HAIL DATA FROM A FIXED NETWORK FOR THE EVALUATION OF A HAIL MODIFICATION EXPERIMENT

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ABSTRACT

Hailfall data collected from a fixed network in northeastern Colorado during three seasons (1960-62) included the estimated impact energy, duration of hailfall, most common stone size, maximum stone size, and number of stones per square inch. These basic data, X , along with the transformations; ln X, $\sqrt{X}$, $\sqrt[3]{X}$, and 1/X, were analyzed by computer methods to determine which parameters could be used in a statistical analysis of hail suppression experiment. The gamma distribution function was fitted to the hailfall data by the method of maximum likelihood. A chi-square goodness of fit test was applied to the data, and one transformation was tested using a sequential analysis technique.

All parameters except impact energy and number of hailstones per square inch were eliminated from the statistical analysis because of bias, non-homogeneity, or sparsity of samples. Transformations which produced the minimum mean coefficient of variation were logarithm of impact energy (ln E) and square root of the number of stones per square inch ($\sqrt{N_{1-6}}$). It was determined that a target-control analysis was not feasible for the analysis of hail suppression experiment. A period of 3 to 5 years is believed necessary to detect changes of 10 to 25 percent in the hail parameters. The gamma distribution function fitted only the $\sqrt{N_{1-6}}$ data. From the results it was concluded that a sequential analysis test alone could not adequately evaluate the effectiveness of a hail modification experiment.

HAILFALL DATA FROM A FIXED NETWORK FOR THE EVALUATION OF A HAIL MODIFICATION EXPERIMENT*

1. Introduction

Development of techniques for cloud modification by cloud seeding has led to a variety of attempts at weather modification, including precipitation increase, hurricane modification, fog dispersal, and hail suppression. The natural variability of meteorological phenomenon is such that detection of any small change which might have been effected artificially is difficult. This difficulty is increased with the variability of the meteorological phenomenon being considered, and leads to frustration for cases in which high variance and low frequencies of occurrence combine to require excessive periods of time to draw valid conclusions concerning the effects of modification attempts.

These difficulties are further compounded for evaluation of attempts of hail suppression because of a lack of basic data on the nature and characteristics of hailfalls. For example, the only statistic concerning hailfalls which is readily available is "days with hail." While this parameter may serve to delimit regional differences in the average annual frequency of hail, it leaves much to be desired as a statistic which would be appropriate for detecting changes in hailfalls which might have been produced artificially.

This paper presents a procedure for the evaluation of hail suppression using data on hailfalls for three seasons (1960-1962) from the fixed hail network operated in northeastern Colorado by Colorado State University. Examination of these data provides an insight into some of the physical properties of the High Plains hailstorm, delimits the hailfall parameters which might be used in the design of a hail modification experiment, and points out the problems inherent in and requirements of a statistical analysis of a hail modification experiment.

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2. Basic Data

From information on hailfalls obtained from the Colorado State University network see Schleusener and Jennings 1960, Schleusener 1961a, and Schleusener and Grant 1961b. The following data were derived and prepared for computer analysis:

From cooperative observers:

Estimated impact energy, ft-lb per ft^2	E_c
Duration of hailfall, minutes	D*
Most common stone size (coded)	MC
Maximum stone size (coded)	MX

From the passive hail indicators:

Impact energy, ft-lb per ft^2	E_i
Number of stones per square inch for (coded) diameters of hailstones	
1 - 6	N_{1-6}
2 - 6	N_{2-6}
3 - 6	N_{3-6}
4 - 6	N_{4-6}

These hail parameters were analyzed from the viewpoint of attempting to identify parameters which, on the basis of physical reasoning, might be changed if cloud seeding were effective in modification of hail.

3. Transformations of Basic Data

Transformations were applied to the basic data, to attempt to produce a minimum absolute dispersion of the individual coefficients of variation about the mean coefficient of variation $(\Sigma|Cv_i-\overline{C}v|)$. In addition to the basic observations, X ,

*D was removed from further analysis because of excessive bias. Most cooperators reported D in 5 to 10 minute increments.

consideration was given to each of the following transformations which were applied to each of the observations:

$$\ln X, \; -\sqrt{X}, \; \sqrt[3]{X} \; \text{and} \; 1/X \; .$$

The selected transformations were $\ln X$ for E_c and E_i and $-\sqrt{X}$ for MC, MX, N_{1-6}, N_{2-6}, N_{3-6}, and N_{4-6}. The value of the mean, standard deviation, and the number of samples was determined for each transformed hail parameter set plus subsets by years, months and geographic locations. The transformations decreased the variance of the parameters considerably but the coefficient of variation continued to be greater than unity in most cases.

4. Tests of Homogeneity of Data

The variables and transformed variables were tested by means of an F test for homogeneity among years and among months by a one way analysis of variance.

From these analyses (Table 1) it was determined that the hypothesis of homogeneity among years for $\ln E_i$ $-\sqrt{MC}$, $-\sqrt{N_{1-6}}$, and $-\sqrt{N_{2-6}}$ must be rejected. The hypothesis of homogeneity among months must also be rejected for $-\sqrt{MC}$ and $-\sqrt{N_{1-6}}$. On this basis and since $-\sqrt{MX}$ (not listed in Table 1) was more variable than $-\sqrt{MC}$, the parameters $-\sqrt{MC}$, $-\sqrt{MX}$ and $-\sqrt{N_{2-6}}$ were rejected from further analysis. The hypothesis of homogeneity among years for $\ln E_i$ when using data from all of the hail network was rejected whereas when using only data from the west triangle of the network it could not be rejected. The hypothesis of homogeneity among years and months for $\ln E_c$, and $-\sqrt{N_{3-6}}$ could not be rejected. The parameters $-\sqrt{N_{3-6}}$ and $-\sqrt{N_{4-6}}$ were rejected from further analysis due to low frequencies of occurrence.

The remaining sets of parameters, consisting of $\ln E_i$, $-\sqrt{N_{1-6}}$ (retained for analysis despite homogeneity considerations noted above), and $\ln E_c$, were arbitrarily divided into subsets from east to west and subsets from north to south. The hypothesis of homogeneity among east-west subsets and north-south subsets could not be rejected; consequently it was decided to add a 60 mile north-south line of indicators through the west half of the network for the 1963 season.

TABLE 1

F Test Results for Significance of Differences

Between Parameter Subsets

Parameter		Among years	Among months	Between east and west			Between north and south		
				60	61	62	60	61	62
$\ln E_c$	F	0.90	0.34	4.00**	2.86*	1.72	3.94**	2.39	2.77**
	df	(2x963)	(2x1011)	(4x324)	(4x504)	(4x283)	(3x261)	(3x348)	(3,502)
$\ln E_i$	F	5.78**	0.72	1.11	9.52**	1.63	1.16	4.13*	1.38
	df	(3x761)	(2x726)	(2x91)	(2x275)	(2x328)	(2x49)	(2x222)	(2x423)
$-\sqrt{MC}$	F	51.00**	19.60**						
	df	(2x1180)	(2x1095)						
$-\sqrt{N_{1-6}}$	F	4.20*	12.90**	1540**	1.30	1.15			
	df	(2x757)	(2x717)	(1x111)	(1x196)	(1x453)			
$-\sqrt{N_{2-6}}$	F	18.55**	2.69	4.0*	4.4*	2.73			
	df	(2x585)	(2x571)	(1x74)	(1x152)	(1x362)			
$-\sqrt{N_{3-6}}$	F	1.18	0.48	0.24	5.70*	0.04			
	df	(2x231)	(2x228)	(1x15)	(1x59)	(1x160)			
$-\sqrt{N_{4-6}}$	F	3.17*							
	df	(2x57)							
$\ln E_i$ (west triangle)	F	1.28							
	df	(2x403)							

NOTES:

* Significant at 0.05 level

** Significant at 0.01 level

5. Effect of Size of Sampling Area on Variability of Parameters

The coefficient of variation was computed as a function of sampling area for the 3 remaining hail parameters. (Figures 1, 2 and 3) From these computations it was determined that the variance of the hail parameters would be changed very little if only half of the sampling area was used. For some of the hail parameters, a reduction in sampling area produced a decrease in variance. From these results and the results of the homogeneity tests on $\ln E_i$ it was concluded that the size of the sampling area could be reduced, and the east half of the indicator network was abandoned prior to the beginning of the 1963 season.

6. Tests for Normality of Parameters

The $-\sqrt{N_{1-6}}$, $\ln E_i$ and $\ln E_c$ data sets and subsets by years, were tested for normality. In nearly all sets and subsets the data were highly skewed right, with the kurtosis ranging from leptokurtic to isokurtic to platykurtic, (Figures 4, 5 and 6).

7. Tests for Independence of Observations

Correlation coefficients were computed between certain combinations of the hail parameters averaged over one region versus the parameters averaged over other regions approximately 25 miles away. The results indicate no significant correlation.

Correlation coefficients were also computed for the hail parameters $\ln E_i$ and $-\sqrt{N_{1-6}}$ between each dented indicator and its neighbor (and mean of neighbors) located approximately 2 and 4 miles away. Although the correlation coefficients were all less than 0.50, the hypothesis of dependence could not be rejected (Table 2).

8. Tests to Estimate the Period of Time Required to Detect Scale Changes in the Hail Parameters

Computations were made to estimate the period of time required to obtain significant differences in the hail parameters, assuming various scale changes in the parameters (Appendix A). These computations assume that negative reports of hail occurrence can be obtained in those cases in which

TABLE 2

Number of Significant and Non-significant Correlations
Between a Dented Hail Indicator and
its Neighbor (Neighbors) Located 2(4) Miles Away

Correlation	2 Miles		4 Miles	
	Neighbors	Neighbor	Neighbors	Neighbor
NS	8	14	13	16
S	4	4	3	3
S*	12	6	8	5

Note: NS = No significant correlation (0.05 level)
S = Significant correlation (0.05 to 0.01 level)
S* = Significant correlation (0.01 level)

complete hail suppression might be attained, that all hail-producing storms would have been subjected to a modification treatment during the period of 15 May to 31 July, and that the average number of hail samples will remain constant. It may be noted that "success" in a hail modification experiment would increase difficulty of statistical analysis because of an increase in zero values.

When these three (questionable) assumptions are made, a period of 3 to 5 years is estimated to detect scale changes of 10 to 25 percent in the hail parameters. (Tables A1, A2, and A3)

9. Gamma Distribution Parameters

Hartley and Lewish (1959) have reported fitting a two-parameter gamma distribution function to non-zero rainfall data. The first step in fitting the gamma distribution function to the data is to estimate the distribution parameters. The most efficient method for estimating these distribution parameters is the method of maximum likelihood. Maximum likelihood estimators are obtained by differentiating the likelihood function and equating the derivative to zero. Greenwood and Durand (1960) have solved the complex equations in this procedure using polynominal approximations. Since this method involves a logarithm of the data, it can be used only with data that are greater than zero. To eliminate zero values from the observed hail data, which are either positive or zero, 0.01 was added to each data sample.

The gamma distribution function was fitted to the following data:

$$E_i,\ \ln E_i,\ \ln E_i \ \text{ for } \ (E_i > 1),\ N_{1-6},\ \sqrt[2]{N_{1-6}},\ \text{and}\ -\sqrt{N_{1-6}}\ .$$

Gamma distribution parameters for these data are given in Table 3.

10. Test for Goodness of Fit

A chi-square goodness of fit test was used to compare the distribution of the hailfall data to a theoretical gamma distribution.

In applying this test, boundaries were selected such that the number of expected values exceeded 5 in each category. It is permissible to set boundaries in any desired manner, provided one is not influenced by the observed frequencies. In applying this test an attempt was made to obtain

TABLE 3

Gamma Distribution Parameters for Hailfall Data

Data	Gamma Parameters	
	Shape Parameter, γ	Scale Parameter, β
E_i	0.8211	16.1812
$\ln E_i$	0.9292	2.053
$\ln E_i \ (E_i > 1)$	3.5488	0.6070
N_{1-6}	1.1587	2.7071
$\sqrt{N_{1-6}}$	4.2209	0.3727
$.95x\sqrt{N_{1-6}}$	4.2210	0.3541
$.90x\sqrt{N_{1-6}}$	4.2209	0.3354
$.85x\sqrt{N_{1-6}}$	4.2210	0.3168
$.75x\sqrt{N_{1-6}}$	4.2210	0.2795
$1.05x\sqrt{N_{1-6}}$	4.2210	0.3913
$1.25x\sqrt{N_{1-6}}$	4.2210	0.4659
$\sqrt[3]{N_{1-6}}$	9.2925	0.1421

TABLE 4

Cell Boundaries and Chi-Square Values for the Goodness of Fit Test of Hailfall Data

Type of Data	Cell Boundaries							Calculated χ^{2*}	Remarks
	Cell 1	Cell 2	Cell 3	Cell 4	Cell 5	Cell 6	Cell 7		
E_i	0.00-1.90**	1.90-2.70	2.70-3.90	3.90-6.70	6.70-13.10	13.10-25.10	25.10-400.1	122.1	
E_i	0.00-1.30	1.30-2.70	2.70-3.90	3.90-7.50	7.50-14.10	14.10-25.1	25.1 -330.1	56.1	2nd Run on E_i
$\ln E_i$	0.00-0.61	0.61-1.01	1.01-1.26	1.26-1.51	1.51-1.86	1.86- 3.46	3.46- 6.01	208.8	
$\ln E_i$	0.00-2.41	2.41-2.91	2.91-3.21	3.21-3.71	3.71-3.91	3.91- 4.71	4.71- 5.91	96.9	2nd Run on $\ln E_i$
$\ln E_i (E_i > 1)$	0.00-0.79	0.79-1.01	1.01-1.26	1.26-1.51	1.51-1.86	1.86- 3.06	3.06- 6.01	76.7	
N_{1-6}	0.00-0.61	0.61-1.01	1.01-1.51	1.51-2.51	2.51-4.51	4.51- 9.01	9.01- 41.51	28.0	
$-\sqrt{N_{1-6}}$	0.00-077	0.77-1.01	1.01-1.26	1.26-1.51	1.51-1.86	1.86- 3.01	3.01- 6.01	18.5	
$-\sqrt{N_{1-6}}$	0.00-0.79	0.79-1.01	1.01-1.26	1.26-1.51	1.51-1.86	1.86- 3.06	3.06- 6.01	6.5	2nd Run on $-\sqrt{N_{1-6}}$

* For 4 degrees of freedom and a .05 probability level, the tabulated χ^2 value is 9.5.

**Cell boundaries are inclusive on the larger value, (i.e. $0.00 < \text{cell } 1 \leq 1.90$).

categories which contained approximately the same number of observations. It is believed that in using this method it was not possible to completely eliminate the influence of the observed frequencies. This factor produced a slight effect on the calculated values of chi-square. Judicious selection of boundaries reduced the chi-square value in all cases, but only for the $-\sqrt{N_{1-6}}$ data did it affect the decision as to the goodness of fit. The boundaries used and the results obtained with this test are presented in Table 4.

If the gamma distribution function adequately fitted the observed data, then the probability of obtaining a chi-square value less than 9.49 is 0.95 with 4 degrees of freedom. Since only the $-\sqrt{N_{1-6}}$ data produced a chi-square value less than 9.49, it was concluded that the gamma distribution did not adequately fit E_i, $\ln E_i$ and $\ln E_i (E_i > 1)$, but that this distribution function did provide a marginal fit for $-\sqrt{N_{1-6}}$ data.

11. Sequential Analysis Testing

Statistical tests such as Student's t, F, and χ^2 set an α level for a null hypotheses (type I error) and n number of samples and lets the β level for a specific alternative hypotheses (type II error) fall where it may. In the sequential analysis test both the α and β levels are set and the observations are tested sequentially. With each new observation one of the following decisions is reached: (1) accept the hypothesis, (2) reject the hypothesis, and (3) continue the experiment by taking an additional observation. The experiment continues until the hypothesis is either accepted or rejected.

In the sequential analysis test, if the distribution function under consideration has two parameters, it is necessary to either test a composite hypothesis, or to reduce it to a simple hypothesis by assuming one of the parameters constant. It may be shown both theoretically (Appendix B) and experimentally that scale changed data maintains a constant shape parameter (Table 3). Therefore, a simple hypothesis can be used when a scale changed data is considered (see Thom 1957, and Wald 1947).

Sequential analysis testing involves plotting a cumulative function of the data against the number of observations as shown diagrammatically in Figure 7.

In this test maximum and minimum values (from the rejection and acceptance lines) corresponding to each observation are compared with the accumulated value of the data. If the cumulative value remains between these limits the test continues. When the cumulative value exceeds the maximum value the hypothesis is rejected with probability α of a type I error. If the cumulative value becomes less than the minimum value the hypothesis is accepted with probability β of a type II error. The slope of the rejection and acceptance lines and the difference between maximum and minimum values are functions of the hypothesis being tested and the α and β levels selected.

In selecting values for α and β, economic considerations (cost of seeding versus potential benefit) suggest the establishment of a low probability of error for type I errors (rejecting a true hypothesis) with a higher probability of error for the type II errors (accepting a false hypothesis).

Results of the sequential analysis test indicate that if the gamma distribution function fits the $-\sqrt{N_{1-6}}$ data, a 5 percent scale change with α and β both 0.01 would require 732 observations to obtain an accept or reject decision. The results calculated for other scale changes with various α and β values are presented in Table 5.

12. Conclusions

1. From the 9 hailfall parameters derived from data on hailfalls collected by the Colorado State University hail network, 6 were eliminated for use in any statistical analysis of hail modification because of bias, non-homogeneity between years, or sparcity of samples. The remaining parameters were E_c, E_i, and N_{1-6}.
2. The transformations which produce the minimum $\Sigma|Cv_i-\bar{C}_v|$ are $\ln E_c$, $\ln E_i$, and $-\sqrt{N_{1-6}}$.
3. A north-south extension of the hail indicator network can be made, and the east half of the network can be abandoned without significantly affecting the statistical properties of the indicator data.
4. The hypothesis of dependence between adjacent indicators spaced 2 to 4 miles apart cannot be rejected, even though the correlation coefficients are less than 0.50.

5. A period of 3 to 5 years is estimated to be required to detect scale changes of 10 to 25 percent in the hail parameters that might be accomplished by modification attempts. However, there are practical difficulties involved in attaining the conditions assumed in the analysis, one of the most difficult being the problem of handling zero values if complete hail suppression were to be attained.
6. Lack of significant correlation between adjacent areas indicates that a target-control analysis is not feasible for attempting to detect significant changes that might result from a hail modification experiment.
7. Of the data collected and the transformations studied, only the $\sqrt{N_{1-6}}$ data can be fitted by a gamma distribution function and it provides only a marginal fit.
8. The sequential analysis test alone could not adequately evaluate the effectiveness of this hail modification experiment.
9. Further work is presently being done to develop procedures not dependent on a fixed network for analysis of effects which might be produced in a hail modification experiment.

TABLE 5

Number of Observations Required in a Hypothetical Case to Reach an Accept or Reject Decision Using a Sequential Analysis Test and $-\sqrt{N_{1-6}}$ Hailfall Data. These Results were Obtained Experimentally by Assuming Certain Indicated Changes were Applied to all of the Data.

Type of Data	Type of Change	Alpha	Beta	No. of Obs. Required	Decision
$-\sqrt{N_{1-6}}$	5% scale decrease	0.01	0.05	405	Accept
$-\sqrt{N_{1-6}}$	10% scale decrease	0.01	0.05	135	Accept
$-\sqrt{N_{1-6}}$	25% scale decrease	0.01	0.05	36	Accept
$-\sqrt{N_{1-6}}$	5% scale decrease	0.01	0.01	732	Accept
$-\sqrt{N_{1-6}}$	10% scale decrease	0.01	0.01	141	Accept

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Appendix A

The following equation estimates the number of replications required to obtain significant results at a particular probability level.

$$r \geq 2 \left(\frac{\sigma}{\delta m}\right)^2 (t_p + t_s)^2$$

where

r = replications required.

σ = population standard deviation.

m = population mean.

δ = scale change.

t_p = value of students' t at significance level p.

t_s = value of students' t at significance level $2(1-p)$ for a two-tailed test.

Using the above equation and assuming various scale changes and significance levels, the number of replications required to obtain significant results for $\ln E_i$, $\ln E_c$ and $-\sqrt{N_{1-6}}$ were computed and are presented in Tables A1, A2 and A3.

Appendix B

The following proof shows that for scale-changed data, the shape parameter, γ , of the gamma distribution function remains constant. The gamma distribution has a probability function given by the equation

$$f(x_i) = \frac{1}{\beta^{\gamma}\Gamma(\gamma)} x^{\gamma-1} e^{-x_i/\beta} . \tag{1}$$

The maximum likelihood estimates of β , the scale parameter, and γ , the shape parameter are

$$\gamma = \left[1 + (1 + 4A/3)^{\frac{1}{2}}\right]/4A \tag{2}$$

and

$$\beta = \bar{x}/\gamma \tag{3}$$

where $\bar{x}$ is the mean of x_i's and

$$A = \log \bar{x} - \frac{1}{n}\sum_{i=1}^{n} \log x_i \tag{4}$$

$$= \log\left(\sum_{i=1}^{n} x_i/n\right) - \frac{1}{n} \log P\ (x_i) \tag{5}$$

for

$$P(x_i) = (x_1)\ (x_2)\ (x_3)\ . . . (x_n) \tag{6}$$

Now assume that a gamma distribution also fits x_i' , where

$$x_i' = x_i\ (1 + d_i) \tag{7}$$

where d_i is some scale or non-scale change and $d_i > -1$.
Substituting (7) in to (5)

$$A' = \log \frac{1}{n}\sum_{i=1}^{n}\left[x_i\ (1 + d_i)\ \right] - \frac{1}{n} \log P\left[x_i\ (1 + d_i)\right] \tag{8}$$

where

$$\log \frac{1}{n}\sum_{i=1}^{n}\left[x_i\ (1 + d_i)\right] = \log\ \left[\frac{1}{n}\sum_{i=1}^{n} x_i + \frac{1}{n}\sum_{i=1}^{n} x_i\ d_i\ \right] \tag{9}$$

$$= \log \bar{x} + \log\ 1 + \sum_{i=1}^{n} x_i\ d_i/ \sum_{i=1}^{n} x_i \tag{10}$$

and since

$$P\left[x_i\ (1+d_i)\right] = P\ (x_i)\ P\ (1+d_i) \tag{11}$$

and

$$\log P\left[x_i\ (1+d_i)\right] = \log P\ (x_i) + \log P\ (1+d_i)\ , \tag{12}$$

consequently substituting (10), (11), and (12) into (8)

$$A' = \log \bar{x} + \log \left(1 + \frac{1}{n\ \bar{x}} \sum_{i=1}^{n} x_i\ d_i\right) - \frac{1}{n} \log P\ (x_i) - \frac{1}{n} \log P\ (1+d_i)\ . \tag{13}$$

Let

$$a = \log \left(1 + \frac{1}{n\ \bar{x}} \sum_{i=1}^{n} x_i\ d_i\right) - \frac{1}{n} \log P\ (1+d_i)\ , \tag{14}$$

then substituting (4) and (14) into (13) gives

$$A' = A + a \tag{15}$$

Now substitute (15) into (2) and (3) gives

$$\gamma' = \left[1 + (1 + 4\ (A+a)/3)^{\frac{1}{2}}\right]/4\ (A+a) \tag{16}$$

and

$$\beta' = \bar{x}'/\gamma' = \left(\bar{x} + \frac{1}{n} \sum_{i=1}^{n} x_i\ d_i\right)/\gamma' \tag{17}$$

For $d_i = d$, a constant

$$\log \left(1 + \frac{1}{n\ \bar{x}} \sum_{i=1}^{n} x_i\ d\right) = \log\ (1+d) \tag{18}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \log\ (1+d) = \log\ (1+d) \tag{19}$$

therefore, substituting (18) and (19) into (14)

$$a = 0\ . \tag{20}$$

And finally substituting (20) into (16) yields the result:

$$\gamma' = \gamma \tag{21}$$

$$\beta' = \bar{x}\ (1+d)/\gamma\ . \tag{22}$$

Experimental tests (see Table 3) verified these results for $\sqrt{N_{1-6}}$ data. The shape parameter, ν , remained constant for the scale changed data while the scale parameter, β , varied as suggested in equation (22) above.

and since

$$P\left[x_i\ (1+d_i)\right] = P\ (x_i)\ P\ (1+d_i) \tag{11}$$

and

$$\log P\left[x_i\ (1+d_i)\right] = \log P\ (x_i) + \log P\ (1+d_i)\ , \tag{12}$$

consequently substituting (10), (11), and (12) into (8)

$$A' = \log \bar{x} + \log \left(1 + \frac{1}{n\,\bar{x}} \sum_{i=1}^{n} x_i\ d_i\right) - \frac{1}{n} \log P\ (x_i) - \frac{1}{n} \log P\ (1+d_i)\ . \tag{13}$$

Let

$$a = \log \left(1 + \frac{1}{n\,\bar{x}} \sum_{i=1}^{n} x_i\ d_i\right) - \frac{1}{n} \log P\ (1+d_i)\ , \tag{14}$$

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$$\beta' = \bar{x}'/\gamma' = \left(\bar{x} + \frac{1}{n} \sum_{i=1}^{n} x_i\ d_i\right)/\gamma' \tag{17}$$

For $d_i = d$, a constant

$$\log \left(1 + \frac{1}{n\,\bar{x}} \sum_{i=1}^{n} x_i\ d\right) = \log\ (1+d) \tag{18}$$

and

$$\frac{1}{n} \sum_{i=1}^{n} \log\ (1+d) = \log\ (1+d) \tag{19}$$

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$$a = 0\ . \tag{20}$$

And finally substituting (20) into (16) yields the result:

$$\gamma' = \gamma \tag{21}$$

$$\beta' = \bar{x}\ (1+d)/\gamma\ . \tag{22}$$

Experimental tests (see Table 3) verified these results for $-\sqrt{N_{1-6}}$ data. The shape parameter, ν , remained constant for the scale changed data while the scale parameter, β , varied as suggested in equation (22) above.