MATHEMATICAL MODEL FOR TRANSIENT FLOW IN POROUS MEDIA

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PROGRESS REPORT

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INTRODUCTION

This study is devoted to the development of a general mathematical model simulating transient ground water movement. The basic equation describing ground water motion in porous media, is a non-linear, second-order partial differential equation, which can be conveniently solved by numerical finite-difference methods. The numerous computations involved in solving finite-difference equations necessitated the use of a digital computer.

The computer program was written so that hydrologic data, hydraulic properties of the aquifer and boundary conditions can be easily introduced into the model.

THE APPLIED EQUATIONS

The partial differential equation for transient saturated flow, in porous media, may be written as

$$\frac{\partial}{\partial x}\left(K\,h\,\frac{\partial H}{\partial x}\,dx\right)dy + \frac{\partial}{\partial y}\left(K\,h\,\frac{\partial H}{\partial y}\,dy\right)dx = \phi\,\frac{\partial h}{\partial t}\,dx\,dy + q \qquad (1)$$

where, h = saturated thickness of the aquifer $(H - z)$ $[L]$

H = water table elevation above a datum $[L]$

Z = impermeable bed elevation above a datum $[L]$

K = hydraulic conductivity (permeability) $[L/T]$

ϕ = storage coefficient [dimensionless]

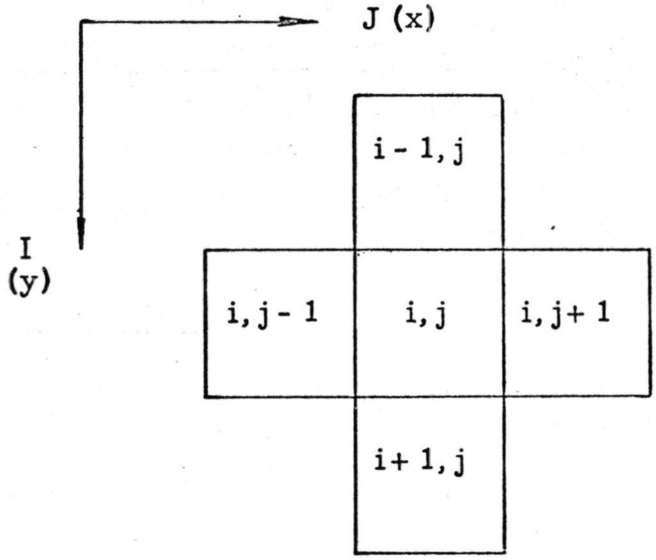
q = net extraction rate $[L^3/T]$

x and y = space coordinates $[L]$

t = time dimension $[T]$

In the derivation of Equation 1, it is assumed that the fluid was incompressible (ρ = constant) and that there was no variation of the storage coefficient with respect to time ($\partial \phi/\partial t = 0$). The third, vertical, dimension for a horizontal case enters the picture with the term h varying with space and time.

Consider five grids in the grid system of Figure 1



$i = 1, 2, \ldots, m$

$j = 1, 2, \ldots, n$

Figure 1

where, I and J = block numbers in the coordinate system (y, x)

i and j = subscripts used to identify grid blocks

m and n = number of blocks in a 2-dimensional model.

By employing a central finite-difference scheme, Equation 1 may be written in the following form 1/ *

$$\left\{ A\, H_{i,j-1} + B\, H_{i-1,j} + \left(-A-B-C-D-\frac{\phi}{\Delta t}\right) H_{i,j} + C\, H_{i,j+1} + D\, H_{i+1,j} \right\}^{t_o + \Delta t} = \left\{ -\frac{\phi}{\Delta t} H_{i,j} + \frac{q_{i,j}}{\Delta x_{i,j}\, \Delta y_{i,j}} \right\}^{t_o} \qquad (2)$$

where, $i = 1, 2, \ldots, m;\ j = 1, 2, \ldots, n$

t_o = starting time or previous time

Δt = time increment

Δx and Δy = areal increments.

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* These numerals refer to corresponding items in the References.

$$A = \frac{(K\,h)_{i,j-1} + (K\,h)_{i,j}}{\Delta x_{i,j}(\Delta x_{i,j-1} + \Delta x_{i,j})}$$

$$B = \frac{(K\,h)_{i-1,j} + (K\,h)_{i,j}}{\Delta y_{i,j}(\Delta y_{i-1,j} + \Delta y_{i,j})}$$

$$C = \frac{(K\,h)_{i,j+1} + (K\,h)_{i,j}}{\Delta x_{i,j}(\Delta x_{i,j+1} + \Delta x_{i,j})}$$

and

$$D = \frac{(K\,h)_{i+1,j} + (K\,h)_{i,j}}{\Delta y_{i,j}(\Delta y_{i+1,j} + \Delta y_{i,j})}$$

The coefficients A, B, C, and D are computed from values obtained in the previous time step.

Equation 2, written for every grid block, may be considered as the flow balance of each grid, involving the four neighboring grids in the given rectangular system. This form of the finite-difference equation is also suitable for a curvilinear grid system similar to Figure 1. The h terms are assumed to be constant during a given time step, and have the value at the end of the previous time period. An iterative procedure would be needed to treat the variations of h with time differently. Boundary conditions, i. e., a constant head (H = constant) or an impermeable barrier ($\partial H/\partial n = 0$), are treated by introducing appropriate values into Equation 2, for the corresponding grids. In general, the known coefficients and quantities appear on the right-hand side of the equation. Equation 2 is a linear algebraic equation in implicit form and as such has to be solved simultaneously. The classical explicit form of finite-difference equations for a linear parabolic partial differential equation has the disadvantage of conditional stability (Δt must be smaller than a certain function of Δx and Δy).[2/]

PROCEDURE

In this investigation, two implicit schemes were employed for the solution of the simultaneous finite-difference equations: 1) Gaussian Elimination Procedure and 2) Alternating Direction Implicit Procedure.

Gaussian Elimination

A finite-difference equation, similar to Equation 2, is set up for each grid of the entire model area and for a particular time step. This system of equations is solved simultaneously by straight forward successive eliminations. The left-hand side of Equation 2 has five unknown H's that have to be solved for time $(t_o + \Delta t)$ using known values on the right-hand side for time (t_o). It is readily seen that the number of equations in a set is equal to the number of grids in the model. Some equations will represent boundary conditions, and hence will have less than five unknowns. The system of $N = m \times n$ linear algebraic equations can be written in matrix notation as:

$$[L]_{N,\,N} \cdot [H]_{N,\,1} = [R]_{N,\,1} \tag{3}$$

where, $[L]_{N,\,N}$ = the matrix of the coefficients of the H's on the left-hand side of Equation 2, the size of which is N x N.

$[H]_{N,\,1}$ = the column vector of H's to be solved for time $(t_o + \Delta t)$, the size of which is N x 1.

$[R]_{N,\,1}$ = the column vector of the known right-hand side of Equation 2 at time (t_o), the size of which is N x 1.

Matrix $[L]_{N,\,N}$ written explicitly (i. e., m = 3, n = 3, N = 9) in the form,

$$\begin{bmatrix} E & D & 0 & C & 0 & 0 & 0 & 0 & 0 \\ B & E & D & 0 & C & 0 & 0 & 0 & 0 \\ 0 & B & E & 0 & 0 & C & 0 & 0 & 0 \\ A & 0 & 0 & E & D & 0 & C & 0 & 0 \\ 0 & A & 0 & B & E & D & 0 & C & 0 \\ 0 & 0 & A & 0 & B & E & 0 & 0 & C \\ 0 & 0 & 0 & A & 0 & 0 & E & D & 0 \\ 0 & 0 & 0 & 0 & A & 0 & B & E & D \\ 0 & 0 & 0 & 0 & 0 & A & 0 & B & E \end{bmatrix}_{N,N}$$

Figure 2

where, A, B, C, and D are as defined for Eq. 2.

E = the value of $(-A-B-C-D-\frac{\phi}{\Delta t})$ in Eq. 2

N = total number of grids in the example model.

This matrix is symmetric, definite and has a dominating main diagonal (as a result of the parabolic type of differential equation applied, Eq. 1). The matrix $[L]_{N,\ N}$ may be rearranged in the following form:

$$\begin{bmatrix} 0 & 0 & 0 & E & D & 0 & C \\ 0 & 0 & B & E & D & 0 & C \\ 0 & 0 & B & E & 0 & 0 & C \\ A & 0 & 0 & E & D & 0 & C \\ A & 0 & B & E & D & 0 & C \\ A & 0 & B & E & 0 & 0 & C \\ A & 0 & 0 & E & D & 0 & 0 \\ A & 0 & B & E & D & 0 & 0 \\ A & 0 & B & E & 0 & 0 & 0 \end{bmatrix}_{N,\ 2m+1}$$

Figure 3

where, m = the number of grids in the short side of the example model. Here, the three central diagonals and the two apart-diagonals are arranged as vertical columns.

A shortcut computer program, BANDSOLVE*, uses the matrix in Figure 3 to solve the given N equations with N unknowns simultaneously. The N computed H's are placed by this procedure in the matrix (column vector) $[R]_{N,\ 1}$. These results are the set of water table elevations for time $(t_o + \Delta t)$.

<u>Alternating Direction Implicit Procedure (ADIP)</u>

The ADIP technique differs from the Gaussian Elimination by a two-half steps solution in two directions of the model problem. The first step is the solution of m systems of n linear algebraic equations for

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*BANDSOLVE algorithm was written in ALGOL by Donald H. Thurnau of Marathon Oil Company, Littleton, Colorado [3] half-time step ($\Delta t/2$) and the second step solves n systems of m equations for the other half-time step. Every equation has three unknowns at most, and each system of equations is solved simultaneously - one at a time. For the Gaussian Elimination technique, every equation has five unknowns at most, and $n \times m = N$ equations are solved simultaneously for a full-time step (Δt). The advantage of ADIP is the diminution of computer storage requirements. Computing time for a 50 grid model is approximately the same for both techniques.

Developing the equations for the ADIP technique, the basic matrix Eq. 3 may be written as, [4]

$$[L]_{N,N} \cdot [H]_{N,1} = ([F]_{N,N} + [V]_{N,N}) \cdot [H]_{N,1} = [R]_{N,1} \tag{4}$$

which gives the two matrix equations,

$$[F] \cdot [H] = [R] - [V] \cdot [H] \tag{5}$$

$$[V] \cdot [H] = [R] - [F] \cdot [H] \tag{6}$$

where, $[F]$ = the matrix of the coefficients of the unknown H's of one direction in the model (rows)

$[V]$ = the matrix of the coefficients of the unknown H's of the second direction in the model (columns).

Matrix $[F]$ = has the form shown in Figure 4:

$$\begin{bmatrix} E & 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & E & 0 & 0 & C & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 & C & 0 & 0 & 0 \\ A & 0 & 0 & E & 0 & 0 & C & 0 & 0 \\ 0 & A & 0 & 0 & E & 0 & 0 & C & 0 \\ 0 & 0 & A & 0 & 0 & E & 0 & 0 & C \\ 0 & 0 & 0 & A & 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & A & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & A & 0 & 0 & E \end{bmatrix}_{N,N}$$

Figure 4

where, A, E, and C are as defined for Figure 2, and matrix $[V]$ has three central diagonals. Matrices $[F]$ and $[V]$ are symmetric, definite, with no more than three nonzero entries per row (tridiagonal) and diagonally dominant. Similarly, the basic finite-difference equation (Equation 2) is broken into two equations analogous to Equations 5 and 6,

$$\left\{A\,H_{i,j-1} + \left(-A-C-\frac{\phi}{\Delta t/2}\right) H_{i,j} + C\,H_{i,j+1}\right\}^{t_o + \Delta t/2} =$$

$$= \left\{-\frac{\phi}{\Delta t/2}\,H_{i,j} + \frac{q_{i,j}}{2\,\Delta x_{i,j}\,\Delta y_{i,j}} - B\,H_{i-1,j} + \right. \tag{7}$$

$$\left. + (B+D)\,H_{i,j} - D\,H_{i+1,j}\right\}^{t_o}$$

$$\left\{B\,H_{i-1,j} + \left(-B-D-\frac{\phi}{\Delta t/2}\right) H_{i,j} + D\,H_{i+1,j}\right\}^{t_o + \Delta t} =$$

$$= \left\{-\frac{\phi}{\Delta t/2}\,H_{i,j} + \frac{q_{i,j}}{2\,\Delta x_{i,j}\,\Delta y_{i,j}} - A\,H_{i,j-1} + \right. \tag{8}$$

$$\left. + (A+C)\,H_{i,j} - C\,H_{i,j+1}\right\}^{t_o + \Delta t/2}$$

where the terms are as defined before.

The standard equation for a grid block situated in a row is Equation 7. The left-hand side of the equation has the three unknown H's which are solved for time ($t_o + \Delta t/2$) by the known right-hand side at time (t_o). Equation 7 is written for every block in each row, hence, there are n equations with n unknowns per row. Every system of equations is solved simultaneously (one row at a time) by the following scheme, based on successive eleminations,

$$\text{a.}\quad W_1 = \frac{C_1}{E_1} \qquad\qquad \text{b.}\quad G_1 = \frac{R_1}{E_1}$$

$$\text{c.}\quad W_j = \frac{C_j}{E_j - A_j W_{j-1}} \qquad\qquad \text{d.}\quad G_j = \frac{R_j - A_j G_{j-1}}{E_j - A_j W_{j-1}} \tag{9}$$

when, j = 2, 3, . . . , n; for a particular value of i;

and a. $H_{i,n} = G_n$ b. $H_{i,j} = G_j - W_j H_{i,j+1}$ (10)

when, $j = (n-1), (n-2), \ldots, 1$; for a particular value of i;
where, A, E, and C are as defined for Figure 2

R = the known value of the right side of Equation 7

W and G = auxiliary values for the computation of the unknown H's.

Equations 9 evaluate two sets of two auxiliary computational elements for a certain row (i) from the start of the row to its end. Then Equations 10 are employed to get the unknown H's of the row, proceeding from the n^{th} element to the first one. This simple solution of simultaneous equations of single-row set is repeated for each of the m rows of the model. Row-by-row sweeping results in new computed H values for time $(t_o + \Delta t/2)$. The second half-time step uses Equation 8 for column-by-column sweeping (n sets of m equations each - solved simultaneously - one at a time). The solution of the single-column set is analogous to the scheme given by Equations 9 and 10, except that terms i and j will be interchanged where $i = 2, 3, \ldots, m$ for a particular j ; B and D are introduced instead of A and C ; and R will be the right-hand side of Equation 8. These computations use quantities computed from the first half-step at time $(t_o + \Delta t/2)$ and will result in solving for the unknown values of H at time $(t_o + \Delta t)$. One time increment is completed by computing the first half-step with rows and the second half-step with columns to give us values of water table elevations for time $(t_o + \Delta t)$.

MODEL DETAILS AND TESTS

Mathematical models have been developed using both the Gaussian Elimination and the Alternating Direction Implicit Procedures. Both horizontal and vertical models have been studied. A brief description of the use and tests conducted on the models follows.

Boundary Conditions

The computer programs written for the two implicit techniques or procedures can handle 2 types of boundary conditions: 1) An impermeable boundary or barrier which prevents flow. This type of boundary results in the coefficients (A, B, C, or D) having values equal to zero for the grid points representing the barrier. 2) Constant head boundary, such as a river hydraulically connected to the aquifer. The water elevation in the river assumed constant and known for a certain period of time will determine the water table elevation in the aquifer nearby. It was assumed water table continuity existed in the vicinity of the constant head boundary and the potential gradient was finite. Likewise, there was to be no restricting limitation for adjustment of the aquifer water table to the constant head. The constant head value along with other known quantities was then transferred to the right side of the finite-difference equation.

Both types of boundary conditions were assigned negative values (-H) in the grid block to distinguish them from the non-boundary grid. The impermeable barrier has an arbitrary negative value, not used elsewhere in the model. The constant head boundary will have its known value, but with a minus sign preceding it.

Input Data

The computer programs prepared to date have been general in nature and not developed specifically for a particular condition. Thus, it is necessary to introduce data into the computer to describe a particular area to be modeled. Required data includes the amount of water in storage, the aquifer properties, and the net application of water to the land for the particular area and period to be modeled. Data must also be supplied on the size of grid to be used (Δx and Δy), the model size (I, J), the time increment (Δt) and the total time (t).

The quantity of water in storage at the initial starting time is introduced by assigning a water table elevation (H) and an impermeable bedrock elevation (Z) to each grid point. These values in conjunction with the aquifer properties including the storage coefficient (ϕ) and the hydraulic conductivity or permeability (K) are used to define the initial conditions and describe the aquifers ability to transmit water. Average or representative values for the above parameters are assigned to each grid point. Impermeable aquifer boundaries or the presence of a stream, constant head boundary, must also be introduced for the proper grids as previously described.

The net application of water to the area during the period to be studied is a function of the following parameters: consumptive use, evapotranspiration, pumping, precipitation, seepage from canals and water applied as irrigation to the land. A net extraction of water (q) is estimated for each grid block and is the algebraic sum of the above parameters. Net flow of water into or out of the aquifer is represented by the respective value, negative or positive, of (q).

Input data for all the variables are punched on IBM cards and read into the computer as called for by the program.

Output Data

Data computed by the program includes values for the water table elevation (H) at various times requested in the program and estimates of the influent or effluent flow for each grid block adjacent to a stream, constant head boundary.

Once the model has been verified, when computed values compare with historical records for a particular period, it will be possible to vary the applied water (q) and note changes in the water table elevations (H) or return flows in the streams. This would allow one to study the individual effect of precipitation, pumping, evapotranspiration or applied irrigation water on the water in storage or river return flow.

Horizontal Model Tests

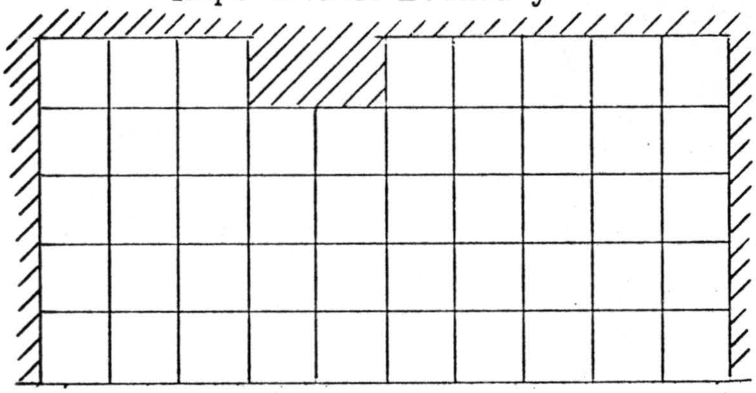
A simple test using 50 grids and hypothetical data was run on a computer to check the validity of the program and to compare the Gaussian Elimination technique with the Alternating Direction Implicit Procedure. The numerical results coincided with analytical solutions using heat flow equation for all time steps taken and were consistent and stable.

Another model test using 50 grids and realistic input data was run for one day time increments and produced data on water table fluctuations and return flow to a river for a period of two years. Daily and monthly results for both techniques were the same. For this trial run, it was assumed that there was no outflow from a grid block when the saturated thickness was very small.

To solve 50 equations (50 unknowns) by the Gaussian Elimination technique for a total time period of two years in increments of one day time-steps required 4.86 minutes on an IBM 7094 computer. Time requirements for larger grids are now being investigated.

One trial run was executed with the Gaussian Elimination technique where two grids of the "active blocks" were occupied by an impermeable boundary as shown here:

Impermeable Boundary



Constant Head Boundary

Figure 5

This run with irregular boundary shapes gave satisfactory results and presumably shows that such model situations can be handled.

Vertical Model Test

To study the limitation of the horizontal model when a water table slope is significant (i. e., aquifer near a stream) a two-dimensional vertical model was tried utilizing the ADIP technique. The basic equation governing unsteady flow is $\nabla^2 H = 0$, except at the free surface (water table) where the equation is $\nabla(KS\nabla H) = \phi \, \partial H/\partial t$. The term S is the length dimension in the X and Z directions. In this case too, the heat equation is nonlinear as the vertical block dimension (Z) varies with time. The Alternating Direction Implicit Procedure was modified to: 1) locate the water table, 2) introduce the storage coefficient in blocks containing the water surface, and 3) discard computations above the free water surface. Results obtained to date indicate that the ADIP computer program should be remodified to handle the discontinuity at the moving water surface and a seepage face near the stream.

RECOMMENDATIONS AND CONCLUSIONS

The hydrologic values of parameters to be used as data in a model solution, are probably the weakest point of an analysis. Hence, it is imperative to exercise particular care in the establishment of these data, and to document them well for assumptions and decisions made. A proposed way to treat these input data is to study the yearly distributions of the components of q, sum them and introduce it in the numerical computations as a monthly percentage of the yearly quantity.

For the vertical model the Gaussian Elimination technique will be adapted to treat the problem of a moving free water surface. This problem will be similar to the penetrating or irregular boundary problem solved in the horizontal model.

The Gaussian Elimination and the Alternating Direction Implicit Procedure are just two techniques that can be used to obtain approximate solutions for the parabolic differential equations. It appears that explicit techniques with unconditional stability [5/] may also be adaptable to the problem. Further study of various techniques is anticipated to obtain an accurate method for solving the equations for a large number of grid points utilizing a minimum amount of computer time.

Field data for a reach of the Arkansas River between LaJunta and Las Animas, Colorado, is now being processed in preparation for a large scale test of the horizontal mathematical model. Approximately 450 grid points will be considered with a one day increment used as the time step. Calculations on change in the water table elevation within each grid and inflow or outflow from each grid along the river to the stream will be made. It will be necessary to compare the computed values with historic water level and river flow data to determine what changes will have to be made in the model or the input data to obtain compatible results.

Work to date indicates that a digital computer program can be prepared for a ground water aquifer system to study the effect of pumping, applied irrigation water, precipitation, and consumptive use upon the ground water in storage and return flows to a river hydraulically connected to the aquifer. If historical data can be matched with computer solutions, indicating a valid model, then the mathematical model could be used extensively in developing water management policies to maximize the use of our water resources.

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