{x_o,x} \qquad x > x_o \tag{3.2.3}$$

could mean that t_o belongs to the first storm period and that the following relation is valid:

$$\pi^*(\tau_1) - x_o > x \quad .$$

In other words, (3.2.3) represents the event that the total amount of precipitation during the first storm period either exceeds the value $(x-x_o)$ or there is no precipitation at all in the considered interval of time T^* .

On the basis of (3.2.1), the mathematical expectation of the random variable 1_x is equal to

$$E(1_x) = \sum_{\nu=1}^{\infty} \nu P(G_\nu^{x_o, x}) \tag{3.2.4}$$

and represents an average number of storm periods such that the corresponding total amount of precipitation does not exceed $(x-x_o)$, while the amount of precipitation for

$$\sum_{\nu=1}^{\infty} \nu P(G_\nu^{x_o, x}) + 1$$

storm periods exceeds $(x-x_o)$. Finally, if $P(u|x)$ denotes the distribution function of the random variable η_x , i.e.,

$$P\{\omega; \eta_x \leq u\} = P(u|x) \qquad u \geq 0$$

then obviously the following is valid:

$$P(u|x) = \sum_{\nu=0}^{[u]} P(G_\nu^{x_o, x}) \tag{3.2.5}$$

where u denotes the greatest integer not greater than u .

Consider the random variable X_ν , where

$$X_\nu = \pi(\tau_\nu,\omega) \tag{3.2.6}$$

Since X_ν is a random variable for every $\nu=1,2,\ldots$, it means that we have a countable family of random variables or a discrete parameter stochastic process

$$\{X_\nu;\ \nu=1,2,\ldots\} \quad . \tag{3.2.7}$$

On the basis of (3.2.6), X_ν represents the total amount of precipitation during exactly ν storm periods. Obviously, for all $\nu=1,2,\ldots$, the following inequality is valid:

$$X_\nu \leq X_{\nu+1} \qquad \forall\nu= 1,2,\ldots$$

Consider now the event

$$\{\omega;X_\nu \leq x\}$$

i.e., the random event that the total amount of precipitation will be less or equal to x (where $x \geq x_o$); then the following theorem is valid:

Theorem 6.

Let $F_\nu(x)$ denote the distribution of the random variable X_ν , for every $\nu=1,2,\ldots$, i.e.,

$$F_\nu(x) = P\{\omega;\ X_\nu \leq x\}$$

Then

$$F_\nu(x) = \sum_{j=\nu}^{\infty} P(G_j^{x_o,x}) \tag{3.2.8}$$

(For proof of this theorem see Appendix of this section).

Let us denote the corresponding density function of the distribution function (3.2.8) by $f_\nu(x)$, i.e.,

$$f_\nu(x) = \frac{\partial F_\nu(x)}{\partial x} \quad .$$

If the derivative exists, then we have:

Theorem 7.

Assume that the following conditions are satisfied:

a) $$\lim_{\Delta x \to o} \frac{\sum_{\tau=2}^{\infty} P(G_\tau^{x,x+\Delta x})}{\Delta x} = 0$$

b) $$\lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} \mid G_{\nu-1}^{x_o,x})}{\Delta x} = \lambda_2(x)$$

then

$$f_\nu(x) = \lambda_2(x) P(G_{\nu-1}^{x_o,x}) \quad . \tag{3.2.9}$$

(For the proof of the theorem see Appendix.)

Let us discuss now the conditions a) and b) of Theorem 7, which represent the fundamental assumptions for further investigations. According to definition, $G_2^{x,x+\Delta x}$ represents the event (random) that exactly two successive points X_ν and $X_{\nu+1}$ will belong to the interval $(x,x+\Delta x)$ where $\nu = 1, 2, \ldots$. Phenomenologically speaking, $G_2^{x,x+\Delta x}$ represents the event that the total amount of precipitation

X_ν during ν storms and $X_{\nu+1}$ during $(\nu+1)$ storms will lie between x and $x+\Delta x$, i.e.,

$$x<X_\nu \leq x+\Delta x \quad \text{and} \quad x<X_{\nu+1} \leq x+\Delta x \quad (\nu=1,2,\ldots)$$

Obviously, $\bigcup_{\tau=2}^{\infty} G_\tau^{x,x+\Delta x}$ represents the event that at least two of the events will occur in $(x,x+\Delta x)$. Condition a) means that the probability

$$P\left(\bigcup_{\tau=2}^{\infty} G_\tau^{x,x+\Delta x}\right) = \sum_{\tau=2}^{\infty} P(G_\tau^{x,x+\Delta x})$$

when $\Delta x \to 0$ is an infinitesimal of a higher order than Δx. With respect to the nature of the precipitation phenomenon, this condition seems very realistic.

Let us dwell now on the second condition of Theorem 7. The expression

$$P(G_1^{x,x+\Delta x} \mid G_{\nu-1}^{x_o,x}) \tag{3.2.10}$$

represents the conditional probability that the total amount of precipitation of the previous $(\nu-1)$ storms is less or equal to x.

Since, generally speaking, (3.2.10) depends on $x, \Delta x$ and ν, it would be realistic to assume that the function λ_2 depends on x and ν, i.e., to assume that

$$\lambda_2 = \lambda_2(x,\nu-1)$$

Under this hypothesis (3.2.9) becomes

$$f_\nu(x) = \lambda_2(x, \nu-1)\, P(G_{\nu-1}^{x_o,x}) \tag{3.2.11}$$

On the basis of (3.2.5) and (3.2.8), it is easily seen that between distribution functions $P(u|x)$ and $F_\nu(x)$ there exists the relationship

$$P(u|x) = 1 - F_{\nu+1}(x) \quad \text{and} \quad [u] = \nu \quad . \tag{3.2.12}$$

Therefore, it is sufficient to calculate only one of the functions (3.2.5) or (3.2.8); the other follows automatically from the relation (3.2.12).

If the domain of definition of the stochastic process $X_t = \pi(t,\omega)$ is a finite interval of time $T^* = (t_o, T)$, then it is necessary to consider the probability that $X_\nu \leq x$ under the condition that $\tau_\nu \varepsilon T^*$, i.e.,

$$F^*_\nu(x|T^*) = P\{\omega;\ X_\nu \leq x | \tau_\nu \varepsilon T^*\} \tag{3.2.13}$$

where $\nu = 1, 2, \ldots$.

Theorem 8.

For every $\nu = 1, 2, \ldots$ and $x \geqslant x_o$, the following is valid

$$F_\nu^*(x|T^*) = \frac{\sum\limits_{j=\nu}^{\infty} \sum\limits_{i=\nu}^{\infty} P(G_j^{x_o,x} \cap E_i^{t_o,T})}{\sum\limits_{i=\nu}^{\infty} P(E_i^{t_o,T})} \quad . \tag{3.2.14}$$

If $f^*_\nu(x|t)$ denotes the corresponding density function, then we have:

Theorem 9.

If the following conditions are satisfied

a) $$\lim_{\Delta x \to o} \frac{\sum\limits_{\tau=2}^{\infty} P(G_\tau^{x,x+\Delta x})}{\Delta x} = 0$$

b) $$\lim_{\Delta x \to o} \frac{\sum\limits_{i=\nu}^{\infty} P(G_{\nu-1}^{x_o,x} \cap G_1^{x,x+\Delta x} \cap E_i^{t_o,T})}{\Delta x} = \lambda_2(x) \sum_{i=\nu}^{\infty} P(G_{\nu-1}^{x_o,x} \cap E_i^{t_o,t})$$

then

$$f_\nu^*(x|T^*) = \frac{\lambda_2(x) \sum_{i=\nu}^{\infty} P(G_{\nu-1} \cap E_i^{t_o,t})}{\sum_{i=\nu}^{\infty} P(E_i^{t_o,t})} \tag{3.2.15}$$

The function (3.2.15) represents the density function of the random variable X_ν under the condition that at least ν storms have occurred in the interval of time (t_o,T)

2. Appendix - Let us prove the theorems which have been considered in this section.

Proof of Theorem 6.

The proof of this theorem is very similar to that of Theorem 2. Indeed on the basis of definition of $G_\nu^{x_o,x}$ (see (2.2.8)), obviously the following relationship is valid:

$$\{\omega; X_\nu \leq x\} = \sum_{j=\nu}^{\infty} \{\omega; \pi^*(\tau_j) \leq x < \pi^*(\tau_{j+1})\}$$

so that by (2.2.8) we have

$$\{\omega; X_\nu \leq x\} = \bigcup_{j=\nu}^{\infty} G_j^{x_o,x}$$

Finally, on the basis of (2.2.9) the following is obtained:

$$P\{\omega; X_\nu \leq x\} = \sum_{j=\nu}^{\infty} P(G_j^{x_o,x})$$

and the assertion follows.

Proof of Theorem 7.

In order to prove the theorem, consider the next relation

$$F_\nu(x+\Delta x) = \sum_{j=\nu}^{\infty} P(G_j^{x,x+\Delta x}) =$$

$$= \sum_{j=\nu}^{\infty} P \bigcup_{\tau=0}^{j} (G_{j-\tau}^{x_o,x} \cap G_\tau^{x,x+\Delta x}) = \sum_{j=\nu}^{\infty} \sum_{\tau=0}^{j} P(G_{j-\tau}^{x_o,x} \cap G_\tau^{x,x+\Delta x}) =$$

$$= \sum_{j=\nu}^{\infty} P(G_j^{x_0,x} \cap G_0^{x,x+\Delta x}) + \sum_{j=\nu}^{\infty} P(G_{j-1}^{x_0,x} \cap G_1^{x,x+\Delta x}) +$$

$$+ \sum_{j=\nu}^{\infty} \sum_{\tau=2}^{j} P(G_{j-\tau}^{x_0,x} \cap G_{\tau}^{x,x+\Delta x})$$

Since

$$\sum_{j=\nu}^{\infty} \sum_{\tau=2}^{j} P(G_{j-\tau}^{x_0,x} \cap G_{\tau}^{x,x+\Delta x}) \leq \sum_{\tau=2}^{\infty} P(G_{\tau}^{x,x+\Delta x})$$

then on the basis of condition a) of theorem, for sufficiently small Δx , the following is valid:

$$F_\nu(x+\Delta x) = \sum_{j=\nu}^{\infty} P(G_j^{x_0,x} \cap G_0^{x,x+\Delta x}) +$$

$$\sum_{j=\nu}^{\infty} P(G_{j-1}^{x_0,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

Consequently, we have

$$\frac{\partial F_\nu(x)}{\partial x} \Delta x = - \sum_{j=\nu}^{\infty} P(G_j^{x_0,x} - G_j^{x_0,x} \cap G_0^{x,x+\Delta x}) +$$

$$+ \sum_{j=\nu}^{\infty} P(G_{j-1}^{x_0,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

On the basis of (2.3.14) and the following relation,

$$(G^{x,x+\Delta x})^c = \bigcup_{\tau=1}^{\infty} G^{x,x+\Delta x} \tag{3.2.16}$$

we obtain (see the proof of Theorem 3):

$$\frac{\partial F_\nu(x)}{\partial x} \Delta x = \sum_{j=\nu}^{\infty} P(G_j^{x_0,x} \cap G_1^{x,x+\Delta x}) +$$

$$+ \sum_{j=\nu}^{\infty} P(G_{j-1}^{x_0,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x) =$$

$$= P(G_{\nu-1}^{x_0,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

where from condition b) of the theorem the assertion follows.

Proof of Theorem 8.

On the basis of (3.2.13), the following is valid

$$F_\nu^*(x|T^*) = \frac{P\{\omega; X_\nu \le x, \tau_\nu \le T\}}{P\{\tau_\nu \le T\}}$$

Therefore, by virtue of Theorem 2 and 6 we have

$$F_\nu(x|T^*) = \frac{P(\bigcup_{j=\nu}^{\infty} G_j^{x_o,x}) \cap (\bigcup_{i=\nu}^{\infty} E_i^{t_o,T})}{P(\bigcup_{i=\nu}^{\infty} E_i^{t_o,t})} =$$

$$= \frac{P\bigcup_{j=\nu}^{\infty} \bigcup_{i=\nu}^{\infty} (G_j^{x_o,x} \cap E_i^{t_o,T})}{\sum_{i=\nu}^{\infty} P(E_i^{t_o,T})}$$

which proves the theorem.

Proof of Theorem 9.

Since the numerator of the expression (3.2.14) depends on x only, it is obviously sufficient to find its derivative only. Toward this end, consider the following relation:

$$\sum_{j=\nu}^{\infty} \sum_{i=\nu}^{\infty} P(G_j^{x_o,x+\Delta x} \cap E_i^{t_o,t}) =$$

$$\sum_{j=\nu}^{\infty} \sum_{i=\nu}^{\infty} \sum_{\tau=o}^{j} P(G_{j-\tau}^{x_o,x} \cap G_\tau^{x,x+\Delta x} \cap E_i^{t_o,t}) =$$

$$= \sum_{j=\nu}^{\infty} \sum_{i=\nu}^{\infty} P(G_j^{x_o,x} \cap G_o^{x,x+\Delta x} \cap E_i^{t_o,T}) + \sum_{j=\nu}^{\infty} \sum_{i=\nu}^{\infty} P(G_j^{x_o,x} \cap G_i^{x,x+\Delta x} \cap E_i^{t_o,t}) +$$

$$\sum_{j=\nu}^{\infty} \sum_{i=\nu}^{\infty} \sum_{\tau=2}^{j} P(G_{j-\tau}^{x_o,x} \cap G_\tau^{x,x+\Delta x} \cap E_i^{t_o,T})$$

Further, on the basis of the following inequality

$$\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty}\sum_{\tau=2}^{\infty} P(G_j^{x_o,x}\cap G_\tau^{x,x+\Delta x}\cap E_i^{t_o,T}) \leq \sum_{\tau=2}^{\infty} P(G_\tau^{x,x+\Delta x})$$

we have

$$\frac{\partial}{\partial x}\{\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty} P(G_j^{x_o,x}\cap E_i^{t_o,T})\}\ \Delta x =$$

$$= -\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty} P(G_j^{x_o,x}\cap E_i^{t_o,T} - G_j^{x_o,x}\cap G_o^{x,x+\Delta x}\cap E_i^{t_o,T}) +$$

$$\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty} P(G_{j-1}^{x_o,x}\cap G_1^{x,x+\Delta x}\cap E_i^{t_o,T}) + 0(\Delta x) =$$

$$= -\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty} P(G_j^{x_o,x}\cap G_1^{x,x+\Delta x}\cap E_i^{t_o,T}) +$$

$$+\sum_{j=\nu}^{\infty}\sum_{i=\nu}^{\infty} P(G_{j-1}^{x_o,x}\cap G_1^{x,x+\Delta x}\cap E_i^{t_o,T}) + 0(\Delta x) =$$

$$= \sum_{i=\nu}^{\infty} P(G_{\nu-1}^{x_o,x}\cap G_1^{x,x+\Delta x}\cap E_i^{t_o,T}) + 0(\Delta x)$$

Therefore, on the basis of condition b) the proof of the theorem follows.

3. CALCULATION OF PROBABILITIES $P(G_\nu^{x_o,x})$

1^o. It is seen that distribution functions of the random variables 1_x and X_ν are expressed over probabilities $P(G_\nu^{x_o,x})$. The same is valid for their density functions. Therefore, an effective obtaining of one-dimensional distribution functions of the stochastic processes l_x and X_ν depends on our capabilities to calculate probabilities $P(G_\nu^{x_o,x})$ for every $x \geqslant x_o$ and $\nu = 0, 1, 2, \ldots$. On the basis of condition a) and b) of Theorem 7, this calculation can be done. To accomplish this objective, it is necessary to prove the following theorem.

Theorem 10.

Assuming that conditions a) and b) of Theorem 7 are satisfied, then the probabilities $P(G_\nu^{x_o,x})$ for every $\nu = 0, 1, 2, \ldots$ are solutions of the following system of differential equations.

$$\frac{\partial P(G_\nu^{x_o,x})}{\partial x} = -\lambda_2(x)[P(G_\nu^{x_o,x}) - P(G_{\nu-1}^{x_o,x})] \tag{3.3.1}$$

To obtain a solution of the system (3.3.1), the same procedure as in the case of Theorem 5 should be applied, and, under the assumption that the following conditions are satisfied:

$$\forall x \geq x_o \quad P(G_\nu^{x_o,x}) = 0 \quad \text{if} \quad \nu < 0$$

we have

$$P(G_\nu^{x_o,x}) = e^{\int_{x_o}^{x} \lambda_2(s)ds} \frac{(\int_{x_o}^{x} \lambda_2(s)ds)^\nu}{\nu!} \tag{3.3.2}$$

The function considered represents a solution of the system of equation (3.3.1). If the condition a) of that theorem is satisfied and the condition b) is modified in the following manner:

$$\lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} \cap G_{\nu-1}^{x_o,x})}{\Delta x} = \lambda_2(x, \nu-1) \tag{3.3.3}$$

then (3.3.1) becomes

$$\frac{\partial P(G_\nu^{x_0,x})}{\partial x} = \lambda_2(x,\nu-1)P(G_{\nu-1}^{x_0,x}) - \lambda_2(x,\nu)P(G_\nu^{x_0,x}) \quad (3.3.4)$$

Solution of this system of differential equations will not be considered in this study. Finally, the particular very important case is if it is assumed that λ_2 does not depend on x, but on the variable ν only, then (3.3.4) becomes

$$\frac{\partial P(G_\nu^{x_0,x})}{\partial x} = \lambda_2(\nu-1)P(G_{\nu-1}^{x_0,x}) - \lambda_2(\nu)P(G_\nu^{x_0,x}) \quad (3.3.5)$$

In the following exposition, the proof of the previous assertions will be given.

Appendix - Let us prove now Theorem 10: Obviously this proof must be very similar to the proof of Theorem 5.

Proof of Theorem 10.

In order to prove the theorem consider the relation

$$P(G_\nu^{x_0,x+\Delta x}) = P\bigcup_{\tau=0}^{\nu}(G_{\nu-\tau}^{x_0,x}\cap G_\tau^{x,x+\Delta x}) =$$

$$= P(G_\nu^{x_0,x}\cap G_0^{x,x+\Delta x}) + P(G_{\nu-1}^{x_0,x}\cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

It is not difficult to see that the following is valid:

$$\frac{\partial P(G_\nu^{x_0,x})}{\partial x}\Delta x = -P(G_\nu^{x_0,x} - G_\nu^{x_0,x}\cap G_0^{x,x+\Delta x}) +$$

$$+ P(G_{\nu-1}^{x_0,x}\cap G_1^{x,x+\Delta x}) + 0(\Delta x) =$$

$$- P[G_\nu^{x_0,x}\cap(G_0^{x,x+\Delta x})^c] + P(G_{\nu-1}^{x_0,x}\cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

Therefore, on the basis of relation (3.2.16) the following is obtained:

$$\frac{\partial P(G_{\nu}^{x_o,x})}{\partial x}\,\Delta x = -\,P(G_{\nu}^{x_o,x} \cap G_{1}^{x_o,x}) + P(G_{\nu-1}^{x_o,x} \cap G_{1}^{x,x+\Delta x}) + O(\Delta x)$$

which proves the theorem.

The solution (3.3.2) of the system of differential equations (3.3.1) is obtained in the same manner as the solution of the system (2.4.1). This result represents the most general expression for the probability of the event $G_{\nu}^{x_o,x}$, $\nu = 0, 1, 2, \ldots$ and $x \geqslant x_o$, if conditions a) and b) of Theorem 7 are satisfied. Obviously these probabilities depend on an unknown function $\lambda_2(x) \geqslant 0$. Therefore, it is of interest to possess a method for its evaluation.

4. DISCUSSION AND APPLICATIONS

As we have seen, an analytic expression for probability $P(G^{x_o,x})$ has been given in the previous section, for every $\nu = 0, 1, 2, \ldots$ and $x \geqslant x_o$. This result is very important since one-dimensional distribution function of the processes 1_x and X_{ν}. Only the question of how the function $\lambda_2(x)$ can be effectively obtained remains open.

In order to answer this question, it is necessary, besides the pure probabilistic definition of this function given by the limit

$$\lim_{\Delta x \to 0} \frac{P(G_{1}^{x,x+\Delta x} \mid G_{\nu-1}^{x_o,x})}{\Delta x} = \lambda_2(x)$$

to possess a phenomenologic interpretation of this function. In order to achieve this goal, consider first the process 1_x. Since by (3.2.1),

$$P\{\omega;\ 1_x = \nu\} = P(G_{\nu}^{x_o,x})$$

then by virtue of (3.3.2) we have

$$P\{\omega;\ 1_x = \nu\} = e^{-\int_{x_o}^{x} \lambda_2(s)ds}\,\frac{\left(\int_{x_o}^{x} \lambda_2(s)ds\right)^{\nu}}{\nu!}$$

Therefore,

$$E(1_x) = e^{-\int_{x_o}^{x} \lambda_2(s)ds} \sum_{\nu=1}^{\infty} \nu \frac{\left(\int_{x_o}^{x} \lambda_2(s)ds\right)^{\nu}}{\nu!} =$$

$$= e^{-\int_{x_o}^{x}\lambda_2(s)ds} \left(\int_{x_o}^{x} \lambda_2(s)ds\right) \sum_{\nu=o}^{\infty} \frac{\left(\int_{x_o}^{x} \lambda_2(s)ds\right)^{\nu}}{\nu!} = \int_{x_o}^{x} \lambda_2(s)ds$$

i.e.

$$E(1_x) = \int_{x_o}^{x} \lambda_2(s)ds \qquad (3.4.1)$$

Since 1_x represents the maximum number of storm periods such that the corresponding total amount of precipitation does not exceed the value $(x-x_o)$, integral (3.4.1) represents the average maximum number of storms whose total amount of precipitation is less than or equal to $(x-x_o)$.

Let us dwell now on the problem of evaluation of the function

$$\Lambda_2(x_o,x) = \int_{x_o}^{x} \lambda_2(s)\, ds \quad .$$

On the basis of definition, it follows that $\lambda_2(x) \geqslant 0$ for all $x \geqslant x_o$. Therefore, the integral considered represents a nondecreasing x function (see Fig. 11).

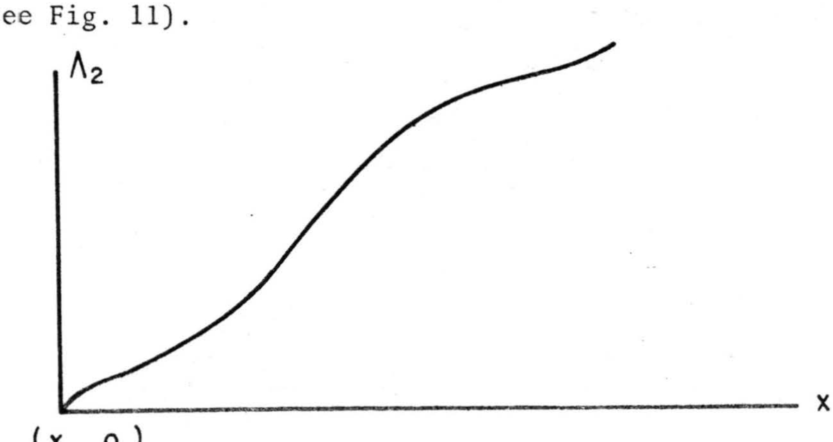

Fig. 11 Graphical presentation of the function $\Lambda_2(x_o,x)$

On the basis of the phenomenological interpretation of the function λ_2, obviously the integrand $\lambda_2 = \lambda_2(x)$ represents some kind of frequency of storm in the interval (x_o, x). In the case when tropical zone is considered, this function can be easily evaluated. Then it is very realistic to assume that the function $\lambda_2(x)$ can have only the two values, $\lambda_{2,1}$ in the case of wet season and λ_{22} in the case of dry season. Under this assumption, the problem becomes very similar to the problem which has been considered in section 5 of Chapter II.

Consider now distribution function $F_\nu(x)$ of the random variable X_ν, $\nu = 1, 2, \ldots$. On the basis of (3.2.8) and (3.3.2) the following relation is valid

$$F_\nu(x) = e^{-\int_{x_o}^{x} \lambda_2(s)ds} \sum_{j=\nu}^{\infty} \frac{1}{\Gamma(j+1)} \left(\int_{x_o}^{x} \lambda_2(s)ds\right)^j \qquad (3.4.2)$$

Since (3.4.2) could be written in the following manner:

$$F_\nu(x) = e^{-\int_{x_o}^{x} \lambda_2(s)ds} \left[\left(e^{-\int_{x_o}^{x} \lambda_2(s)ds} - \sum_{j=o}^{\nu-1} \frac{1}{\Gamma(j+1)} \left(\int_{x_o}^{x} \lambda_2(s)ds\right)^j\right]\right. =$$

$$= 1 - e^{-\int_{x_o}^{x} \lambda_2(s)ds} \sum_{j=o}^{\nu-1} \frac{1}{\Gamma(j+1)} \left(\int_{x_o}^{x} \lambda_2(s)ds\right)^j$$

we have

$$F_1(x) = 1 - e^{-\int_{x_o}^{x} \lambda_2(s)ds}$$

$$F_2(x) = 1 - e^{-\int_{x_o}^{x} \lambda_2(s)ds} \left(1 - \int_{x_o}^{x} \lambda_2(s)ds\right)$$

etc.

By virtue of Theorem 7, the corresponding density function $f_\nu(x)$ of the distribution function $F_\nu(x)$ has the following shape:

$$f_\nu(x) = \frac{\lambda_2(x)}{\Gamma(\nu)} e^{-\int_{x_0}^{x} \lambda_2(s)ds} \left(\int_{x_0}^{x} \lambda_2(s)ds\right)^{\nu-1} \qquad (3.4.3)$$

Therefore, for $\nu = 1$ and $\nu = 2$, the following is obtained:

$$f_1(x) = \lambda_2(x)\, e^{-\int_{x_0}^{x} \lambda_2(s)ds}$$

$$f_2(x) = \lambda_2(x)\, e^{-\int_{x_0}^{x} \lambda_2(s)ds} \int_{x_0}^{x} \lambda_2(s)ds$$

etc.

The mathematical expectation of the random variable X_ν, for $\nu = 1, 2, \ldots$, is given by the following expression:

$$E(X_\nu) = \frac{1}{\Gamma(\nu)} \int_{x_0}^{x} x\, \lambda_2(x)\, e^{-\int_{x_0}^{x} \lambda_2(s)ds} \left(\int_{x_0}^{x} \lambda_2(s)ds\right)^{\nu-1} dx \qquad (3.4.4)$$

5. STOCHASTIC PROCESS Z_ν

One of the most interesting questions concerning the rainfall phenomenon is the problem of the total amount of precipitation during one storm period only. Generally speaking, it could be assumed that this amount depends on the number ν, where ν indicates the serial number of storms, or on ν and the total amount of precipitation in the previous storms, etc.

The purpose of this section is to establish some mathematical models which could help to investigate all of these problems. Toward this end, let us denote by Z_ν the total amount of precipitation during the ν-th storm period. Obviously, then,

$$Z_\nu = X_\nu - X_{\nu-1} \quad \nu = 1, 2, \ldots \tag{3.5.1}$$

(see Fig. 12), and

$$Z_o \equiv 0 \quad .$$

Therefore, we have a countable family of random variables or a discrete parameter stochastic process

$$\{Z_\nu; \quad \nu = 1, 2, \ldots\} \tag{3.5.2}$$

In the following exposition, we shall start to study the simplest case; we will suppose that (3.5.2) is a sequence of independent random variables, i.e., that the stochastic process (3.2.7) is a process with independent increments. Phenomenologically speaking, it is assumed that the total amount of precipitation during the ν-th storm period does not depend on the amount of precipitation during the previous storms, but on the serial number ν only.

For the following exposition, it will be necessary to prove that Z_ν and $X_{\nu-1}$ are independent random variables for every $\nu = 1, 2, \ldots$ Indeed, consider the characteristic function

$$E\{e^{(\alpha X_{\nu-1} + \beta Z_\nu)i}\} =$$

$$E\{e^{[\alpha \sum_{k=1}^{\nu-1} (X_k - X_{k-1}) + \beta Z_\nu] i}\} =$$

$$E\{e^{[\alpha \sum_{k=1}^{\nu-1} Z_k + \beta Z_\nu] i}\}$$

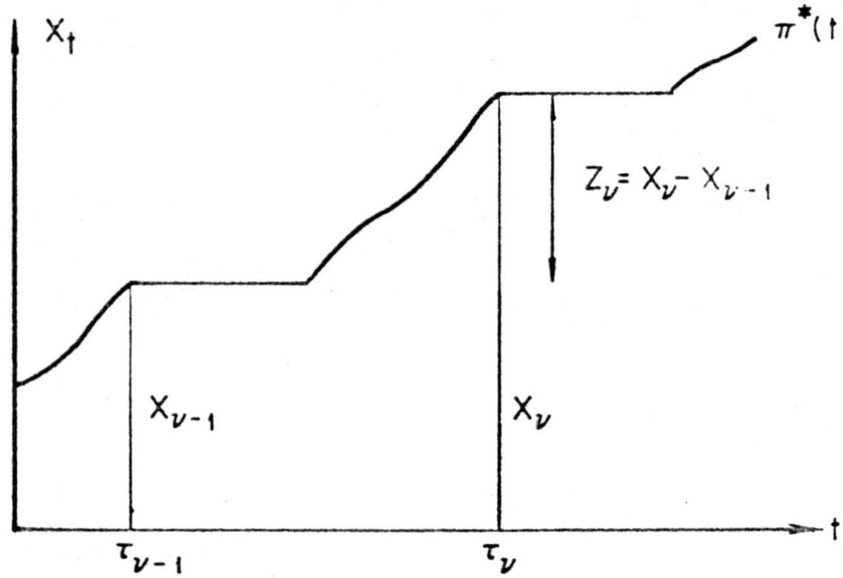

Fig. 12 Graphic Representation of the Physical Meaning of the Process Z_ν

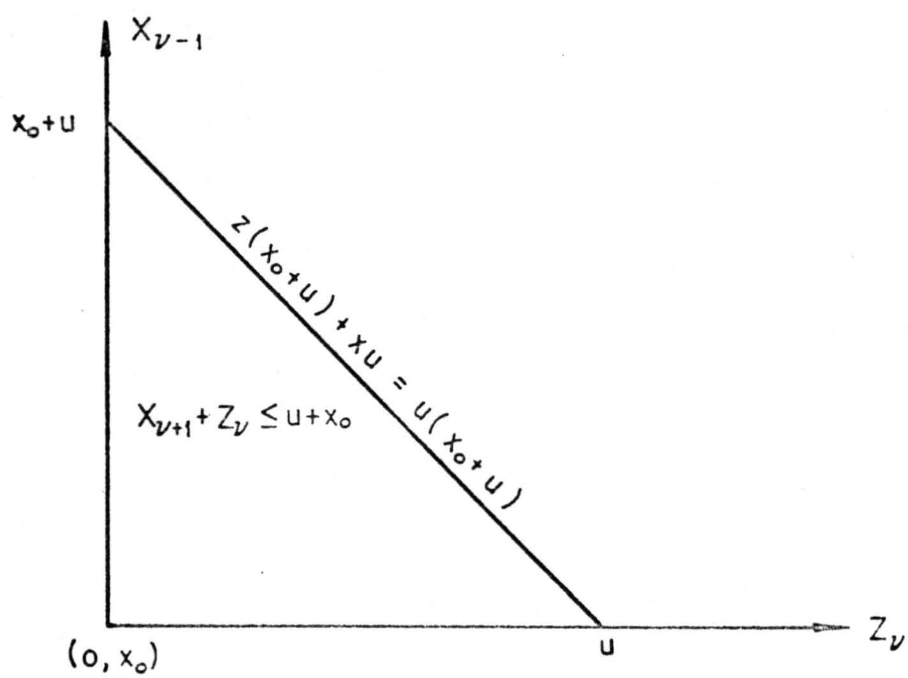

Fig. 13 Graphic Representation of the Relation (3.5.3)

Since, by definition, random variables Z_ν $\nu = 1, 2, \ldots,$ are independent, i.e., every finite class of Borel set $(S_{\tau 1}, S_{\tau 2}, \ldots S_{\tau n})$, the following relation is valid:

$$P \bigcap_{k=1}^{n} \{\omega ;\ Z_k \varepsilon\ S_{\tau_n}\} = \prod_{k=1}^{n} P\{\omega ;\ Z_k \varepsilon\ S_{\tau n}\} \quad ,$$

and we have

$$E\{e^{(\alpha X_{\nu-1} + \beta Z_\nu) i}\} = E\{e^{(\alpha \sum_{k=1}^{\nu-1} Z_k) i}\}\ E\{e^{\beta Z_\nu i}\} =$$

$$E\{e^{\alpha X_{\nu-1} i}\}\ E\{e^{\beta Z_\nu i}\}$$

Let $B_\nu(z)$ denote the one-dimensional distribution function of the stochastic process (3.5.2), i.e.,

$$B_\nu(z) = P\{\omega ;\ Z_\nu \leq z\}$$

and suppose that the following derivative exists

$$b_\nu(z) = \frac{dB_\nu(z)}{dz}$$

for every $z \geqslant 0$. Consider now the sum

$$X_\nu = X_{\nu-1} + Z_\nu \quad ;$$

then by virtue of the previous results we have (See Fig. 13)

$$P\{\omega ;\ X_\nu \leq u + x_o\} = P\{\omega ;\ X_{\nu-1} + Z_\nu \leq u + x_o\} =$$

$$\int_o^x \int_{x_o}^{(x_o+u)(1-\frac{z}{u})} f_{\nu-1}(x)\ b_\nu(z)\ dx\ dz$$

Therefore, we have

$$F_\nu(x_o+u) = \int_0^u \int_{x_o}^{(x_o+u)(1-\frac{z}{u})} f_{\nu-1}(x)\, b_\nu(z)\,dx\,dz \qquad (3.5.3)$$

Differentiating (3.5.2) by u , the following is obtained

$$f_\nu(x_o+u) = \int_0^u f_{\nu-1}\left[(x_o+u)(1-\frac{z}{u})\right] b_\nu(z)dz$$

or, on the basis of (3.2.9),

$$\lambda_2(x_o+u)P(G_{\nu-1}^{x_o,x_o+u}) = \int_0^u \lambda_2\left[(x_o+u)(1-\frac{z}{u})\right] P(G_{\nu-2}^{x_o,(x_o+u)(1-\frac{z}{u})})\, b_\nu(z)dz$$

(3.5.4)

where $b_\nu(z)$ is an unknown function. Therefore, by virtue of (3.3.2), the following is obtained:

$$\lambda_2(x_o+u)\, e^{-\int_{x_o}^{x_o+u}\lambda_2(s)ds} \frac{\left(\int_{x_o}^{x_o+u}\lambda_2(s)ds\right)^{\nu-1}}{(\nu-1)!}$$

$$= \int_0^u \lambda_2(x_o+u)(1-\frac{z}{u})\, e^{-\int_{x_o}^{(x_o+u)(1-\frac{z}{u})}\lambda_2(s)ds} \frac{\left(\int_{x_o}^{(x_o+u)(1-\frac{z}{u})}\lambda_2(s)ds\right)^{\nu-2}}{(\nu-2)!}\, b_\nu(z)dz$$

(3.5.5)

The equation (3.5.4) represents an integral equation. Since, by definition, $\lambda_2(x) \equiv 0$ for $x < 0$, this is a Voltera's integral equation of the first kind with

$$(\nu-1)\lambda_2(x_o+u)(1-\frac{z}{u})\quad e^{-\int_{x_o}^{(x_o+u)(1-\frac{z}{u})}\lambda_2(s)ds} \left(\int_{x_o}^{(x_o+u)(1-\frac{z}{u})}\lambda_2(s)ds\right)^{\nu-2}$$

as the kernel.

Solution of this integral equation gives the one-dimensional density function $b_\nu(z)$ of the stochastic process (3.5.2). In fact, the equation considered represents a particular Voltera's integral equation called "convolution integral equation" (see F. Tricomi (33)). One of the methods for studying such an equation is by Laplace transforms.

Consider now the simplest case, i.e., when

$$\lambda_2(x) = \lambda_2 = \text{const.} \tag{3.5.6}$$

Then on the basis of (3.5.5), the following is valid:

$$\lambda_2 u^{\nu-1} = (\nu-1) \int_0^u e^{-\lambda_2 z}(u-z)^{\nu-2} b_\nu(z)dz$$

where we assume $x_o = 0$. Solution of the equation considered can be easily obtained by differentiating its left and right side $(\nu-1)$ times. If we do that, the following relation is obtained:

$$\lambda_2 = e^{\lambda 2u}\, b_\nu(u)$$

or (3.5.7)

$$b_\nu(z) = \lambda_2 e^{-\lambda_2 z}$$

i.e., an exponential distribution has been obtained.

Obviously, under assumption (3.5.6) the one-dimensional density function $b_\nu(z)$ of the process (3.5.2) does not depend on ν, i.e.,

$$b_\nu(z) = b(z) \quad .$$

Therefore, all Z_ν $\nu = 1, 2, \ldots$ has the same density function given by (3.5.7).

The basic hypothesis in the former exposition was that the following relation is valid:

$$P\{\omega;\ Z_\nu \leq z | Z_1 = z_1, \ldots Z_{\nu-1} = z_{\nu-1}\} = P\{\omega;\ Z_\nu \leq z\}$$

If this relation is not valid, then it is necessary to find new solutions for the problem considered.

6. PROBLEM OF THE EXTREME STORMS

In this part of the study, a particular problem concerned with the maximum and minimum storms in the given interval of time will be considered. In other words, if n storm periods are expected in some interval of time (t_o, T), and z_ν represents the corresponding amount of precipitation during the ν-th storm period, where $\nu = 1, 2, \ldots n$, then one of these n storms has a minimum amount of precipitation and will be denoted by $\underline{Z}_n$, where

$$\underline{Z}_n = \inf_{1 \leq \nu \leq n} Z_\nu \tag{3.6.1}$$

and the other one has a maximum amount,

$$\overline{Z}_n = \sup_{1 \leq \nu \leq n} Z_\nu \tag{3.6.2}$$

Obviously, (3.6.1) and (3.6.2) are random variables which depend on $n = 1, 2, \ldots$; therefore, we have two new families of random variables or two discrete parameter stochastic processes.

$$\{\underline{Z}_n;\ n=1,2,\ldots\} \quad \{\overline{Z}_n;\ n=1,2,\ldots\}$$

Let Q_n distribution function of the random variable $\underline{Z}_n$ i.e.,

$$Q_n(z) = P\{\omega;\ \underline{Z}_n \leqslant z\} =$$

$$= P\{\omega;\ \inf_{1 \leq \nu \leq n} Z_\nu \leq z\}$$

;

then on the basis of the relation (see Halmosh (12))

$$\{\omega;\ \inf_{1 \leq \nu \leq n} Z_\nu > z\} = \bigcap_{\nu=1}^{n} \{\omega;\ Z_\nu > z\}$$

it follows that

$$P\{\omega;\ \inf_{1\leq\nu\leq n} Z_\nu\leq z\} = 1 - P\bigcap_{\nu=1}^{n}\{\omega;\ Z_\nu>z\} =$$

$$1 - P\Big[\bigcap_{\nu=1}^{n}\{\omega;\ Z_\nu\leq z\}\Big]^c = P\bigcup_{\nu=1}^{n}\{\omega;\ Z_\nu\leq z\} \tag{3.6.3}$$

Therefore, the following is valid:

$$Q_n(z) = \sum_{\nu=1}^{n} P\{\omega;\ Z_\nu\leq z\} - \sum_{i\neq j}^{n} P\{\omega;\ Z_i\leq z\ Z_j\leq z\} +$$

$$+ \sum_{i\neq j\neq k}^{n} P\{\omega;\ Z_i<z,\ Z_j\leq z,\ Z_k\leq z\} - \ldots + (-1)^{n-1} P\{\omega; Z_\nu\leq z\ \nu=1,n\}$$

If it is assumed that Z_ν $\nu = 1, 2,\ldots$ are independent random variables, we have then the following:

$$Q_n(z) = 1- \prod_{\nu=1}^{n} P\{\omega; Z_\nu>z\}$$

or, finally,

$$Q_n(z) = 1 - \prod_{\nu=1}^{n} [1-B_\nu(z)] \tag{3.6.4}$$

Let $q_n(z)$ denote the corresponding density function of the distribution function $Q_n(z)$; then obviously the following is valid:

$$q_n(z) = \sum_{k=1}^{n} b_k(z) \prod_{\nu\neq k}^{n} [1-B_k(z)] \tag{3.6.5}$$

Consider now the distribution function $H_n(z)$ of the random variable $\overline{Z}_n$, i.e.

$$H_n(z) = P\{\omega;\ \sup_{1\leq\nu\leq n} Z_\nu\leq z\}$$

Since the following is valid (see Halmosh (12))

$$\{\omega;\ \sup_{1\leq\nu\leq n} Z_\nu \leq z\} = \bigcap_{\nu=1}^{n} \{\omega\ ;\ Z_\nu \leq z\} \qquad ,$$

then we have

$$H_n(z) = P\bigcap_{\nu=1}^{n} \{\omega;\ Z_\nu \leq z\}$$

Consequently, assuming that Z_ν $\nu = 1, 2,\ldots n$ are independent random variables, the following is valid:

$$H_n(z) = \bigcap_{\nu=1}^{n} B_\nu(z) \tag{3.6.6}$$

The corresponding density function $h_n(z)$ is of the following shape:

$$h_n(z) = \sum_{k=1}^{n} b_k \prod_{\nu\neq k}^{n} B_\nu(z) \tag{3.6.7}$$

Suppose now that all random variables Z_ν $\nu = 1, 2,\ldots$ have the same distribution function, i.e., assume that

$$B_\nu(z) = B(z) \qquad \nu=1,2,\ldots$$

then on the basis of (3.6.4) and (3.6.6) we have

$$\begin{aligned} Q_n(z) &= 1 - [1 - B(z)]^n \\ H_n(z) &= B^n(z) \end{aligned} \tag{3.6.8}$$

Therefore, the corresponding density functions $q_n(z)$ and $h_n(z)$ are of the following form:

$$\begin{aligned} q_n(z) &= nb(z)\ [1 - B(z)]^{n-1} \\ h_n(z) &= nb(z)\ B^{(n-1)}(z) \end{aligned} \tag{3.6.9}$$

Finally, if it is assumed that

$$\lambda_2(x) = \lambda_2 = \text{cons} \quad ,$$

then we have seen that

$$b_\nu(z) = \lambda_2 \, e^{-\lambda_2 z} \quad B_\nu(z) = 1 - e^{-\lambda_2 z}$$

for every $\nu = 1, 2, \ldots$ so that (3.6.9) becomes

$$q_n(z) = \lambda_2 n e^{-\lambda_2 n z} \tag{3.6.10}$$

$$h_n(z) = \lambda_2 n e^{-\lambda_2 z}(1 - e^{-\lambda_2 z})^{n-1} \tag{3.6.11}$$

Graphical representation of the functions (3.6.10) and (3.6.11) for different values of n are given in Fig. 14 and Fig. 15.

Let us make a very brief analysis of the density function $h_n(z)$. Since

$$h_n'(z) = \lambda_2^2 \, n(n-1)(1-e^{-\lambda_2 z})^{n-2} \cdot e^{-2\lambda_2 z} - \lambda_2^2 \, n(1-e^{-\lambda_2 z})^{n-1} e^{-\lambda_2 z}$$

obviously

$$h_n'(z) = 0 \quad \text{for} \quad z = \frac{1}{\lambda_2} \ln(n)$$

and therefore

$$h_n\left[\frac{1}{\lambda_2} \ln(n)\right] = \lambda_2 \left(1 - \frac{1}{n}\right)^{n-1}$$

i.e.,

$$\max_z h_n(z) = \lambda_2 \left(1 - \frac{1}{n}\right)^{n-1}$$

If $n \to \infty$, then obviously

$$\max_z h_n(z) \to \frac{\lambda_2}{e}$$

Consider now the mathematical expectation of the random variable $\inf_\nu Z_\nu$ and $\sup_\nu Z_\nu$ for the particular case when corresponding density

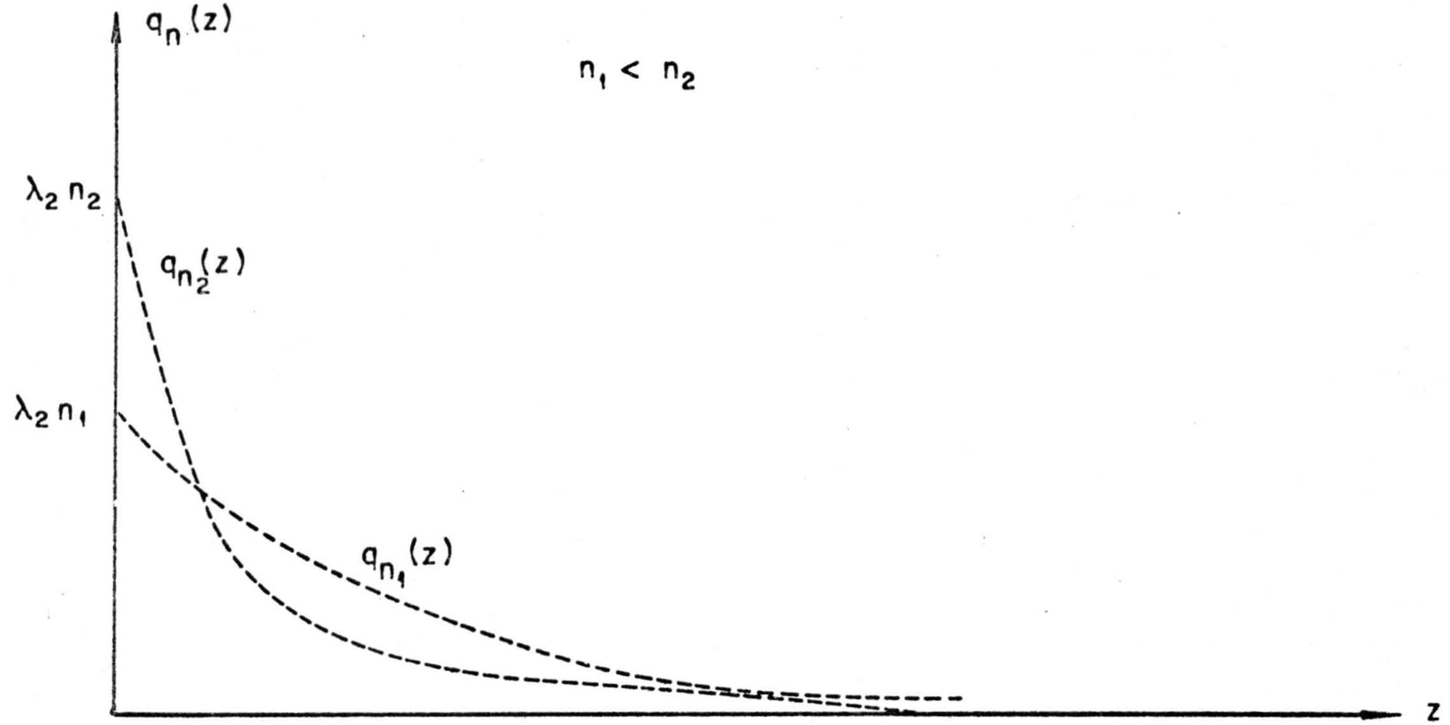

Fig. 14 Graphic Presentation of the Density Function $q(z)$ for Different Values of n

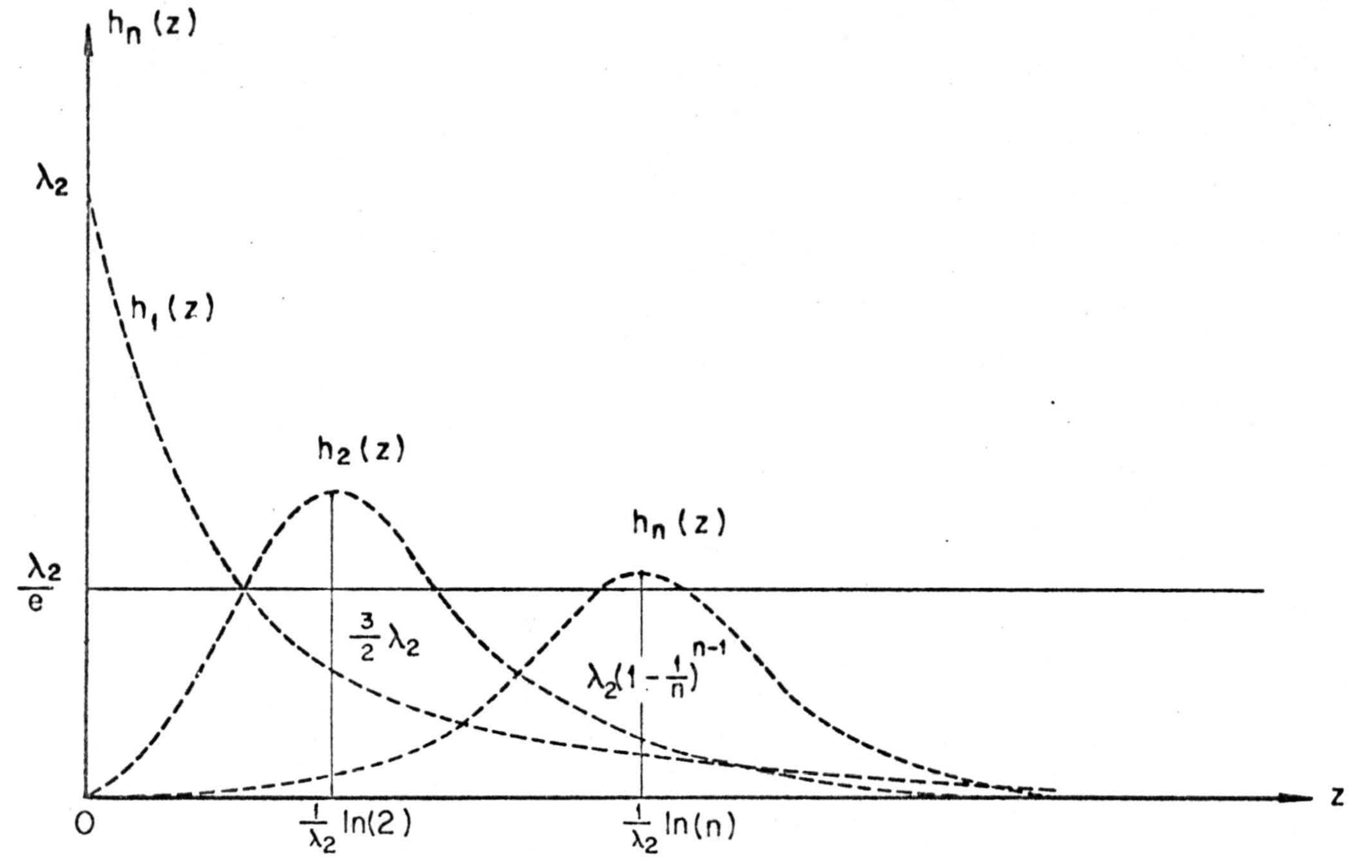

Fig. 15 Graphic Presentation of the Density Function for Different Values of n

functions are given by (3.6.10) and (3.6.11) respectively. On the basis of (3.6.10), we have the following:

$$E(\inf_{1\leq\nu\leq n} Z_\nu) = \lambda_2 n \int_0^\infty z e^{-\lambda_2 n z}\, dz = \frac{1}{\lambda_2 n}$$

On the basis of (3.6.11), the mathematical expectation of the random variable $\sup_{1\leq\nu\leq n} Z$ is

$$E(\sup_{1\leq\nu\leq n} Z_\nu) = \lambda_2 n \int_0^\infty z(1-e^{-\lambda_2 z})^{n-1} e^{-\lambda_2 z}\, dz =$$

$$\lambda_2 n \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} \int_0^\infty z e^{-\lambda_2 z(k+1)}\, dz =$$

$$= \frac{n}{\lambda_2} \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+k)^2} \binom{n-1}{k}$$

and variance

$$D(\sup_{1\leq\nu\leq n} Z_\nu) = \lambda_2 n \int_0^\infty \{z-E(\sup_{1\leq\nu\leq n} Z_\nu)\}^2 (1-e^{-\lambda_2 z})^{n-1} \cdot e^{-\lambda_2 z}\, dz =$$

$$= \frac{4n}{\lambda_2^2} \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+k)^3} \binom{n-1}{k} - \left\{\frac{n}{\lambda_2} \sum_{k=0}^{\nu-1} \frac{(-1)^k}{(1+k)^2} \binom{n-1}{k}\right\}^2$$

In the following table, the values of the mathematical expectation

$$E(\sup_{1\leq\nu\leq n} Z_\nu) = \frac{n}{\lambda_2} \sum_{k=0}^{n-1} \frac{(-1)^k}{(1+k)^2} \binom{n-1}{k}$$

are given for $n = 2, \ldots 9$.

$$n=2 \quad \frac{2}{\lambda_2} \sum_{k=0}^{1} \frac{(-1)^k}{(1+k)^2} \binom{1}{k} = \frac{3}{2}\frac{1}{\lambda_2} = \frac{1.50}{\lambda_2}$$

$$n=3 \quad \frac{3}{\lambda_2} \sum_{k=0}^{2} \frac{(-1)^k}{(1+k)^2} \binom{2}{k} = \frac{11}{6}\frac{1}{\lambda_2} \approx \frac{1.83}{\lambda_2}$$

$$n=4 \quad \frac{4}{\lambda_2} \sum_{k=0}^{3} \frac{(-1)^k}{(1+k)^2} \binom{3}{k} = \frac{25}{12}\frac{1}{\lambda_2} \approx \frac{2.03}{\lambda_2}$$

$$n=5 \quad \frac{5}{\lambda_2} \sum_{k=o}^{4} \frac{(-1)^k}{(1+k)^2} \binom{4}{k} = \frac{132}{60} \frac{1}{\lambda_2} \approx \frac{2.28}{\lambda_2}$$

$$n=6 \quad \frac{6}{\lambda_2} \sum_{k=o}^{5} \frac{(-1)^k}{(1+k)^2} \binom{5}{k} = \frac{49}{20} \frac{1}{\lambda_2} = \frac{2.45}{\lambda_2}$$

$$n=7 \quad \frac{7}{\lambda_2} \sum_{k=o}^{6} \frac{(-1)^k}{(1+k)^2} \binom{6}{k} = \frac{363}{140} \frac{1}{\lambda_2} = \frac{2.59}{\lambda_2}$$

$$n=8 \quad \frac{8}{\lambda_2} \sum_{k=o}^{7} \frac{(-1)^k}{(1+k)^2} \binom{7}{k} = \frac{3043}{1120} \frac{1}{\lambda_2} \approx \frac{2.72}{\lambda_2}$$

$$n=9 \quad \frac{9}{\lambda_2} \sum_{k=o}^{8} \frac{(-1)^k}{(1+k)^2} \binom{8}{k} = \frac{4042}{1428} \frac{1}{\lambda_2} \approx \frac{2.83}{\lambda_2}$$

It is of interest to see how the average amount of precipitation of the maximum storm depends on the parameter λ_2 . In Fig.16 , a graphical representation of the mean

$$E(\sup_{1\leq \nu \leq n} Z_\nu) = \frac{n}{\lambda_2} \sum_{k=o}^{n-1} \frac{(-1)^k}{(1+k)^2} \binom{n-1}{k}$$

is given for different values of n as a function of the parameter λ_2 . Apparently, if λ_2 is larger the average value of precipitation of the maximum storms is smaller. For example, if it is expected that two storms will occur (i.e., n = 2), then obviously, the expected value of the total amount of precipitation of the maximum storm is

$$\text{for } \lambda_2 = 1 \qquad E(\sup_{1\leq \nu \leq n} Z) = 1,50$$

$$\text{for } \lambda_2 = 2 \qquad E(\sup_{1\leq \nu \leq n} Z) = 0,75$$

etc. Therefore, parameter λ_2 is some kind of characteristic of the total amount of precipitation during one storm period.

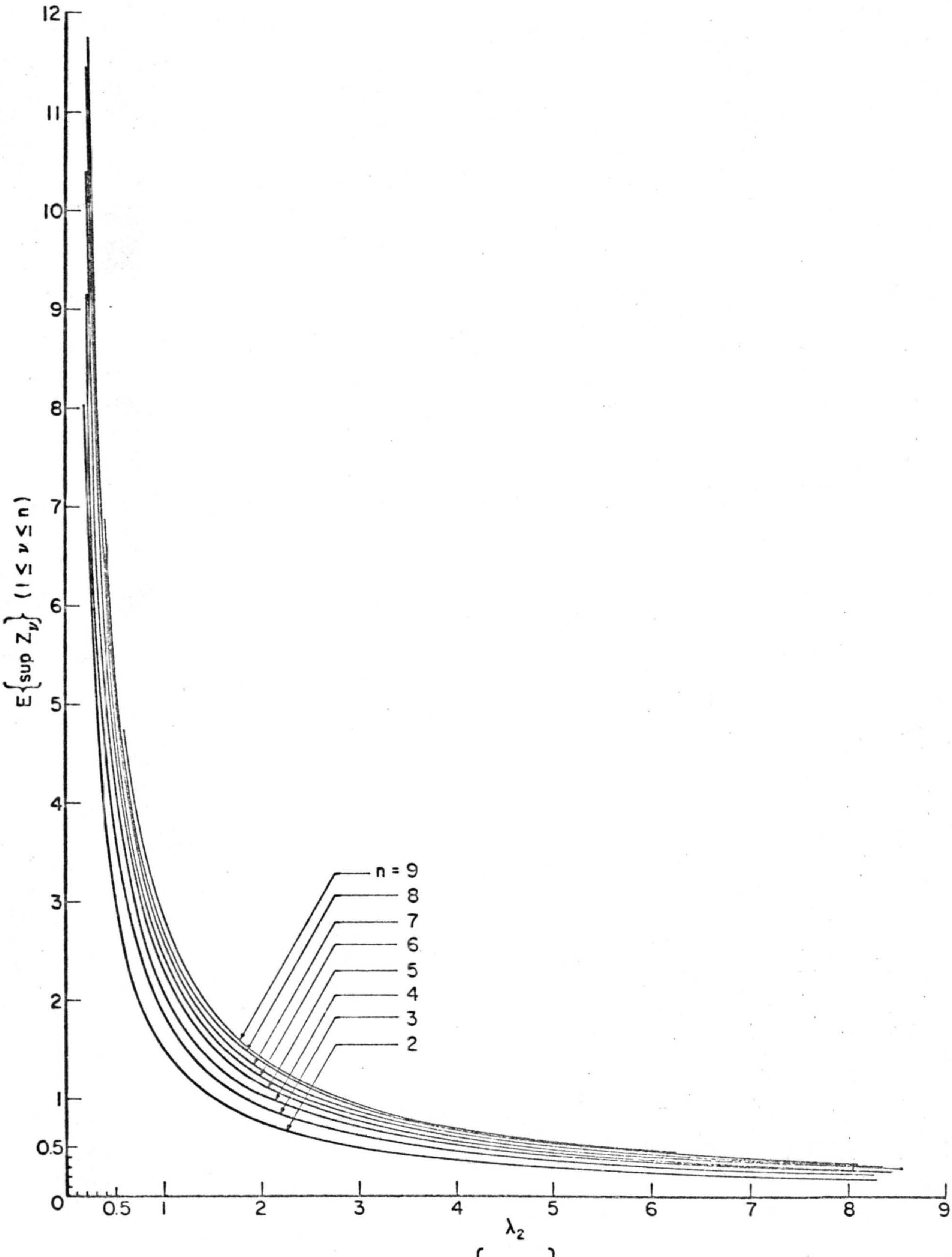

Fig. 16 Graphic Presentation of $E\{\sup Z_\nu\}$ $(1 \le \nu \le n)$ for $n = 2, 3, 4, \ldots, 9$

Chapter IV

1. SOME PREVIOUS CONSIDERATIONS

As has been noted, the general purpose of this paper is to present a new mathematical study of the rainfall phenomenon (not entering into its physical nature) based on the theory of a particular stochastic process of the nondecreasing sample function

$$\{X_t;\ t\varepsilon T^*\} \tag{4.1.1}$$

where

$$X_t = x_o + \int_{t_o}^{t} \xi_s \, ds$$

and ξ_t represents the rainfall intensity at the instant of time t. With respect to the nature of precipitation, we have

$$\forall t\varepsilon T^* \quad \xi_t \geq 0$$

therefore

$$\forall \Delta t > 0 \quad X_t \leq X_t + \Delta t \qquad t, t+\Delta t \varepsilon T^*$$

In the second chapter of this study some "dynamic" properties of the rainfall phenomenon were considered (those characteristics which give us information concerning the frequencies of storm periods in the interval of time under consideration, the distribution function $A_\nu(t)$ of the total elapsed time up to the end of the ν-th storm period, i.e., of $(\tau_\nu - t_o)$, the relationship between numbers of storm periods in two successive intervals of time, etc.)

It is obvious that the dynamic "properties" do not give us information about amount of precipitation, a subject of primary interest.

Consequently, in the third chapter the quantitative characteristics (such as the total amount of precipitation X_ν during exactly ν storm periods or the total amount of precipitation $Z_\nu = X_\nu - X_{\nu-1}$ during the ν-th storm period, where $\nu = 1, 2, \ldots,$ etc.) of the rainfall phenomenon were discussed.

It is obvious that neither dynamic nor quantitative aspects considered separately can furnish us with a complete description of the precipitation phenomenon. For example, information that the total amount of precipitation during ν storms is X_ν is incomplete if nothing is known about time interval $(\tau_\nu - t_o)$.

In order to avoid this inconvenience, one can use the average values; for instance, it is possible to say that the average elapsed time up to the end of the ν-th storm period is

$$E(\tau_\nu) - t_o$$

and the average amount of precipitation during this time is

$$E(X_\nu) - x_o$$

This result gives us some information about the relationship between a dynamic and a quantitative characteristic of the phenomenon considered, but in application it is necessary to know much more. If X_t denotes this amount of precipitation, obviously X_t is a random variable for every $t > t_o$, and therefore it is necessary to study this variable. On the other hand, if an amount of precipitation has been observed, say x , what amount of time would be necessary for this amount to be realized? If T_x denotes this time, then T_x is a random variable for every $x \geqslant x_o$. In the following section, these problems will be studied.

2. ONE-DIMENSIONAL DISTRIBUTION FUNCTION OF THE PROCESS $\pi(t,\omega)$

In this section, the problem of estimating the total amount of precipitation X_t up to time t will be considered. As has been said, X_t is a random variable for every $t > t_o$, so it is impossible to predict a certain value of X_t at any moment of time $t > t_o$. Instead of this, the event

$$B_t(x) = \{\omega;\ X_t \leq x\} \quad \forall x \geq x_o \tag{4.2.1}$$

will be considered. (4.2.1) represents the random event that the total amount of precipitation up to time t will be less than or equal to x . Therefore, the corresponding probability

$$P\{\omega;\ X_t \leq x\} = F_t(x) \tag{4.2.2}$$

represents a one-dimensional distribution function of the stochastic process $X_t = \pi(t,\omega)$ (see Chapter II, sect. 1).

Let us now try to obtain effectively the distribution function (4.2.2). Toward this end, consider (2.2.5); then on the basis of the relations (2.2.7) the following is valid

$$B_t(x) = \bigcup_{\nu=0}^{\infty} (E_\nu^{t_o,t} \cap B_t)$$

i.e.,

$$F_t(x) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_o,t} \cap B_t) \tag{4.2.3}$$

Let us write the function considered in the following manner:

$$F_t(x) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_o,t})\ F^*(x,t) \tag{4.2.4}$$

where the function $F_\nu^*(x,t)$ represents conditional probability that the total amount of precipitation X_t up to time t will be less than or equal to x, under the condition that exactly ν complete storm periods have occurred in time interval (t_o,t), i.e.,

$$F_\nu^*(x,t) = P\{\omega;\ X_t \leq x \mid E_\nu^{t_o,t}\} \tag{4.2.5}$$

Obviously, the following is valid:

$$F_\nu^*(x,t) = \begin{cases} 0 \text{ if } x<x_o \\ 0 \leq F_\nu^* \leq 1 \text{ if } x \geq x_o \end{cases}$$

We have seen that under condition a) and b) of Theorem 3, the calculation of the probability $P(E_\nu^{t_o,t})$ is not difficult, but as far as the calculation of the conditional probability (4.2.5) is concerned, the situation is considerably more difficult. In fact, up to now a method for calculation of this probability in the very general form is not known.

In the following exposition, an attempt will be made to obtain some information about this function. Instead of exact calculation of (4.2.5), a method will be established for obtaining its lower and upper approximations. In this manner, instead of exact distribution function (4.2.4) the two lower and upper approximations

$$F_{1t}(x) \quad \text{and} \quad F_{2t}(x)$$

where

$$F_{2t}(x)$$

will be used.

The approximations considered satisfy the following condition

$$0 \leq F_{1t}(x) \leq F_t(x) \leq F_{2t}(x) \leq 1 \tag{4.2.6}$$

and the upper approximation is a distribution function for $\forall t \geqslant t_o$, namely

$$F_{2t}(-\infty) = 0 \quad F_{2t}(+\infty) = 1$$

and for $\forall \Delta x>0$

$$F_{2t}(x) \leq F_{2t}(x+\Delta x) \qquad \forall x \geq x_o$$

In the next section, the method for obtaining these approximations will be established, and it will be shown that $F_{1t}(x)$ and $F_{2t}(x)$ could be expressed over the probabilities of the events $E_\nu^{t_o,t}$ and $G_j^{x_o,x}$.

3. APPROXIMATIONS OF DISTRIBUTION

Function $F_t(x)$

1°. In the previous exposition it has been pointed out that instead of the distribution function of the stochastic process $X_t = \pi(t,\omega)$ the corresponding approximations will be used. This approach is justified by the fact that the method for an effective calculation of the conditional probability (4.2.5) is not known, and therefore the distribution function $F_t(x)$ cannot be obtained. On the other hand, in the numerous cases important for practical application, the functions $F_{1t}(x)$ and $F_{2t}(x)$ can be easily obtained. Toward this end, let us prove the following theorem:

Theorem 11.

Let $P(E_\nu^{t_o,t}) > 0$ for every $\nu = 0.\ 1,\ 2,\ldots$ and $t > t_o$. Then the following inequality is valid,

$$F_\nu^*(x,t) \geq \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} | E_\nu^{t_o,t})$$

$$F_\nu^*(x,t) \leq \sum_{j=\nu}^{\infty} P(G_j^{x_o,x} | E_\nu^{t_o,t}) \tag{4.3.1}$$

The proof of this theorem is very simple (see Appendix of this section). Phenomenologically speaking, inequality (4.3.1) means the following: first, as one has seen, functions (4.2.5) represent conditional probability that the total amount of precipitation up to time t will be less or equal to t, under the condition that exactly ν complete storm periods have occurred up to the instant of time t. This probability is greater than the probability that the total amount of precipitation during $(\nu+1)$ storms is less than $(x-x_o)$, under the condition that exactly ν complete storms have occurred in the interval of time (t_o,t). Indeed, on the basis of the first of the inequalities (4.3.1), the following is valid:

$$F_\nu^*(x,t) \geq \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} | E_\nu^{t_o,t}) = \frac{1}{P(E_\nu^{t_o,t})} \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} \cap E_\nu^{t_o t}) =$$

$$= \frac{1}{P(E_\nu^{t_o,t})} P\left[\left(\bigcup_{j=\nu+1}^{\infty} G_j^{x_o,x} \right) \cap E_\nu^{t_o,t} \right]$$

According to Theorem 2, we have

$$F_\nu^*(x,t) \geq \frac{1}{P(E_\nu^{t_o,t})} P\{X_{\nu+1} \leq x,\ E_\nu^{t_o,t}\} =$$

$$= P\{X_{\nu+1} \leq x \mid E_\nu^{t_o,t}\}$$

In a similar manner, it is possible to prove that

$$F_\nu^*(x,t) \leq P\{X_\nu \leq x | E_\nu^{t_o,t}\}$$

Finally, on the basis of (4.3.1) we have

$$\left| F_\nu^*(x,t) - \sum_{j=k}^{\infty} P(G_j^{x_o,x} | E_\nu^{t_o,t}) \right| \leq P(G_\nu^{x_o,x} | E_\nu^{t_o,t}) \tag{4.3.2}$$

where $k = \nu$ or $\nu + 1$. Indeed,

$$F_{\nu}^{*}(x,t) - \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) \leq P(G_{\nu}^{x_o,x} | E_{\nu}^{t_o,t})$$

$$F_{\nu}^{*}(x,t) - \sum_{j=\nu}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) \geq - P(G_{\nu}^{x_o,x} | E_{\nu}^{t_o,t})$$

Therefore, we have

$$F_{\nu}^{*}(x,t) - \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) \leq P(G_{\nu}^{x_o,x} | E_{\nu}^{t_o,t})$$

$$\sum_{j=\nu}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) - F_{\nu}^{*}(x,t) \leq P(G_{\nu}^{x_o,x} | E_{\nu}^{t_o,t})$$

and the assertion follows.

On the basis of (4.2.4) and Theorem 11, obviously the following is valid:

$$F_t(x) \leq \sum_{\nu=0}^{\infty} P(E_{\nu}^{t_o,t}) \sum_{j=\nu}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) =$$

$$= \sum_{\nu=0}^{\infty} \sum_{j=\nu}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x})$$

and

$$F_t(x) \geq \sum_{\nu=0}^{\infty} P(E_{\nu}^{t_o,t}) \sum_{j=\nu+1}^{\infty} P(G_j^{x_o,x} | E_{\nu}^{t_o,t}) =$$

$$= \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x})$$

Therefore, we have

$$\sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x}) \leq F_t(x) \leq \sum_{\nu=0}^{\infty} \sum_{j=\nu}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x}) \tag{4.3.3}$$

The left side of the inequality considered is $F_{1t}(x)$ and the right side is $F_{2t}(x)$, i.e.

$$F_{1t}(x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x})$$

$$F_{2t}(x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x}) \tag{4.3.4}$$

The functions considered represent the lower and upper approximation of the distribution function $F_t(x)$, i.e., we do not know the exact value of probability that up to an instant of time $t > t_o$ the total amount of precipitation X_t will be less than or equal to x, (see Fig. 17), but we know that it will lie between $F_{1t}(x)$ and $F_{2t}(x)$.

On the basis of (4.3.4), obviously the following is valid

$$|F_t(x) - F_{it}(x)| \leq F_{2t}(x) - F_{1t}(x) =$$

$$\sum_{\nu=0}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})$$

Therefore, using any one of the function $F_{it}(x)$ instead of $F_t(x)$, the error is less than or equal to the sum

$$\sum_{\nu=o}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})$$

2^o. Appendix - The first of two proofs of Theorem 11 is based on Theorem 6; the second proof is independent of any previous results.

First proof of Theorem 11.

Obviously, any t such that $t \geq t_o$ must belong to some interval $(\tau_\nu, \tau_{\nu+1})$, i.e., where $\nu = 0, 1, 2, \ldots$ and $\tau_o = t_o$. Assume, for instance, that considering that the moment of time t belongs to the interval $(\tau_\nu, \tau_{\nu+1})$, i.e.,

$$\tau_\nu < t \leq \tau_{\nu+1}$$

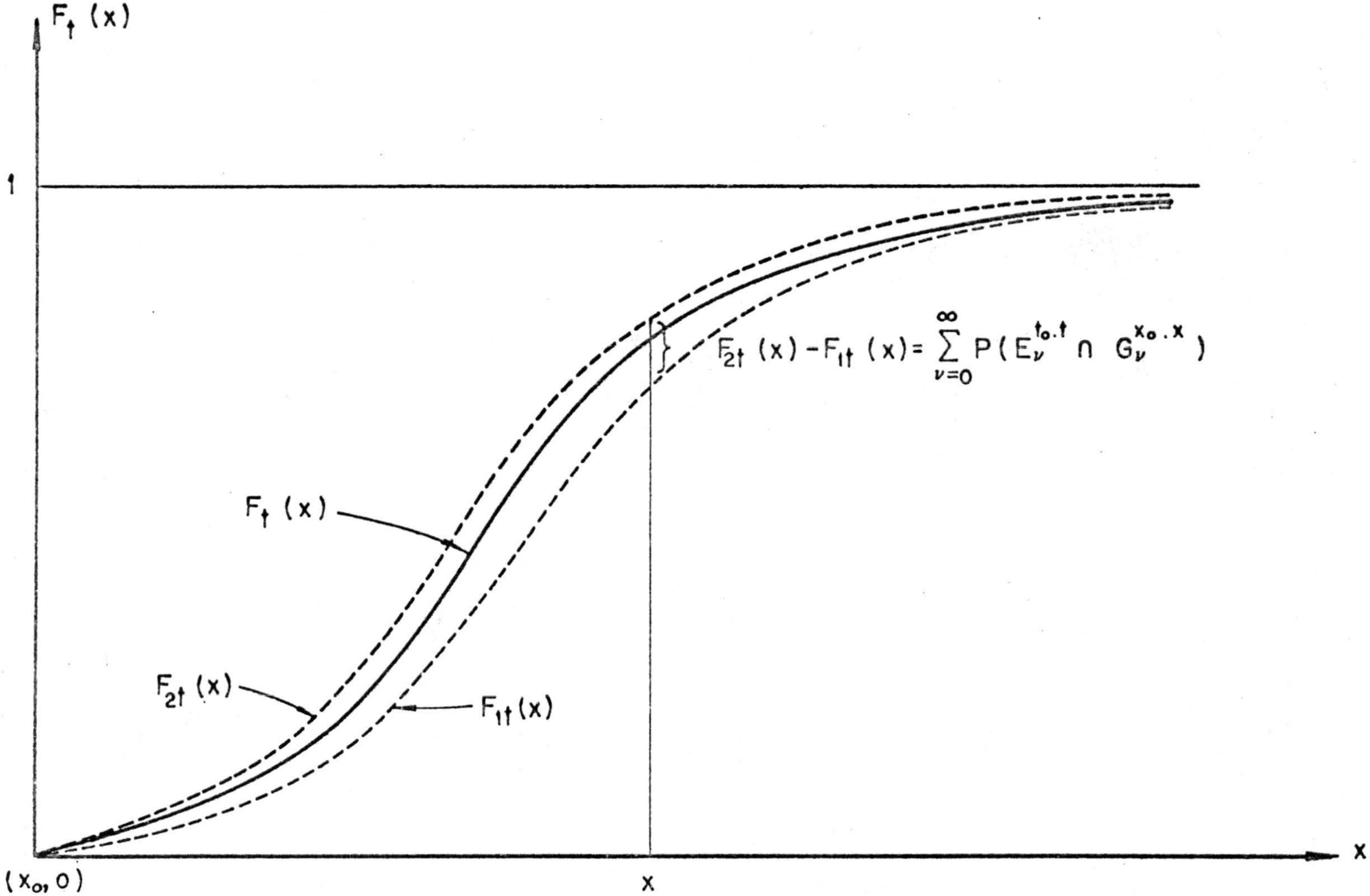

Fig. 17 Graphic Representation of Distribution Function $F_t(x)$ and Its Approximations

then obviously the corresponding amounts of precipitation X_ν , X_t and $X_{\nu+1}$ satisfy the following inequality:

$$X_\nu \leq X_t < X_{\nu+1}$$

Therefore, the next set of relations is valid.

$$\{\omega;\ X_\nu \leq x\} \supseteq \{\omega;\ X_t \leq x\} \supseteq \{\omega;\ X_{\nu+1} \leq x\} \quad ,$$

and consequently,

$$\{\omega;\ X_\nu \leq x\} \cap E_\nu^{t_o,t} \supseteq \{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t} \tag{4.3.5}$$

$$\{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t} \supseteq \{\omega;\ X_{\nu+1} \leq x\} \cap E_\nu^{t_o,t} \tag{4.3.6}$$

Therefore, on the basis of Theorem 6 we have

$$\bigcup_{j=\nu}^{\infty} (G_j^{x_o,x} \cap E_\nu^{t_o,t}) \supseteq \{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t}$$

$$\{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t} \supseteq \bigcup_{j=\nu+1}^{\infty} (G_j^{x_o,x} \cap E_\nu^{t_o,t})$$

i.e.,

$$\bigcup_{\nu=o}^{\infty} \bigcup_{j=\nu}^{\infty} (G_j^{x_o,x} \cap E_\nu^{t_o,t}) \supseteq \bigcup_{\nu=o}^{\infty} \{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t} \tag{4.3.7}$$

$$\bigcup_{\nu=o}^{\infty} \{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t} \supseteq \bigcup_{\nu=o}^{\infty} \bigcup_{j=\nu+1}^{\infty} (G_j^{x_o,x} \cap E_\nu^{t_o,t}) \tag{4.3.8}$$

which proves the theorem.

Note:

Let us try to explain the meaning of approximations $F_{1t}(x)$ and $F_{2t}(x)$. Toward this end, consider relation (4.3.5). Phenomenologically speaking, the left side of (4.3.5) denotes the event that total amount of precipitation during ν storm periods will be less than or equal to x and that exactly ν storms have occurred in the interval of time (t_o,t). Obviously, the following is valid.

$$P\ \{\omega;\ X_\nu \leq x\} \cap E_\nu^{t_o,t} \geq P\ \{\omega;\ X_t \leq x\} \cap E_\nu^{t_o,t}$$

or

$$\sum_{\nu=0}^{\infty} P\{\omega;\ X_\nu \leq x,\ E_\nu^{t_o,t}\} \geq F_t(x) \tag{4.3.9}$$

where

$$\{\omega;\ X_\nu \leq x,\ E_\nu^{t_o,t}\} = \{\omega;\ X_\nu \leq x\} \cap E_\nu^{t_o,t} \quad .$$

The left side of (4.3.9) denotes the probability that the total amount of precipitation of the complete storms in (t_o,t) will be less than or equal to x (see Fig. 8), i.e., we exchange the quantity X_t with X_ν where $\tau_\nu < t < \tau_{\nu+1}$.

The second proof of the theorem can be found in ref. (29).

4. APPROXIMATIONS OF DENSITY FUNCTION $f_t(x)$

4°. Let $f_t(x)$ denote the corresponding density function of the distribution function $F_t(x)$, i.e.

$$f_t(x) = \frac{\partial F_c(x)}{\partial x} \quad .$$

Obviously, then,

$$f_{it}(x) = \frac{\partial F_{it}(x)}{\partial x} \qquad i = 1,2$$

could be assumed as approximations of $f_t(x)$. Of course, among the functions $f_{1t}(x)$, $f_t(x)$, and $f_{2t}(x)$ there is no existing relation of the following form:

$$F_{1t}(x) \leq F_t(x) \leq F_{2t}(x)$$

Let us prove the following theorem:

Theorem 12.

If conditions a) and b) are satisfied, i.e.,

$$\text{a)} \quad \lim_{\Delta x \to 0} \frac{\sum_{\tau=2}^{\infty} P(G_\tau^{x,x+\Delta x})}{\Delta x} = 0$$

$$\text{b)} \quad \lim_{\Delta x \to 0} \frac{\sum_{\nu=0}^{\infty} P(E_k^{t_0,t} \cap G_\nu^{x_0,x} \cap G_1^{x,x+\Delta x})}{\Delta x} = \lambda_2(x,t) \sum_{\nu=0}^{\infty} P(E_k^{t_0,t} \cap G_\nu^{x_0,x})$$

where $k = \nu$, $\nu + 1$, then the following is valid:

$$f_{1t}(x) = \lambda_2(x,t) \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x})$$

$$f_{2t}(x) = \lambda_2(x,t) \sum_{\nu=0}^{\infty} P(E_{\nu+1}^{t_0,t} \cap G_\nu^{x_0,x}) \qquad (4.4.1)$$

The proof of the theorem will be given in the Appendix of this section.

Let us discuss conditions a) and b) which obviously represent the basic hypothesis and as such are of importance in the further investigations. The first of these two conditions was discussed in the previous exposition. Therefore, we shall dwell on the second condition of Theorem 12.

Obviously, it could be written in the following manner:

$$\frac{\sum_{\nu=0}^{\infty} P(E_k^{t_0,t} \cap G_\nu^{x_0,x} \cap G_1^{x,x+\Delta x})}{\Delta x}$$

$$= \sum_{\nu=0}^{\infty} P(E_k^{t_0,t} \cap G_\nu^{x_0,x}) \frac{P(G_1^{x,x+\Delta x} \cap E_k^{t_0,t} \cap G_\nu^{x_0,x})}{\Delta x}$$

so that assuming the following relation is correct

$$\lim_{\Delta x \to o} \frac{\sum_{\nu=o}^{\infty} P(E_k^{t_o,t} \cap G_\nu^{x_o,x} \cap G_1^{x,x+\Delta x})}{\Delta x} =$$

$$\sum_{\nu=o}^{\infty} P(E_k^{t_o,t} \cap G_\nu^{x_o,x}) \lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} \mid E_k^{t_o,t} \cap G_\nu^{x_o,x})}{\Delta x}$$

apparently everything depends on the limit

$$\lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} \mid E_k^{t_o,t} \cap G_\nu^{x_o,x})}{\Delta x} \quad . \tag{4.4.2}$$

The expression

$$P(G_1^{x,x+\Delta x} \mid E_k^{t_o,t} \cap G_\nu^{x_o,x}) \tag{4.4.3}$$

represents the conditional probability that the total amount of precipitation of the $(\nu+1)$ storm period will lie between x and $x+\Delta x$ under condition that in the interval of time (t_o,t) there occurred exactly $k = \nu, \nu + 1$ storm periods, and the total amount of precipitation during previous ν storms is less than or equal to x (and for $(\nu+1)$ storms exceeds x).

Since (4.4.3) depends on ν , t , x and Δx , it would be natural to assume that the function λ_2 depends on ν , t and x , i.e.,

$$\lim_{\Delta x \to o} \frac{P(G_1^{x,x+\Delta x} \mid E_k^{t_o,t} \cap G_\nu^{x_o,x})}{\Delta x} = \begin{cases} \lambda_2^{(1)}(x,t,\nu) & \text{for } k=\nu \\ \lambda_2^{(2)}(x,t,\nu) & \text{for } k=\nu+1 \end{cases}$$

and, consequently, (4.4.1) becomes

$$f_{1t}(x) = \sum_{\nu=o}^{\infty} \lambda_2^{(1)}(x,t,\nu)\; P\,(E_\nu^{t_o,t} \cap G_\nu^{x_o,x})$$

$$f_{2t}(x) = \sum_{\nu=o}^{\infty} \lambda_2^{(1)}(x,t,\nu)\; P\,(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) \tag{4.4.4}$$

The hypothesis that (4.4.2) does not depend on ν leads to the (4.4.1); finally, the assumption that $E_i^{t_o,t}$ and $G_j^{x_o,x}$ are independent events results in

$$\lambda_2(x,t) = \lambda_2(x)$$

(see Appendix) and (4.4.1) becomes

$$f_{1t}(x) = \lambda_2(x) \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t})P(G_\nu^{x_o,x})$$

$$f_{2t}(x) = \lambda_2(x) \sum_{\nu=o}^{\infty} P(E_{\nu+1}^{t_o,t})P(G_\nu^{x_o,x}) \tag{4.4.5}$$

Therefore, on the basis of (2.4.3) and (3.3.2), the following is valid:

$$f_{1t}(x)=\lambda_2(x)e^{-(\int_{x_o}^{x} \lambda_2(s)ds \,+\, \int_{t_o}^{t} \lambda_1(s)ds)}$$

$$\sum_{\nu=o}^{\infty} \frac{(\int_{t_o}^{t}\lambda_1(s)ds)^{\nu}}{\nu!} \; \frac{(\int_{o}^{x}\lambda_2(s)ds)^{\nu}}{\nu!}$$

$$f_{2t}(x)=\lambda_2(x)e^{-(\int_{x_o}^{x} \lambda_2(s)ds \,+\, \int_{t_o}^{t} \lambda_1(s)ds)}$$

$$\sum_{\nu=o}^{\infty} \frac{(\int_{t_o}^{t}\lambda_1(s)ds)^{\nu+1}}{(\nu+1)!} \; \frac{(\int_{x_o}^{x}\lambda_2(s)ds)^{\nu}}{\nu!}$$

2^{o}. Appendix - let us now prove Theorem 12. Toward this end, consider the function $F_{1t}(x)$.

$$F_{1t}(x+\Delta x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j}^{x_o,x+\Delta x}) =$$

$$= \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap \bigcup_{\tau=o}^{j} (G_{j-\tau}^{x_o,x} \cap G_{\tau}^{x,x+\Delta x})) =$$

$$= \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} \sum_{\tau=o}^{j} P(E_{\nu}^{t_o,t} \cap G_{j-\tau}^{x_o,x} \cap G_{\tau}^{x,x+\Delta x})$$

Since,

$$\sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} \sum_{\tau=2}^{j} P(E_{\nu}^{t_o,t} \cap G_{j-\tau}^{x_o,x} \cap G_{\tau}^{x,x+\Delta x}) \leq \sum_{\tau=2}^{j} P(G_{\tau}^{x,x+\Delta x})$$

then by virtue of condition a) of the theorem the following is valid:

$$F_{1t}(x+\Delta x) = \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j}^{x_o,x} \cap G_{o}^{x,x+\Delta x}) +$$

$$= \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j-1}^{x_o,x} \cap G_{1}^{x,x+\Delta x}) + 0(\Delta x)$$

Thus it turns out that

$$\frac{\partial F_{1t}(x)}{\Delta x} \Delta x = - \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j}^{x_o,x}) +$$

$$+ \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j}^{x_o,x} \cap G_{o}^{x,x+\Delta x}) +$$

$$+ \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j-1}^{x_o,x} \cap G_{1}^{x,x+\Delta x}) + 0(\Delta x) =$$

$$= - \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x} - E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_1^{x,x+\Delta x}) +$$

$$+ \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j-1}^{x_o,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

Since we have

$$(G_o^{x,x+\Delta x})^c = \bigcup_{\tau=2}^{\infty} G_{\tau}^{x,x+\Delta x}$$

then the following is valid:

$$P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x} - E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_1^{x,x+\Delta x}) =$$

$$= P[E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap (E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_o^{x,x+\Delta x})^c] =$$

$$= P[E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap (G_o^{x,x+\Delta x})^c] =$$

$$= P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_1^{x,x+\Delta x}) + \sum_{\tau=2}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_{\tau}^{x,x+\Delta x})$$

Therefore, we have

$$\frac{\partial F_{1t}(x)}{\partial x} \Delta x = - \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_j^{x_o,x} \cap G_1^{x,x+\Delta x}) +$$

$$+ \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{j-1}^{x_o,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x) =$$

$$= \sum_{\nu=o}^{\infty} P(E_{\nu}^{t_o,t} \cap G^{x_o,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

from which by condition b) the proof of the first part of the theorem follows. In a similar manner, it is possible to obtain the proof of the second part of the theorem.

On the basis of the Theorem 12 and the following inequality,

$$F_t(x) \le F_{2t}(x) = 1 - \sum_{\nu=1}^{\infty} \sum_{j=o}^{\nu-1} P(E_\nu^{t_o,t} \cap G_j^{x_o,x}) \le 1 ,$$

it follows that

$$F_{2t}(x) \to 1 \quad \forall t \ge t_o$$

if $x \to \infty$, but since $f_{2t}(x) \geqslant 0$ $F_{2t}(x) \uparrow 1$. Finally, it is not difficult to see that

$$F_t(x) \geqslant F_{t+\Delta t}(x) \qquad .$$

Indeed,

$$P\{\omega;\ X_t \le x\} \ge P\{\omega;\ X_{t+\Delta t} \le x\}$$

since $X_t \leqslant X_{t+\Delta t}$, and the assertion follows.

5. CALCULATION PROBABILITIES $P(E_k^{t_o,t} \cap G_\nu^{x_o,x})$

As is seen, the approximations of the density function $f_t(x)$ are expressed over the probabilities

$$P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) \text{ and } P(E_{\nu+1}^{t_o,t} \cap G_\nu^{x_o,x}) \quad \nu=1,1,2,\ldots \qquad (4.5.1)$$

Therefore, for any effective calculation of approximations $f_{1t}(x)$ and $f_{2t}(x)$ it is necessary to possess a method for calculation of (4.5.1).

In the following exposition we shall assume that the conditions are satisfied that

$$\lim_{\Delta t \to o} \frac{P(E_2^{t,t+\Delta t})}{\Delta t} = 0 \qquad \lim_{\Delta x \to o} \frac{P(G_2^{x,x+\Delta x})}{\Delta x} = 0 . \qquad (4.5.2)$$

$$\lim_{\Delta t \to o} \frac{P(E_1^{t,t+\Delta t} | E_k^{t_o,t} \cap G_\nu^{x_o,x})}{\Delta t} = \lambda_1(t) \tag{4.5.3}$$

$$\lim_{\Delta t \to o} \frac{P(G_1^{x,x+\Delta x} | E_k^{t_o,t} \cap G_\nu^{x_o,x})}{\Delta x} = \lambda_2(x) \tag{4.5.4}$$

where $k = \nu, \quad \nu + 1$.

Theorem 13.

Let the function $\rho(x,t,z)$ denote the following sum:

$$\rho(x,t,z) = \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) z^\nu ,$$

then under the assumption that conditions (4.5.2), (4.5.3) and (4.5.4) are satisfied, the following is valid

$$\frac{\partial^2 \rho}{\partial t \partial x} + \lambda_2(x) \frac{\partial \rho}{\partial t} - \lambda_1(t) \frac{\partial \rho}{\partial x} = \lambda_1(t) \lambda_2(x)(1+z)\rho \tag{4.5.5}$$

Proof:

Consider the probability

$$P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) ,$$

then, on the basis of (4.5.2) we have

$$P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x+\Delta x}) = P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x} \cap G_o^{x,x+\Delta x}) +$$

$$+ P(E_\nu^{t_o,t} \cap G_{\nu-1}^{x_o,x} \cap G_1^{x,x+\Delta x}) + 0(\Delta x)$$

Therefore, the following is obtained:

$$\frac{P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x})}{\partial x} \Delta x = -P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x} - E_\nu^{t_o,t} \cap G_\nu^{x_o,x} \cap G_o^{x,x+\Delta x}) +$$

$$+P(E_\nu^{t_o,t}\cap G_{\nu-1}^{x_o,x}\cap G_1^{x,x+\Delta x})+0(\Delta x)=-P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x}\cap G_1^{x,x+\Delta x})+$$

$$+P(E_\nu^{t_o,t}\cap G_{\nu-1}^{x_o,x}\cap G_1^{x,x+\Delta x})+0(\Delta x)$$

so that on the basis of (4.5.4) we have

$$\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial x}=$$

$$\lambda_2(x)[P(E_\nu^{t_o,t}\cap G_{\nu-1}^{x_o,x})-P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})] \qquad (4\ 5.6)$$

In a similar manner, it is possible to obtain

$$\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial t}=$$

$$\lambda_1(t)[P(E_{\nu-1}^{t_o,t}\cap G_\nu^{x_o,x})-P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})] \qquad (4.5.7)$$

therefore,

$$P(E_\nu^{t_o,t}\cap G_{\nu-1}^{x_o,x})=\frac{1}{\lambda_2(x)}\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial x}+P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})$$

$$P(E_{\nu-1}^{t_o,t}\cap G_\nu^{x_o,x})=\frac{1}{\lambda_1(t)}\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial t}+P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})$$

Finally, differentiating (4.5.6) by t we obtain

$$\frac{\partial^2 P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial t\partial x}=\lambda_2(x)\{\lambda_1(t)[P(E_{\nu-1}^{t_o,t}\cap G_{\nu-1}^{x_o,x})-$$

$$P(E_\nu^{t_o,t}\cap G_{\nu-1})-P(E_{\nu-1}^{t_o,t}\cap G_\nu^{x_o,x})+P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})]\}=$$

$$\lambda_1(t)\lambda_2(x)[P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})+P(E_{\nu-1}^{t_o,t}\cap G_{\nu-1}^{x_o,x})+$$

$$+\frac{1}{\lambda_2(x)}\cdot\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial x}-\frac{1}{\lambda_1(t)}\cdot\frac{\partial P(E_\nu^{t_o,t}\cap G_\nu^{x_o,x})}{\partial t}]$$

Multiplying the left and right sides by z^{ν} , we obtain the following partial differential equation:

$$\sum_{\nu=0}^{\infty} \frac{\partial^2 P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})}{\partial t \partial x} z^{\nu} = \lambda_1(t)\ \lambda_2(x) [\ \sum_{\nu=0}^{\infty} P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x}) z^{\nu} +$$

$$\sum_{\nu=1}^{\infty} P(E_{\nu-1}^{t_o,t} \cap G_{\nu-1}^{x_o,x}) z^{\nu} + \frac{1}{\lambda_2(x)} \sum_{\nu=0}^{\infty} \frac{\partial P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})}{\partial x} z^{\nu} -$$

$$\frac{1}{\lambda_1(t)} \sum_{\nu=0}^{\infty} \frac{\partial P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})}{\partial t} z^{\nu}\ ]$$

or

$$\frac{\partial^2 \rho}{\partial t \partial x} = \lambda_1 \cdot \lambda_2\ [(1+z)\rho + \frac{1}{\lambda_2} \frac{\partial \rho}{\partial x} - \frac{1}{\lambda_1} \frac{\partial \rho}{\partial t}]$$

Therefore, we have

$$\frac{\partial^2 \rho}{\partial t \partial x} + \lambda_2(x) \frac{\partial \rho}{\partial t} - \lambda_1(t) \frac{\partial \rho}{\partial x} = \lambda_1(t) \lambda_2(x) (1-z)\rho$$

which proves the theorem.

On the basis of the previous theorem, in order to obtain the probabilities $P(E_{\nu}^{t_o,t} \cap G_{\nu}^{x_o,x})$ for $\nu = 0, 1, 2, \ldots$, it is necessary to solve the partial differential equation of the second order. Its solution will give us the function $\rho(x,t,z)$, from which the probabilities can be obtained. In this manner, we get the approximation $f_{1t}(x)$.

In order to obtain the approximation $f_{2t}(x)$, it is necessary to possess a method for the calculation of the probability

$$P(E_{\nu+1}^{t_o,t} \cap G_{\nu}^{x_o,x}) \quad \nu=0,1,2,\ldots$$

It is not difficult to see that the following is valid:

$$\frac{\partial P(E_{\nu+1}^{t_o,t} \cap G_\nu^{x_o,x})}{\partial t} =$$

$$\lambda_1(t)[P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) - P(E_{\nu+1}^{x_o,x} \cap G_\nu^{x_o,x})]$$

from which it follows that

$$\rho_1(x,t,z) = \sum_{\nu=o}^{\infty} P(E_{\nu+1}^{t_o,t} \cap G_\nu^{x_o,x}) z^\nu$$

where

$$\frac{\partial \rho_1}{\partial t} + \lambda_1 \rho_1 = \lambda \rho_1 \quad .$$

6. SOME PARTIAL DIFFERENTIAL EQUATIONS

In this section of the study we will show that the approximation functions $F_{1t}(x)$ and $F_{2t}(x)$ can be obtained as solutions of partial differential equations of the second order and hyperbolic type. Toward this end, let us prove the following theorem.

Theorem 14.

Assume that the following conditions are satisfied:

a) $$\lim_{\Delta t \to o} \frac{\sum_{\tau=2}^{\infty} P(E_\tau^{t,t+\Delta t})}{\Delta t} = 0$$

b) $$\lim_{\Delta t \to o} \frac{\sum_{\nu=o}^{\infty} P(G_k^{x_o,x} \cap E_\nu^{t_o,t} \cap E_1^{t,t+\Delta t})}{\Delta t} = \lambda_1(x,t) \sum_{\nu=o}^{\infty} P(G_k^{x_o,x} \cap E_\nu^{t_o,t})$$

$(k = \nu,\ \nu + 1)$; then the following is valid:

$$\frac{F_{1t}(x)}{\partial t} = - \lambda_1(x,t) \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t} \cap G_{\nu+1}^{x_o,x})$$

$$\frac{F_{2t}(x)}{\partial t} = - \lambda_1(x,t) \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t} \cap G_\nu^{x_o,x}) \qquad (4.6.1)$$

Proof:

The proof of Theorem (14) can be obtained in a manner similar to the proof of Theorem (8). If we consider the function $F_{1,t}(x)$ first, it is not difficult to see that

$$F_{1,t+\Delta t}(x) = \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_0,t+\Delta t} \cap G_j^{x_0,x}) =$$

$$= \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P[G_j^{x_0,x} \cap \bigcup_{\tau=0}^{j} (E_{\nu-\tau}^{t_0,t} \cap E_{\tau}^{t,t+\ t})]=$$

$$= \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} \sum_{\tau=0}^{j} P(G_j^{x_0,x} \cap E_{\nu-\tau}^{t_0,t} \cap E_{\tau}^{t,t+\Delta t})$$

On the basis of condition a) of the theorem, we have the following:

$$F_{1,t+\Delta t}(x) = \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(G_j^{x_0,x} \cap E_{\nu}^{t_0,t} \cap E_0^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(G_j^{x_0,x} \cap E_{\nu-1}^{t_0,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

and consequently,

$$\frac{\partial F_{1t}(x)}{\partial t} \Delta t = - \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_0,t} \cap G_j^{x_0,x} - E_{\nu}^{t_0,t} \cap G_j^{x_0,x} \cap E_0^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(G_j^{x_0,x} \cap E_{\nu-1}^{t_0,t} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) =$$

$$= - \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} [P\ E_{\nu}^{t_0,t} \cap G_j^{x_0,x} \cap (E_0^{t,t+\Delta t})^c] +$$

$$+ \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu-1}^{t_0,t} \cap G_j^{x_0,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) =$$

$$= - \sum_{\nu=0}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu}^{t_0,t} \cap G_j^{x_0,x} \cap E_1^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=o}^{\infty} \sum_{j=\nu+1}^{\infty} P(E_{\nu-1}^{t_o,t} \cap G_j^{x_o,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta x) =$$

$$= \{- \sum_{j=1}^{\infty} P(E_o^{t_o,t} \cap G_j^{x_o,x} \cap E_1^{t,t+\Delta t}) - \sum_{j=2}^{\infty} P(E_1^{t_o,t} \cap G_j^{x_o,x} \cap E_1^{t,t+\Delta t}) - \ldots\}$$

$$+ \{ \sum_{j=1}^{\infty} P(E_{-1}^{t_o,t} \cap G_j^{x_o,x} \cap E_1^{t,t+\Delta t}) + \sum_{j=2}^{\infty} P(E_o^{t_o,t} \cap G_j^{x_o,x} \cap E_1^{t,t+\Delta t}) + \ldots\} +$$

$$+ 0(\Delta t) = - \sum_{\nu=1}^{\infty} P(E_{\nu-1}^{t_o,t} \cap G_\nu^{x_o,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) =$$

$$= - \sum_{\nu=o}^{\infty} P(E_\nu^{t_o,t} \cap G_{\nu+1}^{x_o,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

Therefore, on the basis of condition b), the first part of the theorem follows. In a similar manner, it is possible to obtain the proof of the second part.

On the basis of Theorem 12 and Theorem 14, it is possible to prove that approximations $F_{1,t}(x)$ and $F_{2,t}(x)$ could be obtained as solutions of corresponding partial differential equations of the second order and hyperbolic type. Proof of this assertion gives the following theorem.

Theorem 15.

If the conditions a) and b) of Theorems 12 and 14 are satisfied, then the following is valid:

$$\begin{aligned} &\frac{\partial F_{1t}(x)}{\partial x \partial t} + \psi_1(x,t) \frac{\partial F_{1t}(x)}{\partial x} + \psi_2(x,t) \frac{\partial F_{1t}(x)}{\partial t} = 0 \\ &\frac{\partial F_{2t}(x)}{\partial x \partial t} + \bar{\Psi}_1(x,t) \frac{\partial F_{2t}(x)}{\partial x} + \bar{\Psi}_2(x,t) \frac{\partial F_{2t}(x)}{\partial t} = 0 \end{aligned} \tag{4.6.2}$$

where

$$\psi_1(x,t) = \lambda_1(x,t) - \frac{\partial\lambda_2(x,t)}{\partial t} \qquad \psi_2(x,t) = \lambda_2(x,t)$$

$$\bar{\psi}_1(x,t) = \lambda_1(x,t) \qquad \bar{\psi}_2(x,t) = \lambda_2(x,t) - \frac{\partial \ln\lambda_1(x,t)}{\partial x}$$

Proof:

First, consider the approximation $F_{1t}(x)$; differentiating by t, the following relation will be obtained:

$$\frac{\partial F_{1t}(x)}{\partial t\partial x} = \frac{\partial\lambda_2(x,t)}{\partial t} \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) +$$

$$+ \lambda_2(x,t) \frac{\partial}{\partial t} \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x})$$

If we set

$$\alpha(x,t) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) \qquad ,$$

then obviously the following is valid:

$$\alpha(x,t+\Delta t) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t+\Delta t} \cap G_\nu^{x_0,x}) =$$

$$\sum_{\nu=0}^{\infty} \sum_{\tau=0}^{\infty} P(E_{\nu-\tau}^{t_0,t} \cap G_\nu^{x_0,x} \cap E_\nu^{t,t+\Delta t})$$

On the basis of condition a) of Theorem 14, we have

$$\alpha(x,t+\Delta t) = \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x} \cap E_0^{t,t+\Delta t}) +$$

$$\sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

Therefore,

$$\frac{\partial\alpha(x,t)}{\partial t}\Delta t = -\sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) + \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=0}^{\infty} P(E_{\nu-1}^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

$$= - \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x} - E_\nu^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=0}^{\infty} P(E_{\nu-1}^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t) = - \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) +$$

$$+ \sum_{\nu=0}^{\infty} P(E_{\nu-1}^{t_0,t} \cap G_\nu^{x_0,x} \cap E_1^{t,t+\Delta t}) + 0(\Delta t)$$

so that, on the basis of condition b) of the Theorem 14, we have

$$\frac{\partial\alpha(x,t)}{\partial t} = - \lambda_1(x,t) \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) +$$

$$+ \lambda_1(x,t) \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x})$$

If we use Theorem 12, then the following is obtained:

$$\frac{\partial\alpha(x,t)}{\partial t} = - \frac{\lambda_1(x,t)}{\lambda_2(x,t)} \frac{\partial F_{2t}(x)}{\partial x} - \frac{\partial F_{1t}(x)}{\partial t}$$

Therefore,

$$\frac{\partial (F_{1t}(x)}{\partial t \partial x} = - \frac{\partial \ln \lambda_2(x,t)}{\partial t} \frac{\partial F_{1t}(x)}{\partial x} + \lambda_1(x,t) \frac{\partial F_{1t}(x)}{\partial t} -$$

$$- \lambda_2(x,t) \frac{\partial F_{1t}(x)}{\partial t}$$

from which the first part of the theorem follows.

In order to prove the second part of the theorem, let us differentiate by x the function

$$\frac{\partial F_{2t}(x)}{\partial t}$$

$$\frac{\partial F_{2t}(x)}{\partial x \partial t} = - \frac{\partial \lambda_1(x,t)}{\partial x} \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) -$$

$$- \lambda_1(x,t) \frac{\partial}{\partial x} \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x})$$

In a manner similar to that in the previous exposition, it can be shown that the following is valid:

$$\frac{\partial}{\partial x} \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) =$$

$$= \frac{\lambda_2(x,t)}{\lambda_1(x,t)} \cdot \frac{\partial F_{2t}(x)}{\partial t} + \frac{\partial F_{2t}(x)}{\partial x}$$

from which the proof of the theorem follows.

Note:

In the case when

$$\lambda_1 = \lambda_1(t) \quad \text{and} \quad \lambda_2 = \lambda_2(x) \qquad ,$$

we have

$$\psi_1 \equiv \lambda_1(t) \qquad\qquad \psi_2 \equiv \lambda_2(x)$$

$$\bar{\psi}_1 \equiv \lambda_{1(t)} \qquad\qquad \bar{\psi}_2 \equiv \lambda_{2(x)}$$

so that the system (4.2.6) reduces to a single partial differential equation of the second order and hyperbolic type,

$$\frac{\partial F_t(x)}{\partial x \partial t} + \lambda_1(t) \frac{\partial F_t(x)}{\partial x} + \lambda_2(x) \frac{\partial F_t(x)}{\partial t} = 0 \qquad (4.6.3)$$

Neither the solution of the system (4.2.6) nor the solution of (4.6.3) will be considered in this study.

7. SOME IMPORTANT NOTES

In previous sections of this chapter, the random variable X_t has been considered to represent the total amount of precipitation up to the moment of time t. As we have seen, X_t is a random variable for every $t > t_o$, such that $0 \leqslant X_t \leqslant X_{t+\Delta t}$; therefore, we have a continuous parameter family of random variables or a continuous parameter stochastic process of nondecreasing sample functions.

As has been shown, the study of some important characteristics of the precipitation phenomenon can be reduced to the study of the corresponding properties of the stochastic process. Therefore, one of the main objectives of this study has been development of the process X_t, its analysis, the calculation of corresponding one-dimensional distribution functions, and their application to the problem of precipitation phenomenon. In fact, the discussion has been limited in this study to those problems of stochastic process which are closely related to the problem of the rainfall phenomenon.

In the following exposition, we will consider one other aspect of the precipitation phenomenon. Toward this end, let us denote by T_x times indispensible for an amount of precipitation x to be reached. Apparently T_x is a random variable for every $x > x_o$, such that

$$0 \leqslant T_x \leqslant T_{x+\Delta x} \qquad \forall \Delta x > 0$$

and therefore represents a stochastic process of nondecreasing sample functions.

$$\{T_x;\ x > x_o\}$$

If $F^*_x(t)$ represents the corresponding one-dimensional distribution function, i.e.,

$$F^*_x(x) = P\{\omega;\ T_x \leq t\}$$

where $t > t_o$, then obviously the following is valid:

$$P\{\omega;\ T_x \leq t\} = P\{\omega;\ X_t > x\} =$$
$$1 - P\{\omega;\ X_t \leq x = 1 - F_t(x) \qquad (4.7.1)$$

(see Fig. 18).

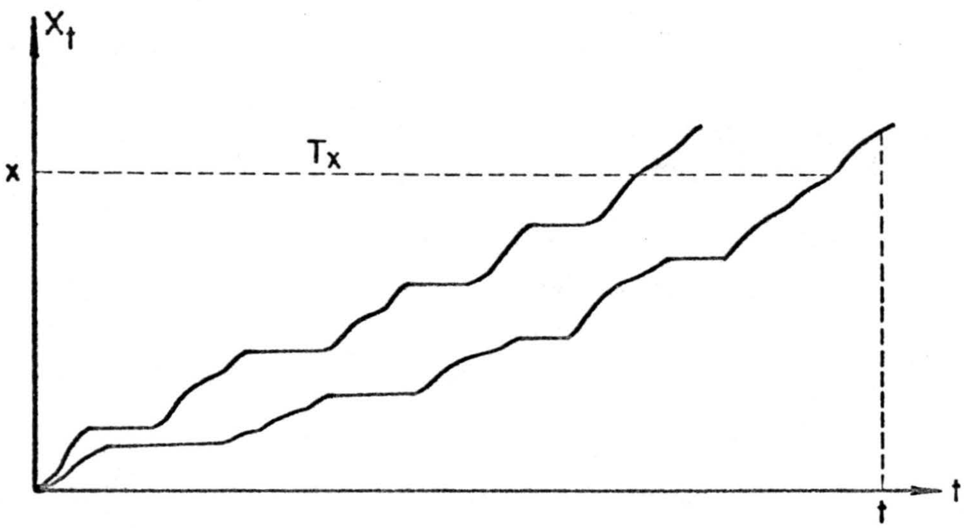

Fig. 18 Graphical interpretation of the relation (4.7.1)

$$F^*_x(t) = 1 - F_t(x)$$

If $f^*_x(t)$ is the corresponding density function, i.e.,

$$f^*_x(t) = \frac{\partial F^*_x(t)}{\partial t} \quad ,$$

then obviously

$$f^*_x(t) = - \frac{\partial F_t(x)}{\partial t} \quad .$$

Further, by virtue of the following inequality

$$1 - F_{1,t}(x) \geq F_x^*(t) \geq 1 - F_{2,t}(x)$$

we have that the upper $F_{2x}^*(t)$ and lower $F_{1x}^*(x)$ approximations of $F_x(t)$ are of the following form:

$$F_{1x}^*(t) = 1 - F_{2t}(x) \tag{4.7.2}$$

$$F_{2x}^*(t) = 1 - F_{1t}(x) \tag{4.7.3}$$

If one denotes

$$f_{ix}^*(t) = \frac{\partial F_{ix}^*(t)}{\partial t} \qquad i=1.2 \qquad ,$$

approximations of the density function $f_x^*(t)$ are

$$\begin{aligned} f_{1.x}^*(t) &= -\frac{\partial F_{2.t}(x)}{\partial t} \\ f_{2.x}^*(t) &= -\frac{\partial F_{1.t}(x)}{\partial t} \end{aligned} \tag{4.7.4}$$

Therefore, by virtue of (4.6.1) we have

$$\begin{aligned} f_{1.x}^*(t) &= \lambda_1(x,t) \sum_{\nu=0}^{\infty} P(E_{\nu+1}^{t_0,t} \cap G_\nu^{x_0,x}) \\ f_{2.x}^*(t) &= \lambda_1(x,t) \sum_{\nu=0}^{\infty} P(E_\nu^{t_0,t} \cap G_\nu^{x_0,x}) \end{aligned} \tag{4 7.5}$$

etc.

Chapter V

APPLICATIONS

In this part of the paper, mathematical methods described in previous sections will be applied to an analysis of rainfall data collected over 54 years at the Austin, Texas precipitation station during the period 1914-1967. Because the chief aim of this section is not complete analysis of the rainfall data but to use the data to test theoretical results, only some aspects of precipitation phenomenon will be considered.

The most desirable data for this purpose would be the continuous long-term precipitation records. These, however, were not available, and, as an alternative, the daily rainfall records have been studied. Although these records do not give complete information about the rainfall phenomenon, they can serve as valuable tools for an orientational investigation.

In the following exposition data concerning the number of storms and the termination time of the first and second storm period will be objects of an analysis. Since we have daily precipitation records instead of the number of storm periods the number of stormy days in an interval of time will be investigated.

Toward this end, consider the distribution of the stormy days during the period of the first five days in January (Table 1). Since this period of time is sufficiently small, the effect of seasonal variations is practically negligible so that the function $\lambda_1(t)$ could be assumed as a constant λ_1 . On the basis of the records (see Table 1) an estimation of the parameter $\lambda_1 \simeq 0.181$, hence taking for $t = 5$

(five days) by virtue of (2.5.7) we obtain theoretical (in Table 1 expected) values. In a similar manner the distributions for the first ten and for the first fifteen days have been obtained (see Tables 2 and 3). In Fig. 19 a graphical presentation of these distributions is given.

The values of the parameter λ_1 for the first, second and third distributions are 0.181, 0.200 and 0.208, respectively. Therefore, it could be assumed that influence of the seasonal variation is negligible during these fifteen days.

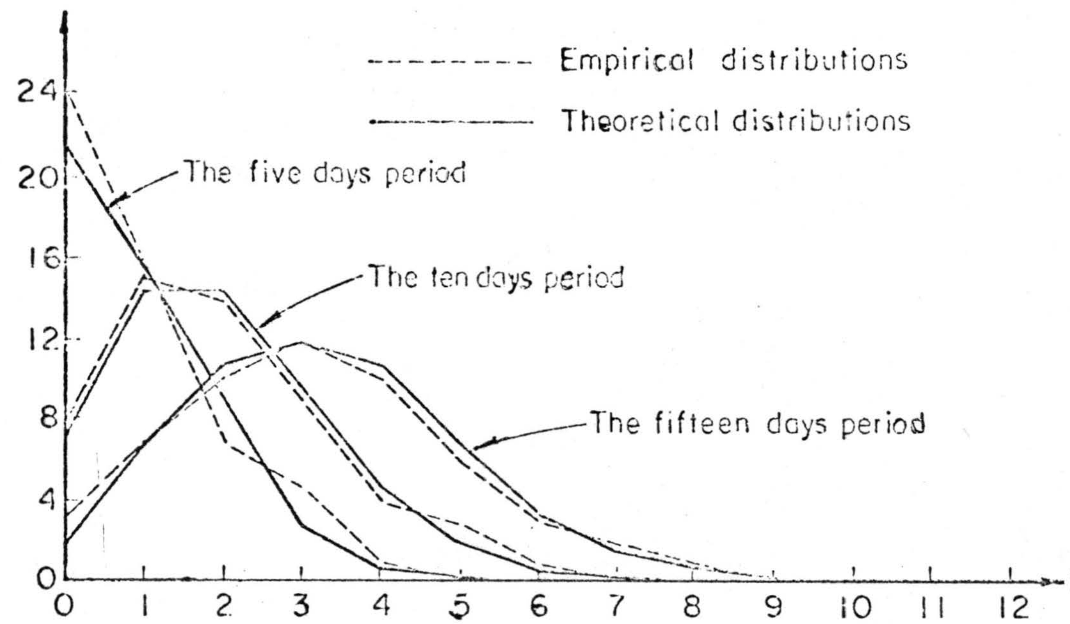

Fig. 19 Theoretical and empirical distributions of the number of stormy days

Under the assumption that for January an estimation of $\lambda_1 \approx 0.2$, it is very easy to obtain the corresponding distributions for τ_1 and τ_2. In Table 4 observed and expected frequency distributions for τ_1 and τ_2 calculated on the basis of (5.3) for $\lambda_1 \approx 0.2$ are presented. In Figs. 20 and 21 a graphical presentation of these distributions is given.

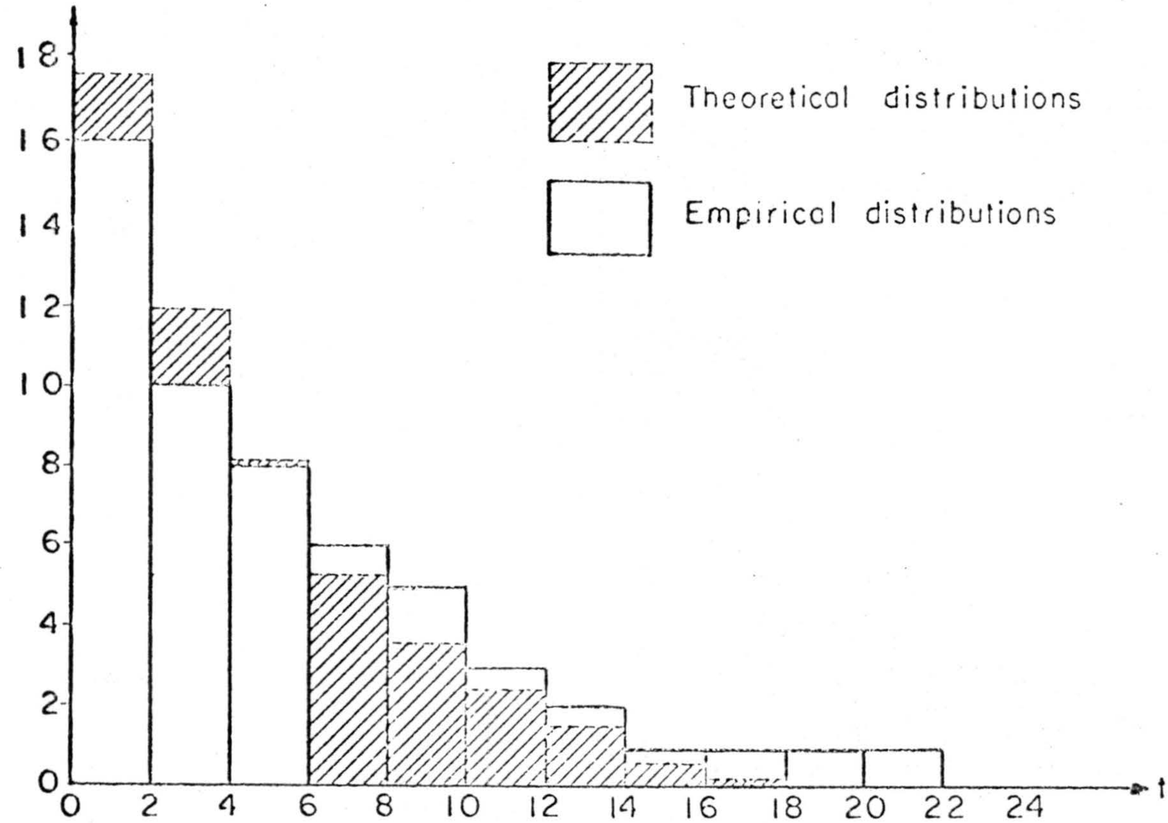

Fig. 20 Graphical representation of expected and observed distributions of random variable τ_1

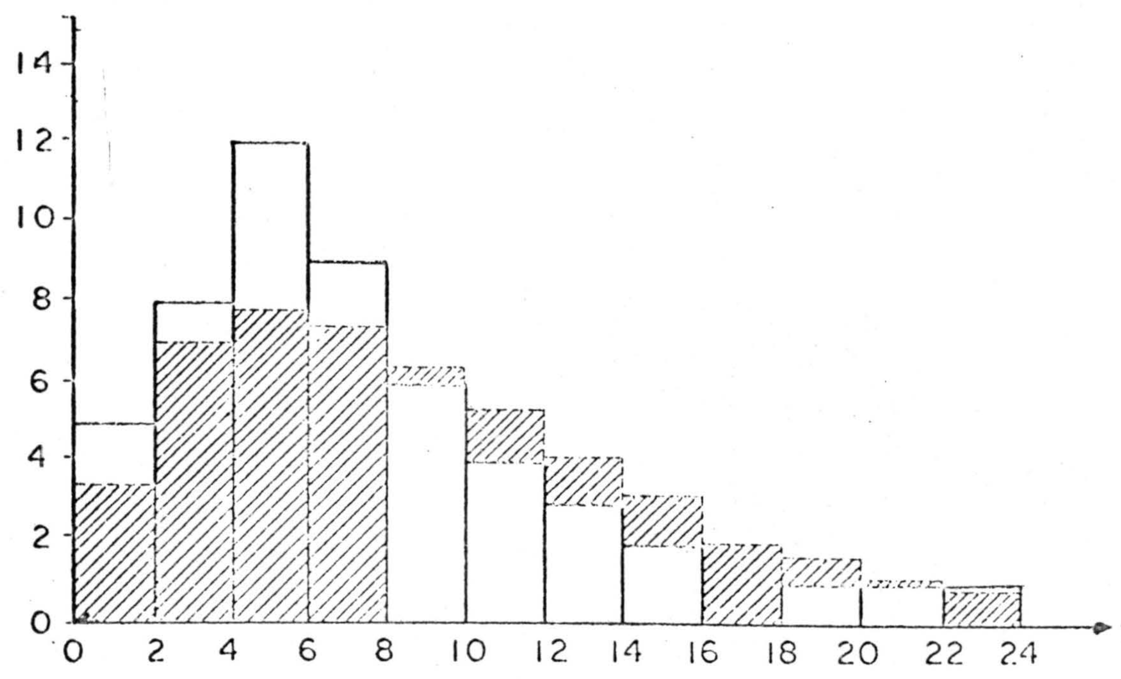

Fig. 21 Graphical representation of expected and observed distributions of random variable τ_2

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TABLES

Table 1. Number of stormy days during the first five days in January for period 1914-1967. $\lambda_1 \approx 0.181$

Number of stormy days	0	1	2	3	4	5
Observed	25	16	7	5	1	0
Expected	21.95	15.98	8.89	2.67	0.60	0.00

Table 2. Number of stormy days during the first ten days in January for period 1914-1967. $\lambda_1 \approx 0.2$

Number of stormy days	0	1	2	3	4	5	6	7	8	9	10
Observed	8	15	14	9	4	3	1	0	0	0	0
Expected	7.306	14.618	14.618	9.741	4.871	1.949	0.648	0.179	0.048	0.00	0.00

Table 3. Number of stormy days during the first fifteen days in January for period 1914-1967. $\lambda_1 \approx 0.208$

Number of stormy days	0	1	2	3	4	5	6	7	8	9	10	11	12
Observed	3	7	10	12	9	6	3	2	1	0	0	0	0
Expected	1.992	6.572	10.843	11.934	9.844	6.496	3.575	1.685	0.697	0.178	0.005	0.000	0.000

Table 4.

	Frequency distributions of random variable τ_1		Frequency distributions of random variable τ_2	
	observed	expected	observed	expected
0 - 2	16	17.803	5	3.328
2 - 4	10	11.934	8	7.000
4 - 6	8	8.013	12	7.824
6 - 8	6	5.318	9	7.452
8 - 10	5	3.596	6	6.416
10 - 12	3	2.408	4	5.312
11 - 14	2	1.614	3	4.168
14 - 16	1	1.080	2	3.220
16 - 18	1	0.729	2	2.464
18 - 20	1	0.486	0	1.840
20 - 22	1	0.324	1	1.356
22 - 24			1	1.017
24 - 26			1	0.728
	54	53.305	54	52.185