UNSTEADY FREE SURFACE FLOW
IN A STORM DRAIN

General and Analytical Study

by

Vujica M. Yevdjevich

Sponsored by
The U.S. Bureau of Public Roads
Division of Hydraulic Research

Engineering Research
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ABSTRACT

This report represents an analytical investigation of unsteady free surface flow in a storm drain. As a preliminary general study its broad scope is the outline of problems, the selection of mathematical tools and procedures, and the elaboration of a general approach for further studies by hydraulic model and by digital computer investigations in order to develop a set of routing methods for storm drain floods. Each method of this set should be feasible to the particular conditions of data available of flood hydrographs and storm drain characteristics, as well as of precision of computation.

The initial and boundary conditions, applications, and the general approach selected are briefly enumerated in the introduction. The two partial differential equations for unsteady flow are derived and discussed. These basic mathematical tools serve as the starting equations for the computation of all high-order approximations of unsteady flow. Characteristic curves are derived and discussed. The integration of differential equations by method of finite differences is treated in detail, with special attention given to boundary problems.

The coefficients of differential equations are analyzed. Particular attention is given to geometric characteristics of conduit drains, to velocity distribution coefficients, to flow resistance and to lateral flows, in order to treat their functions which are introduced in the two partial differential equations.

The initial and boundary data are analyzed: for flow before the storm flood enters the drain; for inflow hydrographs; for data of junction problems when main drains meet; for outflow conditions; and for data of general boundary conditions.

The end of the report contains the specific aims of the future research program, methods and procedures to be used (especially hydraulic

studies, digital computer studies and comparative studies for the results of these two procedures), significance and characteristics of the future research, and finally the facilities either available now, or to be installed in the near future in order to enable the research program to be carried out.

Description of Research Project

"UNSTEADY FLOW IN A STORM DRAIN"

Part One

THE BROAD PROBLEM

Construction of highways in urban areas (and sometimes elsewhere) requires disposal of stormwater by means of underground storm drains because property values and other considerations prohibit carrying stormwater in open channels. These systems frequently include picking up stormwater contributed by areas outside the right-of-way. The usual design procedure is to compute sizes of pipe by the so-called "rational method." When the highway is depressed the highway department usually attempts to exclude all water falling outside of the depressed section so that the size of the system collecting water for the highway itself (and usually requiring pumping) will be a minimum.

Storm drains for depressed highways sometimes are miles in length (West Route in Chicago for example is about 6 miles) this producing a watershed that is very long in relation to its width. There is good reason to doubt that the rational method is reliable in such a case, (nor for that matter has the rational method been scientifically proved in any case). A flood-routing procedure beginning with routing of overland flow to inlets is generally conceded to be the logical approach especially since digital computers would permit investigation of various storm patterns both as to time and areal distribution in testing the probable functioning of a given system and indicated modifications. Such a procedure would make it possible to know where every cubic foot of water was at any time so that opportunities for temporary storage reducing the peak load could be investigated. Major economies in initial cost might result and are worthwhile exploring since the usual storm-drain system for a depressed highway will cost around $500,000 per mile.

To my knowledge no one has developed a procedure for routing stormwater through a storm drain by any except grossly approximate methods.

The problem then is to study the hydrodynamics of unsteady flow in storm drains with the objective of developing a sound procedure adopted to a digital computer, verifying the procedure by hydraulic model tests and field measurements as may seem necessary. The ultimate purpose is to provide a working design method applicable to any situation where storm drains are used for removal of storm water. However, there are many variations possible in the physical set-up as will be shown in the following paragraphs.

Part Two

AN OUTLINE OF THE FACTORS INVOLVED IN THE HYDRODYNAMICS ANALYSIS

The inflow hydrographs to the storm-drain system will not be considered as part of the hydrodynamics of the storm drain as that is a separate problem. It can be assumed that methods of computing inflow hydrographs will be provided. The system will also be assumed to consist of a single continuous line of pipe with inflow from inlets, or from laterals collecting flow from a series of inlets all located on the highway right-of-way. The right-of-way may include large interchange areas in which case lateral inflow may be substantial in relation to flow in the main drain and conceivably may be large enough to require analysis as a system by itself. For purpose of analysis it may be assumed that flow entering system at any point will have no momentum in the direction of the outflow pipe.

Conduit may be circular or of any shape commonly used, either precast or monolithic concrete, generally will increase in size in downstream direction, changes in size being made at manholes open to atmospheric pressure, and crown-lines will match up except in case where a drop manhole occurs. The latter would be equivalent to a free outlet for system upstream. In large drains especially those of monolithic construction, conduit may be continuous with manhole rising at one side in which case transitions will be used for changes in size.

The slope of the main drain will change usually with breaks at manholes but could be constructed on a vertical curve. Slopes may be subcritical or supercritical and can be very steep, slopes of 3-5% sometimes occurring in main drains. The latter may produce augmented rates of discharge. A single line may involve a wide range of slopes, the usual situation involving steep slopes on upstream reaches becoming mild on downstream end. A break to a steeper slope, however, is also possible.

Alignment commonly will be straight or with relatively small deflections at manholes. Curved alignment is possible but rare. As a rule the main drain will not involve abrupt changes in direction such as 90^0 except at a connection to existing interceptor which case should be given special treatment which is beyond the scope of this problem.

Design criteria ordinarily provide that conduit will not flow under pressure for the design storm. But it should be possible to compute what will happen in the main drain when any part does flow full. Outflow may be either free, or subject to back pressure from stream or conduit into which drain discharges, or from water in wet well of a pumping station. In the latter case flow may be subject to surges created by sudden stoppage of pumps due to power failure.

Manholes are commonly constructed either round or square with or without a stream-lined invert conforming to invert of conduit; section through manhole normal to direction of flow will be as large or larger than cross section of conduit. Common practice is to bring all laterals in at manholes and may be at any elevation at or above flow line of main drain. The laterals for individual inlets are brought increasingly in at a T or Y connection (inflow from one inlet is usually so small relative to flow in main drain that momentum in downstream direction may be neglected).

<u>Inflow hydrographs</u> may have a single peak, or more than one peak. A situation will also occur where a second storm follows so closely after the first that only a part of the volume from the first storm will have been discharged from the system, when the inflow from the second storm begins.

Part Three

LIMITATIONS OF ANALYTICAL STUDY FOR FIRST YEAR

The numerous possible variations in boundary conditions, inflow hydrographs, and outlet conditions require that the analytical study contemplated for the first year be limited so that initial solution for the more simple cases will be possible.

During the first year the study will be limited to hydrodynamic analysis of a single storm drain on straight alignment with single-peak hydrographs (not necessarily identical) introduced at discrete points along the line, and a free outlet.

The conduit shall be considered to be circular in cross section (other cross sections may be introduced if feasible), changing in size at manholes, with crown lines matched up and changes in slope at manholes but not necessarily at every manhole.

The conduit shall be considered to be smooth concrete with resistance factor Darcy-Weisbach "f" varying as a function of the Reynolds Number of the flow in accordance with latest results from full-scale tests made for the Florida State Road Department and Public Roads at the St. Anthony Falls Hydraulic Laboratory. In the event this requirement complicates the solution unduly, then an average value of "f" for each size may be used.

Only the case of free-water surface at atmospheric pressure is to be studied initially. Flows as introduced to line shall be considered to have no momentum in direction of outflow line. Both subcritical and supercritical slopes shall be studied but not as steep as to augment the rate of discharge. Disturbances created by discontinuity of boundary at manholes shall be given consideration based on assumption that manhole is an abrupt enlargement over entire periphery of conduit except at flow line and distance across manhole in direction of flow is not more than 3 pipe diameters.

The hydrodynamic analysis shall be made having in mind conversion of the results to solution by a digital computer. The contract will provide for employment of a consultant on machine computation to assist in that development. It is hoped that the end result will be a program whereby the outflow hydrograph for the simple case herein described may be printed out for any set of inflow hydrographs which do not cause the line to flow under pressure at any point (i.e. to flow full).

The analysis is quite likely to require experimental verification and establishment of certain constants by empirical tests. The study should outline the tests and how they should be made, but no experimental work is to be included under the initial contract.

Carl F. Izzard, Chief
Division of Hydraulic Research
U.S. Bureau of Public Roads

August 1960
Washington 25, D. C.

UNSTEADY FREE SURFACE FLOW IN A STORM DRAIN

by

Vujica M. Yevdjevich

I. INTRODUCTION

A. PROBLEM

The problem is to study the hydrodynamics of unsteady free surface flow in storm drains with the objective of developing a set of routing procedures adapted to a digital computer, helping or verifying the procedures by hydraulic model tests as may seem necessary. The ultimate purpose is to provide working design methods applicable to any situation where storm drains are used for removal of storm water.

The first-year investigation contained in this report, is limited to general and analytical studies which will be the basis for an advanced research program in subsequent years. The aim of this report is to outline problems in detail, to set-up the basic mathematical tools, to discuss the initial and boundary conditions, and to select the general approach to be followed in the next phases of this research program.

B. INITIAL CONDITIONS ASSUMED FOR THE ANALYSIS

1. The depth of water in storm drain is small prior to storm inflow and a steady low flow regime is assumed; later a second storm may occur while the storm drain is still partly filled due to the previous storm.

2. Storm inflow hydrographs along the storm drain are given either as simple hydrographs of any shape or as composed hydrographs of successive individual storm hydrographs. Each inlet point (i) has an

individual discharge hydrograph Q_i (t) with the shape, peaks and time of peaks of the hydrograph different from inlet to inlet, and they depend on catchment area of each inlet, the storm characteristics, and the direction and speed of storm movement.

C. BOUNDARY CONDITIONS

1. Storm drain consists of a single continuous line of pipe (except when the problem of junction of two main drains is discussed).

2. Storm water inflows at discrete points, from inlet to inlet, which are located along the storm drain.

3. Inflow discharge at an inlet has momentum which is negligible in the direction of the outflow pipe.

4. The conduit is circular (representing all other drain shapes).

5. Conduit dimensions change at manholes, open to atmospheric pressure, which might be or might not be inlet points. No inlet points are located outside the manholes.

6. The conduit has the matching crown-lines at manholes, except in case where a drop manhole occurs. When a drop occurs, the hydraulic characteristics of those manholes are known as boundary conditions, namely the outflow rating curve of the outlet for system upstream is given.

7. The hydraulics of transitions at manholes is known as a head loss function of the main pipe discharge, the inlet discharge if any, and a water stage. This head loss is to be considered as a singular resistance loss at manhole points.

8. The slope of the storm drain is constant between manholes, with changes in slope at manholes. If the storm drain is constructed on a vertical curve, it is composed of many constant slopes which change at given

distances along the general vertical curve. The points at which slopes change will be equivalent to manholes without inflows and with a singular head loss.

9. Slopes can vary, with both subcritical and supercritical flow, reaching slopes up to 3-5 percent.

10. Alignment of storm drain is generally straight, but sometimes with small deflection at manholes, with the head loss due to deflection included in the general singular loss at the manhole.

11. The flow regime is a free surface water motion during all the movement of storm flood wave through the drain.

12. Outflow at the end of the storm drain is free, with a given outflow stage-discharge relationship (rating curve or family of rating curves).

13. The Darcy-Weisbach f factor will be used to define the flow resistance for both the rough and smooth conduits.

D. APPLICATIONS

The free surface unsteady water movement through pipes, tunnels, storm drains, and all other conduits, either of circular or any other shapes, is applicable to many problems, including the problem outlined in the "Description of Research Project".

Some of these problems are:

1. Removal of rainfall water through storm drains in highway and urban drainage problems.

2. Computation of free surface wave movement along water power tunnels and conduits. This leads, generally, to a computation of inflow hydrograph in a forebay of downstream water power station, when the outflow hydrograph of upstream water power station at the entrance of the free

surface tunnel is known. It is a computation of hydrograph transformation along the tunnel with free surface flow.

3. Analysis of tunnels and conduits either as pools for peaking power or as storage in the case of water pumping, with unsteady free surface water flow along them.

4. Passage of flash-floods of small water courses through diversion tunnels or conduits.

5. Computation of unsteady flow along canals which have cross sections close to a circular shape.

6. Study of movement of splashing water waves along semi-circular flumes to drift logs.

7. Computation of splashing water waves in cleaning sewage drains, pipes, and tunnels.

8. Computation of propagation of water waves from a sea or a lake along storm drains which enter into these bodies of water.

There are other potential applications of this free surface flow in hydraulic engineering.

The results of this study of unsteady free surface water movement along circular drains can be applied, with due modifications, to all free surface unsteady movements in canals and regular artificial channels.

E. GENERAL APPROACH SELECTED

1. The general approach to the problem of unsteady free surface flow in storm drains was selected as follows:

   a. A hydrodynamic analysis of the problem was pursued with a minimum of basic assumptions and of neglected factors.

b. All assumptions or neglected factors were discussed with the characteristics of storm drains in view.

c. Simplifications to suit the accuracy of available data, and needed precision of results, will be introduced in a later stage of the study, when the evaluation of their effect could be made.

The derivations of basic hydrodynamic equations have been extensive in this study, regardless of the fact that many of them repeat the lines of derivations in the already classical studies. They have been pursued for two reasons:

(1) To give the complete analytical background of methods for the computation of unsteady free surface flow in storm drains.

(2) To modify and adapt the analytical expressions to the specific characteristics of storm drains.

2. In treating the unsteady free surface flow in a conduit or a channel, the following facts are emphasized here:

a. Any existing mathematical expression which describes the unsteady free surface flow is based on some assumptions, which means that there is always a difference between the mathematically derived unsteady flow patterns, and the real patterns.

b. Existing methods or methods which will be developed in the future for the computation of unsteady free surface flow are only approximations of the real flow, and the degree to approximation is a basic question which should be determined for each individual method. There is little value in discussing the merits of individual methods without determining the degree of approximation, or the accuracy of the computed unsteady flow patterns from initial patterns, when compared with the real flow patterns.

c. Selection of the computation method for unsteady free surface flow should depend on the degree of approximation which is justified economically (or from any other point of view). This implies that the following problems have to be answered:

(1) What is the degree of approximation for each method?

(2) What degree of approximation is justified, both from the point of view of an analysis of the case at hand, and from the economy of computation?

3. The problem of unsteady free surface flow in conduits or channels may be systematically approached from two different directions. They are:

a. A very simple method, generally a rough approximation of the real flow, is adopted for the computation of unsteady flow. This method is considered as a low-degree approximation. By adding the other factors, i.e., flow resistance, acceleration factors, or similar, new methods which are more accurate are derived, and so on, from lower-degree approximations to higher-degree approximations.

b. The complete hydrodynamic equations are the closest existing mathematical approximation to the physics of the unsteady free surface flow. Any computation of unsteady flow by these equations is assumed to be the highest order approximation possible at the present status of fluid dynamics and applied mathematics in hydrodynamics. By neglecting some factors or by simplifying the initial and boundary conditions, and quantities which describe these conditions, the lower-order approximations are derived. As the accuracy of computations decreases by an increase of simplifications and neglect of factors, the practical problem is in determination of the lower-order but simple method of approximation, which satisfies requirements imposed by other considerations.

4. This second direction is pursued in this analytical study and will be pursued in the studies which will be its sequence. The procedures to follow are:

   1. Regardless of which mathematical expressions are used to describe the unsteady free surface flow, there are always several assumptions, which introduce the first departures between the real flow and the flow described by the mathematical tool. The effects of these assumptions on the flow patterns are discussed.

   2. Mathematical tools in the form of the two partial differential equations (often called the De Saint Venant's partial differential equation of unsteady free surface flow), as continuity equation and momentum equation, are derived in the most general form, in order to stress the physics of the unsteady free surface flow, and to show the variables and quantities entering into equations and having the effect on the flow patterns.

   3. These general equations are adapted to storm drains in order to derive suitable methods for computation of unsteady free surface flow along such conduits under different conditions.

   4. Initial and boundary conditions, already defined for the storm drain problem, are discussed whenever they influence the computation method.

   5. Methods of integrating the two partial differential equations are discussed for selected initial and boundary conditions, and the numerical computational methods in using digital computers are analyzed shortly.

   6. Specific hydraulic problems related to the storm drains are studied in general in this study, but will be studied in detail later in future research.

7. Simplified methods will be derived in a later stage of the study or existing ones will be discussed, which will serve the preliminary design of storm drains. They will be analyzed in the light of errors introduced by errors in basic data, of the errors inherent to the methods themselves, and in the light of tolerable errors.

The analytical solution of the two partial differential equations for unsteady free surface water flow is impossible for the conduits and channels under the natural conditions. As soon as the simplifications which enable an analytical solution are introduced in equations, the departures from the real conditions are so great that the results become invalid in most cases. The methods of approximate integrations have been thus imposed.

Ninety years, from 1871 to 1961, of application of the two basic partial differential equations of unsteady free surface flow for practical purposes in canals, channels, and conduits has resulted in many methods of solution, both graphical and numerical, with different degrees of approximation to exact solutions. As the amount of work to be done was very large in numerical methods, the graphical methods have dominated the field until recently. As the graphical procedures are tedious and time consuming in practice, they are being replaced by the unsteady flow routing methods based either on the simple continuity equation alone (water storage equation), or on it and on a simplified momentum equation. A very large number of these approximate methods has been developed (ref 1), and most of them are used currently.

Two relatively recent developments have influenced greatly the treatment of unsteady flows: 1) Appropriate numerical procedures, generally based on using methods of finite differences for integration of differential equations; and 2) Computing machines of varying characteristics, which are suitable to carry out fast and large numerical computations with relatively low cost. The inovations and progress in both of these directions have enabled the use of procedures which have been outlined a long time ago, but were considered as impractical 2-5 decades ago. Among these procedures of integration are both: a) The method of finite differences in solving the two partial differential equations for unsteady free surface flow under complex conditions; and b) The method of finite differences applied to the four characteristic differential equations as an equivalent set to the two partial differential equations. The use of finite differences (graphical or numerical methods) to integrate the two partial differential equations through use of their equivalent characteristic curves (and sometimes straight lines), is usually called method of characteristics. By using the names "Method of finite differences" for the first case, and "Method of characteristics" for the second case, they should be understood in the above sense, regardless that the method of characteristics is also a method of finite differences.

There are two other problems for which a general approach is taken in this study, namely: the selection of a routing method in the case of water flow in two directions in a storm drain, or of a junction problem which must be included in the computation, and the selection of the routing method in the light of accuracy of computation, which is justified.

As the inflow hydrographs at inlets along the storm drain may be with different phases of the occurrence of peak discharges and of different rising limb times, it can happen that the water temporarily may flow in both directions of the storm drain, upstream and downstream. In the similar way, junctions between storm drains in the level creates the interdependence of the unsteady flow in the system of the storm drains. For this purpose the flood routing methods which cannot take into consideration that case (as the flood routing methods based on the simple storage equation) theoretically are not feasible for the computation of unsteady free surface flow in storm drains. The two partial differential equations can treat this case.

As the inlet hydrographs along the storm drain are subject to errors by making assumptions and by carrying out their computations, and as the design storms are also of a limited accuracy when compared with the real storms in nature, there is a limit of accuracy economically justified in flood routing methods. This accuracy corresponds to the precision of basic data (in this case to accuracy of inflow hydrographs along the storm drain). Greater accuracy than this is then not justified in hydraulic computation. The problem seems for the moment to be beyond the analytical approach to unsteady flow problem, and is not considered here in this analytical investigation, but it will be treated later in the subsequent research program, when the set of routing methods and their conditions of application will be discussed.

II. DERIVATION OF THE TWO PARTIAL DIFFERENTIAL EQUATIONS FOR UNSTEADY FREE SURFACE FLOW IN CONDUITS

A. DERIVATION OF CONTINUITY EQUATION (LAW OF CONSERVATION OF MASS)

At a given time t the cross section area of unsteady free surface flow at the section x (fig. 1) is A. At the section $x + dx$ at the same time t the area is $A + \frac{\partial A}{\partial x} dx$. The mass of water between these two sections (slice 1-2-3-4-1) is $\rho A dx + \frac{1}{2} \rho \frac{\partial A}{\partial x} dx^2$. By neglecting the second order differential term, the mass is $\rho A dx$. Assuming that the lateral outflow or inflow is given as q, discharge per unit length of conduit, with q positive for outflow and negative for inflow, then the change of mass with time is

$$\frac{d}{dt} (\rho A dx) = - \rho q dx \tag{1}$$

For incompressible fluid ρ = constant.

For a particle it is

$$\frac{d(dx)}{dt} = \frac{(v + \frac{\partial v}{\partial x} dx)dt - vdt}{dt} = \frac{\partial v}{\partial x} dx$$

where v is the particle velocity along a stream line.

Using the symbol $\frac{d\cdot}{dt} = \frac{\partial \cdot}{\partial t} + v \frac{\partial \cdot}{\partial x}$ for the total derivative of a particle along its stream line, applied to dA/dt, then because

$$\frac{1}{VA} \iint_A v dA = 1 \text{ , and } \frac{1}{A} \iint_A \frac{\partial v}{\partial x} dA = \frac{\partial V}{\partial x} \text{ ,}$$

V = mean velocity in a cross section,

it is

$$\frac{\partial A}{\partial t} + V \frac{\partial A}{\partial x} + A \frac{\partial V}{\partial x} + q = 0 \qquad (2)$$

or

$$\frac{\partial A}{\partial t} + \frac{\partial (VA)}{\partial x} + q = 0 \qquad (3)$$

with $Q = VA$.

Equation 3 is sometimes written as

$$\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} + q = 0 \qquad (3a)$$

B. DERIVATION OF MOMENTUM (DYNAMIC) EQUATION

Using Newton's second law, or law of momentum

$$\frac{d(mv)}{dt} = F \qquad (4)$$

where m is the mass, v velocity of the particle, and F the resultant force of all forces acting on the particle. Replacing the particle by the elementary slice of water (fig. 1) between section x and $x + dx$, and the particle velocity v by the mean velocity V in the cross section, in that case the following velocity distribution coefficients must be introduced

$$\alpha = \frac{1}{AV^3} \iint_A v^3 dA \text{ , and } \beta = \frac{1}{AV^2} \iint_A v^2 dA$$

to take care of the replacement of particle velocity v by the mean cross section velocity V.

The coefficients α and β depend on velocity distribution across a cross section A (fig. 2), and therefore depend on the shape and area of cross section.

Equation 4 will be applied along the direction of the conduit bottom (fig. 1), in which case dx is to be replaced by $dx/\cos\psi$, and all acting forces, external and internal (gravity force), are projected to this direction.

Gravity force or its tangential component T along the bottom taking positive sign with the direction of slope (fig. 4), is $T = \rho g A dx \sin\psi$.

Friction force F_f , with the head loss dH_f along the conduit length $dx/\cos\psi$, with $dH_f/dx = S_f$ = friction slope, can be expressed as:

$$F_f = -\rho g A S_f dx$$

Pressure forces, fig. 2, 3, and 4, can be expressed as follows:

$$F_p = \int_0^H \rho g (H-y)\, B_y\, dy \text{ , with}$$

$$\frac{\partial F_p}{\partial x} = \int_0^H \rho g \frac{\partial H}{\partial x} B_y\, dy + \int_0^H \rho g (H-y) \frac{\partial B_y}{\partial x}\, dx dy \text{ ,}$$

or

$$\frac{\partial F_p}{\partial x} dx = \rho g A \frac{\partial H}{\partial x} dx + F_{x1} + F_{x2} \text{ ,}$$

so that the resultant pressure force in the direction of conduit bottom becomes

$$-\rho g A \frac{\partial H}{\partial x} dx \cos\psi \text{ .}$$

In this case, equation 4, with

$$m = \rho dx \iint_0^A dA = \rho A dx$$

becomes for direction of conduit bottom slope

$$\frac{d}{dt}\left(\frac{\rho dx}{\cos\psi}\iint_0^A v dA\right) + \rho \frac{dx}{\cos\psi}\iint_0^A \left(\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x}\right) dA =$$

$$= \rho g A \; dx \; \sin\psi - \rho g A S_f \; dx - \rho g A \frac{\partial H}{\partial x} dx \cos\psi \; . \tag{5}$$

With

$$\iint_0^A v dA = VA \; , \; \cos\psi \approx 1 \; , \; \sin\psi \approx \mathrm{tg}\;\psi \approx S_o$$

bottom slope along dx, and taking into account equation 1, equation 5 becomes

$$- \rho q V dx + \rho dx \iint_0^A \frac{\partial v}{\partial t} dA + \rho \frac{dx}{2}\iint_0^A \frac{\partial v^2}{\partial x} dA =$$

$$= \rho g A dx \; (S_o - S_f - \frac{\partial H}{\partial x}) \tag{6}$$

with

$$\iint_0^A \frac{\partial v}{\partial t} dA = A\beta \frac{\partial V}{\partial t} ; \text{ and } \iint_0^A \frac{\partial v^2}{\partial x} dA = A \frac{\partial}{\partial x}(\alpha V^2)$$

Equation 6, after arranging becomes

$$\frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\alpha V^2}{2g}\right) + \frac{\partial H}{\partial x} - S_o + S_f - \frac{\beta Vq}{gA} = 0 \tag{7}$$

where

$$S_o = -\partial z/\partial x \text{ (fig. 1)}$$

with

$$\frac{\partial H_e}{\partial x} = \frac{\partial}{\partial x}\left(H + z + \frac{\alpha V^2}{2g}\right) \tag{8}$$

then the general form of momentum equation is

$$\frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial H_e}{\partial x} + S_f - \frac{\beta Vq}{gA} = 0 \tag{9}$$

designating $S_a = \frac{\beta}{g}\frac{\partial V}{\partial t}$ = acceleration slope,

$S_e = \frac{\partial H_e}{\partial x}$ = slope of energy line

and $S_q = -\frac{\beta Vq}{gA}$ = slope due to lateral outflow

(or lateral inflow)

then

$$S_a + S_e + S_f + S_q = 0 \tag{10}$$

or multiplying equation 10 by dx , and with $dH_a = S_o\,dx$, $dH_e = S_e\,dx$, $dH_f = S_f dx$, and $dH_q = S_q\,dx$,

then

$$dH_a + dH_e + dH_f + dH_q = 0 \tag{11}$$

or the sum of all slopes or of all head changes along dx-length is zero. Equation 7 is often written as

$$\beta \frac{\partial V}{\partial t} + \alpha V \frac{\partial V}{\partial x} + g \frac{\partial H}{\partial x} = g(S_o - S_f) \quad \frac{\beta\, Vq}{A} \tag{7a}$$

and generally the last term $\beta\, Vq/A$ is neglected, as well as $\beta \approx \alpha \approx 1$ is assumed.

C. DISCUSSION OF BASIC ASSUMPTIONS USED IN DERIVATION OF THE TWO PARTIAL DIFFERENTIAL EQUATIONS

The general approach in deriving equations 2 and 7 assumes that the flow is gradually varied unsteady free surface water movement. This means that the changes of variables, $\partial H/\partial t$ or $\partial A/\partial t$, and $\partial V/\partial t$, $\partial H/\partial x$ or $\partial A/\partial x$, and $\partial V/\partial x$ are relatively small, in order that this basic assumption could be justified.

The basic and general assumptions underlying the development and the applicability of equations 3 and 7 are:

1. Vertical acceleration can be neglected in comparison with the horizontal acceleration (or better, the acceleration normal to conduit in comparison with the acceleration along the conduit), because of the gradual change of depth and discharge with time and with distance. The steeper a wave is, the less justified becomes this assumption. It is quite inapplicable in the case of water surges (bores and depressions).

2. Flow is gradually varied, or the vertical components of velocities are considered small in comparison with the longitudinal components of velocities.

3. Flow patterns are the same in vertical planes parallel to longitudinal axis of the channel (in the case of curvilinear channels the vertical cylindrical surfaces parallel to the longitudinal axis have the same flow pattern), or the influence of the channel sides and its curvature on flow patterns can be neglected.

4. Velocity distribution along a vertical in unsteady flow is the same as that in steady flow, or the velocity-distribution coefficients α and β in equation 7 are constants, for given values of discharge, depth and velocity, or the unsteady flow does not influence these coefficients. Since this assumption depends on rate-of-change of velocities with time and distance, it is justified only in the case of a small rate-of-change.

5. Friction resistance in unsteady flow is the same as that in steady flow, which assumption is justified only if the rate-of-change of velocities with respect to time and distance is small.

There are no data in the literature that show the numerical effect of these factors either individually or as the group on the computed or observed waves along the river channel, so evidence is lacking for justification of these five assumptions in terms of the specific characteristics of a wave, of channel and of lateral inflow or outflow. Only global comparison, between the observed wave and the computed wave by using equations 3 and 7 exist. In a very gradually varied unsteady flow the total influence of all above factors is relatively small. It is therefore, justified to neglect them in this case.

The effect of the above assumptions will be studied in detail during the future research program. It is anticipated that a hydraulic model conduit sufficiently long and with a large diameter will be available for research purposes. This model will give the detailed hydraulic results to be used in a digital computer as very accurate input and boundary data. Inflow hydrographs, wave profiles for given times, and hydrographs at the characteristic places along the conduit will be recorded very accurately for different storm floods. Inflow hydrographs will be input data for a digital computer. Equations 2 and 7 will be used for computing very precisely the wave profiles for given times and wave hydrographs for given places (same times and same places for which the recording is made in hydraulic model studies).

The comparison between wave profiles and wave hydrographs recorded in the hydraulic model and computed by digital computer will produce a general picture of the effect caused by above assumptions. During this analysis the eventual departures between the two sets of data, created only by recording error in hydraulic model and by computational errors in computer, must be evaluated and taken into consideration. The comparison is planned to be carried out for different rate-of-change of flows.

It is expected that the first assumption, of a negligible vertical acceleration (assumption under 1.), will produce departures among two sets of results which increase with an increase of rate-of-change of hydrograph. A relationship $D(\Delta Q/\Delta t)$, with D = departure and $\Delta Q/\Delta t$ = rate-of-change of discharge hydrograph, would give a general picture how the first assumption influences the computed wave movement. The second assumption is implicitely included in the effect of first assumption, and its effect will also increase with an increase of rate-of-change of discharge hydrograph.

It is a fact, that the mathematical tools available for the computation of unsteady free surface flow are more accurate either for a very gradually varied flow (by using the two De Saint Vernant partial differential equations) or for a steep surge (by using equations for treatment of travelling bores and steep depressions), than it is the case for a steep wave between these two extremes. The anticipated results obtained from hydraulic model and digital computer studies are considered as the potential research data. The new mathematical tools may be developed in order to improve the computational accuracy of steep waves which are still far from the surges, but can not be considered as the gradually varied flow. The storm floods in storm drains in highways and in urban drainage problems are very often just in this transition region of wave shapes.

The effect of assumption under 3.) will be avoided by selecting a straight line conduit, and carrying out the hydraulic experiments in such a way, that the lateral oscillations of the body of water in conduit would not occur during wave movement.

The effects of differences for velocity distribution coefficients and for flow resistance factors between the unsteady and steady flow patterns are difficult to assess without basic studies. The steeper a wave, the more influence the constantly changing boundary layer has on the flow resistance and velocity distribution. These differences increase with an increase of rate-of-change of discharge hydrograph. When the results of hydraulic model and digital computer studies are compared, the effect of these differences will be combined with the effect of first and second assumptions. To isolate the effects, special hydraulic studies should be carried out prior to comparison of two sets of results.

The efforts in determining the effects of the above assumptions, which are involved in the derivation of the two partial differential equations, are worth undertaking both from the theoretical as well as from practical point of view. The current engineering design is always based on an approximate computation procedure, and any improvement in this direction is a replacement of a lower-degree approximation by a higher-degree approximation. This replacement should always be justified from the practical point of view. Information about the degree of approximation for different procedures is as vital as the procedures themselves. The difficulties with the storm flood routing methods actually used in storm drain drainage design is the lack of information on the degree of approximation that is attained when the method is applied. The analysis of effect of above basic assumptions should be a substantial contribution to the estimation of degrees of approximation, both for the most advanced mathematical methods of computing flood wave propogation, and for any other flood routing method, either existing or to be developed for future practical applications.

From the theoretical point of view any new detailed study of the degree of approximation attained by the application of De Saint Venant equations under different conditions will be a contribution to the body of knowledge for unsteady free surface flow in conduits and channels.

D. BRIDGE BETWEEN TWO PARTIAL DIFFERENTIAL EQUATIONS

The continuity equation involves the cross-section area, while the momentum equation is based on the rate-of-change of energy line, or of water surface position, plus the dynamic head. For irregular conduits with changing bottom slope and irregular cross section shape and area, the bridge between these two partial differential equations introduces the first complexity in the mathematical analysis. The rate-of-change of cross-section characteristics, as related to the bottom position, and the rate-of-change of bottom slope with distance, when expressed in analytical form, generally provide the bridge between the two equations. Some assumptions and simplifications for cross section, and for bottom position are necessary to enable analytical treatment of equations 3 and 9.

To bridge equations 2 and 7 , or 2 and 7a, the general area function in the form $A(H,x,t)$ should be available, where the variable t designates the change of the contour position with time (movable boundary). Assuming that the conduit contours are fixed (in some movable alluvial beds, this assumption is only approximately satisfied) the area function becomes $A(H,x)$. There are two general cases:

1. The conduit is prismatic, so that $A(H)$, or the area is independent of x. The simplest equation is $A = BH$, a rectangular prismatic channel or conduit, with constant width B, and which is usually used for the theoretical analysis of unsteady free surface flow in channels. A fit to natural channels is the power function $A = pH^s$, with p and s constants. The circular drains of a given diameter D have a complex (arccosine) area function, with area only dependent on depth H.

2. The conduit or channel is non-prismatic, with $A(H,x)$. The power function $A = pH^s$ is applicable for some channels, with p and s being functions of x. The converging or diverging circular conduits belong to this category of conduits.

The bridge between equations 2 and 7 cannot be made, unless the relation of A and H is defined along the conduit.

A storm drain is assumed to be prismatic conduit between the two successive manholes, so that $A(H)$ is valid for that reach, or the area is independent of x. In this case

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial H}\frac{\partial H}{\partial t} = B\frac{\partial H}{\partial t} \tag{12}$$

and

$$\frac{\partial A}{\partial x} = \frac{\partial A}{\partial H}\frac{\partial H}{\partial x} = B\frac{\partial H}{\partial x} \tag{13}$$

with

$$B(H) = \partial A/\partial H \tag{14}$$

Introducing the expressions of equations 12 and 13, equation 2 becomes

$$\frac{1}{V}\frac{\partial H}{\partial t} + \frac{A}{VB}\frac{\partial V}{\partial x} + \frac{\partial H}{\partial x} + \frac{q}{VB} = 0 \qquad (15)$$

so that equations 15 and 7 which describe the unsteady free surface flow are given in dimensionless form.

It must be assumed also that the function of q is known in advance, which in a general form is given as $q(H,x,t)$. The variable t is necessary if there are any changes in the contour or the time (slow opening of gates, or valves, and slow breaches of levees, in the case of channels or conduits). The term with q should not be neglected for storm drains in general, because they may have the lateral spillways along them (lateral continuous inflows also).

E. NAME AND MEANING OF DIFFERENT TERMS IN THE TWO PARTIAL DIFFERENTIAL EQUATIONS

The four terms in equation 2 when multiplied by dxdt give dimension of volume. In the order of sequence, they have the following physical meanings: 1) storage of rate-of-rise of level (rate- of-change of area A with time); 2) wedge storage (due to the difference of depths at the beginning and the end of the elementary reach dx by a change of area A along the conduit); 3) prism storage; and 4) storage (positive or negative) due to lateral inflow or outflow. The six terms in equation 7, in the order of sequence have the following physical meanings: 1) acceleration term (ratio of accelerations, or ratio of the change of velocity with time and the acceleration of gravity, also called acceleration-head term, velocity-hydrograph inclination, localized acceleration gradient); 2) rate-of-change of velocity head (also called dynamic head, velocity-head term, energy grade line inclination, instantaneous energy gradient), or the slope created by the change of velocity head along the conduit; 3) rate-of-change of depth (depth-taper or depth-change term), or the slope created by the change of depth along the conduit; 4) bottom slope; 5) friction slope; and 6) part of the gradient on the energy line created by the lateral outflow or inflow.

This equation is differently expressed in different papers: dimensionless as in equation 7, or with dimension of head, acceleration, momentum, energy, or other dimension.

F. SELECTION OF DEPENDENT VARIABLES

The partial differential equations are simplest in the case when the dependent variables are the mean velocity V and the depth H, with length x and time t being independent variables. In order to get a discharge hydrograph at a place x of the drain, the depth hydrograph $H(t)$, and velocity hydrograph $V(t)$ are first obtained in this case. Then the area hydrograph $A(t)$ is determined from depth hydrograph. The discharge hydrograph is then $Q(t) = V(t)\,A(t)$.

If the discharge hydrographs at different places should be the final result of the computation of a storm flood movement through drains, it might be more convenient to use discharge Q and depth H as dependent variables, instead velocity V and depth H, though the partial differential equations come out to be more complex, and therefore, the computational procedures by digital computer (programming and computation time) might be somewhat longer than in using V and H as dependent variables.

The problem should be solved by the digital computer, and it might come out that for one type of problems the dependent variables V and H are feasible, while for the other type the use of dependent variables Q and H may give a better approach.

III. DERIVATION AND DISCUSSION OF CHARACTERISTIC CURVES

A. DERIVATION OF CHARACTERISTIC CURVES*

The partial differential equations for unsteady free surface flow in conduits, with two dependent (V,H) and two independent variables (x,t) have the general form

$$A_1 \frac{\partial V}{\partial x} + B_1 \frac{\partial V}{\partial t} + C_1 \frac{\partial H}{\partial x} + D_1 \frac{\partial H}{\partial t} + E_1 = 0 \tag{16}$$

$$A_2 \frac{\partial V}{\partial x} + B_2 \frac{\partial V}{\partial t} + C_2 \frac{\partial H}{\partial x} + D_2 \frac{\partial H}{\partial t} + E_2 = 0 \tag{17}$$

with coefficients A, B, ... E as functions of V, H, x, t .

Equations 16 and 17 are linear in relation to partial derivatives, but the coefficients are functions of dependent variables also. Equations 16 and 17 are called quasilinear partial differential equations.

Equations 15 and 7a, in order to be comparable with equations 16 and 17 respectively, have the form

$$\frac{A}{VB} \frac{\partial V}{\partial x} + \frac{\partial H}{\partial x} + \frac{1}{V} \frac{\partial H}{\partial t} + \frac{q}{VB} = 0 \tag{18}$$

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* This derivation follows closely the derivation of characteristic curves (applied to the case of this report), given in the book: R. Courant and K. O. Friedrichs, Supersonic Flow and Shock Waves, Interscience Publishers, Inc., New York, 1948, ref. 2.

For all other references on characteristic curves see Bibliography on Flood Routing Methods, ref. 1, or ref. 3.

$$\frac{\alpha V}{g}\frac{\partial V}{\partial x} + \frac{\beta}{g}\frac{\partial V}{\partial t} + \frac{\partial H}{\partial x} + S_f - S_o - \frac{\beta Vq}{gA} = 0 \tag{19}$$

in this case:

$$A_1 = \frac{A}{VB}; \quad B_1 = 0; \quad C_1 = 1; \quad D_1 = \frac{1}{V}; \quad E_1 = \frac{q}{VB}$$

$$A_2 = \frac{\alpha V}{g}; \quad B_2 = \frac{\beta}{g}; \quad C_2 = 1; \quad D_2 = 0; \quad E_2 = S_f - S_o - \frac{\beta Vq}{gA}$$

As E_1 is different from E_2, equations 18 and 19 are not homogeneous. As E_1 and E_2 in a general case are not equal, and as E_2 is different from zero, even if $q = 0$, equations 18 and 19 are not reducible, or the roles of dependent and independent variables are not interchangeable. In other words, the hodograph transformation of the (x,t)-plane into the (V,H)-plane is not applicable.

The solution of equations 18 and 19 gives the two functions $V(x,t)$, and $H(x,t)$.

A linear combination of two derivatives: $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial t}$ of function $f(x,t)$ is a derivation of f in a direction given by $dx{:}dt = a{:}b$. For the case $x(\sigma)$ and $t(\sigma)$ representing a curve with the parameter σ, then $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial t}$ is a derivative along the curve, if $\frac{\partial x}{\partial \sigma} : \frac{\partial t}{\partial \sigma} = a{:}b$. A direction is called characteristic, if the derivatives of V and those of H combine to derivatives in the same direction, so that the coefficients of differential equations 16 and 17 for such two directions and for derivatives in these two directions become functions of x,t only. In other words, the quasilinear partial differential equations become linear for the two characteristic directions. There are two such different directions, and they depend on the point x,t, as well as on the values V,H at this point.

Assuming that such a direction is given by $\frac{\partial x}{\partial \sigma} : \frac{\partial t}{\partial \sigma}$, then the following relation is obtained by a procedure given in above mentioned main reference (ref 2, p. 41)

$$a \left(\frac{\partial t}{\partial \sigma} \right)^2 - 2b \frac{\partial x}{\partial \sigma} \frac{\partial t}{\partial \sigma} + c \left(\frac{\partial x}{\partial \sigma} \right)^2 = 0 \qquad (20)$$

where

$$a = [AC] \; ; \; 2b = [AD] + [BC] \; ; \; c = [BD]$$

with the abbreviation $[XY] = X_1 Y_2 - X_2 Y_1$. In the case of unsteady free surface flow of equations 18 and 19

$$a = \frac{A}{VB} - \frac{\alpha V}{g} ; \quad 2b = - \frac{\alpha + \beta}{g} ; \quad c = - \frac{\beta}{gV}$$

As $ac - b^2$ is always smaller than zero, because in this case

$$ac - b^2 = - \frac{\beta A}{gBV^2} - \frac{1}{4g^2} \left(\alpha - \beta \right)^2$$

the system of equations 18 and 19 is called hyperbolic, so that their name is <u>quasilinear hyperbolic partial differential equations of unsteady free surface flow with two dependent and two independent variables</u>.

Designating the slope of a characteristic direction as $\xi = \frac{\partial x}{\partial \sigma} : \frac{\partial t}{\partial \sigma}$, then equation 20 becomes a quadratic equation of ξ :

$$c \, \xi^2 - 2b \, \xi + a = 0 \qquad (21)$$

with two different real solutions : ξ_+ and ξ_- , but generally with $\xi_+ \neq \xi_-$. The two characteristic directions in the (x,t)-plane are given by two equations

$$\left(\frac{dx}{dt} \right)_1 = \xi_+ \; , \text{ and } \left(\frac{dx}{dt} \right)_2 = \xi_- \qquad (22)$$

at the point (x,t) .

The two solutions of equation 21 are

$$\xi_+ = \frac{\alpha + \beta}{2\beta} V - \sqrt{\frac{(\alpha - \beta)^2 V^2}{4\beta^2} + \frac{gA}{\beta B}} \tag{23}$$

$$\xi_- = \frac{\alpha + \beta}{2\beta} V + \sqrt{\frac{(\alpha - \beta)^2 V^2}{4\beta^2} + \frac{gA}{\beta B}} \tag{24}$$

The two roots of equations 21, or equation 23 and equation 24, are therefore functions of V, H, x, t, and they depend on the individual functions $V(x,t)$ and $H(x,t)$.

The above equations 22 are two separate ordinary differential equations of the first order. Each of them define one-parametric family of characteristic curves (simply called the characteristics) in the (x,t)-plane. These two families will be designated as C_+ and C_-.

Equations 23 and 24 show that ξ_+ and ξ_- do not depend on E_1 and E_2, or they do not depend directly on q, S_o and S_f (lateral flows, conduit slope, and resistance to flow), but indirectly through the dependent variables V and H.

Taking the mean depth $H_m = A/B$ into account, and putting as a first approximation that $\alpha = \beta = 1$, equations 23 and 24 become

$$\xi_+ = V - \sqrt{gH_m} = V - C_o \tag{25}$$

$$\xi_- = V + \sqrt{gH_m} = V + C_o \tag{26}$$

with $C_o = \sqrt{gH_m}$ an approximate expression of theoretical celerity of a very small water disturbance in a large canal.

With the same idea of the above approximation, ξ_+ is positive and ξ_- is negative for subcritical flow, while both ξ_+ and ξ_- are positive for supercritical flow.

The two families of characteristic curves, C_+ and C_-, are often represented in the form $\delta(x,t)$ = constant, and $\gamma(x,t)$ = constant respectively, and they form a curvilinear net. For each constant value of δ or γ then a characteristic curve is defined, (fig. 5).

The introduction of new parameters, γ, δ instead x,t in such a way that δ is constant along the curves C_+ and γ is constant along C_- is very useful. Through any point (x,t) the two characteristic curves C_+ and C_- pass, and their parameters γ, δ are then characteristic parameters. If these parameters are introduced, then

$$\frac{\partial x}{\partial \gamma} = \xi_+ \frac{\partial t}{\partial \gamma} \text{ along } C_+, \quad \text{and } \frac{\partial x}{\partial \delta} = \xi_- \frac{\partial t}{\partial \delta} \text{ along } C_- .$$

Using similar procedure as in the case of deriving equation 20, two new differential equations are obtained in the form (ref 2, p. 43)

$$T\,\xi_+ \frac{\partial V}{\partial \gamma} + (a - M\xi_+) \frac{\partial H}{\partial \gamma} + (K - N\xi_+) \frac{\partial x}{\partial \gamma} = 0 \tag{27}$$

$$T\,\xi_- \frac{\partial V}{\partial \delta} + (a - M\xi_-) \frac{\partial H}{\partial \delta} + (K - N\xi_-) \frac{\partial x}{\partial \delta} = 0 \tag{28}$$

in which $T = \left[AB\right]$, $M = \left[BC\right]$, $K = \left[AE\right]$, and $N = \left[BE\right]$.

Using the coefficients of equations 18 and 19 these values are:

$$T = \frac{\beta A}{gVB} \; ; \; M = -\frac{\beta}{g} \; ; \; K = \frac{A}{VB}(S_f - S_o) - \frac{q}{gB}(\alpha + \beta) \; ; \; N = -\frac{\beta q}{gVB}$$

The following four characteristic equations are therefore, available:

$$I_+)\quad \frac{\partial x}{\partial \gamma} - \xi_+ \frac{\partial t}{\partial \gamma} = 0 \tag{29}$$

$$I_-)\quad \frac{\partial x}{\partial \delta} - \xi_- \frac{\partial t}{\partial \delta} = 0 \tag{30}$$

$$II_+)\quad \frac{\beta A}{gVB}\,\xi_+ \frac{\partial V}{\partial \gamma} + \left(\frac{A}{VB} - \frac{\alpha V}{g} + \frac{\beta}{g}\xi_+\right)\frac{\partial H}{\partial \gamma} +$$

$$+ \left[\frac{A}{VB}(S_f - S_0) - \frac{q}{gB}(\alpha + \beta) + \frac{\beta q}{gVB}\xi_+\right]\frac{\partial x}{\partial \gamma} = 0 \tag{31}$$

$$II_-)\quad \frac{\beta A}{gVB}\,\xi_- \frac{\partial V}{\partial \delta} + \left(\frac{A}{VB} - \frac{\alpha V}{g} + \frac{\beta}{g}\xi_-\right)\frac{\partial H}{\partial \delta} +$$

$$+ \left[\frac{A}{VB}(S_f - S_o) - \frac{q}{gB}(\alpha + \beta) + \frac{\beta q}{gVB}\xi_-\right]\frac{\partial x}{\partial \delta} = 0 \tag{32}$$

which hold for every solution $V(x,t)$ and $H(x,t)$, and refer to its characteristics C_+ and C_-, while ξ_+ and ξ_- are given by equations 23 and 24 respectively. Equations 29 through 32 is a system of four differential equations for four variables V, H, x, t as functions of parameters γ and δ. These four equations are simple because each equation contains the derivatives with respect to only one of independent parameters, and the coefficients do not depend on the independent parameters (canonical hyperbolic differential equations).

Introducing the simplifying assumptions in equations 29 through 32 and in equations 23 and 24, as:

$C_o = \sqrt{gH_m} = \sqrt{g\frac{A}{B}} = \sqrt{g\,\epsilon H}$ (with $H_m = \epsilon H$, where H

is water depth), $q = 0$, $\alpha = \beta = 1$, then equations 29 through 32 become

$$I_+) \quad \frac{\partial x}{\partial \gamma} - (V - C_o) \frac{\partial t}{\partial \gamma} = 0 \tag{33}$$

$$I_-) \quad \frac{\partial x}{\partial \delta} - (V + C_o) \frac{\partial t}{\partial \delta} = 0 \tag{34}$$

$$II_+) \quad (V - C_o) \left(\frac{\partial V}{\partial \gamma} - \frac{g}{C_o} \frac{\partial H}{\partial \gamma} \right) + g\,(S_f - S_o) \frac{\partial x}{\partial \gamma} = 0 \tag{35}$$

$$II_-) \quad (V + C_o) \left(\frac{\partial V}{\partial \delta} + \frac{g}{C_o} \frac{\partial H}{\partial \delta} \right) + g\,(S_f - S_o) \frac{\partial x}{\partial \delta} = 0 \tag{36}$$

The derivatives $\partial H/\partial \gamma$ and $\partial H/\partial \delta$ may be replaced by $\partial C_o/\partial \gamma$ and $\partial C_o/\partial \delta$, in which case equations 35 and 36 become

$$II_+) \quad (V - C_o) \frac{\partial}{\partial \gamma} (V - \frac{2}{\epsilon} C_o) + g\,(S_f - S_o) \frac{\partial x}{\partial \gamma} = 0 \tag{35a}$$

$$II_-) \quad (V + C_o) \frac{\partial}{\partial \delta} (V + \frac{2}{\epsilon} C_o) + g\,(S_f - S_o) \frac{\partial x}{\partial \delta} = 0 \tag{36a}$$

In an idealized channel, with $\epsilon = 1$, and with horizontal bottom ($S_o = 0$) and frictionless walls ($S_f = 0$) the above equations 35a and 36a become

$$II_+) \quad d\,(V - 2\,C_o) = 0 \tag{35b}$$

$$II_-) \quad d\,(V + 2\,C_o) = 0 \tag{36b}$$

In this last case, equations 33 and 34 become

$$I_+) \quad \left(\frac{dx}{dt}\right)_1 = V - C_o \tag{33b}$$

$$I_-) \quad \left(\frac{dx}{dt}\right)_2 = V + C_o \tag{34b}$$

Equations 33b through 36b are often used in the application of characteristics to unsteady free surface flow in large channels, but the approximation thus introduced departs appreciably from the actual wave patterns.

B. DISCUSSION OF CHARACTERISTIC CURVES

The coefficients of equations 18 and 19 contain beside the variables V, H, the following quantities: A, B, q, α, β, S_f, and S_o, and g being a constant. The Darcy-Weisbach formula for resistance losses is $S_f = \frac{f}{4R}\frac{V^2}{2g}$, with f = Darcy-Weisbach friction coefficient, and R = hydraulic radius. The coefficient f, in general, is a function of Reynolds number, but for rough pipes of given roughness it is a constant for sufficiently great Reynolds numbers. In this case: A, B, R, f, α, β, and g are generally functions of H and x only. The quantities A, B, R, f, α, β, q, and S_o do not contain derivaties of H and V, but are functions of V, H, x and t.

The main feature of characteristic curves is the replacement of the original system of the two partial differential equations, equations 18 and 19, by the characteristic system of the four differential equations, equations 29 through 32. According to the derivation (ref 2) every solution of the original system satisfies this characteristic system, and the converse is also true, that every solution of the characteristic system, equations 29 through 32, satisfies the original system, equations 18 and 19, provided that Jacobian

$$\frac{\partial x}{\partial \gamma}\frac{\partial t}{\partial \delta} - \frac{\partial x}{\partial \delta}\frac{\partial t}{\partial \gamma} = (\xi_+ - \xi_-)\frac{\partial t}{\partial \gamma}\frac{\partial t}{\partial \delta}$$

does not vanish.

In the case the differential equations 18 and 19 are linear, then ξ_+ and ξ_- are known functions of x, t, so that equations I, 29 and 30, are not coupled with equations II, 31 and 32. In this case equations I determine two families of characteristic curves, C_+ and C_-, independent of the solution. The linearization of equations 18 and 19 introduces such a departure from the real flow phenomena, that this case will not be pursued here in this analytical study.

If $E_1 = E_2 = 0$, and if $A_1, \ldots\ldots D_2$ depend on V, H only, which would be a very rough approximation to the real flow conditions, the situation is similar, namely the differential equations are reducible, the slopes ξ_+ and ξ_- are known functions of V and H, and equations II are independent of x and t. The same case is when E_1 and E_2 do not vanish but depend on V and H only. This last case is applicable to equations 18 and 19 under the condition that the conduit is prismatic and the bottom slope S_o is constant, because all coefficients $A_1 \ldots\ldots E_2$ may be considered as dependent only on V and H. The characteristic curves in the (V, H)-plane, designated Γ_+ and Γ_-, as the images of the characteristic curves C_+ and C_- in the (x,t)-plane, are independent of the special solution V(x,t), H(x,t) considered. However, the assumptions made above to convert equations 18 and 19 into the reducible equations are already an approximation to the real unsteady free surface flow.

As the purpose of this study is a determination of the effects of different assumptions or of neglect of factors, any restriction in the basic general differential equations would mean a departure from the basic approach already selected for this study.

The initial value problem, or the initial conditions are of a major importance in the theory of hyperbolic differential equations. A curve must be known with all values along it in the (x,t)-plane. In the most general case, and for the unsteady free surface flow in conduits, either a velocity hydrograph $V(t)$, or better a discharge hydrograph $Q(t)$, is known for a given x-value, or a wave profile along the conduit as $H(x)$ for a given t is available. The flow conditions should be known, so as soon as $Q(t)$ is known, that the functions $V(t)$ and $H(t)$ may be determined, or as soon as $H(x)$ is known that $V(x)$ can be determined. In the first case, the initial conditions are given along a vertical straight line, for which all values V, H, x, t, are known, and in the second case a horizontal straight line gives the initial conditions with the corresponding values V, H, x, t known along it.

As soon as the initial line is known, the problem is to determine in the neighborhood of this line a solution $V(x,t)$, $H(x,t)$ of equations 18 and 19, which takes on the prescribed values V, H on the line. It is assumed that the line initially known has no characteristic direction, which in this particular case of unsteady flow is a right assumption.

Using the characteristic form, equations 29 through 32, of the partial differential equations 18 and 19 the integration problem can be treated as the corresponding problem for ordinary differential equations.

As equations 29 through 32 are given in parametric form, with γ, δ characteristics parameters, the line of initial values may be considered as the image of the special line: $\gamma + \delta = 0$, because the characteristic parameters γ and δ were introduced with reference to a curve on which $\gamma = \delta$, and now it is necessary only to replace δ by $-\delta$. The initial conditions can be formulated for the differential equations 29 through 32 in the (γ, δ)-plane. A method of iterations (ref 2, p. 49) enables the determination of values V, H, x, t at a point (γ, δ) which is in one-sided neighborhood of the initial line: $\gamma + \delta = 0$, (fig. 6). The solution thus obtained is a solution of equations 29 through 32.

The iteration process shows that the values V, H, x, t at the point $P(\gamma, \delta)$ depend only on the initial values at the segment of the line $\gamma + \delta = 0$ between the points $(-\delta, \delta)$ and $(\gamma_1 - \gamma)$ indicated in fig. 6. In the (x,t)-plane, (fig. 7) it means that the values V, H at the point $P(x,t)$ depend only on the values V, H of the segment from x_1 to x_2, and since the curves δ = constant and γ = constant determine two characteristic curves C_+ and C_-, the interval x_1 to x_2 on the line between these characteristic curves passing through the point $P(x,t)$ is called the domain of dependence.

On the other hand, if a point $R(x,0)$ is selected in the initial line, (fig. 7) with the initial values V, H, then the characteristic curves C_+ and C_- through the point R determine the range of influence. Only the values V, H at the points (x,t) inside the range of influence depend on the initial values V, H of the point R, the outside points do not.

If the values of first and higher partial derivatives of V and H are continuous along the initial line, then they are continuous also in all the points in the (x,t)-plane. If, however, there are some places of discontinuity either at the initial line ($\partial V/\partial x$, $\partial V/\partial t$, $\partial H/\partial x$, $\partial H/\partial t$ or higher partial deri-catives are not continuous, which mean that some disturbances exist), or the discontinuities are introduced at some points in the (x,t)-plane, then the discontinuities in derivatives occur only along characteristics passing through the discontinuity points on the initial line. In a common way of expression, the discontinuities in first or higher partial derivatives of V and H propagate along the characteristic lines in the (x,t)-plane. These discontinuities propagate along one or both of the two characteristics through the point of the source of discontinuity, and they can never disappear. The discontinuities refer only to the derivatives of V, H, but not to the discontinuities in V, H themselves, which propagate as surges (bores or depressions).

The characteristic form, equations 29 through 32, of the differential equations 18 and 19 are especially useful for numerical solutions. If the differential equations are replaced by equations for finite differences, the numerical solutions can be carried out with little labor, and especially if the digital computers are used for these computations.

The characteristic curves, particularly in their simplified form, are quite useful in analyzing the properties of the solutions, and in studying the initial and boundary conditions.

IV. SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS FOR UNSTEADY FREE SURFACE FLOW IN CONDUITS

A. INTEGRATION OF DIFFERENTIAL EQUATIONS

Three methods are available for the integration either of the two partial differential equations 18 and 19, or of their four equivalent characteristic equations 29 through 32: analytical integration, graphical integration of finite differences, and numerical integration by finite differences.

For the analytical method of integration, the initial and boundary conditions (i.e., hydrographs, conduit shape, lateral inflows or outflows) must be expressed in simple analytical forms and introduced in equations 18 and 19 . The complexity of these expressions as well as that of equations 18 and 19 make the analytical integration impossible except in cases of extreme simplifications which mean a substantial departure from the real flow patterns. The method of analytical integration is, therefore, outside the procedures planned for this study, except for some very approximate preliminary design methods to be eventually developed in the future research activities of this study.

The method of finite differences, either graphical or numerical, is based on a replacement of increments dV, dH, dx, dt, dA, dB . . . . . by their finite differences ΔV, ΔH, Δx, Δt, ΔA, ΔB . . . . . The partial derivatives $\partial V/\partial x$, $\partial V/\partial t$, $\partial H/\partial x$, and $\partial H/\partial t$ are replaced by ratios of finite differences $\Delta V/\Delta x$, $\Delta V/\Delta t$, $\Delta H/\Delta x$, and $\Delta H/\Delta t$. Now equations 18 and 19 become

$$\frac{A_o}{V_o B_o} \frac{\Delta V}{\Delta x} + \frac{\Delta H}{\Delta x} + \frac{1}{V_o} \frac{\Delta H}{\Delta t} + \frac{q_o}{V_o B_o} = 0 \tag{37}$$

$$\frac{\alpha_o V_o}{g} \frac{\Delta V}{\Delta x} + \frac{\beta_o}{g} \frac{\Delta V}{\Delta t} + \frac{\Delta H}{\Delta x} + (S_f - S_o)_0 - \frac{\beta_o V_o q_o}{g A_o} = 0 \tag{38}$$

Where the finite differences Δx and Δt of independent variables are selected in some way, ΔV, and ΔH are the changes of dependent variables for Δx, Δt, and A_o, B_o, V_o, q_o, $(S_f - S_o)_0$, α_o, β_o are the mean values of these quantities for both Δx and Δt.

For selected Δx, and Δt at the values V_2, H_2 are assumed first at the end of Δx, Δt. With the initial values V_1, H_1 known, the mean values $V_o = (V_1 + V_2)/2$, $A_o = (A_1 + A_2)/2$, $B_o = (B_1 + B_2)/2$, $q_o = (q_1 + q_2)/2$, $2(S_f - S_o)_0 = (S_f - S_o)_1 + (S_f - S_o)_2$, α_o, β_o, are determined. Equations 37 and 38 for these coefficients known and for Δx and Δt selected give the values ΔV, ΔH. If $V_1 + \Delta V = V_2$, and $H_1 + \Delta H = H_2$, then the assumed values V_2, H_2 are correct. If not, the iterative process is carried out until the right values V_2, H_2 are obtained.

The procedure of supplementing the finite differences method consists of dividing the conduit in reaches Δx_1, Δx_2, ... Δx_n, either equal or unequal, which division depends on the type of conduit. In the case of storm drains the inlet points (manholes), the junction points, and the points where any changes of quantities A, B, S_o, f, q, etc., occur, determine the subdivision of the conduits into reaches. The selection of Δt intervals, equal or unequal, is a special problem to deal with.

There is a limitation for the selection of Δt once Δx is selected. As it is shown in figure 7, the domain of dependence must be taken into consideration, namely for a selected $\Delta x = x_2 - x_1$, the Δt should be so selected, as to have the point P inside the domain of dependence. In this case, one can be sure that the changes in variables V, H and other quantities outside the reach $\Delta x = x_2 - x_1$ have no bearing on the corresponding values at the point P. This limitation makes the use of characteristic curves advantageous, that the selection of Δx, Δt would satisfy in all cases the requirement of the new points in the (x,t)-plane being selected inside the domain of dependence. The approximate relation $\Delta x = (V \pm \sqrt{gH_m})\Delta t$ determines the relation between

Δx and Δt . It can be used to check numerically, that Δt is inside the domain of dependence, and it depends upon the selected mesh of points in the (x,t)-plane. The Δt must be sufficiently small that it falls within the domain of dependence.

The smaller are the values of Δx and Δt , the greater is the computational work, and also the greater is the accuracy (taking into account the effects of rounding errors in the computer), because the assumption of linear change of variables V, H and of the other quantities inside the finite differences Δx , Δt becomes more justified than in the case of large values of Δx , Δt . The accuracy of background data for conduit characteristics, wave shape and lateral flows, however, determines the economical low limit Δx and Δt . The more accurate the background data, the smaller can be the finite differences for a more accurate routing procedure.

The computed end values of V, H and other quantities for Δx_1 and Δt_1 become the values at the beginning of finite differences Δx_2 and Δt_2 . The iterative (or trial-and-error) or direct computation procedure, inherent to the finite differences method, depend on the numerical set-up selected. The points selected in the (x,t)-plane make a mesh, and the choice of the most appropriate mesh, which gives the greatest accuracy for a given amount of computational work, represents one of the pivot problems of the numerical solution by finite differences.

The difficulties in applying the iterative procedure in the classical numerical computations by a desk computer have shifted this method of finite differences in the past in two directions, namely towards the use of:
a) Characteristic curves in the form of simplified four differential equations which replace the equations 18 and 19; b) Graphical procedures, by using the (x,t)-plane for characteristic lines, and the (V,H)-plane for the results of this graphical integration. The four characteristic differential equations are also expressed in finite differences form, but they are nearly always used in the very simplified form, which means neglecting some factors, or assuming simpler conduit shapes and flow resistance than they actually are.

B. COMPARISON OF METHODS OF FINITE DIFFERENCES AS APPLIED TO TWO PARTIAL DIFFERENTIAL EQUATIONS, AND TO FOUR CHARACTERISTIC DIFFERENTIAL EQUATIONS

As it was stressed in the introduction of this report, the advent of digital computers (and also analog computers), to eliminate the tedious and expensive labor of the iterative procedure of finite differences method, and the development of numerical methods in solving partial differential equations, have changed the conditions previously existing in the application of numerical procedures in solving the differential equations. Some of advantages of the method of characteristics, especially the use of its graphical procedure and the finite differences, have disappeared, and the direct use of the two partial differential equations of unsteady free surface flow in conduits, expressed in finite differences form of equations 37 and 38, with the appropriate numerical methods and the digital computer, has become as attractive (ref 4, 5, 6), as the use of method of characteristics.

As both sets of equations, set of equations 18 and 19 expressed also in finite differences form as equations 37 and 38, and set of four equations, equations 29 through 32, expressed also in finite differences form, may be used for obtaining the solutions $V(x,t)$ and $H(x,t)$, and as both sets of equations have been programmed and solved by digital computers (or analog computers), the practical questions arise as to the choice between two sets for the use in this study.

The finite difference method applied to the two basic partial differential equations of unsteady free surface flow in conduits is planned to be used to determine the effects of different factors in equations 18 and 19 during the wave movement along drains. This method is planned to be used also to determine the order of magnitude of individual terms in these equations under different initial and boundary conditions. The purpose is developing a set of wave routing procedures for practical use. The preference at the initial stages of the future research program of this study will be given to the use of the two partial differential equations instead of to that of four equivalent characteristic differential equations. This is considered especially appropriate in the case of the most general case of equations 18 and 19, with no neglect of terms, or no simplification of expressions. It is assumed that equations 18 and 19 in their finite difference form will be simpler to use in digital (or analog) computers, than it is the case with the four equations 29 through 32, with the simultaneous use of equations 23 and 24, in their finite difference form.

It might be the case, however, that some approximate methods of unsteady flow computations, when several assumptions and simplifications are introduced for equations 18 and 19, may be simpler if the method of characteristics is used instead that of the two partial differential equations. This problem will remain open for the present status of the study, and will be investigated when the phase of digital (or analog) computer studies would be carried out.

C. PRACTICAL ASPECTS IN THE APPLICATION OF FINITE DIFFERENCE METHOD

The finite difference method applied to the two partial differential equations of unsteady free surface flow in conduits has, in summary, the following characteristics:

1. The increments are replaced by finite differences, and partial derivatives by quotients of the corresponding finite differences.

2. The variation of variables and all parameters inside the finite differences Δx, Δt is assumed to be linear.

3. The changes of parameters A, B, R, f, S_o, q, α, β, as functions of V, H, x, t are known as boundary and initial conditions.

4. The initial conditions of unsteady flow are clearly defined.

5. The selection of Δx is a function of the accuracy of available data, and the economic justification of the precision of results. The selection of Δt depends on the selection of Δx, in order that the domain of dependence should cover the selected new points in the (x,t)-plane, for which the next computation should determine V, H values.

6. The selection of the mesh of points in the (x,t)-plane is an important part of numerical methods to be applied, in order that the numerical procedure becomes feasible and practical.

The basic principle of applying the method of finite differences is to carry out the solution step-wise in time-units of length Δt. To show this procedure, the (x,t)-plane with points determined by the finite differences will be used, (fig. 8), and especially adapted to the unsteady free surface flow in storm drains.

As the initial conditions, for $t = 0$, the values V and H should be known all along the storm drain, (fig. 8), or the values V, H should be known along x-axis. In the same time, the inflow hydrographs at inlet-points, I_1, I_2, should be given. In other words, on the verticals I_1, I_2 . . . . . the inflow discharges Q are given. At each inlet, therefore, there is a continuity equation $Q_i + Q_u = Q_d$ to be satisfied, where Q_i = inflow discharge from the inlet, Q_u = the discharge flowing from upstream drain, in Q_d = the discharge flowing at the beginning of downstream drain, (fig. 9). The other condition at inlet point, namely the relation of stages, would be satisfied also, after the hydraulic characteristics of an inlet type manholes have been specified.

It is supposed, that the inlet inflows at the moment $t = 0$ are zero, or that values V, H on the x-axis are not affected by the inflows. In other words, it may be assumed that the water flow through storm drain before the storm starts is steady low flow produced by the general drainage of adjacent ground water in order to keep the highway area or the other areas dry. This assumption is generally true in city drainage problems, when the storm floods are usually superimposed on sewage flow in drains.

To perform the computations by the method of finite differences, a rectangular, a staggered, or any other mesh of points may be selected. For the purpose of this analytical study and as an example the staggered net point lattice, or the staggered mesh is selected as a feasible point net (ref 4, 5, 6, and 7), (fig. 8). The values V, H are given along x-axis ($t = 0$) for all points at distances Δx. For two points 1, 2 the values are V_1, H_1, and V_2, H_2. In the middle of two points, point 0, the values are approximately

$$V_o = (V_1 + V_2)/2 \text{, and } H_o = (H_1 + H_2)/2 \tag{39}$$

The partial derivatives replaced by quotients of finite differences along t-axis from the point 0 to point 3, Δt distant from $t = 0$, are approximately:

$$\frac{\Delta V}{\Delta t} = \frac{V_3 - V_o}{\Delta t} ; \quad \frac{\Delta H}{\Delta t} = \frac{H_3 - H_o}{\Delta t} \tag{40}$$

and along x-axis

$$\frac{\Delta V}{\Delta x} = \frac{V_2 - V_1}{\Delta x} ; \quad \frac{\Delta H}{\Delta x} = \frac{H_2 - H_1}{\Delta x} \tag{41}$$

Equations 37 and 38 for Δx, Δt, and their coefficients determined at the point 0, with $A_o = (A_1 + A_2)/2$; $B_o = (B_1 + B_2)/2$; $q_o = (q_1 + q_2)/2$, α_o and β_o averaged in the same way, and $(S_f - S_o)_0 = \left[(S_f - S_o)_1 + \right.$ $\left. + (S_f - S_o)_2\right]/2$, give the two unknown values V_3 and H_3 at the point 3:

$$V_3 = V_o - \frac{\alpha_o}{\beta_o} V_o (V_2 - V_1) \frac{\Delta t}{\Delta x} - \frac{g (H_2 - H_1)}{\beta_o} \frac{\Delta t}{\Delta x} - \frac{g}{\beta_o} (S_f - S_o)_0 \Delta t + \frac{V_o q_o}{A_o} \tag{42}$$

$$H_3 = H_o - \frac{A_o}{B_o}(V_2 - V_1)\frac{\Delta t}{\Delta x} - V_o(H_2 - H_1)\frac{\Delta t}{\Delta x} - \frac{q_o \Delta t}{B_o} \tag{43}$$

with $q_o = 0$ these two equations become simple.

The criterion for convergence of the finite difference scheme as Δx and Δt tend to zero is that the point 3 should be inside the domain of dependence determined by the characteristic lines C_+ and C_- through points 1, 2, (fig. 8). Using the expressions, equations 25 and 26, for the slopes of characteristic lines, the above criterion may be simply written as an approximation

$$\Delta t \leq \frac{\Delta x}{2\sqrt{gA_o/B_o}} \tag{44}$$

The problem of selecting the type of net point lattice (rectangular, staggered, centered net point scheme, or any other type) is a special problem to be dealt with during the analysis of digital computer procedures and is treated here only briefly for illustrative purposes (see ref 4, 5, and 6).

D. BOUNDARY PROBLEMS

There are four types of problems in a storm drain system, which can be considered as boundary problems: 1) The most upstream inlet point problem, with the discharge hydrograph Q(t) given; 2) The problem of ordinary inlet points, with discharge hydrograph Q(t) given, as well as the hydraulic head loss relationship at the inlet structure; 3) The junction problem, where two main storm drains meet at the level, with $Q_1 + Q_2 = Q_3$, and $H_1 = H_2 = H_3$, the discharge and level relationships for 3 close cross sections to the theoretical junction cross section; and 4) The problem of free outlet boundary conditions, given by discharge-stage relationship at the conduit end.

1. Most Upstream Inlet

The first boundary problem is shown schematically at the extreme left in the figures 8 and 9, as inlet I_1. The inflow discharges are given at discrete points Δt apart along the vertical for $x = 0$.

According to the properties of characteristic curves, for the subcritical flow and at the point $(x = 0, t)$ one characteristic curve is directed to the right and another to the left. It means that only one boundary condition is necessary. For the supercritical flow both characteristic curves are directed to the right, so that two boundary conditions are necessary. For all points along the vertical

$$Q = VA \tag{45}$$

so that the relation of V, H is determined. For the points 4, 5, 6, and 7 the partial derivatives are replaced by quotients as

$$\frac{\Delta V}{\Delta t} = \frac{V_6 - V_7}{\Delta t}; \quad \frac{\Delta H}{\Delta t} = \frac{H_6 - H_7}{\Delta t}; \quad \frac{\Delta V}{\Delta x} = \frac{2(V_5 - V_7)}{\Delta x}; \quad \frac{\Delta H}{\Delta x} = \frac{2(H_5 - H_7)}{\Delta x} \tag{46}$$

In some cases it would be better to use $\Delta V / \Delta t = 2(V_6 - V_4)/\Delta t$, and $\Delta H / \Delta t = 2(H_6 - H_4)/\Delta t$, but in this case with given $Q_6 = V_6 A_6$, and $Q_7 = V_7 A_7$, it is better to use the quotients of equation 46.

Equations 37 and 38 give for $q = 0$

$$\frac{2A_7}{V_7 B_7} \frac{V_5 - V_7}{\Delta x} + \frac{2(H_5 - H_7)}{\Delta x} + \frac{1}{V_7} \frac{H_6 - H_7}{\Delta t} = 0 \tag{47}$$

$$\frac{2\alpha_7 V_7}{g} \frac{V_5 - V_7}{\Delta x} + \frac{\beta_7}{g} \frac{V_6 - V_7}{\Delta t} + \frac{2(H_5 - H_7)}{\Delta x} + (S_f - S_o)_7 = 0 \tag{48}$$

with

$$Q_6 = V_6 A_6 \tag{49}$$

$$Q_7 = V_7 A_7 \tag{50}$$

There are four equations 47 through 50 for four unknown V_6, H_6, V_7, H_7, because V_5 and H_5 are known, and all other quantities are functions of known V, H.

For the supercritical flow the discharge-stage rating curve at the beginning of the drain gives the other boundary condition.

2. Current Inlet

For the ordinary inlet points, as the second boundary problem, it is necessary to know the head loss at the inlet as the function of two discharges, Q_u (upstream), Q_i (inlet discharge), and one depth (H_u) so that

$$\Delta H_i = F(Q_u, Q_i, H) \tag{51}$$

for a given inlet box, with the inlet discharge assumed to enter the inlet box under the water surface.

The values V, H are known for the points 9, 11 of fig. 8. It is feasible to consider the section Δx of 9-11 as three reaches: $\Delta x/2$ from 9 to left of 10 (10L), the reach of length zero at the inlet, and $\Delta x/2$ from the right of 10 (10R) to 11. There are now 8 unknowns:

H_{10L}, H_{10R}, H_{12L}, H_{12R}, V_{10L}, V_{10R}, V_{12L}, V_{12R}, with

the following equations:

$$H_{10L} + h_{10} = H_{10R} + \Delta H_{10} \tag{52}$$

$$H_{12L} + h_{12} = H_{12R} + \Delta H_{12} \tag{53}$$

$$V_{10L} A_{10L} + Q_{10} = V_{10R} A_{10R} \tag{54}$$

$$V_{12L} A_{12L} + Q_{12} = V_{12R} A_{12R} \tag{55}$$

For the points 9-10 L-12 L , the partial derivatives are replaced by the quotients

$$\frac{\Delta V}{\Delta x} = \frac{2(V_{10L} - V_9)}{\Delta x}; \quad \frac{\Delta H}{\Delta x} = \frac{2(H_{10L} - H_9)}{\Delta x}; \quad \frac{\Delta V}{\Delta t} = \frac{V_{12L} - V_{10L}}{\Delta t}; \quad \frac{\Delta H}{\Delta t} =$$

$$= \frac{H_{12L} - H_{10L}}{\Delta t}$$

The similar ratios are obtained for points 10R - 11-12R. Equations 37 and 38 for $q = 0$ give four additional equations

$$\frac{2A_{10L}}{V_{10L} B_{10L}} \frac{V_{10L} - V_9}{\Delta x} + \frac{2(H_{10L} - H_9)}{\Delta x} + \frac{1}{V_{10L}} \frac{H_{12L} - H_{10L}}{\Delta t} = 0 \tag{56}$$

$$\frac{\alpha_{10L} V_{10L}}{g} \frac{V_{10L} - V_9}{\Delta x} + \frac{\beta_{10L}}{g} \frac{V_{12L} - V_{10L}}{\Delta t} + \frac{2(H_{10L} - H_9)}{\Delta x} +$$

$$+ (S_f - S_o)_{10L} = 0 \tag{57}$$

$$\frac{2A_{10R}}{V_{10R}B_{10R}} \frac{V_{11}-V_{10R}}{\Delta x} + \frac{2(H_{11}-H_{10R})}{\Delta x} + \frac{1}{V_{10R}} \frac{H_{12R}-H_{10R}}{\Delta t} = 0 \tag{58}$$

$$\frac{\alpha_{10R}V_{10R}}{g} \frac{V_{11}-V_{10R}}{\Delta x} + \frac{\beta_{10R}}{g} \frac{V_{12R}-V_{10R}}{\Delta t} + \frac{2(H_{11}-H_{10R})}{\Delta x} +$$

$$+ \ (S_f - S_o)_{10R} = 0 \tag{59}$$

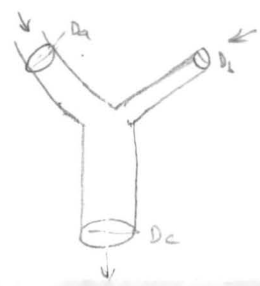
The eight equations, 52 through 59, make it possible to obtain eight unknowns. In equations 52 and 53 $h_{10} = h_{12} = D_2 - D_1$, and ΔH_{10} and ΔH_{12} are given by equation 51, or they are functions of V_{10L}, H_{10L}, and Q_{10}, or V_{12L}, H_{12L}, and Q_{12} respectively. For the solution of these eight equations by a digital computer, they should be arranged in such a manner as to facilitate the programming and the computation.

3. Junction

The third boundary problem of storm drain is the junction problem. In the case the vertical line representing a junction of two drains, with the crown lines matching for all three drains at the junction, then for three branches a, b, and c the following conditions must be satisfied at the junction:

$$H_a = H_b + (D_a - D_b) = H_c - (D_c - D_a) \tag{60}$$

when the diameters of three conduits are $D_b < D_a < D_c$, and



$$Q_a + Q_b = Q_c \text{ , or } V_a A_a + V_b A_b = V_c A_c \tag{61}$$

Assume that the vertical line I_2 in fig. 8 and the inlet box in fig. 9 represent the junction of three storm drains. The values of V , H are supposed to be known at the points 9, 11, and values V_a , V_b , V_c , H_a , H_b , and H_c at the point 8 are also known. The 6 values V, H at the point 10, and 6 values at the point 12 make 12 unknowns. Equations 60 and 61 for the points 10 and 12 give six equations, and the application of equations 37 and 38 for the three branches at the points 10 and 12 give the other six equations. The solution of this set of 12 equations give the 12 unknowns. Due to the fact that equation 60 gives a simple relation among H_a , H_b , and H_c at each point , the 12 equations are easily reduced to the 8 equations with the 8 unknowns. The quotients $\Delta H/\Delta t$ and $\Delta V/\Delta t$ might be used either between the points 10 and 12, 8 and 10, or 8 and 12, in order to approximate at the best way the corresponding partial derivatives for each branch. The arrangements of eight equations thus obtained would be made according to the feasibility of digital computer programming and computation.

4. Outlet Section

The fourth boundary problem is defined by the drain outlet conditions. In this general study a free conduit outlet will be assumed, though any other hydraulic outlet relationship may be equally treated, if it is already known.

For subcritical flow, one characteristic curve is directed to the right, another one to the left. One boundary condition is necessary. For the supercritical flow both characteristic curves are directed to the right, and therefore, no boundary condition is necessary. The initial stage value at the end is a sufficient data.

The outflow rating curve, as a discharge-stage relationship is supposed to be known at the end of the conduit. It is assumed here, that the subcritical flow occurs at the end of the pipe, so that a stage is associated with a given discharge, or that $Q(H)$ is given. In the case the unsteady flow influences the discharge-stage relation, a family of rating curves $Q(H_1, H_2)$ must be available.

Assume that the outlet cross section, (fig. 9) is represented by the last vertical in the (x,t)-plane, (fig. 8) and that the V, H-values are known at the points 13 and 14. To determine the V, H-values at the points 15 and 16, the following equations can be used:

$$Q_{15}(H_{15}) = 0, \text{ or } Q_{15}(H_{14}, H_{15}) = 0 \tag{62}$$

$$Q_{16}(H_{16}) = 0 \text{ or } Q_{16}\left(H_{16}, \frac{H_{16}+H_{17}}{2}\right) = 0 \tag{63}$$

$$\frac{2A_{15}}{V_{15}B_{15}} \frac{V_{15}-V_{14}}{\Delta x} + \frac{2(H_{15}-H_{14})}{\Delta x} + \frac{1}{V_{15}} \frac{H_{16}-H_{15}}{\Delta t} = 0 \tag{64}$$

$$\frac{2\alpha_{15}V_{15}}{g} \frac{V_{15}-V_{14}}{\Delta x} + \frac{\beta_{15}}{g} \frac{V_{16}-V_{15}}{\Delta t} + \frac{2(H_{15}-H_{14})}{\Delta x} + (S_f-S_o)_{15} = 0 \tag{65}$$

It is feasible, (ref 7), to use the quotients $\frac{V_{16}-V_{13}}{2\Delta t}$ and $\frac{H_{16}-H_{13}}{2\Delta t}$ instead of $\frac{V_{16}-V_{15}}{\Delta t}$ and $\frac{H_{16}-H_{15}}{\Delta t}$.

Equations 62 through 65 give four equations for four unknowns V_{15}, V_{14}, H_{15}, and H_{14}. The discharges Q_{15} and Q_{16} are replaced in equations 62 and 63 by $Q_{15} = V_{15}A_{15}$, and $Q_{16} = V_{16}A_{16}$ respectively.

5. Other Boundary Problems

The lateral flow q_o of equations 37 and 38 are neglected in the treatment of four boundary problems. If, however, there are continuous inflow (drainage) or outflow (spillways) along the drains, the use of q_o corresponding does not change the solutions of boundary problems, but only adds the new factor, which complicates the programming and increases the computational work.

In the case there is a water level drop at an inlet box for all discharges, the upstream part of the drain can be treated independently from its downstream part, and the outflow hydrograph from the upstream part becomes the inlet hydrograph (first boundary problem) of the downstream part. It can occur, that in the low flows there is a level drop at the inlet box, so that the upstream levels are independent from the downstream flow conditions, but for the highest stages the interdependence may be created by a backwater effect. These conditions would impose a new boundary limit, namely the level when the backwater starts to affect the upstream part of the drain, and different programming must be carried out, so as to switch from one condition to the other as soon as the boundary limit stage would be passed in one or another direction.

The boundary problems are very important when the digital computers are used for unsteady free surface flow computations, and their detailed analysis and solutions for the best programming and cheapest computations will be the subject of the future studies of unsteady flow by digital computer.

V. COEFFICIENTS FOR DIFFERENTIAL EQUATIONS

A. COEFFICIENTS

The coefficients of the two partial differential equations 18 and 19 are:

$$\frac{A}{VB} , \frac{1}{V} , \frac{q}{VB} , \frac{\alpha V}{g} , \frac{\beta}{g} , \text{ and } S_f - S_o - \frac{\beta Vq}{gA}$$

All quantities should be expressed by four variables, V, H, x, t , plus some constants, in order that equations 18 and 19 can be integrated by finite difference methods. Therefore: A, B, q, α, β, S_o , S_f , must be expressed as functions of these four variables.

B. GEOMETRIC CHARACTERISTICS OF CONDUITS

Four quantities: A (cross section area), B (cross section width at water surface level), R (hydraulic radius), and S_o (bottom slope) determine the general geometric characteristics of a storm drain.

Storm drains may be circular or of any other shape. In the case of circular drains the quantities A, B, R, may be expressed as a function of diameter and water depth in the conduit. However, this procedure becomes very cumbersome in the case of the other storm drain shapes. The graphical or numerical relationship is usually given for non-circular shapes. As this study will be conducted for the circular conduits, the other drain shapes will not be treated here. It is a current procedure to determine the approximate analytical expressions A(H) , B(H), R(H) for a drain type, so that this expression could be used in digital computer operations, with the minimum occupation of its storage space.

1. Area Function

Instead of using the water depth H, the dimensionless ratio $h = H/D$, and a ratio $a = A/A_f$ may be used, with A_f = area of a pipe flowing full, with $A_f = D^2\pi/4$. It follows from fig. 10

$$A = \frac{D^2}{8}(\beta - \sin\beta) = \frac{D^2}{4} \arccos\left(1-2\frac{H}{D}\right) - \left(\frac{D}{2} - H\right)\sqrt{(D-H)H} \qquad (66)$$

or

$$a = \frac{A}{A_f} = \frac{1}{\pi} \arccos(1-2h) - \frac{2}{\pi}(1-2h)\sqrt{h(1-h)} \qquad (67)$$

$$A = \frac{D^2}{4}\left[\arccos(1-2h) - 2(1-2h)\sqrt{h(1-h)}\right] \qquad (68)$$

Figure 11 gives the relative cross section area for different relative depths $h = H/D$ of the pipe.

2. Width Function

The width B, (fig. 10), can be expressed in both ways, absolute as B, or relative as $b = B/D$, so that

$$B = D \sin\frac{\beta}{2} = 2D\sqrt{\frac{H}{D}\left(1 - \frac{H}{D}\right)} = 2D\sqrt{h(1-h)} \qquad (69)$$

$$b = 2\sqrt{h(1-h)} \qquad (70)$$

3. Hydraulic Radius Function

The wetted perimeter P is

$$P = \frac{D\beta}{2} = D \arccos\left(1 - \frac{2H}{D}\right) = D \arccos(1-2h) \qquad (71)$$

Using equations 66 and 71 the hydraulic radius is

$$R = \frac{A}{P} = \frac{D}{4} - \frac{(D-2H)\sqrt{(D-H)H}}{2D\ \arccos\,(1-2H/D)} \tag{72}$$

$$R = \frac{D}{4}\left[1 - \frac{2(1-2h)\sqrt{h\,(1-h)}}{\arccos\,(1-2h)}\right] \tag{73}$$

or with the relative value $r = R/(D/4)$, where $D/4$ is the hydraulic radius of full pipe flow

$$r = 1 - \frac{2(1-2h)\sqrt{h(1-h)}}{\arccos\,(1-2h)} \tag{74}$$

The relative values r are given in fig. 11 as function of $h = H/D$. The relative value R/H is given as

$$\frac{R}{H} = \frac{1}{4h} - \frac{(1-2h)\sqrt{h(1-h)}}{2h\ \arccos\,(1-2h)} \tag{75}$$

4. Presentation of Geometric Characteristics

The cross section characteristics at any point of a storm drain are, therefore, given for area, width and hydraulic radius as functions of conduit diameter and water depth: in the form of analytical expressions (exact or approximate), in graphical form, or in a numerical (tabular) form. The use of one of these forms will depend on the computational method and device selected. The above formulae show that A, B, R are functions of both D, H. As D is a constant for a given drain section, all these three cross section characteristics are function of depth H only.

The practical problem arises when these functions have to be used in computation by digital computer. They are generally approximated by simpler expressions which fit sufficiently well the numerical values, (ref 8).

The area, width, and hydraulic radius functions are complex to use. They will be, therefore, replaced by power series form, so that they will be easily programmed in the case a digital computer is used for storm flood computations. The work would be so directed as to minimize the necessary storage occupation in a computer by the geometric properties of drains, in order to leave storage capacity for inflow hydrographs and other elements not easily subject to an analytical interpretation.

The approximations for above functions, equations 68, 69, and 73 will be dealt with in the future programming for the integration of equations 18 and 19 by digital computer and finite difference method.

5. Drain Slope

The slope S_o will be selected always a constant for a given Δx, or the selection of finite difference Δx will be made so as to never have the change of S_o inside that difference. The slope S_o is positive when the bottom is inclined toward the direction of water flow, and negative for the opposite case.

C. VELOCITY DISTRIBUTION COEFFICIENTS

Two velocity distribution coefficients α and β appear in equations 18 and 19, and are defined as:

$$\alpha = \frac{\int v^3 \, dA}{V^3 \; A} \tag{76}$$

where v is the stream line velocity of the elementary cross section area dA, V = mean velocity, and A = total flow area. This coefficient, sometimes called energy coefficient, Coriolis coefficient or kinetic energy correction factor, comes from the requirements that the energy of flow is the same when the mean velocity replaces the velocities of incremental areas, so that $\alpha V^3 A$ takes care of that requirement.

$$\beta = \frac{\int v^2 dA}{V^2 A} \tag{77}$$

so that $\beta V^2 A$ takes care of the requirement that the momentum of flow is the same when the mean velocity replaces the velocities of incremental cross section areas. This coefficient is sometimes called the momentum coefficient, Boussinesq coefficient, or momentum correction factor.

Putting

$$v = V\ (1 - \nu) \tag{78}$$

and

$$\xi = \frac{1}{A} \int \nu^2\, dA \tag{79}$$

the approximate values of α and β are, (ref 9),

$$\alpha = \frac{1}{A} \int (1 + \nu)^3\ dA = 1 + \frac{3}{A} \int \nu^2 dA = 1 + 3\,\xi \tag{80}$$

$$\beta = 1 + \frac{1}{A} \int \nu^2 dA = \frac{2 + \alpha}{3} = 1 + \xi \tag{81}$$

Another approximation is used, (ref 10) , in the case of a logarithmic velocity distribution with

$$\epsilon = \frac{v_{max}}{V} - 1 \tag{82}$$

so that

$$\alpha = 1 + 3\,\epsilon^2 - 2\epsilon^3 \tag{83}$$

$$\beta = 1 + \epsilon^2 \tag{84}$$

The experimental data indicate, (ref 10) that the α - value varies in the limits 1.03 to 1.36 for fairly straight prismatic channels. The value of α decreases in general with an increase of the channel size, and for channels with considerable depth.

For pipes in full flow, and with the logarithmic distribution of velocities, the analytical expressions for α and β are, (ref 11):

$$\alpha = 1 + 2.93\ f - 1.55\ f^{3/2} \tag{85}$$

and

$$\beta = 1 + 0.98\, f \tag{86}$$

where f is Darcy-Weisbach friction coefficient. As f is the function of Reynolds number for smooth pipe, α and β are, therefore, functions also of Reynolds number.

The values of α change for fully flowing pipes from 1.028 for $f = 0.01$ to 1.13 for $f = 0.05$, and the values of β change less, from 1.01 for $f = 0.01$ to 1.05 for $f = 0.05$.

For the partly flowing conduits α and β are functions of depth. It is expected that α and β depend also on the absolute value of pipe diameter, apart from the depth or depth-diameter ratio. So

$$\alpha\ (f, H, D)\ ;\ \beta\ (f, H, D)\ .\ .\ . \tag{87}$$

with F approximately constant for a given rough pipe, and function of Re for smooth pipes. A recent survey of literature has not produced the data or information on the velocity distribution coefficient functions for partly filled pipes.

In general, the coefficients α and β are taken as unity. This approximation does not affect substantially the flow computations. A different approach is planned for this study in connection with the general approach outlined in the introduction part of this report.

One of the purposes of this basic study is the analysis of errors and departures among computed unsteady free surface flows and the true observed flow patterns, by using the statistical methods of theory of errors. The assumption of $\alpha = 1$ and $\beta = 1$, instead of using their real functions, equation 87, will introduce the departures between the two compared flow patterns. In order to analyze the sources of errors, and their order of magnitude, the efforts will be made to determine the functions of α and β, equation 87, and to use them in equations 18 and 19 for the highest order approximation treatment of unsteady free surface flow in storm drains. The later neglect of these functions by assuming $\alpha = 1$ and $\beta = 1$, and the differences detected will give the effects of velocity distribution coefficients on flow patterns under different initial and boundary conditions.

The basic remarks directed towards different methods of computing the unsteady free surface flows are in the lack of any real experimental or computational data which enable one to assess the errors with a good scientific approach.

It will be necessary, however, to introduce an approximation also in this most general treatment of velocity distribution coefficients and of their effects. It consists in assuming that the coefficients determined for steady flow are the same as those for the unsteady flow. This assumption will be carried out, if data (either available in literature or obtained easily on the future hydraulic model study) would not be obtained during the period of the study. This assumption will have a relatively small effect, because the small differences of α and β in unsteady and in steady flow should be applied to a relatively small effect of α and β coefficients on unsteady flow patterns.

D. FLOW RESISTANCE

The Darcy-Weisbach formula for the flow in conduits will be used. The standard formula for the full flowing pipe is

$$S_f = \frac{f}{D}\,\frac{V^2}{2g} \tag{88}$$

with f = Darcy-Weisbach friction coefficient, and $V^2/2g$ the kinetic-energy head. As the kinetic-energy head is $\alpha V^2/2g$, the velocity distribution coefficient α is, therefore, included in f. When α is only function of f, this fact does not matter, but if α has a different function, as for example given by equation 87, then f is dependent on the other factors also than in the case α and f were separated.

For partly filled pipes, a modified formula will be used here, replacing D by 4R, with R = hydraulic radius, so that

$$S_f = \frac{f}{4R}\,\frac{\alpha V^2}{2g} \tag{89}$$

with R given by equation 72 or equation 73, α by equation 87, and f is the friction coefficient dependent on relative roughness of conduit, and Reynolds number Re (for smooth pipes only).

There are several formulae in literature for f for fully flowing pipes, but appropriate expressions are lacking for partly flowing conduits.

The values for f of Karman-Prandtl for smooth full flowing pipes are

$$\frac{1}{\sqrt{f}} = 2 \log \, Re \sqrt{f} - 0.8 \tag{90}$$

For full flowing tamped concrete pipes (ref 12) the values of f experimentally determined start to depart from the curve of equation 90 at $Re = 2 \times 10^5$, but are nearly constant for values $Re > 2 \times 10^5$. For concrete pipes cast in steel forms and vibrated with good joints the conduits are smooth, so that f decreases with an increase of Re, but the f(Re)-line is somewhat higher than the line of equation 90. The fitted line has the expression

$$\frac{\sqrt{8}}{\sqrt{f}} = 2.7 + 5.75 \log \frac{Re\sqrt{f}}{4\sqrt{8}} \tag{91}$$

which approximates well the experimental data. This refers to fully flowing pipes. Experimental data for partly flowing pipes are lacking.

For the purpose of this study two kinds of pipe roughness will be used, in order to develop for both the corresponding programs for digital computer:

1. Rough pipes, with f = constant for greater values of Re than a minimum value (around $Re = 1 \times 10^5$). Using equation 89 the variability of α will be excluded from the potential variability of f with the depth H.

2. Smooth pipes, with f(Re), as in equations 90 and 91, where Re for partly full pipes is defined in an appropriate manner.

In order to introduce the accurate relationship of equation 89 into the basic differential equation 19, the functions $f(Re) = f(V,R,T)$ with T = temperature and R = hydraulic radius; $\alpha(f, H, D)$, and $R(D,H)$ must be clearly defined and substituted in equation 89. In this case, the friction slope becomes a function of five variables: S_f (f, V, D, H, T). For smooth pipes, with $Re = VR/\nu(T)$, where $\nu(T)$ = dynamic viscosity as function of temperature, and using equations 73, 87, and 91 the function S_f (f, V, D, H,T) becomes very complex and feasible to treat only by a digital computer during the integration of partial differential equations by finite difference methods.

The complexity of this resistance law imposes the solution of finite difference equations by an interative (trial-and-error) procedure in most cases, especially for the boundary problems.

It is necessary to investigate the flow resistance at the manholes or inlet boxes for free surface flow. There are limited data in the literature for any type of inlet boxes with free surface flow. There are sufficient data for flow resistance at inlet boxes working under water pressure.

The bulletin No. 41 of the Engineering Experiment Station of the University of Missouri, "Pressure Changes at Storm Drain Junctions", treats only the storm drains with flow under water pressure. As in this study of "Unsteady Free Surface Flow in a Storm Drain" it is supposed that there is only surface flow, a surface level drop takes place at manholes instead of a pressure drop. It has not been possible as yet to use results of the University of Missouri study for this research project.

A survey of literature has not as yet resulted in a feasible method of treating the hydraulic losses of surface flow at manholes, with assumed matching arrangements and with lateral inflow. The adequate solution of this problem seems to be very important for accurate storm flood computation. It might be that the problem would be solved satisfactorily only when the model study would be undertaken in the next phase of study, with this problem as a secondary model study attached to the general model study of storm flood movement through drains.

In order to simulate the actual conditions in the computation of unsteady free surface flow by digital computer, and to compare the results with the wave movement in a conduit, determined by hydraulic model, the simple inlet boxes will be designed according to actual practices, and their hydraulic characteristics will be determined. As an inlet box represents the junction of a small drain with the main drain, the flow resistance in the form of a level drop ΔH at inlet box, will be a function of two types of parameters, geometric and hydraulic. For a fixed shape the geometric characteristics may be reduced to one or two parameters, for example L_1 and L_2, while the hydraulic parameters must be three: upstream discharge Q_u entering the inlet box; lateral discharge Q_i given by inlet hydrograph; and the upstream (or downstream) water level at the manhole, so that

$$\Delta H = F(L_1, L_2, Q_u, Q_i, H_u) \tag{92}$$

In order to solve the problem of manhole hydraulics for surface flow in drains, a characteristic shape of manholes used in highway or urban drainage systems will be adopted.

The hydraulic investigation of selected inlet box, being a part of this study, is considered as a prerequisite for any good comparison among hydraulic model investigations and digital computer analysis of unsteady flow.

The hydraulics of local resistances or of singular losses, which are due to the changes of cross section, to the type of transition from one to another cross section, or even due to the change of their shape, then to the change of direction of storm drain, is considered as known and the relationships will be obtained either from the most recent literature on these subjects or from measurements on a hydraulic model.

The above flow resistance formulae as well as discussion and program for future detailed hydraulic measurements on model conduit refer to steady free surface flow. As discussed previously, the question of a difference in flow resistance between an unsteady and a steady flow may be answered only by special hydraulic investigations, both analytical and experimental. It is anticipated that the research facilities and the future research program will allow - at least partly - an insight in this problem.

E. LATERAL FLOWS

The boundary and initial conditions of a storm drain, selected for this study, do not foresee the continuous lateral spill of water either into the storm drain or out of it. In these cases, therefore, $q = 0$. The point inlet inflows are treated with the finite differences $\Delta x = 0$. If, however, the spill-over out of the storm drains is designed, or if water can spill over an edge or drain into the storm drain, the function $q(H,x,t)$ should be known as a boundary and initial condition.

VI. INITIAL AND BOUNDARY DATA

A. WATER FLOW CONDITIONS IN DRAIN BEFORE THE STORM FLOOD BEGINS

The antecedent flow conditions to the storm flood inflows into a drain may be the following:

1. The storm drain is dry;
2. The flow is steady, but with a low flow covering the bottom of the drain;
3. The flow is unsteady of a previous storm flood on which the new storm flood (generally to be assumed larger than the previous storm flood) is superimposed.

The dry drain imposes a flood wave front movement along the conduit. The two partial differential equations 18 and 19 do not apply to that condition, because they are developed for gradually varied flow, while the wave front on a dry bed is a rapidly varied flow, a special type of progressing surge. The movement of flood wave on a dry bed will be considered as a particular problem to be investigated separately later in the broad framework of basic and applied research for unsteady free surface flow in storm drain.

The steady flow prior to storm flood inflow will be assumed in further studies. A small steady discharge will create a sufficient flow depth in conduit, that the storm flood wave movement at its beginning can be still considered as gradually varied flow.

The third case may be assumed, under conditions that the wave characteristics (V, H as function of x for a given t, for example) are known prior to the new storm flood inflows.

B. INFLOW HYDROGRAPHS

All inflow hydrographs should be known as the initial data. Their accuracy will determine the precision with which the hydrographs along the storm drain will be computed in comparison with the actual hydrographs.

The steepness of storm flood rise or of recession part of inflow hydrographs has a bearing on the selection of finite difference Δt . The steeper the discharge hydrograph, the smaller should be the finite difference Δt , at least at the parts of hydrograph with steep rate-of-change $\Delta Q/\Delta t$.

The general case of unsteady flow in a storm drain should be analyzed independently of the type of inflow hydrographs at the inlet points, so that no specific shape of hydrograph will be assumed for the analytical approach, except in eventual examples.

C. JUNCTION DATA

When the junction problem must be included in the unsteady flow computation, all data should be available for the drain branches at the junction. If the drainage system has many storm drains which join together from place to place, with the confluence at water levels, all branches should be included into the unique solution, because their wave flows are dependent. In such cases, only the digital computers, with large storage space and a relatively fast computation capacity, are able to carry out economically the simultaneous solution for wave movement and development along all drain branches.

D. OUTFLOW CONDITIONS

The outflow rating curve at the storm drain end in unsteady flow is not in general a unique relation of discharge and water depth at a given place at outlet, because the changing slope of water surface at the outlet makes that relationship dependent also on this third variable. This problem is to be clarified for each case, especially as to the effect which any third variable (in this case: slope, another depth or level, and level difference) may have on the discharge change for a constant value of a stage. This general case will enable, however, the treatment of any outflow conditions, when outflow discharge depends on two other variables (for example, the backwater effect of a pool at the storm drain end).

The free outlfow from a drain was the subject of some experimental studies, but under steady flow conditions. It is an open question how much the rating curve changes in unsteady flow due to changing surface slope. This boundary condition at the outlet poses the question of what is the departure of rating curves in steady and unsteady free surface flow in drains. The future experimental conduit will permit study of this problem.

E. GENERAL BOUNDARY CONDITION DATA

The general scheme of boundary conditions for a storm drain is that all inlets and changes of cross section areas and shapes, matching arrangements, direction of storm drains and similar are located at the discrete points along the drain, known in advance. Between the two adjacent points the drain data are uniform, with slope, cross section area and shape, and resistance factor constant and with no lateral inflow or outflow.

Where there is a drop at a manhole, and where there is never a tailwater effect carried from downstream section to upstream section through the drop, the storm flood computation of the upstream portion of storm drain is independent of all downstream conditions, but it is dependent on the data concerning the outflow rating curve at the drop. The outflow hydrograph from the upstream section of the drain is the inflow hydrograph to the downstream portion of the system. These drop points will make the computation procedure simple, and should be identified and taken into consideration as the important boundary condition data.

Where there is a tailwater effect at the drop from downstream to upstream section, this effect has to be taken into account either through the rating curve $Q = f(H_a, H_d)$, with H_a = upstream level, and H_d = downstream level at the drop, or a submergence factor can be introduced, which represents the ratio of the real outflow at the drop to the outflow which could occur with out the submergence (tailwater effect). The submergence factor for the largest discharge during the surface flow, or for the highest water levels at the drop will determine if the drop could be considered or not as a boundary condition data of storm drain from the point of view of unsteady surface flow, or the drop is only a singular resistance in a unique system. Some conventional criteria for this condition cannot be avoided in the practical applications.

The boundary condition should be analyzed prior to unsteady flow computation for the occurrence of passage from supercritical to subcritical flow or vice versa, both in steady and unsteady surface flow in the storm drain. The places where there is always a passage from subcritical to supercritical flow or vice versa (hydraulic jumps) in unsteady flow will be considered as dividing points for unsteady flow computations, similarly, as for the drops without or with the submergence effect.

VII. RESEARCH PROGRAM, SIGNIFICANCE AND FACILITIES

A. SPECIFIC AIMS

The specific aims of the future research program are:

1. Development of a set of flood routing methods of unsteady free surface flow in a storm drain. Each method of the set should be a feasible procedure for given conditions of storm floods and drain characteristics. This set of methods should cover as wide a range of flow conditions as possible, in computing the depth, the velocity or discharge hydrographs and wave profile at any point or along a system of storm drains.

2. The long term goals of both the basic research and the applied research of unsteady free surface flow in conduits or in special storm drains are a better understanding of flow phenomena, and the development of design criteria and methods for storm drains.

3. To conduct analytical or experimental study of hydrodynamics aspects of unsteady free surface flow in conduits, which could have any effect on flood routing methods to be developed for storm drains.

4. The developed set of flood routing methods should be based on the use of a digital computer, with Fortran programming, in order to use for this purpose any available digital computer manufactured in the United States.

B. METHOD OF PROCEDURES

The research program and procedures used in carrying out the research plan for the unsteady free surface flow in storm drains, foreseen for the fiscal years 1962, 1963, and 1964 are divided into three parts:

1. Hydraulic experimental studies;

2. Digital computer studies, and

3. General comparative studies with final results.

1. Hydraulic Experimental Studies

As the flood hydrographs change slowly in a smooth conduit, an experimental pipe (about 800 feet long) is planned to be installed for hydraulic experiments. The research program will consist of the following problems and procedures:

a. The study of the relationship between boundary roughness and steady free surface flow in a storm drain. The purpose of this study is to determine the effect of boundary roughness, Reynolds number (especially of depth of flow) on the Darcy-Weisbach coefficient f. Results of the experimental study will be used as input data for the digital computer.

b. The study of the relationship between head loss, discharges and water levels at manholes in a storm drain. The first phase will consider the elementary type of manhole only. The experimental results will be entered as input data into the digital computer.

c. The study of the relationship between depths of free surface flow and discharge at the conduit outlet. The study will consider both steady and unsteady flow, to inquire for an eventual difference of rating curves between steady and unsteady free surface flow. Rating curves will be developed for different conduit conditions, and they will be used as boundary conditions for the studies by digital computer.

d. The study of other flow phenomena involving unsteady free surface flow, such as: the amplification of flood waves in channels of steep slopes; the instability of flow when the pipe is flowing nearly full; the passage from supercritical to subcritical flow by the hydraulic jump and the instability of position of the hydraulic jump in unsteady flow; and the study of other similar problems will be carried out.

e. The study of the velocity distribution in drains flowing partly full to determine accurately the velocity distribution coefficients, α and β, for their use as input data into the digital computer.

f. The study of the other flow phenomena observed by using either transparent windows in the drain or transparent sections of pipe. observed flow phenomena would be checked or simulated by digital computer. Reproduction of the flow phenomena would be essential to establishing identity of phenomena examined by model drain studies and digital computer analysis.

g. The research schedule will include the use of three or more slopes of the storm drain, the use of three or more boundary roughnesses, and, for reasons of economy initially one pipe diameter only, or the slope and boundary roughness will be variables, while the pipe diameter will be first constant, but later two additional small pipes may be added to the experimental set-up in order to make diameter the third variable.

h. On the basis of the aforementioned studies to simulate floods in a storm drain by introducing inflow hydrographs at the extreme and at at several inlet points along the model drain. The inflow hydrographs--discharge as a function of time--will be accurately recorded by appropriate devices. The movement and development of flood waves along the storm drain and at the pipe outlet will also be accurately recorded. The recorded inflow hydrographs will be used as input data for the digital computer. The recorded hydrographs at any point in the storm drain or wave profiles along the drain for different times will serve as a basis for comparing with and checking the hydrograph determined for the same point or wave profile at a given time determined by the computer.

2. Digital Computer Studies

The main purpose of using the digital computer studies is to investigate the feasibility of using the two partial differential equations of unsteady free surface flow as the basic mathematical tool for routing of flood waves through storm drains.

The influence of different factors as well as the order of magnitude of different terms in the two partial differential equations will be investigated by computer for different inflow hydrograph and storm drain characteristics.

The selection between digital and analog computer for the research purposes is planned to be studied also.

The advantages of using the digital computer in integrating the two partial differential equations of unsteady free surface flow are:

a. Economy of computation.

b. Speed of predicting or computing the flood waves along the storm drain, and

c. Increased accuracy, but which should correspond to the level of accuracy of the background data.

The program for the digital computer is planned to be carried out simultaneously with the hydraulic model studies. The results obtained by the hydraulic study will be used in developing the program for the computer studies. The errors created in rounding the numbers in digital computer will be studied also.

Two different procedures for integration of the two partial differential equations are planned to be carried out by computer during the study. First, the method of finite differences, as applied to the two partial differential equations of unsteady flow, with different increments (finite differences) of time and length of storm drain will be used. Computations by this method will be compared with the results obtained by hydraulic studies. The second method of integrating will be in applying the method of finite differences to the four characteristic equations, as equivalent equations derived from the two partial differential equations. By using the four characteristics, they will be developed in the most general terms, and then to investigate by the computer which factors, especially the velocity distribution coefficients, could be neglected.

The following is planned to be solved by using a digital computer:

a.) To determine the order of magnitude of different terms in the two partial differential equations for different characteristics of conduit and inflow hydrographs.

b.) To determine the effects of different factors in the two partial differential equations of unsteady free surface flow, equations 18 and 19, on the flood wave developments along a storm drain.

c.) To solve some of the hydrodynamic problems, which are not yet solved by experimental or mathematical methods. One of these problems is the criterion when a wave in a storm drain either does not change or amplifies under different conduit characteristics.

d.) To supply the basic data to assess the accuracy and reliability of different existing flood routing procedures, or to develop a new set of flood routing procedures for storm drains, using the digital computer.

3. Comparative Studies

As soon as the first results from both hydraulic experiments and from the computer will be available, the comparative studies will be carried out simultaneously.

The purpose of these studies is to generalize the results, but also to direct if necessary the hydraulic experimental studies, for additional problems, and to guide the computer programming and study, so that the basic results can be improved. The comparative studies will use the theory of errors and other tools of mathematical statistics and probability, in order to derive a set of flood routing methods feasible for storm drains.

C. SIGNIFICANCE AND CHARACTERISTICS OF THE RESEARCH

In order to drain highways or urban areas during the storm precipitation, and to avoid flooding of highways and cities of all consequences for the given intensity, duration and probability of occurrence of rainfall, by using the large storm drains, four following problems should be solved simultaneously:

1. To determine the inflow hydrographs into inlet points on the highways or streets.
2. To shape and design curb inlets so that they will not impede the desired conveyance of flood waters.
3. To design primary and secondary storm drains in order to reduce flooding on the highways or streets for the design inflow hydrographs.
4. To evacuate the water from the outlet of the main storm drain either by gravity flow or by use of pumps.

The significance of the research program of this study is to find, by using basic and applied research, the feasible methods to solve the problem outlined in item 3. In this case the design inflow hydrographs, geometry of the curb inlets and outflow conditions at main storm drain outlets are assumed to be known, and they will be supposed in this study as to be a known information.

Only the unsteady free surface flow in the storm drain will be studied. The reasons for this are as follows:

a. The main storm drain should be located as near to the highway or street surface as feasible to minimize the stress of the overburden and to avoid additional cost in excavation and in reinforcing of the storm drain.

b. The maximum discharge for free surface flow in conduits is approximately 0.9 of the conduit diameter. If the same discharge should be conveyed in a pipe flowing full and under pressure, the slope of the energy line must exceed the slope of the pipe.

c. If the maximum discharge should exceed the discharge for 0.9 D and the energy line slope, the conduit slope, this results in flooding of the lowest part of the highway or street.

As soon as the flow in the pipe becomes flow under pressure the outflow discharge is approximately equal to the sum inflow of all inlets.

The main significance of this study is to supply the design methods as the final result of the basic and applied research for highway or urban storm drains as well as all other water drainage systems in which flood waves occur. The new methods should replace, where justified or feasible, the current flood routing methods based on simple differential equation.

Due to the fact that the storm drains are made of smooth concrete, the velocities are generally sufficiently great, that the acceleration terms $\partial V/\partial t$, and $\partial V/\partial x$ in equation 19 are not negligible in unsteady water flow through storm drains. This assumption leads to the selection already made that the storm flood routing through storm drains should be primarily based on the two De Saint Venant partial differential equations. The analysis of the order of magnitude of acceleration terms in momentum equation for some specific cases, with the purpose of justifying the above assumption, should be useful and significant.

Though the basic significance will be in defining a convenient set of flood routing methods based on the two partial differential equations, the methods based on the storage equations only (continuity equation), or the methods which are considered as transitions from methods based either on the two or on one differential equation, will be analyzed also as for their applicability in the case of unsteady flow in a storm drain.

To determine the dimensions of a storm drain there are two approaches when the unsteady flow is computed by a storm wave routing method:

1.) The method is so simplified that the diameter or other cross section dimension can be computed directly;

2.) The dimensions of drain are first assumed then the computation of a storm flood by a routing method is carried out along the drain. If the dimensions come out to be either small or large, the new dimensions are assumed, and the storm wave analysis is repeated until the right dimensions are obtained.

When the two partial differential equations are used as the basis for flood routing, only the second approach seems, as by the actual status of unsteady flow theory and practice, possible. The first approach is, however, the goal which should not be overlooked.

The current flood routing methods start from any hydrograph shape, and determine, mostly dividing the hydrograph by time unit Δt in many parts, the transformed hydrograph for a position Δx-distance downstream or upstream from the initial position. As some design storms are one-sharp peak hydrograph, they can be approximated sometimes by an analytical expression. Though the analytical integration of the two partial differential equations is excluded from this study, there is also a potential storm drain routing method, though very approximate, which can be based on routing of parameters (generally three) of the analytical equation of hydrograph along the storm drain, instead of routing the elements of the hydrograph. This method would be suitable for fast computations in preliminary design, if it is shown by future research program to be practically feasible. The significance of this research program will be in determining the feasibility of such a method to storm drain computations.

The expectation is that the future studies are likely to produce a set of potential methods for computation of unsteady flow in a storm drain. Only the comparison by actual computations, by hydraulic model study (or by eventual observations in the nature), and by the use of the digital computer will answer the question under which conditions and with which accuracy each new or existing method should be performed.

D. FACILITIES AVAILABLE

The hydraulic test drain pipe is to be located near the new Hydraulics Laboratory of Colorado State University now under construction. The 30-inch diameter conduit will be approximately 800 feet in length. It will be located along the side of a ridge with a relatively steep but straight side slope. Thus, it will be possible, by holding the entrance end of the pipe fixed while the outlet end is free, to move to vary the slope from 0% to 5%.

Discharge in the conduit will be from a forebay supplied by gravity flow from nearby Horsetooth Reservoir.

Entrance conditions between forebay and test conduit are to be designed such that steady flow conditions will be established within a relatively short reach of the conduit. Thus, permitting a maximum length of pipe for flow analysis and measurement.

The flow hydrographs will be introduced into the main channel flow by means of a supply pipe and laterals. Sections of plexiglass or windows in the test conduit will permit visual observation and photographic analysis of the flow phenomena. The unsteady free surface flow will be measured and recorded by means of appropriate measuring devices. Velocity profiles will be determined.

The conduit flow will be wasted into nearby College Lake and returned during the night by pump and pipe to Horsetooth Reservoir. This system permits a wide range in the discharge demand on the water supply.

The digital computer IBM 1620 located at the computing center on the main campus of the University will be used for the proposed study.

The IBM 7090 as a much faster computer will be available for the research activities of the research staff of Colorado State University, under very favorable conditions. This computer will be used for complex and bulky computation, when IBM 1620 would not be appropriate.

REFERENCES

1. Yevdjevich, V. M., 1960, Flood routing methods. Discussion and Bibliography, prepared by U.S. Geological Survey for U. S. Soil Conservation Service, Project No. 4, Washington, D. C.

2. Courant, R., and Friedrichs, K. O., 1948, Supersonic flow and shock waves, Interscience Publishers, Inc., New York.

3. Stoker, J. J., 1957, Water waves, Interscience Publishers, Inc., New York.

4. Stoker, J. J., 1953, Numerical solution of flood prediction and river regulation problems; Report I; Derivation of basic theory and formulation of numerical methods of attack: New York University, Institute of Mathematical Sciences, Report No. IMM-NYU-200, October.

5. Isaacson, E., Stoker, J. J., and Troesch, B. A., 1954, Numerical solution of flood prediction and river regulation problems (Ohio-Mississippe floods), Report II; New York University, Institute of Mathematical Sciences, Report No. IMM-NYU-205.

6. Isaacson, E., Stoker, J. J., and Troesch, B. A., 1956, Numerical solution of flood prediction and river regulation problems, Report III; New York University, Institute of Mathematical Sciences, Report No. IMM-NYU-235.

7. Isaacson, E., Stoker, J. J., and Troesch, B. A., 1958, Numerical solution of flow problems in rivers; Am. Soc. Civil Engineers Proc., Hydraulic Div. Jour., V. 84, HY5, paper 1810, Oct., p.1-8.

8. Hastings, Cecile Jr., Approximations for digital computers, Princeton University Press, New Jersey (copyright 1955 by the Rand Corporation).

9. Kozeny, Josef, 1953, Hydraulik (Hydraulics), Springer-Verlag, Vienna, p. 61.

10. Chow, Ven Te, 1959, Open-Channel Hydraulics, McGraw-Hill Book Company, Inc., New York, p. 27 to 29.

REFERENCES - continued

11. Streeter, V. L., 1950, Steady flow in pipes and conduits, Chapter VI, p. 401, of the handbook "Engineering Hydraulics", edited by Hunter Rouse, John Wiley and Sons, Inc., New York.

12. Straub, L. G., Bowers, C. E., Pilch Meir, 1960, Resistance to flow in two types of concrete pipe, Technical Paper No. 22, Series B, St. Anthony Falls Hydraulic Laboratory, Minneapolis, Minnesota.

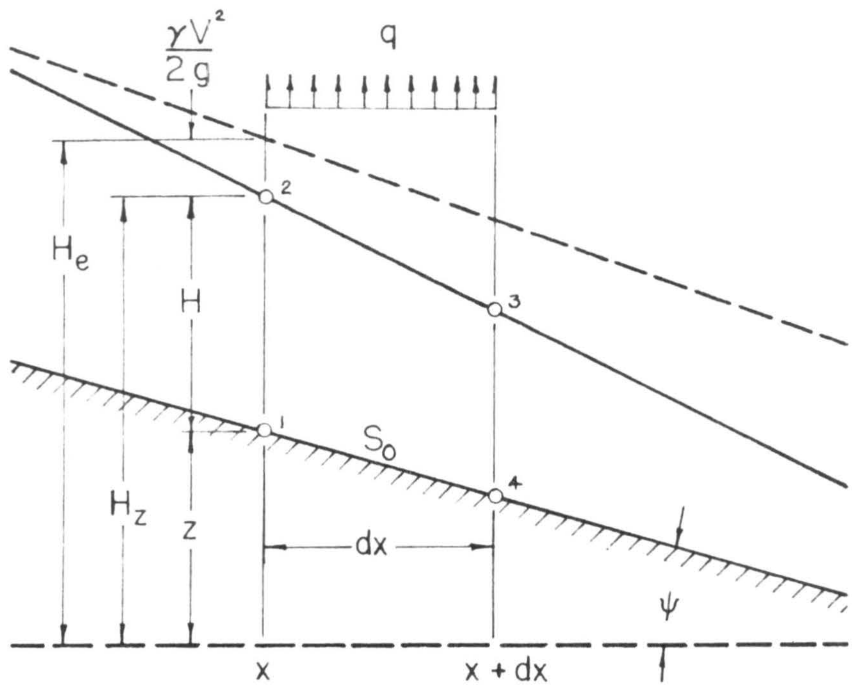


Fig. 1. Unsteady free surface flow in a conduit, schematic representation for derivation of continuity and momentum equations.

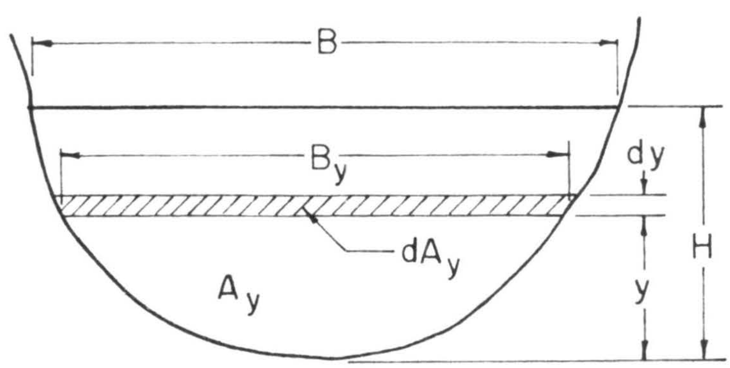


Fig. 2. Geometric elements of a conduit with free surface flow.

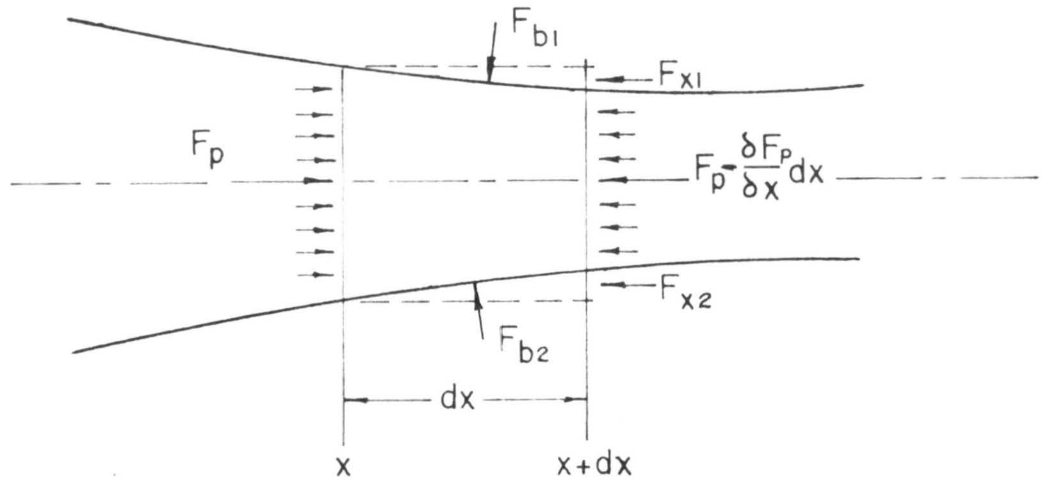


Fig. 3. Forces acting on an incremental slice in free surface flow in a conduit.

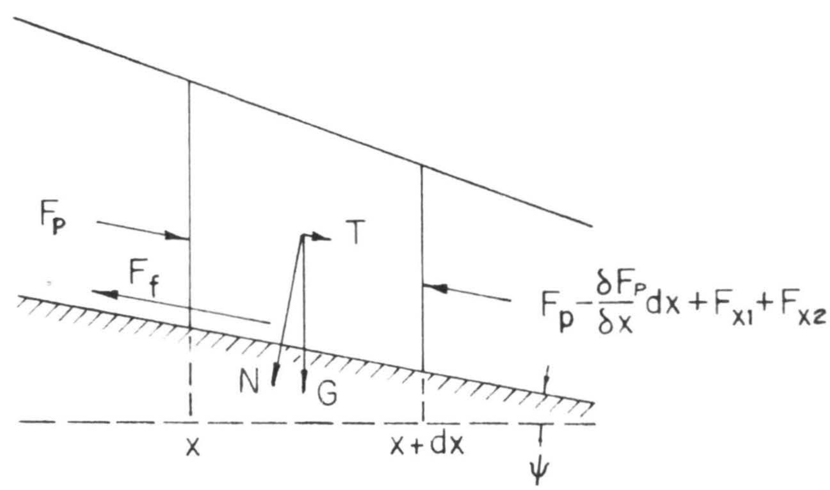


Fig. 4. Forces acting in the direction of slope of conduit bottom.

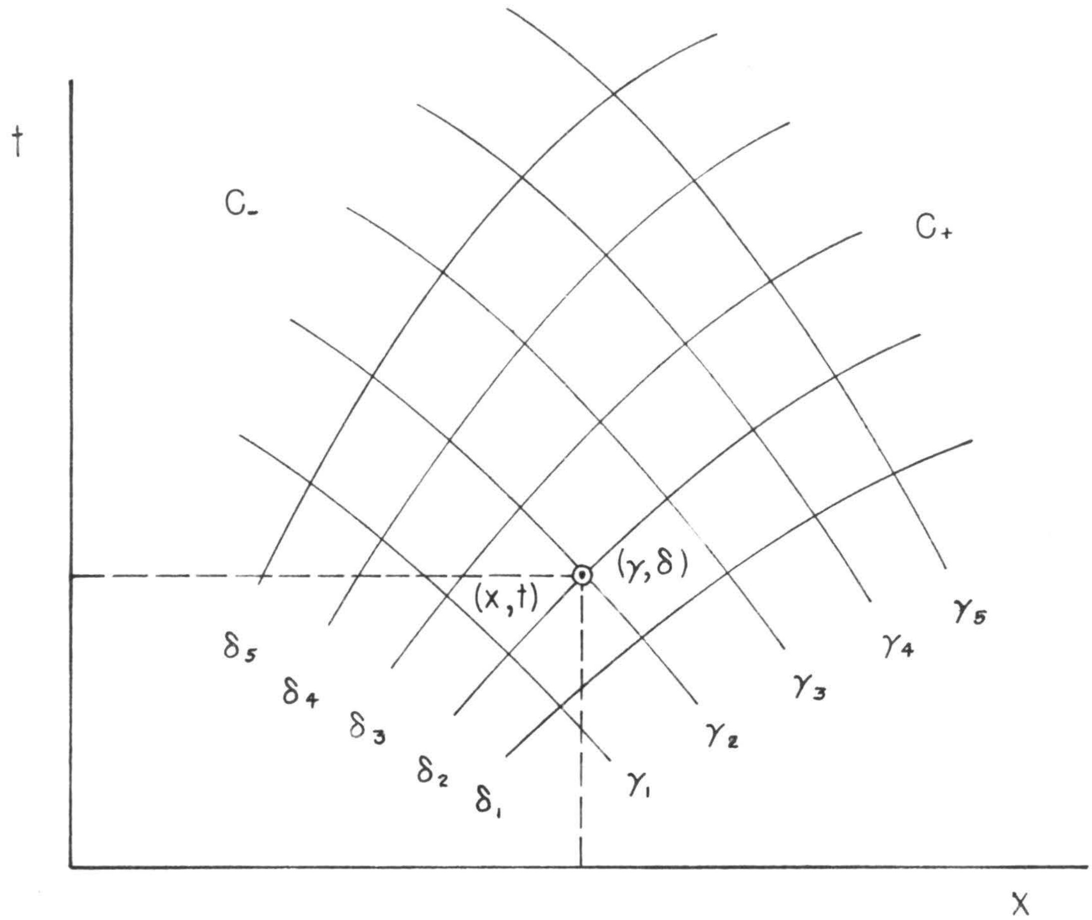


Fig. 5. Net of characteristic curves C_+ and C_- in (x,t)-plane.

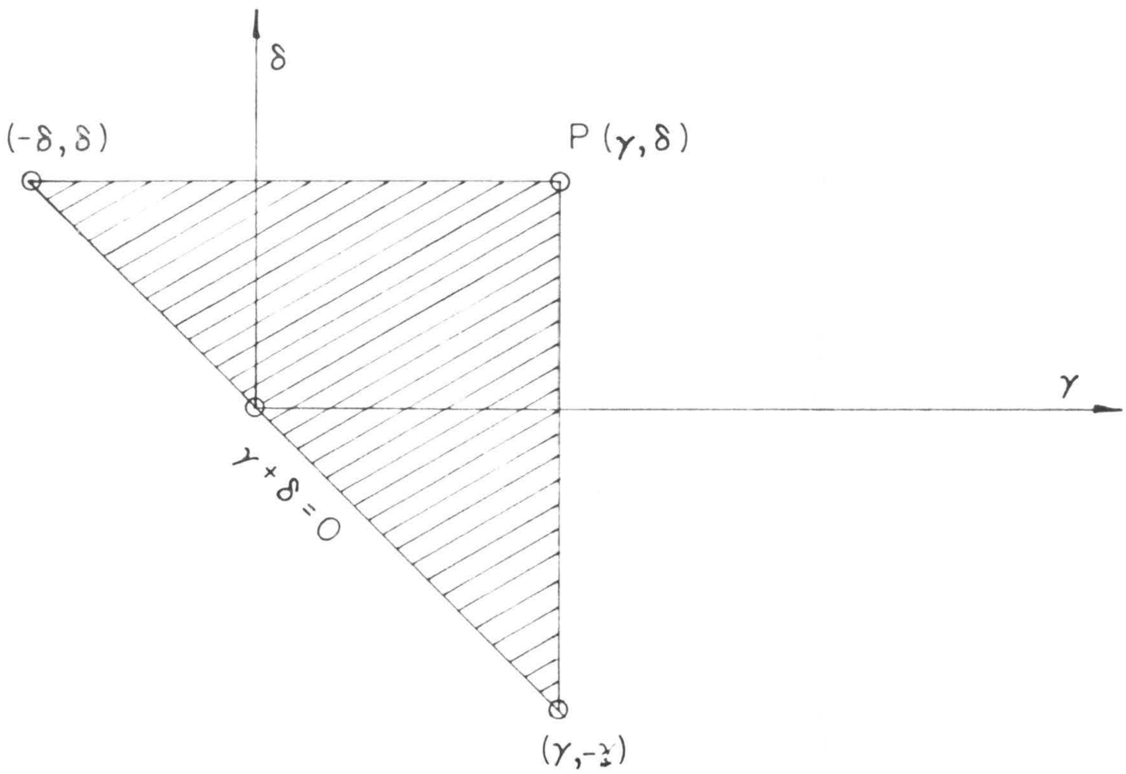


Fig. 6. Characteristic curve $\gamma + \delta = 0$ in (γ, δ)-plane.

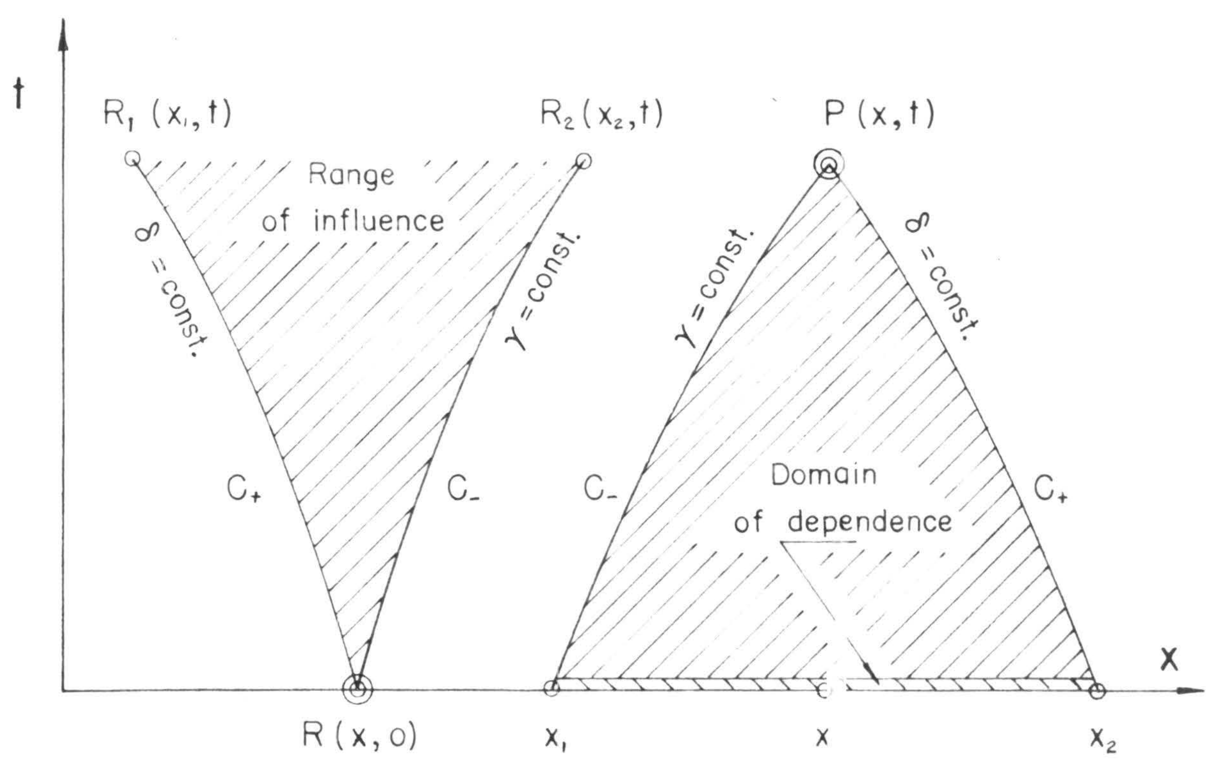


Fig. 7. Domain of dependence and range of influence of characteristic curves in (x,t)-plane.

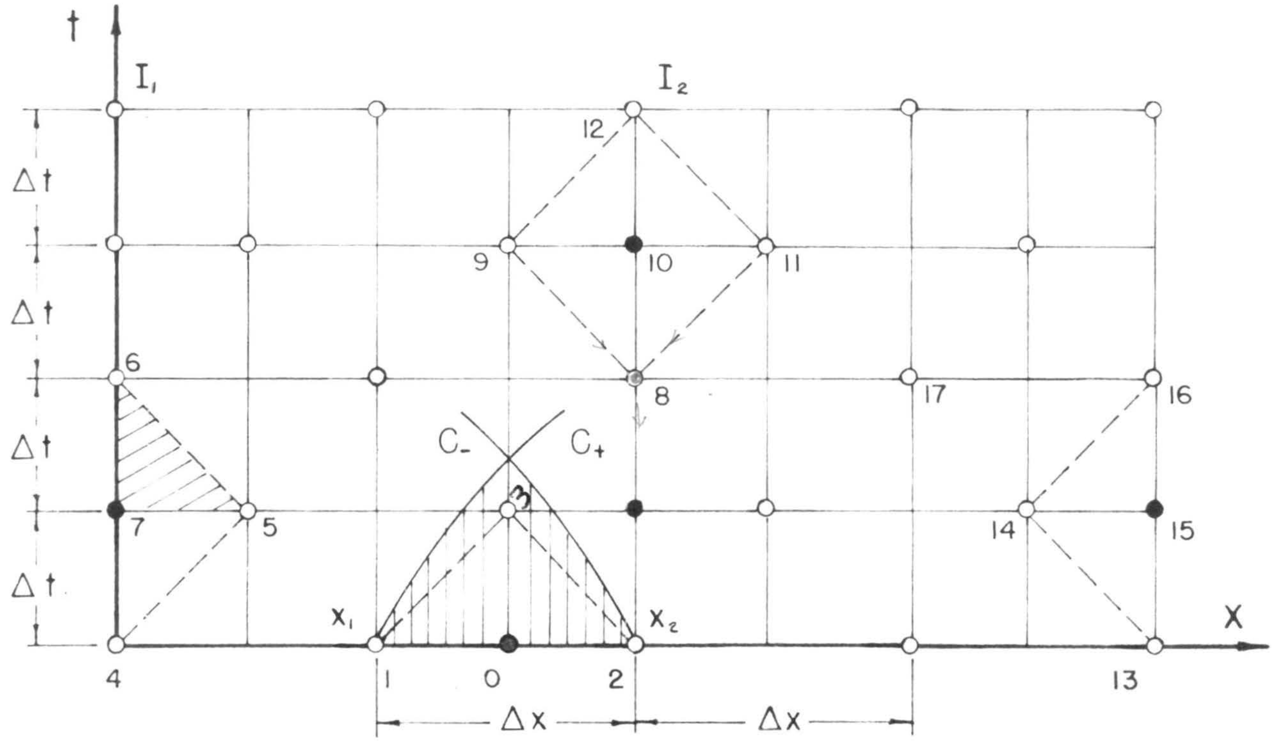


Fig. 8. Staggered net of points Δx and Δt apart in (x,t)-plane, for the analysis of finite difference method, of a general case and of boundary problems.

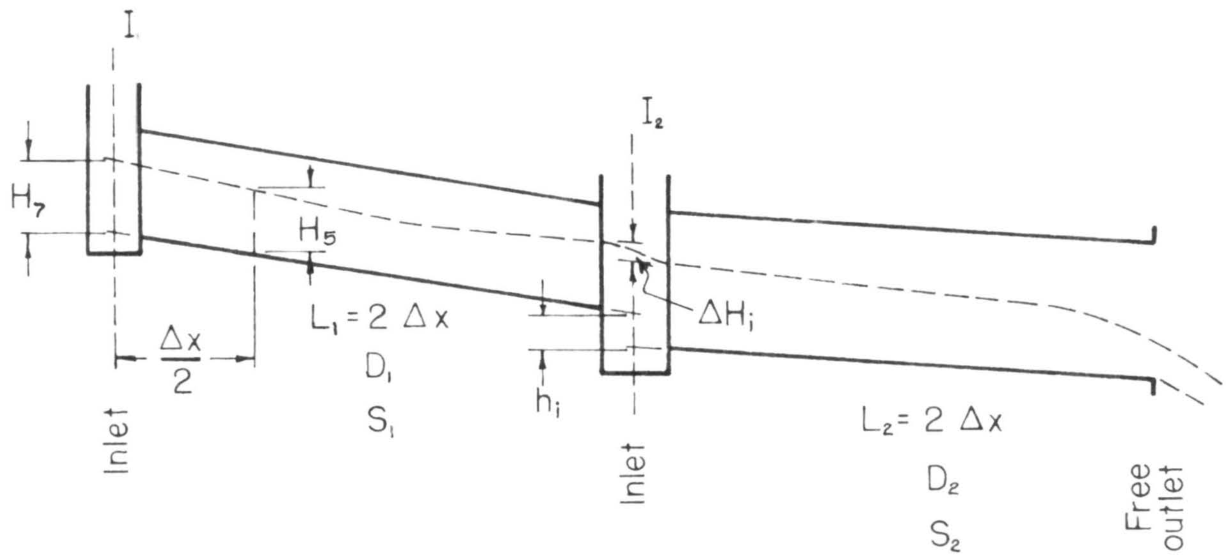


Fig. 9. Schematic representation of circular storm drain with unsteady free surface flow, with an initial inlet, an intermediate inlet (manhole) and the free outlet.

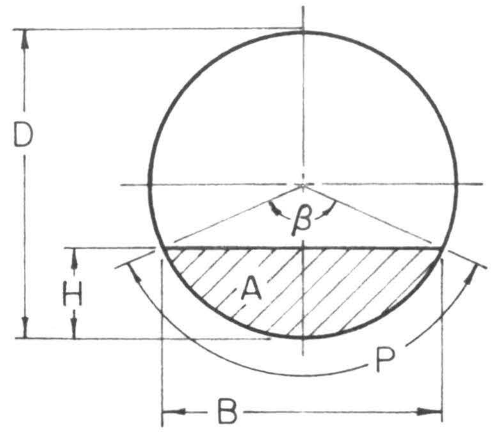


Fig. 10. Geometric elements of a circular drain with free surface flow.

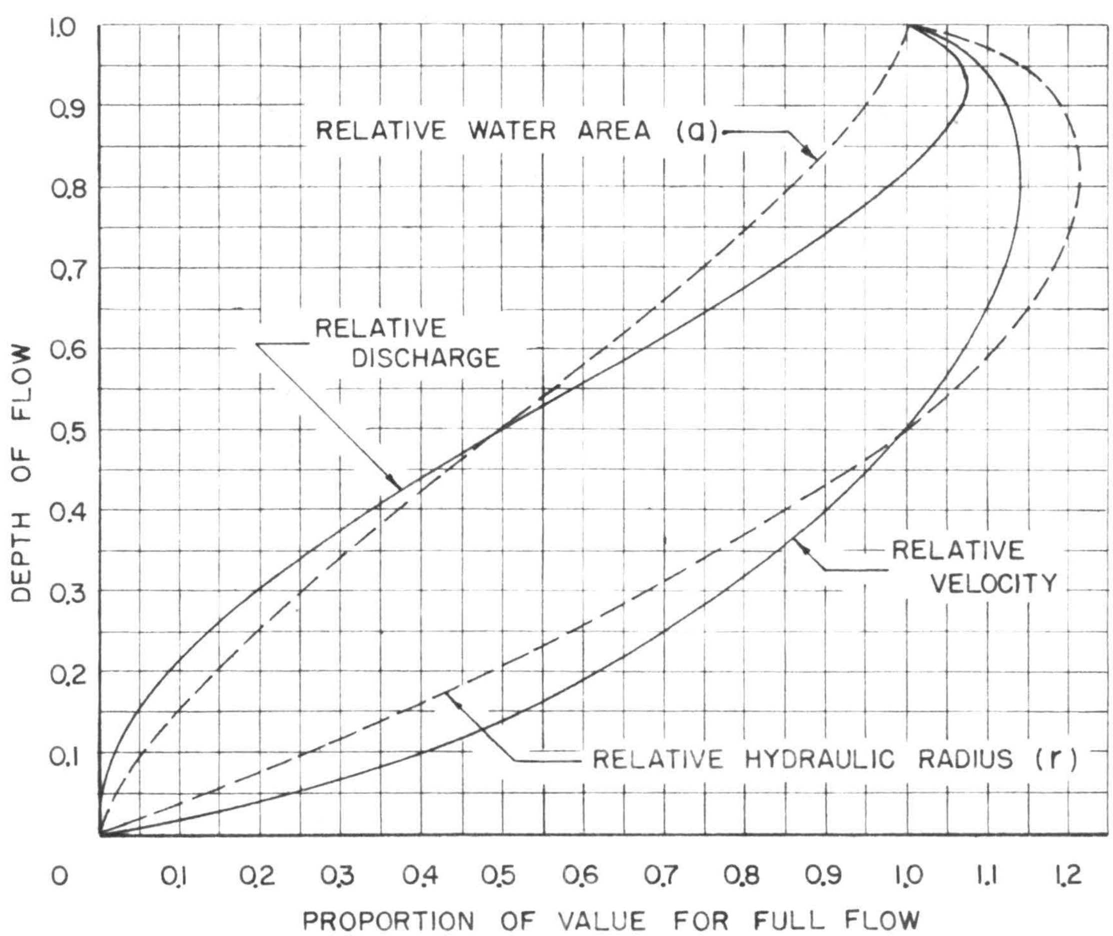


Fig. 11. Relative values (absolute values in relation to values of fully flowing conduit) for area, hydraulic radius, velocity and discharge in drains for any depth of flow (from Concrete Pipe Handbook).