DISSERTATION

INTERSECTIONS OF ψ CLASSES ON HASSETT SPACES OF RATIONAL CURVES

Submitted by

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ABSTRACT

INTERSECTIONS OF ψ CLASSES ON HASSETT SPACES OF RATIONAL CURVES

Hassett spaces are moduli spaces of weighted stable pointed curves. In this work, we consider such spaces of curves of genus 0 with weights all $\frac{1}{q}$, $q \in \mathbb{Z}^+$. These spaces are interesting as they have different universal families and different intersection theory when compared with $\overline{M}_{0,n}$. We develop closed formulas for intersections of ψ-classes on such spaces. In our main result, we encode the formula for top intersections in a generating function obtained by applying an exponential differential operator to the Witten-potential.

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DEDICATION

I dedicate this thesis to my mother.

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Chapter 1

Introduction

The moduli space of algebraic curves of genus 0 with n marked points, $\overline{M}_{0,n}$ (with the Deligne-Mumford compactification [1]) has been an important topic of research in algebraic geometry. These spaces provide an algebro-geometric tool to study how pointed rational curves vary in families, and are of fundamental importance in areas like Gromov-Witten theory and topological quantum field theories [2].

In [3], Hassett constructed a new class of modular compactifications $\overline{M}_{0,\mathcal{A}}$ of the moduli space $M_{0,n}$ of smooth curves with n marked points parameterized by an input datum $\mathcal{A}$, consisting of a collection $\mathcal{A} = (a_1, \ldots, a_n)$ of weights $a_i \in \mathbb{Q} \cap (0, 1]$ such that $a_1 + \ldots + a_n > 2$. We call these spaces $\overline{M}_{0,\mathcal{A}}$ the Hassett spaces of rational curves.

A lot of work is being done on Hassett spaces including developing its tautological intersection theory and weighted Gromov-Witten theory, e.g. in [4], [5] and [6]. In this work, we contribute to the tautological intersection theory of a special case of such spaces- Hassett spaces of rational curves with weights all $\frac{1}{q}$, $q \in \mathbb{Z}^+$, denoted $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$.

For this work, the following notations are used: a ψ class on $\overline{M}_{0,n}$ is denoted as ψ_i; a ψ class on $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$ is denoted as $\bar{\psi}_i$, and the pullback of a ψ class under the reduction morphism from $\overline{M}_{0,n}$ to $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$ is denoted $\hat{\psi}_i$.

First, we develop results for a special case of such spaces- Hassett spaces of rational curves with weights all $\frac{1}{2}$, denoted $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$. These spaces provide for interesting spaces for combinatorial results in its intersection theory because of the symmetry of weights and its connections with intersection theory for $\overline{M}_{0,n}$. These spaces are interesting also because they are fine moduli spaces, are isomorphic to $\overline{M}_{0,n}$, but have different universal families and different intersection theory. Exploring these differences and developing some results in its tautological intersection theory is the first contribution of this work.

In our first result 3.1.1, we develop a closed closed formula (3.4) for the monomials in $\hat{\psi}$ classes for $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$ in terms of cycles on $\overline{M}_{0,n}$. This closed formula is derived using the relation (2.9) between the $\hat{\psi}$ classes and ψ classes on $\overline{M}_{0,n}$, in which ψ_i is corrected by all boundary divisors where the i-th mark is on a twig that gets contracted when pushed forward to $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$. The proof uses this relation to obtain the $\hat{\psi}$ monomials as monomials in ψ classes and boundary divisors on $\overline{M}_{0,n}$. So, the summands in the resulting expansion correspond to modified ψ monomials on certain boundary strata on $\overline{M}_{0,n}$ that are the intersections of these boundary divisors. The dual graphs of these strata are all 'forked' graphs- graphs with a 'central' node and some 'forks', e.g. figure (3.5). We then establish a bijection between summands in the expansion corresponding to these graphs that we call '$\mathcal{P}$'-graphs and the unordered partitions of $[n]$, such that cardinality of each subset in the partition is either 2 or 1. The resulting formula (3.4) has the pullback of monomials in $\bar{\psi}$ classes on $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$ as a sum of the intersections of monomials in ψ classes and boundary stratum corresponding to $\mathcal{P}$-graph on $\overline{M}_{0,n}$. Then we derive two corollaries (3.2.1 and 3.2.2) of this result to calculate the top intersections. These give the top intersections of $\hat{\psi}$ classes as a sum of top intersections of ψ classes on $\overline{M}_{0,n}, \overline{M}_{0,n-1}, \overline{M}_{0,n-2}, \ldots$ with some multiplicities. We point out here that our corollaries (3.2.1 and 3.2.2) can also be deduced from theorem 7.9 in [4]. For our work, we develop specific and explicit closed formulas for our special case of all weights $\frac{1}{2}$ and base our combinatorial analysis closely on the structure of dual graphs.

The main theorem 3.3.1 of this special case of weights all $\frac{1}{2}$ encodes the closed formula (3.2.1) for top intersections in a generating function $G(\mathbf{t})$ obtained by applying a differential operator to the Witten-potential $F(\mathbf{t})$ [7]. This operator $\hat{\mathcal{L}}$ takes the form of an exponential partial differential operator and provides a very nice compact way to describe these top intersections. The proof of this formula also is based on a bijection between the 'forked' graphs and the summands in the coefficient of the appropriate term in $\hat{\mathcal{L}}\left(F(\mathbf{t})\right)$. But in this, the bijection is not with graphs corresponding to the partitions of $[n]$, but with '$\mathcal{P}_k$-graphs' that are defined by replacing the i-th mark with k_i on a $\mathcal{P}$-graph. Here k_i is the exponent of $\hat{\psi}_i$ in the $\hat{\psi}$ monomial (3.3.2). As expected, there is a surjection between $\mathcal{P}$-graphs and $\mathcal{P}_k$-graphs. For the proof, we write a new version of our closed formula in terms of these $\mathcal{P}_k$-graphs (3.15). Then we show a bijection between the summands in this formula and the summands in the coefficient of the appropriate term in $\hat{\mathcal{L}}\left(F(\mathbf{t})\right)$. And the resulting coefficient, as a sum of all these summands, corresponds to the top intersections of $\hat{\psi}$ classes.

Then we develop the generalized versions of these results for the case of weights all $\frac{1}{q}$. In our first result 4.1.1 here, we develop a closed formula (4.1) for the monomials in $\hat{\psi}$ classes for $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$ in terms of cycles on $\overline{M}_{0,n}$. This closed formula is derived using the same relation (2.9) between the $\hat{\psi}$ classes and ψ classes on $\overline{M}_{0,n}$. For the proof of this theorem we use a theorem 2.2 in [8]. This theorem that applies to ω classes on $\overline{M}_{g,n}$ gives our result as a special case, where the number of half-edges on a fork of D_i is restricted to a maximum of q; in the case of ω classes, there is no such restriction. And our partitions that are defined differently according to this restriction take care of this difference. Rest of the proof can be thus read exactly from [8]. The summands in the resulting formula correspond to modified ψ monomials on certain boundary strata on $\overline{M}_{0,n}$ that are the intersections of the boundary divisors as in the case of $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$. The difference is that the dual graphs of these strata are now 'forked' graphs with number of half-edges on a fork varying from 2 and q. We then establish a bijection between summands in the expansion corresponding to these graphs that we call '$\mathcal{P}$'-graphs and the unordered partitions of $[n]$, such that cardinality of each subset in the partition is between q and 1. The resulting formula (4.1) has the pullback of monomials in $\bar{\psi}$ classes on $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$ as a sum of the intersections of monomials in ψ classes and boundary stratum corresponding to $\mathcal{P}$-graph on $\overline{M}_{0,n}$. Then we derive a corollary (4.2.1) of this result to calculate the top intersections. These give the top intersections of $\hat{\psi}$ classes as a sum of top intersections of ψ classes on $\overline{M}_{0,n}, \overline{M}_{0,n-1}, \overline{M}_{0,n-2}, \ldots$ with some multiplicities. We point out again here that our corollary (4.2.1 can also be deduced from theorem 7.9 in [4]. For our work, we develop specific and explicit closed formulas for our special case of all weights $\frac{1}{q}$ and base our combinatorial analysis closely on the structure of dual graphs.

The main theorem 4.3.1 of this generalized case weights all $\frac{1}{q}$ encodes the closed formula (4.2.1) for top intersections in a generating function $G(\mathbf{t})$ obtained by applying a differential operator to the Witten-potential $F(\mathbf{t})$ [7]. This operator $\hat{\mathcal{L}}$ now takes the form of an exponential partial differential operator that is more sophisticated and complex. The proof of this formula again is based on a bijection between the 'forked' graphs and the summands in the coefficient of the appropriate term in $\hat{\mathcal{L}}\left(F(\mathbf{t})\right)$.

The dissertation is organized as follows. In chapter 2, we give the background required for this work which consists of a brief introduction to $\overline{M}_{0,n}$, ψ classes and Hassett Spaces, with some relevant facts and lemmas on these topics. In chapter 3, we prove our first result which gives the closed formula for the intersections of $\hat{\psi}$ classes on $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$. In section 3.2, we give the results for top intersections, and encode the formula for top intersections in the generating function that we obtain by applying a partial differential operator to the Witten-potential.

In chapter 4, we generalize our results to the case of weights all $\frac{1}{q}$. We prove our first generalized result in this chapter which gives the closed formula for the intersections of $\hat{\psi}$ classes on $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$. In section 4.2, we give the results for top intersections, and encode the formula for top intersections in the generating function that we obtain by applying a more sophisticated and generalized partial differential operator to the Witten-potential.

Chapter 2

Background

For background on for this work, the author has mainly used [2], [9], [7], [10] and introductory sections of [11]. Here we will recall some selected facts we explicitly use in this work.

2.1 $\overline{M}_{0,n}$

$\overline{M}_{0,n}$ denotes the moduli space of stable, n pointed rational curves, with at worst nodal singularities. The **boundary** of $\overline{M}_{0,n}$ is defined to be the complement of $M_{0,n}$ in $\overline{M}_{0,n}$. It consists of all points parameterizing nodal stable curves.

Given a rational, stable n-pointed curve $(C, p_1, \ldots, p_n)$, its **dual graph** is defined to have:

- a vertex for each irreducible component of C;
- an edge for each node of C, joining the appropriate vertices;
- a labeled half edge for each mark, emanating from the appropriate vertex.

Figure below gives an example of the dual graphs of some strata in $\overline{M}_{0,5}$.

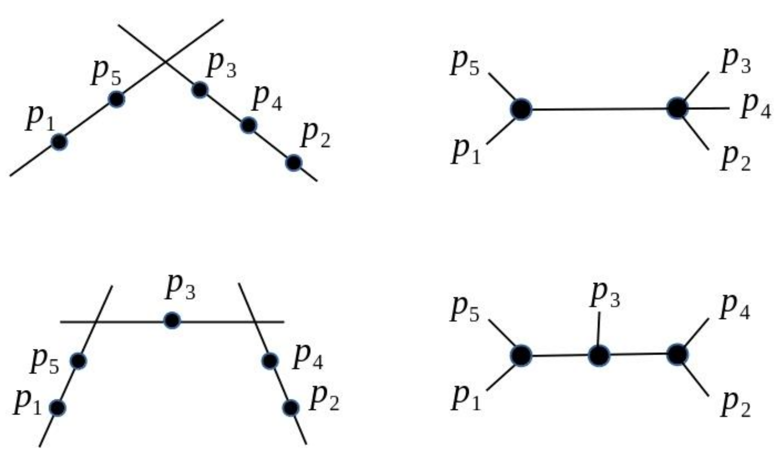


Figure 2.1: On the left are the boundary strata of $\overline{M}_{0,5}$, and the corresponding dual graphs are on the right.

The **closures** of the codimension 1 boundary strata of $\overline{M}_{0,n}$ are called the **irreducible boundary divisors**; they are in one-to-one correspondence with all ways of partitioning $[n] = A \cup A^c$ with the cardinality of both A and A^c strictly greater than 1. We denote $D(A) = D(A^c)$ the divisor corresponding to the partition A, A^c.

2.2 ψ classes

For $i = 1, \ldots, n$, we define the class $\psi_i \in A^1(\overline{M}_{0,n})$. Let $\mathbf{L}_i \to \overline{M}_{0,n}$ be a line bundle whose fiber over each point $(C, p_1, \ldots, p_n)$ is canonically identified with $T^*_{p_i}(C)$. The line bundle $\mathbf{L_i}$ is called the i-th **cotangent** (or **tautological**) line bundle. Then

$$\psi_i := c_1(\mathbf{L_i}) \tag{2.1}$$

where c_1 is the first Chern class of the line bundle $\mathbf{L_i}$.

Some properties of ψ classes on $\overline{M}_{0,n}$ that we use are the following lemmas. Interested reader can find their proofs in [7]. Here $[n] = I \cup I^c$ with the cardinality of both I and I^c strictly greater than 1.

Lemma 2.2.1. *Consider the gluing morphism $gl_I : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,n}$. Assume that $i \in I$ and denote by $\pi_1 : \overline{M}_{0,I\cup\star} \times \overline{M}_{0,I^c\cup\bullet} \to \overline{M}_{0,I\cup\star}$ the first projection. Then:*

$$gl^*(\psi_i) = \pi_1^*(\psi_i). \tag{2.2}$$

Lemma 2.2.2. *Consider the forgetful morphism $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$. Then, for every $i = 1, \ldots, n$,*

$$\psi_i = \pi_{n+1}^*(\psi_i) + D(\{i, n+1\}). \tag{2.3}$$

Lemma 2.2.3. *For any choice of i, j, k distinct, we have the following equation in $A^1(\overline{M}_{0,n})$:*

$$\psi_i = \sum_{i\in I,\ j,k\notin I} D(I). \tag{2.4}$$

Lemma 2.2.4 (String Equation)**.** *Consider the forgetful morphism* $\pi_{n+1} : \overline{M}_{0,n+1} \to \overline{M}_{0,n}$*. Then*

$$\pi_{n+1*}\left(\prod_{i=1}^{n} \psi_i^{k_i}\right) = \sum_{j|k_j\neq 0} \psi_j^{k_j-1} \prod_{i\neq j} \psi_i^{k_i}. \tag{2.5}$$

Lemma 2.2.5. *Let* $\sum k_i = n-3$*. Then*

$$\int_{\overline{M}_{0,n}} \prod_{i=1}^{n} \psi_i^{k_i} = \binom{n-3}{k_1, \ldots, k_n} \tag{2.6}$$

where the integral sign denotes push-forward to the class of a point.

2.3 Hassett spaces

In [3], Hassett constructed a new class of modular compactifications $\overline{M}_{g,\mathcal{A}}$ of the moduli space $M_{g,n}$ of smooth curves with n marked points parameterized by an input datum $(g, \mathcal{A})$. Here g is the genus of the curves and $\mathcal{A} = (a_1, \ldots, a_n)$ is the **weight data** of weights $a_i \in \mathbb{Q} \cap (0, 1]$ satisfying the inequality $2g - 2 + a_1 + \ldots + a_n > 0$.

$\overline{M}_{g,\mathcal{A}}$ that we call Hassett space parameterizes curves $(C, p_1, \ldots, p_n)$ with n marked non-singular points on C that are $\mathcal{A}$**-stable** if the the following two conditions are fulfilled.

1. The twisted canonical divisor $K_C + a_1p_1 + \ldots + a_np_n$ is ample.

2. A subset $p_{i_1}, \ldots, p_{i_k}$ of the marked points is allowed to coincide only if the inequality $a_{i_1} + \ldots + a_{i_k} \leq 1$ holds.

For $g = 0$, the stability condition means that a rational n-pointed curve $(C, p_1, \ldots, p_n)$ is $\mathcal{A}$-stable if on every irreducible component of C the number of nodes plus the sum of the weights of the marks lying on the component is strictly greater than 2, with $a_1 + \ldots + a_n > 2$.

In the case $(a_1, \ldots, a_n) = (1, \ldots, 1)$, this condition is nothing but the traditional notion of an n-marked stable curve, and so the compactification $\overline{M}_{g,\mathcal{A}}$ is exactly the well-known Deligne-Mumford compactification $\overline{M}_{g,n}$ of $M_{g,n}$.

Definition 2.3.1. *Given two weight data $\mathcal{A}, \mathcal{B}$, we say that $\mathcal{B} \leq \mathcal{A}$ if for every i, $b_i \leq a_i$. Then there exists a regular* **reduction** *morphism:*

$$c_{\mathcal{B},\mathcal{A}} : \overline{M}_{0,\mathcal{A}} \to \overline{M}_{0,\mathcal{B}} \tag{2.7}$$

s.t. $c_{\mathcal{B},\mathcal{A}}(C, p_1, \ldots, p_n)$ is obtained by contracting twigs that become unstable when the weights of the points are "lowered" from a_i to b_i.

Moduli spaces of weighted stable rational curves also have psi classes, which are defined in the same way as in $\overline{M}_{0,n}$. A ψ class on $\overline{M}_{0,\mathcal{A}}$ will be denoted as $\bar{\psi}_i$.

Lemma 2.3.1. *Consider the reduction morphism $c : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$. For $i = 1, \ldots, n$, we have:*

$$\psi_i = c^*(\bar{\psi}_i) + \sum_{\substack{I \ni i, \\ \sum_{j \in I} a_j \leq 1}} D(I). \tag{2.8}$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{U}}_{0,n} \arrow[r, "C"] \arrow[d, shift right, "\pi"'] & \overline{\mathcal{U}}_{0,\mathcal{A}} \arrow[d, shift right, "\bar{\pi}"'] \\
\overline{M}_{0,n} \arrow[r, "c"] \arrow[u, shift right, bend right, "\sigma_i"'] & \overline{M}_{0,\mathcal{A}} \arrow[u, shift right, bend right, "\bar{\sigma}_i"']
\end{tikzcd}$$

Here $\overline{\mathcal{U}}_{0,n}$ and $\overline{\mathcal{U}}_{0,\mathcal{A}}$ are the universal families over $\overline{M}_{0,n}$ and $\overline{M}_{0,\mathcal{A}}$ respectively; π and $\bar{\pi}$ are the forgetful morphisms, and σ_i and $\bar{\sigma}_i$ are the i-th tautological sections of the corresponding universal families; c and C are the reduction morphisms. Let $S_i = Im(\sigma_i)$ and $\bar{S}_i = Im(\bar{\sigma}_i)$. Then,

$$\psi_i = \pi_\star(-{S_i}^2)$$

$$\bar{\psi}_i = \bar{\pi}_\star(-\bar{S}_i^2)$$

Now, $C^{\star}(\bar{S}_i) = S_i + \sum_I E_I$, such that $i \in I, j \in I$ if $\sum a_j \leq 1$, and E_I is the exceptional divisor in

$$Blow_{\cap_{i\in I}\sigma_i}$$

such that $\pi_\star(E_I) = D(I)$. Then,

$$\begin{aligned}
c^\star\bar{\psi}_i = c^\star\bar{\pi}_\star(-\bar{S}_i^2) = \pi_\star C^\star(-\bar{S}_i{}^2) &= \pi_\star(-(S_i + \sum_I E_I)^2) \\
&= \pi_\star(-(S_i^2 + 2\sum_I S_i E_I + \sum_I E_I^2)) \\
&= \psi_i - 2\sum_I D(I) + \sum_I D(I) \\
&= \psi_i - \sum_I D(I) \\
&= \psi_i - \sum_{\substack{I \ni i, \\ \sum_{j\in I} a_j \leq 1}} D(I)
\end{aligned}$$

□

Informally, for the pullback of a $\bar{\psi}_i$, a ψ_i is corrected by all boundary divisors where the i-th mark is on a twig that gets contracted by c. In particular we will use the above special case of reduction morphism for this work.

Definition 2.3.2. *We define the $\hat{\psi}_i$ class as the pullback of a $\bar{\psi}$ class under the reduction morphism $c : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$.*

$$\hat{\psi}_i := c^\star\bar{\psi}_i$$

Corollary 2.3.1. *For the reduction morphism* $c : \overline{M}_{0,n} \to \overline{M}_{0,\mathcal{A}}$*, where* $\mathcal{A} = \left\{\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right\}$*. For* $i = 1, \ldots, n$*, we have:*

$$\hat{\psi}_i = \psi_i - \sum_{j, j \neq i} D\left(\{i, j\}\right). \tag{2.9}$$

Proof. This follows from (2.8) by observing that $a_i + a_j = 1$ for all i, j when $a_i = \frac{1}{2}\ \forall\ i$.

□

Chapter 3

Closed Formula for intersections of ψ-classes on $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$

For the work in this chapter, $\mathcal{A} = (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$ for $\overline{M}_{0,\mathcal{A}}$, and we denote it by $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$. In this chapter, we develop the closed formula for integrals of $\hat{\psi}$ monomials corresponding to the $\bar{\psi}$ monomials on $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$.

We denote by $\mathcal{P} = \{P_1, P_2, .., P_j, S_1, S_2, ..., S_i\}$ an unordered partition of $[n]$, such that cardinality of each subset in the partition is either 2 or 1. Further, if the cardinality of such subsets is 2, we denote them with P_j and if the cardinality of such subsets is 1, we denote them with S_j . Denote by P_{j1} and P_{j2} the elements of P_j. Denote by F the set of all P_j's, and by S the set of all S_j's. Denote by $\mathfrak{P}$ the set of all such partitions $\mathcal{P}$.

3.1 Cycle intersections

Definition 3.1.1. *Given a $\mathcal{P} \in \mathfrak{P}$, we define the graph $\Gamma_{\mathcal{P}}$ as follows:*

1. *The $\Gamma_{\mathcal{P}}$ has a 'central' node, with $|F|$ number of edges with nodes on ends*

2. *Attach to each non-'central' node two half edges forming a 'fork'; to the 'central' node, attach $|S|$ number of half-edges*

3. *Label a half-edges on forks with P_{j1} and P_{j2}, and half-edges on central node with S_j's.*

So, each P_j corresponds to a fork and S_j's to half-edges on the central node. We call this a $\mathcal{P}$-graph. $|F|$ gives the number of forks on the graph. Each such graph is a dual graph of a stratum in $\overline{M}_{0,n}$.

Clearly, the set of all $\mathcal{P}$*-graphs* as defined above are in bijection with the set of all partitions $\mathcal{P} \in \mathfrak{P}$.

Definition 3.1.2. *Given a $\hat{\psi}$-monomial $m = \hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\hat{\psi}_3^{k_3}...\hat{\psi}_r^{k_r}$, a decorated $\mathcal{P}$-graph $\Gamma_{\mathcal{P}}^d$is obtained by coloring a half-edge corresponding to point $t \in P_j$ or $t \in S_j$ if $k_t \neq 0$.*

Example 3.1.1. *Given $m = \hat{\psi}_1^2\hat{\psi}_2$ on $\overline{M}_{0,6}$,*

for $\mathcal{P} = \{1,2\},\{3\},\{4\},\{5\},\{6\}$, we get the decorated graph $\Gamma_{\mathcal{P}}^d$ as in figure 3.1;

for $\mathcal{P} = \{1,3\},\{2,4\},\{5\},\{6\}$, we get the decorated graph $\Gamma_{\mathcal{P}}^d$ as in figure 3.2,

and for $\mathcal{P} = \{1,3\},\{5,6\},\{2\},\{4\}$, we get the decorated graph $\Gamma_{\mathcal{P}}^d$ as in figure 3.3.

Figure 3.1: $\Gamma_{\mathcal{P}}^d$

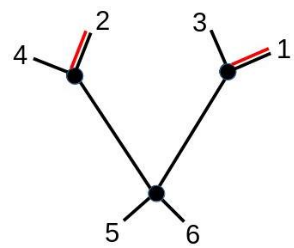


Figure 3.2: $\Gamma_{\mathcal{P}}^d$

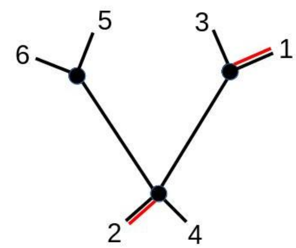


Figure 3.3: $\Gamma_{\mathcal{P}}^d$

Definition 3.1.3. *Consider the decorated $\mathcal{P}$-graph in figure (3.4). In $\overline{M}_{0,n}$, this represents the dual graph of a boundary stratum that is the image of the following gluing morphism:*

$$gl : \overline{M}_{0,P_1 \cup \star P_1} \times \ldots \overline{M}_{0,P_s \cup \star P_s} \times \overline{M}_{0,S \cup \bullet P_1 \ldots \cup \bullet P_s} \to \overline{M}_{0,n}$$

where for each P_j, the half-edge $\bullet P_j$ is the mark on the node pulled back from the factor corresponding to the central node, and $\star P_j$ is the mark on the node pulled back from the factor corresponding to the respective fork. For this decorated $\mathcal{P}$-graph, define the following ψ-function :

$$\phi_{\mathcal{P}}(\psi) = \psi_{\bullet P_1}^{k_{i_1}+k_{i_2}-1} \ldots \psi_{\bullet P_{\frac{t}{2}}}^{k_{i_{t-1}}+k_{i_t}-1} \psi_{\bullet P_{\frac{t}{2}+1}}^{k_{i_{t+1}-1}} \ldots \psi_{\bullet P_s}^{k_{i_s}-1} \psi_{i_{s+1}}^{k_{s+1}} \ldots \ldots \psi_{i_r}^{k_{i_r}} \tag{3.1}$$

where $\psi_{\bullet P_j}$ is the ψ-class at the $\bullet P_j$ mark on the factor corresponding to the central node.

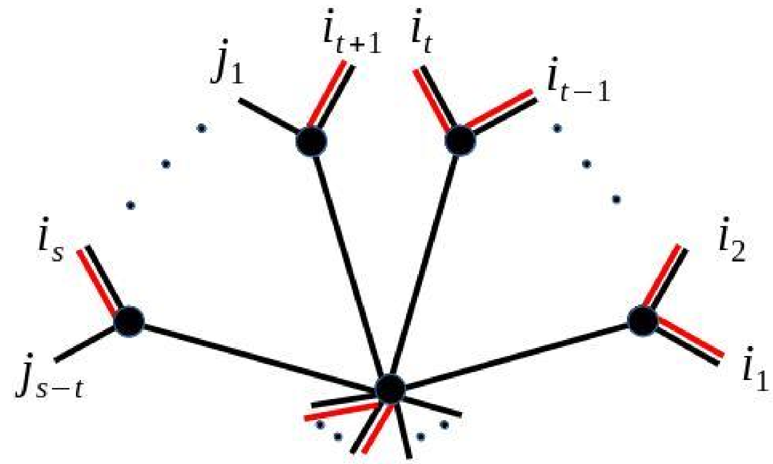
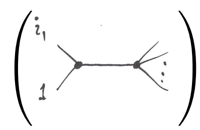
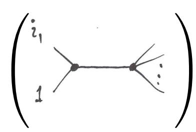
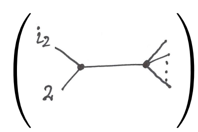


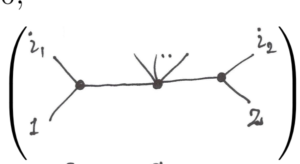
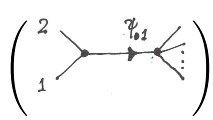
Figure 3.4: A decorated $\mathcal{P}$-graph with s colored half-edges on forks

Lemma 3.1.1. *Let D_i be a divisor $D(\{i,j\})$ where $j \in [n] \setminus \{i\}$; then on $\overline{M}_{0,n}$, $\prod_{i=1}^{s} D_i$, $s \leq n-3$, is supported on a $\mathcal{P}$-graph $\Gamma_{\mathcal{P}}$. And the number of forks on $\Gamma_{\mathcal{P}}$ can vary from $\lfloor \frac{s}{2} \rfloor$ to $min(\lfloor \frac{n}{2} \rfloor, s)$.*

Proof. We prove by induction. D_1 =  is clearly a $\mathcal{P}$-graph. Now,

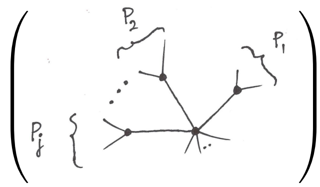
$D_1 D_2 =$  

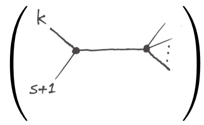
$=$

- $0,$ if $|\{1, i_1\} \cap \{2, i_2\}| = 1$
-  , if $|\{1, i_1\} \cap \{2, i_2\}| = \emptyset$
- $-$  , if $\{1, i_1\} = \{2, i_2\}$

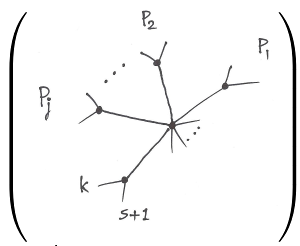
So, all the graphs that support $D_1 D_2$ are $\mathcal{P}$-graphs. Now suppose $\prod_{i=1}^{s} D_i$ is supported on a $\mathcal{P}$-graph, and suppose that $\mathcal{P}$-graph has j number of forks P_i's, and denote by S the set of all half-edges not on forks. Then,

$$(\prod_{i=1}^{s} D_i)D_{s+1} =$$



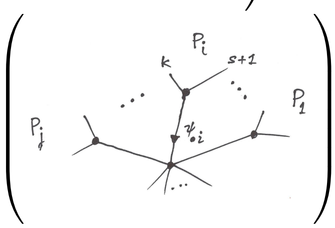


$$= \begin{cases} 0, & \text{if}\{s+1,k\} \nsubseteq P_i, S \;\forall\, i \end{cases}$$



$$, \qquad \text{if}\{s+1,k\} \subseteq \; S$$

$$-$$



$$, \qquad \text{if}\{s+1,k\} \subseteq \; P_i \text{ for some } i$$

So, all graphs that we get for non-zero intersections are in fact $\mathcal{P}$-graphs. For the number of forks on the graphs, there are 2 cases to consider : 1) $s > \lfloor \frac{n}{2} \rfloor$, and 2) $s \leq \lfloor \frac{n}{2} \rfloor$. In the first case, there are at least $s - \lfloor \frac{n}{2} \rfloor$ forks with both half-edges colored, and maximum number of forks is $\lfloor \frac{n}{2} \rfloor$; as $s < n$, minimum number of forks is $\lfloor \frac{s}{2} \rfloor$. In the second case, the minimum number of forks is $\lfloor \frac{s}{2} \rfloor$ and the maximum s. So, the number of forks on $\Gamma_{\mathcal{P}}$ can vary from $\lfloor \frac{s}{2} \rfloor$ to $min(\lfloor \frac{n}{2} \rfloor, s)$.

□

Lemma 3.1.2. *With $D_i = \sum_j D(\{i,j\})$ on $\overline{M}_{0,n}$, and $\bullet P_j$ as in Definition 3.1.3, then for $\overline{M}_{0,(\frac{1}{2})^n}$*

$$\hat{\psi}_i^{k_i} = \psi_i^{k_i} - \sum_j \psi_{\bullet_{\{i,j\}}}^{k_i - 1} D(\{i,j\}) \tag{3.2}$$

Proof. Using Corollary 2.3.1,

$$\hat{\psi}_i^{k_i} = (\psi_i - \sum_j D(\{i,j\}))^{k_i}$$

$$= \psi_i^{k_i} + \ldots + (-1)^{k_i} \sum_j D(\{i,j\})^{k_i}$$

$$= \psi_i^{k_i} + \ldots + (-1)^{k_i} (-1)^{k_i-1} \psi_{\bullet_{\{i,j\}}}^{k_i-1} \sum_j D(\{i,j\})$$

$$= \psi_i^{k_i} - \sum_j \psi_{\bullet_{\{i,j\}}}^{k_i-1} D(\{i,j\})$$

The second equality happens as all the terms in the expansion except the first and the last vanish due to dimension reasons. And it's only the self intersections that are non-zero in the last term in the expansion of $(\psi_i - \sum_j D(\{i,j\}))^{k_i}$. The third and fourth equalities follow from the k_i self intersections of $\sum_j D(\{i,j\})$. □

For the result in the above lemma, we will use the following notation for brevity.

$$\hat{\psi}_i^{k_i} = \psi_i^{k_i} - \psi_{\bullet_i}^{k_i-1} D_i \tag{3.3}$$

where

$$D_i := \sum_j D(\{i,j\})$$

and

$$\psi_{\bullet_i}^{k_i-1} D_i := \sum_j \psi_{\bullet_{\{i,j\}}}^{k_i-1} D(\{i,j\})$$

Also, again for brevity, if the $\mathcal{P}$-graph in figure (3.4) corresponds to partition $\mathcal{P} = \{P_1, P_2, .., P_s, S_1, S_2, ..., S_q\}$, then we will denote $\psi_{\bullet P_j}$ also as ψ_{P_j} in what follows.

Theorem 3.1.1. *With $\mathcal{P}, F, S, P_j, P_{j1}, P_{j2}, S_i$ as defined above, for $n \geq 5$ we have for $\overline{M}_{0,\left(\frac{1}{2}\right)^n}$:*

$$\hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n} = \sum_{\mathcal{P} \in \mathfrak{P}} (-1)^{|F|} [\Gamma_{\mathcal{P}}] \prod_{S_i \in S} \psi_{S_i}^{k_{S_i}} \prod_{P_j \in F} \psi_{P_j}^{k_{P_{j1}+P_{j2}-1}} \tag{3.4}$$

where $[\Gamma_{\mathcal{P}}]$ is the class of boundary stratum in $\overline{M}_{0,n}$ with dual graph $\Gamma_{\mathcal{P}}$.

Proof. Let $D_i = \sum_j D(\{i,j\})$. Omitting $\hat{\psi}$'s with 0-exponents, and assuming WLOG that the first r $\hat{\psi}$'s remain with nonzero-exponents,
$\hat{\psi}_1^{k_1}\psi_2^{k_2}.\hat{.}.\hat{\psi}_n^{k_n} = \hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\hat{\psi}_3^{k_3}...\hat{\psi}_r^{k_r}$. Then,

$$\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\hat{\psi}_3^{k_3}...\hat{\psi}_r^{k_r} = (\psi_1 - D_1)^{k_1}(\psi_1 - D_2)^{k_2}\dots(\psi_r - D_r)^{k_r}$$

using relation (2.9)

$$= (\psi_1^{k_1} - \psi_{\bullet_1}^{k_1-1}D_1)(\psi_2^{k_2} - \psi_{\bullet_2}^{k_2-1}D_2)\dots(\psi_r^{k_r} - \psi_{\bullet_r}^{k_r-1}D_r)$$

using relation (3.3)

$$= (-1)^s\sum_{s=0}^{r}\psi_{\bullet_{i_1}}^{k_{i_1}-1}\psi_{\bullet_{i_2}}^{k_{i_2}-1}\dots\psi_{\bullet_{i_s}}^{k_{i_s}-1}D_{i_1}D_{i_2}\dots D_{i_s}\psi_{i_{s+1}}^{k_{i_{s+1}}}\dots\psi_{i_r}^{k_{i_r}} \tag{3.5}$$

with $1 \le i_j \le r$ and $i_1 < i_2 < \dots < i_s$. Now, each term in the expansion of the expression above is supported on a $\mathcal{P}$-graph from lemma 3.1.1.

Pick a $\mathcal{P}$-graph with s number of colored half-edges on forks and create the corresponding $\Gamma_{\mathcal{P}}^d$. There are two possibilities: 1) at least one fork has both half-edges uncolored, or 2) all forks of $\Gamma_{\mathcal{P}}^d$ have at least one colored half-edge. Let the second type of graph (figure 3.5) have t number of forks with both half-edges colored as shown below.

Intersect both types of $\mathcal{P}$-graph with $\phi_{\mathcal{P}}(\psi)$ as defined in (3.1):

$$\phi_{\mathcal{P}}(\psi) = \psi_{\bullet_1}^{k_{i_1}+k_{i_2}-1}\dots\psi_{\bullet_{t-1}}^{k_{i_{t-1}}+k_{i_t}-1}\psi_{\bullet_{(t+1)}}^{k_{i_{t+1}-1}}\dots\psi_{\bullet_s}^{k_{i_s}-1}\psi_{i_{s+1}}^{k_{s+1}}\dots\dots\psi_{i_r}^{k_{i_r}} \tag{3.6}$$

Claim : The intersection of $\phi_{\mathcal{P}}(\psi)$ with the first type of graphs gives 0.
Proof WLOG, suppose the first type of $\mathcal{P}$-graph have i_1 and i_2 on a fork with $k_{i_1} = k_{i_2} = 0$. Then $k_{i_1} + k_{i_2} - 1 = -1$ and

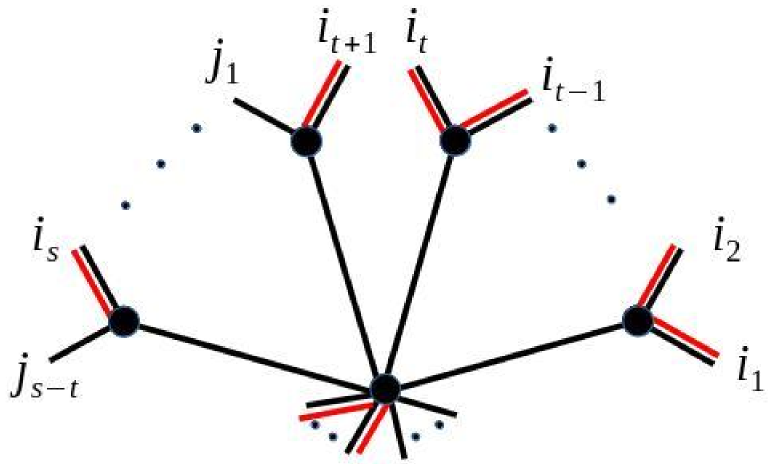


Figure 3.5: A decorated $\mathcal{P}$-graph with no uncolored fork

$$\phi_{\mathcal{P}}(\psi) = \psi_{\bullet_1}^{-1} \dots \psi_{\bullet_{t-1}}^{k_{i_{t-1}}+k_{i_t}-1} \psi_{\bullet_{(t+1)}}^{k_{i_{t+1}-1}} \dots \psi_{\bullet_s}^{k_{i_s}-1} \psi_{i_{s+1}}^{k_{s+1}} \dots \dots \psi_{i_r}^{k_{i_r}}$$

is 0 as negative power of a ψ class by standard convention is 0.

Now, consider the second type of $\mathcal{P}$-graph (figure 3.5).

Claim: $\phi_{\mathcal{P}}(\psi) . \Gamma_{\mathcal{P}}^d$, where $\Gamma_{\mathcal{P}}^d$ is the second type of graph, uniquely determines a term in the expansion (3.5) above.

Proof: Define a map from the set of $\mathcal{P}$-graphs to the terms in the expansion (3.5) as follows. For each i_l in $\Gamma_{\mathcal{P}}^d$ where i_l is a colored half-edge on a fork, assign a D_{i_l}, to form the product $\prod_{l=1}^{s} D_{i_l}$. And for each fork on the graph with half-edges i_l and i_q, assign $\psi_{\bullet_m}^{k_{i_l}+k_{i_q}-1}$ and form their product; the result is $\phi_{\mathcal{P}}(\psi)$. Then $\phi(\psi). \prod_{l=1}^{s} D_{i_l}$ is precisely the term in the expansion (3.5) that $\Gamma_{\hat{\mathcal{P}}}^d$ maps to. Further, reversing the process, we get the preimage of a term in expansion (3.5) the unique $\mathcal{P}$-graph (figure 3.5). So, the map is in fact a bijection. And,

$$\phi_{\mathcal{P}}(\psi) . \Gamma_{\mathcal{P}}^d =$$

$$(-1)^{s+\frac{t}{2}} (\Gamma_{\mathcal{P}}) \psi_{\bullet_1}^{k_{i_1}+k_{i_2}-1} \dots \psi_{\bullet_{t-1}}^{k_{i_{t-1}}+k_{i_t}-1} \psi_{\bullet_{(t+1)}}^{k_{i_{t+1}-1}} \dots \psi_{\bullet_s}^{k_{i_s}-1} \psi_{i_{s+1}}^{k_{s+1}} \dots \dots \psi_{i_r}^{k_{i_r}} \tag{3.7}$$

$$= (-1)^{|F|}\left[\Gamma_{\mathcal{P}}\right]\prod_{S_i\in S}\psi_{S_i}^{k_{S_i}}\prod_{P_j\in F}\psi_{P_j}^{k_{P_{j1}+P_{j2}-1}}$$

$\square$

3.2 Numerical intersections

In this section, we develop two corollaries of theorem (3.1.1) to develop two versions of a closed formula for top intersections of $\hat{\psi}$-classes on $\overline{M}_{0,n}$. Then we encode this formula in a generating function obtained by applying a differential operator to the Witten-potential. As pointed earlier, these corollaries (3.2.1 and 3.2.2) can also be deduced from theorem 7.9 in [4]. For our work, we develop specific and explicit closed formulas here and base our combinatorial analysis closely on the structure of dual graphs.

Corollary 3.2.1. *With $\mathcal{P}, F, P_j$,*
P_{j1}, P_{j2}, S_i as defined above, for $n \geq 5$ we have:

$$\int_{\overline{M}_{0,n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\dots\hat{\psi}_n^{k_n} = \sum_{\mathcal{P}\in\mathfrak{P}}(-1)^{|F|}\binom{|\mathcal{P}|-3}{\langle k_{P_{i1}}+k_{P_{i2}}-1\rangle,\langle k_S\rangle} \tag{3.8}$$

where $\sum k_i = n-3$, and for a $\mathcal{P} = \{P_1, P_2, \dots, P_s, S_1, \dots, S_q\}$,
$\langle k_{P_{i1}}+k_{P_{i2}}-1\rangle = k_{P_{11}}+k_{P_{12}}-1, \dots, k_{P_{s1}}+k_{P_{s2}}-1$, and $\langle k_S\rangle = k_{S_1}, \dots, k_{S_q}$.

Proof. The proof of this is same as for theorem (3.1.1) except in the last part of evaluation of $\phi_{\mathcal{P}}(\psi).\Gamma_{\mathcal{P}}^d$. Here when $\sum k_i = n-3$ this evaluation gives

$$\phi_{\mathcal{P}}(\psi).\Gamma_{\mathcal{P}}^d =$$

$$= (-1)^{s+\frac{t}{2}}\binom{n-3-(s-\frac{t}{2})}{k_{i_1}+k_{i_2}-1,\dots,k_{i_{t+1}}-1,\dots,k_{i_s}-1,k_{i_{s+1}},\dots,k_{i_r}}$$

$$= (-1)^{|F|} \binom{n-3-|F|}{k_{i_1}+k_{i_2}-1, \ldots, k_{i_s}+k_{j_s}-1, k_{i_{s+1}}, \ldots, k_{i_r}}$$

as $k_{i_{r+1}} = \ldots = k_{i_n} = 0$

$$= (-1)^{|F|} \binom{|\mathcal{P}|-3}{\langle k_{P_{i1}}+k_{P_{i2}}-1\rangle, \langle k_S\rangle}$$

$\square$

We can reduce the complexity of computation of (3.8) if we can remove the partitions $\mathcal{P}$'s from $\mathfrak{P}$ whose graphs evaluate to 0 when intersected with $\phi(\psi)$ as described above. Also we can collect together terms corresponding to permutations of the set $\{j_1, j_2, ..., j_{s-t}\}$ as all these terms evaluate to the same value as $k_{j_i} = 0$ for all these j_i.

Form a new set $\mathfrak{P}'$ in the following way : Make the powerset $\mathfrak{R}$ of $[r]$, where r denotes the number of ψ's with non-zero exponent in the ψ-monomial. For each set $\mathcal{R} \in \mathfrak{R}$, create all subsets $\mathcal{P}'$ of $\mathcal{R}$ whose elements are subsets of $\mathcal{R}$ of cardinality 2 or 1 with upper bound of number of subsets of cardinality 2 fixed at $\lfloor \frac{n}{2} \rfloor$. Call $\mathfrak{P}'$ the set of all $\mathcal{P}'$. This set $\mathfrak{P}'$ can also be obtained from $\mathfrak{P}$ via the following map: Given a partition $\mathcal{P}$, project to a $\mathcal{P}'$ by forgetting all points $S_i \in \mathcal{P}$ and in a $P_j \in \mathcal{P}$ forget a point P_{ji} if $k_{P_{ji}} = 0$. More formally, $\mathcal{P}' = \{P'_1, P'_2, .., P'_i, S'_1, S'_2, ..S'_j\}$ where $P'_i = P_i$ if $k_{Pi1} > 0$ and $k_{Pi2} > 0$; $S'_i = P_i \setminus \{P_{ij}\}$ if $k_{P_{ij}} = 0$. This is an onto map. Each $P_j^{'}$ has cardinality 2, and each $S_i{}'$ has cardinality 1. Denote by $\mathcal{P}'^c$ the set $[r] \setminus P'_1 \cup ... \cup P'_i \cup S'_1 \cup S'_2 \cup .. \cup S'_j$.

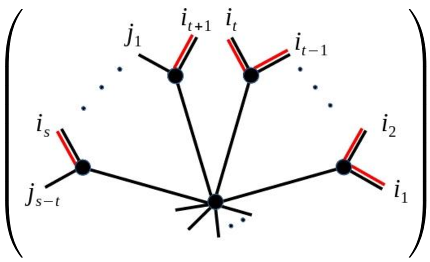
Corollary 3.2.2. *With $\mathfrak{P}'$ as defined above, for $n \geq 5$ we have:*

$$\begin{aligned}\int_{\overline{M}_{0,n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\psi}_3^{k_3} ... \hat{\psi}_r^{k_r} = \\ \sum_{\mathcal{P}' \in \mathfrak{P}'} (-1)^{s+\frac{t}{2}} \frac{(n-r)!}{(n-r-s+t)!} \binom{n-3-(s-\frac{t}{2})}{k_{i_1}+k_{i_2}-1, \ldots, k_{i_{t+1}}-1, \ldots, k_{i_s}-1, k_{i_{j_1}}, \ldots, k_{j_q}}\end{aligned} \tag{3.9}$$

where $t = 2|\{P_i^{'}\}|$, $s = |\{S_i{}'\}| + t$ and $\{i_{j_1}, .., i_{j_q}\} = \mathcal{P}'^c$

Proof. Corresponding to a partition $\mathcal{P}$, form a corresponding decorated graph for $\mathcal{P}^{'}$ by uncoloring any half-edges on the central node, and 'forgetting' the j_i's on uncolored half-edges on the forks as discussed above. This corresponds to $\mathcal{P}' = \{P_1', P_2', \ldots, P_{t/2}', S_1', \ldots, S_{s-t}'\}$, where P_i' correspond to nodes with both half-edges colored, and $S_i' = i_{t+i}$ in the decorated graph of $\mathcal{P}$ in figure (3.5). Now consider intersection of this graph with $\phi_{\mathcal{P}}(\psi)$ as defined earlier in (3.1) :

$\phi_{\mathcal{P}}(\psi).$



$$\cong \overline{M}_{0,3} \times \overline{M}_{0,3}, \ldots, \overline{M}_{0,3} \times \phi(\psi).\overline{M}_{0,n-3-(s-\frac{t}{2})}$$

$$= (-1)^{s+\frac{t}{2}} \binom{n-3-(s-\frac{t}{2})}{k_{i_1}+k_{i_2}-1, \ldots, k_{i_{t-1}}+k_{i_t}-1, k_{i_{t+1}}-1, \ldots, k_{i_s}-1, k_{i_{j_1}}, \ldots, k_{j_q}}$$

As there are $\frac{(n-r)!}{(n-r-s+t)!}$ ways of choosing the uncolored half-edges on the forks, corresponding to j_i's, the term evaluates to

$$= (-1)^{s+\frac{t}{2}} \frac{(n-r)!}{(n-r-s+t)!} \binom{n-3-(s-\frac{t}{2})}{k_{i_1}+k_{i_2}-1, \ldots, k_{i_{t+1}}-1, \ldots, k_{i_s}-1, k_{i_{j_1}}, \ldots, k_{j_q}}$$

$\square$

3.3 Generating Function for the top intersections

We start with the generating function- Witten-potential([7]). The correlation functions are defined as intersection numbers on the moduli space of stable n-pointed curves (here for genus 0) as

$$\langle \tau_{k_1} \ldots \tau_{k_n} \rangle := \int_{\overline{M}_{0,n}} \psi_1^{k_1} \psi_2^{k_2} \ldots \psi_n^{k_n}$$

Collecting all tau's with equal exponent, we can write $\langle \tau_{k_1} \ldots \tau_{k_n} \rangle =$
$\langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \ldots \tau_m^{s_m} \rangle$. Now, define $\mathbf{s} = (s_0, s_1, \ldots)$, and $\langle \tau^{\mathbf{s}} \rangle := \langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \ldots \tau_m^{s_m} \rangle$.

So, for each sequence $\mathbf{s}$, there is a correlation function $\langle \tau^{\mathbf{s}} \rangle$; and $|\mathbf{s}| := \sum s_i$ is the number of marks n. For the generating function, all these correlation functions are collected and used as coefficients in a formal power series. Using notation $\mathbf{t}^{\mathbf{s}} = \prod_{i=0}^{\infty} t_i^{s_i}$, and $\mathrm{s}! = \prod_{i=0}^{\infty} s_i!$, the generating function is

$$F(\mathbf{t}) := \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \langle \tau^{\mathbf{s}} \rangle \tag{3.10}$$

$$= (1)\frac{t_0^3 t_1}{3!} + (1)\frac{t_0^4 t_2}{4!} + (2)\frac{t_0^3 t_1^2}{3!2!} + (1)\frac{t_0^5 t_3}{5!} + (3)\frac{t_0^4 t_1 t_2}{4!} + (6)\frac{t_0^3 t_1^3}{3!3!} + \ldots\ldots$$

where the coefficients of appropriate terms give the intersection numbers $\int_{\overline{M}_{0,n}} \psi_1^{k_1} \psi_2^{k_2} \ldots \psi_n^{k_n}$. Observe that the total codimension of the integrand in $\langle \tau^{\mathbf{s}} \rangle$ is $\sum i s_i$, so $|s| - 3 = \sum i s_i$. With this generating function, the String equation for $\overline{M}_{0,n}$ is the differential equation -

$$\frac{\partial}{\partial t_0} F = \frac{t_0^2}{2} + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} F$$

Definition 3.3.1. *Define a new generating function*

$$G(\mathbf{t}) := \sum_{\mathbf{s}} \frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} \langle \hat{\tau}^{\mathbf{s}} \rangle \tag{3.11}$$

where $\langle \hat{\tau}^{\mathbf{s}} \rangle = \int_{\overline{M}_{0,(\frac{1}{2})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\ldots} \hat{\psi}_n^{k_n}$ *as defined earlier,* $\sum k_i = n - 3$.

So, $G(\mathbf{t})$ has as coefficients of the monomials $\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}$ the intersection numbers $\int_{\overline{M}_{0,\mathcal{A}}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \hat{\ldots} \hat{\psi}_n^{k_n}$ for any value of n and any values of k_i's, with $\sum k_i = n - 3$.

Theorem 3.3.1. *With $G(\boldsymbol{t})$ as defined above,*

$$G(\boldsymbol{t}) = \hat{\mathcal{L}}(F(\boldsymbol{t})) - \frac{t_0^3}{3!} + 2\frac{t_0^3 t_1}{3!} \tag{3.12}$$

where $\hat{\mathcal{L}} = : e^{-\mathcal{L}} :$, and $: e^{-\mathcal{L}} :$ denotes the operator with normal ordering[1]*, and*

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=0}^{\infty} t_i t_j \frac{\partial}{\partial t_{i+j-1}} \tag{3.13}$$

Before we prove the above theorem, consider the $\hat{\psi}-$ monomial $\int_{\overline{M}_{0,\mathcal{A}}} \hat{\psi}_1^{n-3}$. It is the coefficient of $t-$monomial $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ in $G(\mathbf{t})$. When we apply the operator $\hat{\mathcal{L}} = 1 - \mathcal{L} + : \frac{\mathcal{L}^2}{2!} : - \ldots$ to $F(\mathbf{t})$, only the terms with the following $t-$monomials in $F(\mathbf{t})$ contribute to the term with $t-$monomial $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$: $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ and $\frac{t_0^{(n-2)} t_{n-4}}{(n-2)!}$.

The first term of $\hat{\mathcal{L}}$ is 1, which when acts on $F(\mathbf{t})$ produces the $t-$monomial $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ as it is; the second operates as follows:

$$-t_0 t_{n-3} \frac{\partial}{\partial t_{n-4}} \left(\frac{t_0^{(n-2)} t_{n-4}}{(n-2)!} \right) = -(n-1)\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}.$$

No other term in $F(\mathbf{t})$ contributes to the coefficient of the $t-$monomial $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$.

As the monomial $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ has as coefficient $\int_{\overline{M}_{0,n}} \psi_1^{n-3}$, and the monomial $\frac{t_0^{(n-2)} t_{n-4}}{(n-2)!}$ has as coefficient $\int_{\overline{M}_{0,n-1}} \psi_1^{n-4}$ in $F(\mathbf{t})$, the coefficient of $\frac{t_0^{(n-1)} t_{n-3}}{(n-1)!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ is $\int_{\overline{M}_{0,n}} \psi_1^{n-3} - (n-1)\int_{\overline{M}_{0,n-1}} \psi_1^{n-4}$ which equals
$\int_{\overline{M}_{0,(\frac{1}{2})^n}} \hat{\psi}_1^{n-3}$ from corollary (3.2.1).

---

[1]by normal ordering of the operator, we mean that we treat the t_i's and $\frac{\partial}{\partial t_i}$'s as commuting variables, and bring all t_i's to the left of $\frac{\partial}{\partial t_i}$'s. E.g., if $\mathcal{J} = t_i t_j \frac{\partial}{\partial t_{i+j-1}}$,

$$\mathcal{J}^2 = t_i t_j \frac{\partial}{\partial t_{i+j-1}} t_i t_j \frac{\partial}{\partial t_{i+j-1}},$$

but

$$: \mathcal{J}^2 : = t_i^2 t_j^2 \frac{\partial}{\partial t_{i+j-1}} \frac{\partial}{\partial t_{i+j-1}}$$

Observe that both the contributions correspond to the two types of $\mathcal{P}$-graphs that make non-zero contributions to $\int_{\overline{M}_{0,\mathcal{A}}} \hat{\psi}_1^{n-3}$ in corollary (3.2.1).The first are of type with no forks; the second of type with one fork.

Definition 3.3.2. *For each $\mathcal{P}$-graph, the corresponding $\mathcal{P}_k$-graph is defined by replacing each $i \in [n]$ on the $\mathcal{P}$-graph by k_i.*

Clearly the map $\{\mathcal{P}\text{-}graphs\} \to \{\mathcal{P}_k\text{-}graphs\}$ is a surjection.

Lemma 3.3.1. *In genus 0, for a given $\langle \tau_{k_1} \ldots \tau_{k_n} \rangle = \int_{\overline{M}_{0,n}} \psi_1^{k_1} \psi_2^{k_2} \ldots \psi_n^{k_n}$ as defined above, let* **s** $= (s_0, s_1, \ldots s_n)$*, and $\langle \tau_{k_1} \ldots \tau_{k_n} \rangle = \langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \ldots \tau_n^{s_n} \rangle$. Consider a $\mathcal{P}_k$-graph with m forks with q distinct k_i's appearing on the forks; let such k_i's be $\{k_1, k_2, \ldots k_q\}$. Let l_i be the number of times a given k_i appears on any fork on the $\mathcal{P}_k$-graph. Then the number of $\mathcal{P}$-graphs that map to this $\mathcal{P}_k$-graph is given by:*

$$\frac{1}{|Aut(\hat{\mathcal{P}}_k)|} \frac{s_{k_1}!}{(s_{k_1} - l_1)!} \frac{s_{k_2}!}{(s_{k_2} - l_2)!} \cdots \frac{s_{k_q}!}{\left(s_{k_q} - l_q\right)!} =: C_{\mathcal{P}_k} \tag{3.14}$$

where $|Aut(\hat{\mathcal{P}}_k)|$ is the number of automorphisms of the subgraph of $\mathcal{P}_k$-graph obtained from removing half-edges on the central node.

Proof. Consider a $\mathcal{P}_k$-graph with m number of forks s.t q number of k_i's appear on the forks as defined above. Then if all n half-edges are given ordering, the number of $\mathcal{P}$-graphs where half-edges are ordered would be

$$s_0! s_1! s_2! \ldots s_n!$$

Now, we divide by the permutations of half-edges on the central node to get

$$\frac{s_0!}{(s_0 - l_1)!} \frac{s_1!}{(s_1 - l_1)!} \frac{s_2!}{(s_2 - l_2)!} \cdots \frac{s_n!}{(s_n - l_n)!}$$

Now, restrict l_i to be the number of times a given k_i appears on any fork on the $\mathcal{P}_k$-graph. As only q number of k_i's appear on the forks, $l_i = 0$ for $i > q$, so

$$\frac{s_0!}{(s_0-l_1)!}\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!}=\frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!}$$

Further we need to divide by permutations of half-edges on the forks. Let $j_1, j_2, \ldots, j_f$ be the number of forks with the same set of k_i's on them; and let d be the number of forks with both k_i's same on that fork. Then, we divide by $2^d\,(j_1!j_2!\ldots j_f!)$ to get

$$\left(\frac{1}{2^d\,(j_1!j_2!\ldots j_f!)}\right)\frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!}$$

Observe that the number $\left(2^d\,(j_1!j_2!\ldots j_f!)\right)$ is the number of automorphisms of the subgraph of $\mathcal{P}_k$-graph consisting of only the forks; denote this subgraph as $\hat{\mathcal{P}}_k$. Then the number of $\mathcal{P}$-graphs that map to this $\mathcal{P}_k$-graph can be rewritten as :

$$\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}\frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!}$$

□

The reason for organizing $C_{\mathcal{P}_k}$ as in (3.14) will become clear in the proof of theorem (3.3.1).

Lemma 3.3.2. *With the definitions and notations above, corollary (3.2.1) can be rewritten as :*

$$\begin{aligned}&\int_{\overline{M}_{0,(\frac{1}{2})^n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\ldots\hat{\psi}_n^{k_n}=\\&\sum_{\mathcal{P}_k\in\mathfrak{Q}}(-1)^m C_{\mathcal{P}_k}\int_{\overline{M}_{0,(n-m)}}\psi_{\bullet_1}^{k_{i_1}+k_{i_2}-1}\ldots\psi_{\bullet_m}^{k_{i_{2m-1}}+k_{i_{2m}}-1}\psi_{i_{2m+1}}^{k_{2m+1}}\ldots\ldots\psi_{i_r}^{k_{i_r}}\end{aligned}\tag{3.15}$$

where m is number of forks on the $\mathcal{P}_k$-graph and $C_{\mathcal{P}_k}$ is the number of $\mathcal{P}$-graphs that map to this $\mathcal{P}_k$-graph, and $\mathfrak{Q}$ is the set of all $\mathcal{P}_k$-graphs.

Proof. This version of closed formula for $\int_{\overline{M}_{0,(\frac{1}{2})^n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\ldots\hat{\psi}_n^{k_n}$ is just a reorganization of (3.8) using $\mathcal{P}_k$-graphs instead of $\mathcal{P}$-graphs. As the map

$\{\mathcal{P}\text{-}graphs\}\to\{\mathcal{P}_k\text{-}graphs\}$ is a surjection, we get all the terms in (3.8). □

Now, for a general $\mathcal{P}_k$-graph with m number of forks shown below, define the following operator (which appears in $\hat{\mathcal{L}}$):

$$\mathcal{D}_{\mathcal{P}_k} := t_{k_{i_1}} t_{k_{i_2}} \dots t_{k_{i_{2m-1}}} t_{k_{i_{2m}}} \frac{\partial}{\partial t_{k_{i_1}+k_{i_2}-1}} \dots \frac{\partial}{\partial t_{k_{i_{2m-1}}+k_{i_{2m}}-1}} \tag{3.16}$$

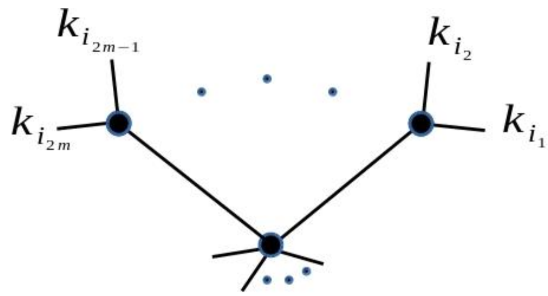


Figure 3.6: $\mathcal{P}_k$-graph with m forks

and the following term in (3.15) :

$$(-1)^m C_{\mathcal{P}_k} \int_{\overline{M}_{0,(n-m)}} \psi_{\bullet_1}^{k_{i_1}+k_{i_2}-1} \dots \psi_{\bullet_m}^{k_{i_{2m-1}}+k_{i_{2m}}-1} \psi_{i_{2m+1}}^{k_{2m+1}} \dots\dots \psi_{i_r}^{k_{i_r}} \tag{3.17}$$

By construction, the terms (3.16) are in bijection with the $\mathcal{P}_k$-graphs. Furthermore, the term (3.16) arises in $\hat{\mathcal{L}}$ as a summand in : $(-1)^m \frac{\mathcal{L}^m}{m!}$: with some multiplicity. As part of the proof of theorem, we will see that this multiplicity is $(-1)^m \frac{1}{|Aut(\hat{\mathcal{P}}_k)|}$ with $|Aut(\hat{\mathcal{P}}_k)|$ as defined in Lemma (3.3.1). And the term (3.17) is a summand in (3.15) corresponding to this $\mathcal{P}_k$-graph.

Strategy of proof of theorem (3.3.1): we will show that for a general t-monomial $\frac{{t_0}^{s_0} {t_1}^{s_1} \dots {t_l}^{s_l}}{s_0! s_1! \dots s_l!}$, its coefficients in $G(\mathbf{t})$ and $\hat{\mathcal{L}}(F(\mathbf{t})) - \frac{t_0^3}{3!}$ are equal. And that both equal $\int_{\overline{M}_{0,(\frac{1}{2})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \dots \hat{\psi}_n^{k_n}$. To show that, we show a bijection between $\mathcal{P}_k$ graphs and the terms (3.17) which are the summands in the formula (3.15). To show this bijection, we pick a $\mathcal{P}_k$ graph, and find the term (3.16) in $\hat{\mathcal{L}}$ (with some multiplicity); then we find the term $\tilde{T}_{\mathcal{P}_k}$ in $F(\mathbf{t})$, such that the term (3.16) when applied to $\tilde{T}_{\mathcal{P}_k}$ gives the t-monomial $\frac{{t_0}^{s_0} {t_1}^{s_1} \dots {t_l}^{s_l}}{s_0! s_1! \dots s_l!}$ with coefficient the intersection number defined in (3.17). As that exactly is the summand in (3.15) corresponding to the chosen $\mathcal{P}_k$-graph, this proves the theorem in one direction. In the other direction, we pick a term (3.17) which is a summand in the coefficient of t-monomial $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!}$, and show that this maps to the same $\mathcal{P}_k$-graph.

Before the proof, here is an example that illustrates the idea.

Example 3.3.1. *Consider*

$$\int_{\overline{M}_{0,\left(\frac{1}{2}\right)^{61}}} \hat{\psi}_1^1 \ldots \hat{\psi}_8^1 \hat{\psi}_9^2 \ldots \hat{\psi}_{13}^2 \hat{\psi}_{14}^3 \hat{\psi}_{15}^3 \hat{\psi}_{16}^4 \hat{\psi}_{17}^4 \hat{\psi}_{18}^4 \hat{\psi}_{19}^5 \hat{\psi}_{20}^5 \hat{\psi}_{21}^6 \hat{\psi}_{22}^6$$

The corresponding t-monomial in $G(\boldsymbol{t})$ is

$$\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!} = \frac{t_0{}^{39} t_1{}^8 t_2{}^5 t_3{}^2 t_4{}^3 t_5{}^2 t_6{}^2}{39!8!5!2!3!2!2!} =: T_{\mathcal{P}_k}$$

Now consider the following $\mathcal{P}_k$-graph with $m = 7$ forks as in the following figure.

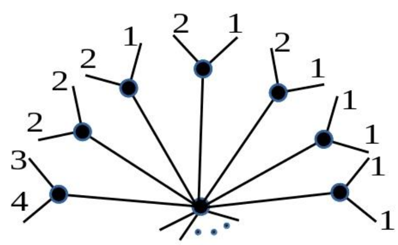


Figure 3.7: $\mathcal{P}_k$-graph with $m = 7$ forks

The corresponding operator (3.16) is

$$\mathcal{D}_{\mathcal{P}_k} = t_1^7 t_2^5 t_3 t_4 \frac{\partial^3}{\partial t_2^3} \frac{\partial^2}{\partial t_1^2} \frac{\partial}{\partial t_3} \frac{\partial}{\partial t_6}$$

The coefficient of this term in $: (-1)^7 \frac{\mathcal{L}^7}{7!} :$ *is given by*

$$(-1)^7 \left(\frac{1}{2^7 7!}\right) \left(2^4 \binom{7}{3,2,1,1}\right) = (-1)^7 \left(\frac{1}{2^3 3! 2!}\right)$$

The corresponding unique term in $F(\boldsymbol{t})$ is

$$\frac{{t_0}^{39}{t_1}^3{t_2}^3{t_3}^2{t_4}^1{t_5}^2{t_6}^3}{39!3!3!2!1!2!3!} =: \tilde{T}_{\mathcal{P}_k}$$

In $\hat{\mathcal{L}}(F(\boldsymbol{t}))$, the corresponding term is

$$(-1)^7\left(\frac{1}{2^3 3!2!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,\mathcal{D}_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})$$

where

$$\langle\tau^{\mathbf{s}}\rangle = \int_{\overline{M}_{0,\left(\frac{1}{2}\right)^{54}}} \hat{\psi}_1^1\hat{\psi}_2^1\hat{\psi}_3^1\hat{\psi}_4^2\hat{\psi}_6^2\hat{\psi}_7^3\hat{\psi}_8^3\hat{\psi}_9^4\hat{\psi}_{10}^5\hat{\psi}_{12}^5\hat{\psi}_{13}^6\hat{\psi}_{14}^6\hat{\psi}_{15}^6$$

Observe that the coefficient $(-1)^7\left(\frac{1}{2^3 3!2!}\right)$ of $\mathcal{D}_{\mathcal{P}_k}$ in $:(-1)^7\frac{\mathcal{L}^7}{7!}:$ is exactly $(-1)^m\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}$ as claimed earlier. Now,

$$(-1)^7\left(\frac{1}{2^3 3!2!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,D_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})$$

$$= (-1)^7\left(\frac{1}{2^3 3!2!}\right)\left(\frac{8!}{(8-7)!}\frac{5!}{(5-5)!}\frac{2!}{(2-1)!}\frac{2!}{(2-1)!}\frac{3!}{(3-1)!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

$$= (-1)^7\left(\frac{1}{2^3 3!2!}\right)\left(\frac{8!5!2!2!3!}{2!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

$$= (-1)^7\left(\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}\right)\left(\frac{8!5!2!2!3!}{2!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

$$= (-1)^7 C_{\mathcal{P}_k}(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

which term in $\hat{\mathcal{L}}(F(\boldsymbol{t}))$ has as coefficient of $T_{\mathcal{P}_k}$ exactly the term (3.17) which is the summand in (3.15) corresponding to the chosen $\mathcal{P}_k$-graph.

Proof. (of theorem (3.3.1))

Consider a general term in $G(\mathbf{t})$ with the corresponding t-monomial $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$, with coefficient $\int_{\overline{M}_{0,(\frac{1}{2})^n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\ldots\hat{\psi}_n^{k_n}$. Now, we will show that we get all the terms of formula (3.15) in $\hat{\mathcal{L}}(F(\mathbf{t})) = (1- :\mathcal{L}: + :\frac{\mathcal{L}^2}{2!}: - \ldots + :(-1)^m\frac{\mathcal{L}^m}{m!}: + \ldots)(F(\mathbf{t}))$ and that each term corresponds to a $\mathcal{P}_k$-graph.

1. Pick a $\mathcal{P}_k$-graph with no fork. Associated operator (3.16) in $\hat{\mathcal{L}}$ is 1, the first term in $\hat{\mathcal{L}}$, which when applied to $F(\mathbf{t})$ results in coefficient 1 for $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$.

2. Pick a $\mathcal{P}_k$-graph with one fork. WLOG, assume a $\mathcal{P}_k$-graph with k_1, k_2 on a single fork, with k_1, k_2 not simultaneously 0.

Case 1 : $k_1 \neq k_2$. Then, the corresponding operator (3.16) as $t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}}$. In $\hat{\mathcal{L}}$, this term has coefficient -1. The term in $F(\mathbf{t})$ that it operates on to produce $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$ has t-monomial -

$$\frac{{t_0}^{s_0}\ldots{t_{k_1}}^{s_{k_1-1}}\ldots{t_{k_2}}^{s_{k_2-1}}\ldots{t_{k_l}}^{s_{k_l}}}{s_0!\ldots s_{k_1-1}!\ldots s_{k_2-1}!\ldots s_l!}t_{k_1+k_2-1}$$

The result of applying in $-t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}}$ in $\hat{\mathcal{L}}$ to $F(\mathbf{t})$ is the following :

$$-t_{k_1}t_{k_2}\frac{\partial}{\partial t_{k_1+k_2-1}}\left(\frac{{t_0}^{s_0}\ldots{t_{k_1}}^{s_{k_1-1}}\ldots{t_{k_2}}^{s_{k_2-1}}\ldots{t_{k_l}}^{s_{k_l}}}{s_0!\ldots s_{k_1-1}!\ldots s_{k_2-1}!\ldots s_l!}t_{k_1+k_2-1}\right)$$

$$= -s_{k_1}s_{k_2}\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$$

So, the coefficient contributed by this operator to $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ is

$$-s_{k_1}s_{k_2}\int_{\overline{M}_{0,(n-1)}}\psi_{\bullet_1}^{k_1+k_2-1}\psi_3^{k_3}\ldots\ldots\psi_r^{k_r}$$

and $C_{\mathcal{P}_k} = s_{k_1}s_{k_2}$ is the number of $\mathcal{P}$-graphs that map to this kind of $\mathcal{P}_k$-graph.

Case 2 : $k_1 = k_2$. In this case we get the corresponding term (3.16) as $t_{k_1}^2\frac{\partial}{\partial t_{2k_1-1}}$. The coefficient of this term in $\hat{\mathcal{L}}$ is $-\frac{1}{2}$. When $-\frac{1}{2}t_{k_1}^2\frac{\partial}{\partial t_{2k_1-1}}$ is applied to $F(\mathbf{t})$, the only terms that produces $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_l}^{s_l}}{s_0!s_1!\ldots s_l!}$ is

$$\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots{t_{k_1}}^{s_{k_1-2}}\ldots{t_{k_l}}^{s_{k_l}}}{s_0!\ldots s_{k_1-2}!\ldots s_l!}t_{2k_1-1}$$

And

$$-\frac{1}{2}t_{k_1}^2\frac{\partial}{\partial t_{2k_1-1}}(\frac{{t_0}^{s_0}{t_1}^{s_1}\dots {t_l}^{s_l}}{s_0!s_1!\dots s_l!}\frac{{t_0}^{s_0}{t_1}^{s_1}\dots {t_{k_1}}^{s_{k_1-2}}\dots {t_{k_l}}^{s_{k_l}}}{s_0!\dots s_{k_1-2}!\dots s_l!}t_{2k_1-1})$$

$$= -\frac{1}{2}s_{k_1}(s_{k_1}-1)\frac{{t_0}^{s_0}{t_1}^{s_1}\dots {t_l}^{s_l}}{s_0!s_1!\dots s_l!}$$

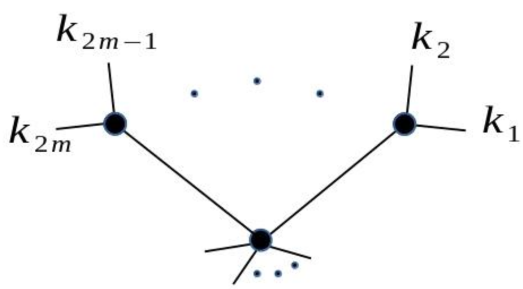
So, the coefficient contributed by this operator to $\frac{{t_0}^{s_0}{t_1}^{s_1}\dots {t_l}^{s_l}}{s_0!s_1!\dots s_l!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ is

$$-\frac{1}{2}s_{k_1}(s_{k_1}-1)\int_{\overline{M}_{0,(n-1)}}\psi_{\bullet_1}^{2k_1-1}\psi_2^{k_2}\dots\dots\psi_r^{k_r}$$

and $C_{\mathcal{P}_k}=\frac{1}{2}s_{k_1}(s_{k_1}-1)$ is the number of $\mathcal{P}$-graphs that that map to to this kind of $\mathcal{P}_k$-graph.

In both cases, the terms contribute $-C_{\mathcal{P}_k}\int_{\overline{M}_{0,(n-1)}}\psi_{\bullet_1}^{k_1+k_2-1}\psi_3^{k_3}\dots\dots\psi_{i_r}^{k_{i_r}}$ to the coefficient of t-term $\frac{{t_0}^{s_0}{t_1}^{s_1}\dots {t_l}^{s_l}}{s_0!s_1!\dots s_l!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$. So, we get both terms in 3.15 corresponding to two $\mathcal{P}_k$-graphs with one fork. Also, observe that if $k_1=k_2=0$, the term $-t_0^2\frac{\partial}{\partial t_{-1}}$ contributes nothing.

Now consider a $\mathcal{P}_k$-graph with m number of forks. WLOG, let the k_i's on the forks be $\{k_1,k_2,\dots,k_{2m}\}$ as shown in figure below.Let l_i be the number of times a given k_i appears on any fork on the $\mathcal{P}_k$-graph, and let $j_1,j_2,\dots,j_f$ be the number of forks with the same set of k_i's on them; and let d be the number of forks with both k_i's same on that fork.



Then the corresponding operator (3.16) is

$$t_{k_1}t_{k_2}\dots t_{k_{2m-1}}t_{k_{2m}}\frac{\partial}{\partial t_{k_1+k_2-1}}\dots\frac{\partial}{\partial t_{k_{2m-1}+t_{k_{2m}}-1}}:=\mathcal{D}_{\mathcal{P}_k}$$

The coefficient of this term in $\hat{\mathcal{L}}$ as a summand in : $(-1)^m\frac{\mathcal{L}^m}{m!}$: is given by

$$(-1)^m\left(\frac{1}{2^m m!}\right)\left(2^{(m-d)}\binom{m}{l_1, l_2, \ldots, l_n}\right) = (-1)^m\left(\frac{1}{2^d l_1! l_2! \ldots, l_n!}\right)$$

The corresponding term in $F(\mathbf{t})$ is

$$\frac{{t_0}^{s_0}\ldots {t_{k_1}}^{\left(s_{k_1}-l_{k_1}\right)}\ldots {t_{k_{2m}}}^{\left(s_{k_{2m}}-l_{k_{2m}}\right)}}{s_0!\ldots(s_{k_1}-l_{k_1})!\ldots(s_{k_{2m}}-l_{k_{2m}})!\ldots s_{k_n}!}t_{k_1+k_2-1}\ldots t_{k_{2m-1}+k_{2m}-1} =: \tilde{T}_{\mathcal{P}_k}$$

In $\hat{\mathcal{L}}(F(\mathbf{t}))$, the corresponding term is

$$(-1)^m\left(\frac{1}{2^d l_1! l_2! \ldots, l_n!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,\mathcal{D}_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})$$

where $\langle\tau^{\mathbf{s}}\rangle$ is the appropriate ψ-monomial on $\overline{M}_{0,\left(\frac{1}{2}\right)^{(n-m)}}$ that appears as coefficient of $\tilde{T}_{\mathcal{P}_k}$ in $F(\mathbf{t})$. Observe that the coefficient $(-1)^m\left(\frac{1}{2^d l_1! l_2! \ldots, l_n!}\right)$ of $D_{\mathcal{P}_k}$ in $:(-1)^m\frac{\mathcal{L}^m}{m!}:$ is exactly $(-1)^m\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}$ as claimed earlier. Now,

$$(-1)^m\left(\frac{1}{2^d l_1! l_2! \ldots, l_n!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,\mathcal{D}_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})$$

$$\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!} =: C_{\mathcal{P}_k}$$

$$= (-1)^m\left(\frac{1}{2^d l_1! l_2! \ldots, l_n!}\right)\left(\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

$$= (-1)^m\left(\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}\right)\left(\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

$$= (-1)^m C_{\mathcal{P}_k}\,(\langle\tau^{\mathbf{s}}\rangle)\,T_{\mathcal{P}_k}$$

which term in $\hat{\mathcal{L}}(F(\mathbf{t}))$ has as coefficient of $T_{\mathcal{P}_k}$ exactly the term (3.17) which is the summand in (3.15) corresponding to the chosen $\mathcal{P}_k$-graph.

So, one direction is proved. To show bijection in the other direction, we pick a summand in the coefficient of $\frac{{t_0}^{s_0}{t_1}^{s_1}\ldots {t_l}^{s_l}}{s_0!s_1!\ldots s_l!} = T_{\mathcal{P}_k}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ that comes from term $:\frac{(-1)^m}{m!}\mathcal{L}^m:$. Let this summand come from the following summand in : $\frac{(-1)^m}{m!}\mathcal{L}^m$:

$$(-1)^m\left(\frac{1}{2^d l_1!l_2!\ldots,l_n!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,\mathcal{D}_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})$$

where

$$\mathcal{D}_{\mathcal{P}_k}=t_{k_1}t_{k_2}\ldots t_{k_{2m-1}}t_{k_{2m}}\frac{\partial}{\partial t_{k_1+k_2-1}}\cdots\frac{\partial}{\partial t_{k_{2m-1}+t_{k_{2m}}-1}},$$

$\langle\tau^{\mathbf{s}}\rangle$ is uniquely determined by $\tilde{T}_{\mathcal{P}_k}$, and $\tilde{T}_{\mathcal{P}_k}$ is uniquely determined by

$$\mathcal{D}_{\mathcal{P}_k}(\tilde{T}_{\mathcal{P}_k})=T_{\mathcal{P}_k}$$

As this $\mathcal{D}_{\mathcal{P}_k}$ is in bijection with the $\mathcal{P}_k$-graph as in the figure by construction, we get the term in (3.15) corresponding to this $\mathcal{P}_k$-graph as the chosen summand in $\hat{\mathcal{L}}(F(\mathbf{t}))$. So the coefficient of $\frac{t_0{}^{s_0}t_1{}^{s_1}\ldots t_l{}^{s_l}}{s_0!s_1!\ldots s_l!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ equals the coefficient of $\frac{t_0{}^{s_0}t_1{}^{s_1}\ldots t_l{}^{s_l}}{s_0!s_1!\ldots s_l!}$ in $G(\mathbf{t})$.

□

Chapter 4

Closed Formula for intersections of ψ-classes on $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$

For the work in this chapter, $\mathcal{A} = (\frac{1}{q}, \frac{1}{q}, \ldots, \frac{1}{q})$, where $q \in \mathbb{Z}^+$ for $\overline{M}_{0,\mathcal{A}}$, and we denote it by $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$. In this chapter, we develop the closed formula for integrals of $\hat{\psi}$ monomials corresponding to the $\bar{\psi}$ monomials on $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$.

4.1 Cycle intersections

We denote by $\mathcal{P} = \{P_1, P_2, .., P_m, S_1, S_2, \ldots, S_c\}$ an unordered partition of $[n]$, such that cardinality of any subset in the partition is between 1 and q. Further, S_i have cardinality 1, and P_i have cardinality other than 1. And for a P_i with cardinality z_i, $2 \leq z_i \leq q$, denote by $P_{i,1}, P_{i,2}, \ldots, P_{i,z_i}$ the elements of P_i. Further, for $\mathcal{P}$ as above, denote by F the set of all P_j's, and by S the set of all S_j's such that $|F| = m$ and such that $|S| = c$. Let $|\mathcal{P}|$ denote the number of subsets in $\mathcal{P}$. Denote by $\mathfrak{P}$ the set of all such partitions $\mathcal{P}$. For a P_j, and $i \in P_j$ define

$$\alpha_j := \sum_{i \in P_j} k_i$$

Theorem 4.1.1. *With $\mathcal{P}, F, S, P_j$ and α_j as defined above, for $n \geq 5$ we have for $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$:*

$$\hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n} = \sum_{\mathcal{P} \in \mathfrak{P}} [\Gamma_{\mathcal{P}}] \prod_{S_i \in \mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_j \in \mathcal{P}} \frac{\psi_{\bullet P_j}^{\alpha_j}}{\left(-\psi_{\bullet P_j} - \psi_{\star P_j}\right)} \tag{4.1}$$

where $[\Gamma_{\mathcal{P}}]$ is the class of boundary stratum in $\overline{M}_{0,n}$ with dual graph $\Gamma_{\mathcal{P}}$. And for each P_j, the half-edge $\bullet P_j$ is the mark on the node pulled back from the factor corresponding to the central node, and $\star P_j$ is the mark on the node pulled back from the factor corresponding to the respective fork, as defined in (3.1.3).

Proof. Let

$$D_i = \sum_{\substack{I \ni i, \\ \sum_{j\in I} a_j \leq 1}} D(I)$$

So, the summands in the above are all the divisors that have on one node half-edges that can vary from 2 to q in number and i mark is on that edge. Omitting $\hat{\psi}$'s with 0-exponents, and assuming WLOG that the first r $\hat{\psi}$'s remain with nonzero-exponents, so that $\hat{\psi}_1^{k_1}\psi_2^{k_2}.\hat{.}.\hat{\psi}_n^{k_n} = \hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\hat{\psi}_3^{k_3}...\hat{\psi}_r^{k_r}$. Then,

$$\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\hat{\psi}_3^{k_3}...\hat{\psi}_r^{k_r} = (\psi_1 - D_1)^{k_1}(\psi_1 - D_2)^{k_2}\dots(\psi_r - D_r)^{k_r} \tag{4.2}$$

using relation (2.9).

From here, we can proceed with the above relation to prove our result as was done for the relation in the case of weights all $\frac{1}{2}$ in proof of theorem (3.4). But instead we choose to use a result already established for ω classes in theorem 2.2 in [8]. This theorem that applies to ω classes on $\overline{M}_{g,n}$ gives our result above as a special case. In essence, while the number of half-edges on a D_i for our case is restricted to a maximum of q, in the case of ω classes, there is no such restriction. Rest of the computation remains the same. So, below we give the required background and this theorem without proofs.

We define ω classes, also called stable ψ classes, which are just pull-backs of ψ classes from spaces of curves with only one mark as follows.

Definition 4.1.1. *Let $g, n \geq 1, i \in [n]$, and let $\rho_i : M_{g,n} \to M_{g,\{i\}}$ be the rememberful morphism, i.e. a composition of forgetful morphisms for all but the i-th mark. Then we define*

$$\omega_i := \rho^\star \psi_i \tag{4.3}$$

in $R^1(\overline{M}_{g,n})$.

We call any boundary stratum where all the genus is concentrated at one vertex of the dual graph a stratum of rational tails type. The following Lemma for the case of $\overline{M}_{g,n}$ gives an explicit relation between ω and ψ classes.

Lemma 4.1.1. *Let $g, n \geq 1, i \in [n]$. Then:*

$$\psi_i := \omega_i + \sum_{I \ni i} D(I) \tag{4.4}$$

where $D(I)$ is a divisor of rational type, and I is subset of marks on the rational tail.

This Lemma means that ψ_i is obtained from ω_i by adding all divisors where the i-th mark is contained in one of the two components. Now, we compare the definition (4.4) of ω_i with the definition of $\hat{\psi}_i$ in (2.8) from Lemma 2.8 which we reproduce below.

$$\psi_i = \hat{\psi}_i + \sum_{\substack{I \ni i, \\ \sum_{j \in I} a_j \leq 1}} D(I) \tag{4.5}$$

We see that the only difference between the ω classes and the $\hat{\psi}$ classes is the extra constraint $\sum_{j \in I} a_j \leq 1$ for $D(I)$ in the case of $\hat{\psi}$ classes, and of course the difference in genus and corresponding stability conditions.

Denote by $\tilde{\mathcal{P}} = \{P_1, P_2, .., P_m, S_1, S_2, \ldots, S_c\}$ an unordered partition of $[n]$, such that cardinality of any subset in the partition is greater than or equal to 1. Further, let S_i have cardinality 1, and P_i have cardinality other than 1. And for a P_i with cardinality z_i, $z_i \geq 2$, denote by $P_{i,1}, P_{i,2}, \ldots, P_{i,z_i}$ the elements of P_i. Further, for $\mathcal{P}$ as above, denote by F the set of all P_j's, and by S the set of all S_j's such that $|F| = m$ and such that $|S| = c$. Let $|\mathcal{P}|$ denote the number of subsets in $\mathcal{P}$. Denote by $\tilde{\mathfrak{P}}$ the set of all such partitions $\tilde{\mathcal{P}}$. For a P_j, and $i \in P_j$ define

$$\alpha_j := \sum_{i \in P_j} k_i$$

Theorem 4.1.2. *[theorem 2.2 in [8]] With $\mathcal{P}, F, S, P_j$ and α_j as defined above, and $g, n \geq 1$ we have for $\overline{M}_{g,n}$:*

$$\omega_1^{k_1}\omega_2^{k_2}\dots\omega_n^{k_n} = \sum_{\tilde{\mathcal{P}}\in\tilde{\mathfrak{P}}} [\Gamma_{\tilde{\mathcal{P}}}] \prod_{S_i\in\tilde{\mathcal{P}}} \psi_{S_i}^{\alpha_i} \prod_{P_j\in\tilde{\mathcal{P}}} \frac{\psi_{\bullet P_j}^{\alpha_j}}{\left(-\psi_{\bullet P_j} - \psi_{\star P_j}\right)} \tag{4.6}$$

where $[\Gamma_{\tilde{\mathcal{P}}}]$ is the class of boundary stratum in $\overline{M}_{0,n}$ with dual graph $\Gamma_{\tilde{\mathcal{P}}}$. And for each P_j, the half-edge $\bullet P_j$ is the mark on the node pulled back from the factor corresponding to the central node, and $\star P_j$ is the mark on the node pulled back from the factor corresponding to the respective fork, as defined in (3.1.3).

The proof of the above theorem in [8] starts by expanding the left hand side as follows.

$$\omega_1^{k_1}\omega_2^{k_2}\dots\omega_n^{k_n} = (\psi_1 - B_1)^{k_1}(\psi_1 - B_2)^{k_2}\dots(\psi_n - B_n)^{k_n} \tag{4.7}$$

where

$$B_i = \sum_{I\ni i} D(I)$$

and then establishes the following:

1. All the intersections in (4.7) are supported on dual graphs that are the forked graphs with 'central' node with concentrated genus, and the forks corresponding to rational tails, with appropriate decorations of ψ classes

2. The forks on such supporting forked graphs can take any number of half-edges greater than 1. So, a supporting forked graph would be as below.

3. Corresponding to any such forked graphs, we can get more than just one intersection. This is because there are many non-transverse intersections possible when the number of half-edges on a divisor is greater than 2.

4. If we associate each of the forked graphs with a partition $\tilde{\mathcal{P}}$ as defined above, the intersections we get in the expansion of (4.7) is given by:

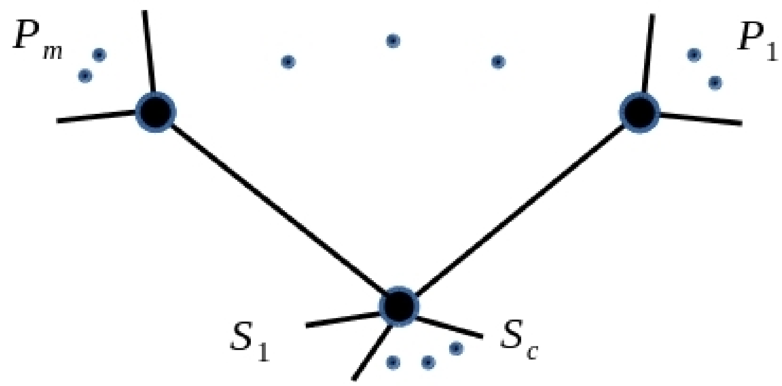


Figure 4.1: $\mathcal{P}$-graph with m forks

$$[\Gamma_{\tilde{\mathcal{P}}}] \prod_{S_i \in \tilde{\mathcal{P}}} \psi_{S_i}^{\alpha_i} \prod_{P_j \in \tilde{\mathcal{P}}} \frac{\psi_{\bullet P_j}^{\alpha_j}}{(-\psi_{\bullet P_j} - \psi_{\star P_j})}$$

5. Summing over all the $\tilde{\mathcal{P}} \in \tilde{\mathfrak{P}}$ gives the result (4.6)

So, the difference between (4.2) and (4.7) is the difference between D_i and B_i, and the genus on the central node of the resulting dual graphs. But the combinatorics of the intersections is the same in both cases. So, using the combinatorial result in (4.7) and restricting the partitions to have the maximum cardinality of q gives our result.

In formula (4.1), the denominator of the rational function is intended to be expanded as a geometric series in $\frac{\psi_{\bullet P_j}}{\psi_{\star P_j}}$. So, for each $P_j \in \mathcal{P}$, we have

$$\frac{\psi_{\bullet P_j}^{\alpha_j}}{-\psi_{\bullet P_j} - \psi_{\star P_j}} = -\psi_{\bullet P_j}^{\alpha_j - 1} + \psi_{\bullet P_j}^{\alpha_j - 2} \psi_{\star P_j} - \psi_{\bullet P_j}^{\alpha_j - 3} \psi_{\star P_j}^2 + \dots \tag{4.8}$$

The sum in (4.8) is finite since we defined negative powers of ψ to vanish. We also observe that if $\alpha_j = 0$, the right-hand side of (4.8) equals 0. Hence the formula is supported on forked-graphs representing strata where each fork has at least one point i with strictly positive k_i .

□

Here is an example that illustrates the result of the above theorem.

Example 4.1.1. *Over $\overline{M}_{0,\left(\frac{1}{5}\right)^{11}}$, consider the monomial:*

$$\hat{\psi}_1^1 \hat{\psi}_2^1 \hat{\psi}_3^2 \hat{\psi}_4^3$$

And consider the following $\mathcal{P}$-graph with 2 forks as in the following figure.

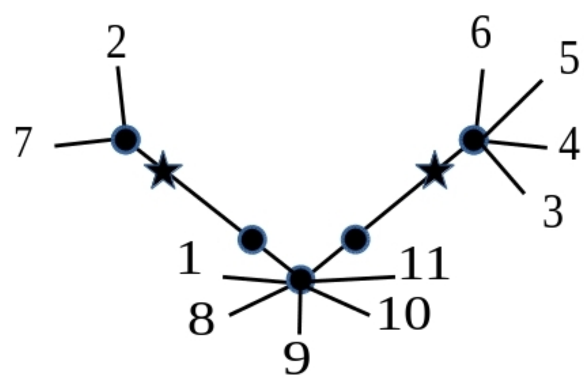


Figure 4.2: $\mathcal{P}$-graph with 2 forks

Corresponding to this dual graph, the term in the sum in (4.1) is given by:

$$[\Gamma_{\mathcal{P}}]\, {\psi_1}^1 . {\psi_{\bullet P1}}^0 \left(-\psi_{\bullet P_2}^4 + \psi_{\bullet P_2}^3 \psi_{\star P_2} - \psi_{\bullet P_2}^2 \psi_{\star P_2}^2 + \ldots\right)$$

$$= [\Gamma_{\mathcal{P}}]\, {\psi_1}^1 . {\psi_{\bullet P1}}^0 \left(-\psi_{\bullet P_2}^4 + \psi_{\bullet P_2}^3 \psi_{\star P_2} - \psi_{\bullet P_2}^2 \psi_{\star P_2}^2\right)$$

$$= [\Gamma_{\mathcal{P}}]\, {\psi_1}^1 \left(-\psi_{\bullet P_2}^4 + \psi_{\bullet P_2}^3 \psi_{\star P_2} - \psi_{\bullet P_2}^2 \psi_{\star P_2}^2\right)$$

The second equality is due to the fact that the terms in the infinite series after the first three terms vanish due to dimensionality reasons.

4.2 Numerical intersections

Corollary 4.2.1. *With $\mathcal{P}, F, S, P_j, z_i$ and α_j as defined above, for $n \geq 5$ and $\sum k_i = n-3$, we have for $\overline{M}_{0,\left(\frac{1}{q}\right)^n}$:*

$$\int_{\overline{M}_{0,\left(\frac{1}{q}\right)^n}} \hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\dots\hat{\psi}_n^{k_n} = \sum_{\mathcal{P}\in\mathfrak{P}} (-1)^{n+|\mathcal{P}|}\binom{|\mathcal{P}|-3}{Ind\,(\mathcal{P})} \tag{4.9}$$

where $Ind\,(\mathcal{P}) = k_{P_1}, k_{P_2}, \dots k_{P_m}$ with

$$k_{P_i} := k_{P_{i,1}} + k_{P_{i,2}} + \dots + k_{P_{i,z_i}} - z_i + 1 \tag{4.10}$$

Proof. This statement follows from formula (4.1), by noticing the following two facts:

1. For any partition $\mathcal{P} \in \mathfrak{P}$, by dimension reasons the only monomial that has nonzero evaluation on $[\Gamma_{\mathcal{P}}]$ is
$$\prod_{S_i\in\mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i\in\mathcal{P}} (-1)^{|P_i|-1}\psi_{\bullet P_i}^{\alpha_i-|P_i|+1}\psi_{\star P_i}^{|P_i|-2} \tag{4.11}$$

2. For any $n \geq 3$, $i \in [n]$,
$$\int_{\overline{M}_{0,n}} \psi_i^{n-3} = 1$$

So, all evaluations for the classes $\psi_{\star P_i}^{|P_i|-2}$ in (4.11) contribute a factor of 1 to the overall evalation of (4.11) on $[\Gamma_{\mathcal{P}}]$. It follows that, for every $\mathcal{P} \in \mathfrak{P}$,

$$\begin{aligned}
&\int_{[\Gamma_{\mathcal{P}}]} \prod_{S_i\in\mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i\in\mathcal{P}} (-1)^{|P_i|-1}\psi_{\bullet P_i}^{\alpha_i-|P_i|+1}\psi_{\star P_i}^{|P_i|-2} \\
&= \int_{\overline{M}_{0,|\mathcal{P}|}} (-1)^{\sum|P_i|-|F|} \prod_{S_i\in\mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i\in\mathcal{P}} \psi_{\bullet P_i}^{\alpha_i-|P_i|+1} \\
&= \int_{\overline{M}_{0,|\mathcal{P}|}} (-1)^{n-|S|-|F|} \prod_{S_i\in\mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i\in\mathcal{P}} \psi_{\bullet P_i}^{\alpha_i-|P_i|+1}
\end{aligned}$$

$$= \int_{\overline{M}_{0,|\mathcal{P}|}} (-1)^{n-|\mathcal{P}|} \prod_{S_i \in \mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i \in \mathcal{P}} \psi_{\bullet P_i}^{\alpha_i - |P_i| + 1}$$
$$= \int_{\overline{M}_{0,|\mathcal{P}|}} (-1)^{n+|\mathcal{P}|} \prod_{S_i \in \mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i \in \mathcal{P}} \psi_{\bullet P_i}^{\alpha_i - |P_i| + 1}$$

And the above expression when evaluated for all $\mathcal{P} \in \mathfrak{P}$ gives the formula in (4.9):

$$\int_{\overline{M}_{0,\left(\frac{1}{q}\right)^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \dots \hat{\psi}_n^{k_n}$$

$$= \sum_{\mathcal{P} \in \mathfrak{P}} \int_{\overline{M}_{0,|\mathcal{P}|}} (-1)^{n+|\mathcal{P}|} \prod_{S_i \in \mathcal{P}} \psi_{S_i}^{\alpha_i} \prod_{P_i \in \mathcal{P}} \psi_{\bullet P_i}^{\alpha_i - |P_i| + 1}$$
$$= \sum_{\mathcal{P} \in \mathfrak{P}} (-1)^{n+|\mathcal{P}|} \binom{|\mathcal{P}| - 3}{Ind\,(\mathcal{P})}$$

□

Example 4.2.1. *Over $\overline{M}_{0,\left(\frac{1}{5}\right)^{11}}$, consider the following numerical intersection:*

$$\int_{\overline{M}_{0,\left(\frac{1}{5}\right)^{11}}} \hat{\psi}_1^1 \hat{\psi}_2^1 \hat{\psi}_3^2 \hat{\psi}_4^3 \hat{\psi}_5^1$$

And consider the following $\mathcal{P}$-graph with 2 forks as in the following figure.

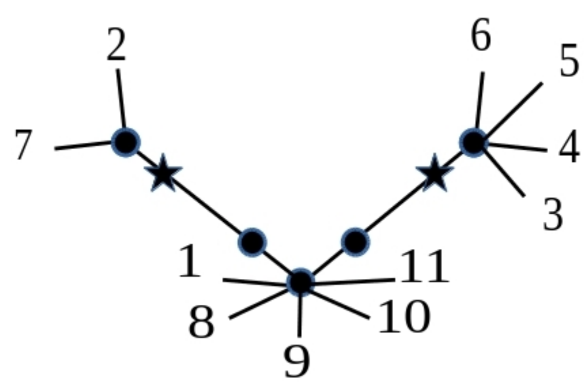


Figure 4.3: $\mathcal{P}$-graph with 2 forks

Corresponding to this dual graph, the term in the sum in (4.9) is given by:

$$[\Gamma_{\mathcal{P}}]\,{\psi_1}^1.{\psi_{\bullet P1}}^0\left(-\psi^5_{\bullet P_2}+\psi^4_{\bullet P_2}\psi_{\star P_2}-\psi^3_{\bullet P_2}\psi^2_{\star P_2}\right)$$

$$=[\Gamma_{\mathcal{P}}]\,{\psi_1}^1.\left(-\psi^5_{\bullet P_2}+\psi^4_{\bullet P_2}\psi_{\star P_2}-\psi^3_{\bullet P_2}\psi^2_{\star P_2}\right)$$

$$=[\Gamma_{\mathcal{P}}]\,{\psi_1}^1.\left(-\psi^3_{\bullet P_2}\psi^2_{\star P_2}\right)$$

$$=-\int_{\overline{M}_{0,7}}{\psi_1}^1.\left(\psi^3_{\bullet P_2}\right)$$

$$=-\binom{7}{1,3}$$

4.3 Generating Function for the top intersections

Here again, as in section 3.3, we start with the generating function- Witten-potential([7])- $F(\mathbf{t})$ with :

$$F(\mathbf{t}):=\sum_{\mathbf{s}}\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}\langle\tau^{\mathbf{s}}\rangle \tag{4.12}$$

and

$$\langle\tau_{k_1}\dots\tau_{k_n}\rangle:=\int_{\overline{M}_{0,n}}\psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n}$$

And our goal is to define an analogous generating function for top intersections on $\overline{M}_{0,(\frac{1}{q})^n}$.

Definition 4.3.1. *Define a new generating function*

$$G(\mathbf{t}):=\sum_{\mathbf{s}}\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}\langle\hat{\tau}^{\mathbf{s}}\rangle \tag{4.13}$$

where $\langle\hat{\tau}^{\mathbf{s}}\rangle=\int_{\overline{M}_{0,(\frac{1}{q})^n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\dots\hat{\psi}_n^{k_n}$ *as defined earlier, and* $\sum k_i=n-3$*.*

So, $G(\mathbf{t})$ has as coefficients of the monomials $\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!}$ the intersection numbers $\int_{\overline{M}_{0,(\frac{1}{q})^n}}\hat{\psi}_1^{k_1}\hat{\psi}_2^{k_2}\dots\hat{\psi}_n^{k_n}$ for any value of n and any values of k_i's, with $\sum k_i=n-3$.

Theorem 4.3.1. *With $G(\boldsymbol{t})$ as defined above,*

$$G(\boldsymbol{t}) = \hat{\mathcal{L}}(F(\boldsymbol{t})) - \frac{t_0^3}{3!} \tag{4.14}$$

where $\hat{\mathcal{L}} = : e^{\mathcal{L}} :$, and $: e^{\mathcal{L}} :$ denotes the operator with normal ordering[2]*, and*

$$\mathcal{L} = \sum_{s=2}^{q} (-1)^{s-1} \sum_{i_1, i_2, \ldots, i_s = 0}^{\infty} \frac{1}{s!} (t_{i_1} \ldots t_{i_s}) \frac{\partial}{\partial t_{i_1+i_2+\ldots+i_s-s+1}} \tag{4.15}$$

Definition 4.3.2. *For each $\mathcal{P}$-graph, the corresponding $\mathcal{P}_k$-graph is defined by replacing each $i \in [n]$ on the $\mathcal{P}$-graph by k_i.*

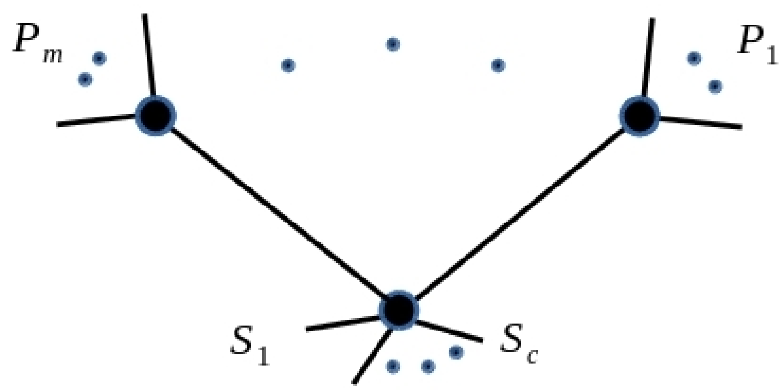


Figure 4.4: $\mathcal{P}$-graph with m forks

[2]by normal ordering of the operator, we mean that we treat the t_i's and $\frac{\partial}{\partial t_i}$'s as commuting variables, and bring all t_i's to the left of $\frac{\partial}{\partial t_i}$'s. E.g., if $\mathcal{J} = t_i t_j \frac{\partial}{\partial t_{i+j-1}}$,

$$\mathcal{J}^2 = t_i t_j \frac{\partial}{\partial t_{i+j-1}} t_i t_j \frac{\partial}{\partial t_{i+j-1}},$$

but

$$: \mathcal{J}^2 : = t_i^2 t_j^2 \frac{\partial}{\partial t_{i+j-1}} \frac{\partial}{\partial t_{i+j-1}}$$

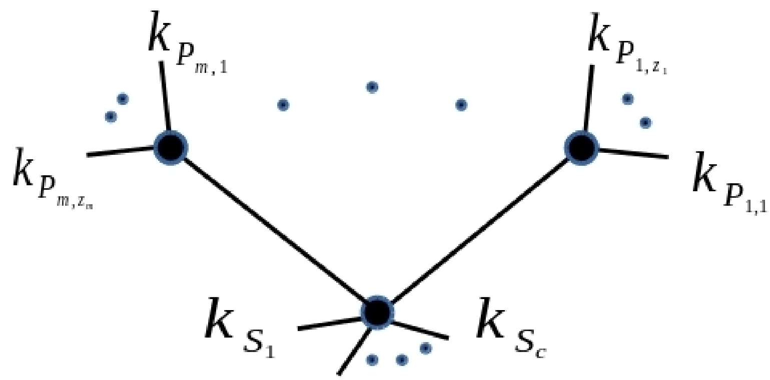


Figure 4.5: $\mathcal{P}_k$-graph with m forks

Let $\mathfrak{Q}$ be the set of all $\mathcal{P}_k$-graphs. We call a fork on $\mathcal{P}$-graph a P_i-fork, and the corresponding fork on a $\mathcal{P}_k$-graph a Q_i-fork. Observe that Q_m and Q_n, $m \neq n$, can have same set of k_i's on them. Clearly the map $\{\mathcal{P}\text{-}graphs\} \to \{\mathcal{P}_k\text{-}graphs\}$ is a surjection.

Lemma 4.3.1. *In genus* 0*, for a given* $\langle \tau_{k_1} \dots \tau_{k_n} \rangle = \int_{\overline{M}_{0,n}} \psi_1^{k_1} \psi_2^{k_2} \dots \psi_n^{k_n}$ *as defined above, let* **s** $= (s_0, s_1, \dots s_n)$*, and* $\langle \tau_{k_1} \dots \tau_{k_n} \rangle = \langle \tau_0^{s_0} \tau_1^{s_1} \tau_2^{s_2} \dots \tau_n^{s_n} \rangle$*. Consider a* $\mathcal{P}_k$*-graph with* m *forks with* q *distinct* k_i*'s appearing on the forks; let such* k_i*'s be* $\{k_1, k_2, \dots k_q\}$*. Let* l_i *be the number of times a given* k_i *appears on any fork on the* $\mathcal{P}_k$*-graph. Then the number of* $\mathcal{P}$*-graphs that map to this* $\mathcal{P}_k$*-graph is given by:*

$$\frac{1}{|Aut(\hat{\mathcal{P}}_k)|} \frac{s_{k_1}!}{(s_{k_1} - l_1)!} \frac{s_{k_2}!}{(s_{k_2} - l_2)!} \dots \frac{s_{k_q}!}{\left(s_{k_q} - l_q\right)!} =: C_{\mathcal{P}_k} \tag{4.16}$$

where $|Aut(\hat{\mathcal{P}}_k)|$ *is the number of automorphisms of the subgraph of* $\mathcal{P}_k$*-graph obtained from removing half-edges on the central node.*

Proof. Consider a $\mathcal{P}_k$-graph with m number of forks s.t q number of k_i's appear on the forks as defined above. Then if all n half-edges are given ordering, the number of $\mathcal{P}$-graphs where half-edges are ordered would be

$$s_0! s_1! s_2! \dots s_n!$$

Let l_i be the number of times a given k_i appears on any half-edge on the $\mathcal{P}_k$-graph. Now, we divide by the permutations of half-edges on the central node to get

$$\frac{s_0!}{(s_0-l_1)!}\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!}$$

Now, restrict l_i to be the number of times a given k_i appears on any fork on the $\mathcal{P}_k$-graph. As only q number of k_i's appear on the forks, $l_i = 0$ for $i > q$, so

$$\frac{s_0!}{(s_0-l_1)!}\frac{s_1!}{(s_1-l_1)!}\frac{s_2!}{(s_2-l_2)!}\cdots\frac{s_n!}{(s_n-l_n)!} = \frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!}$$

Further we need to divide by permutations of half-edges on the forks. Let r_{ij} be the number of times a k_j appears on fork P_i. Then we divide by the order of the stabilizer of S_{z_i}, the permutation group for each Q_i-fork. Further let $z_i = \sum_j r_{ij}$, so that z_i gives the cardinality of P_i for fork P_i. And $l_j = \sum_i r_{ij}$. Then we need to divide by the order of the stabilizer of S_m, the permutation group for m Q_i-forks. Let there be f distinct forks and let each distinct Q_j-fork be repeated m_j times. Then,

$$\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)\frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!}$$

Observe that the number $\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)$ is the number of automorphisms of the subgraph of $\mathcal{P}_k$-graph consisting of only the forks; denote this subgraph as $\hat{\mathcal{P}}_k$. Then the number of $\mathcal{P}$-graphs that map to a $\mathcal{P}_k$-graph can be rewritten as :

$$\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}\frac{s_{k_1}!}{(s_{k_1}-l_1)!}\frac{s_{k_2}!}{(s_{k_2}-l_2)!}\cdots\frac{s_{k_q}!}{\left(s_{k_q}-l_q\right)!} =: A_{\mathcal{P}_k}S_{\mathcal{P}_k} =: C_{\mathcal{P}_k}$$

where

$$A_{\mathcal{P}_k} = \frac{1}{|Aut(\hat{\mathcal{P}}_k)|},$$

and

$$S_{\mathcal{P}_k} = \frac{s_{k_1}!}{(s_{k_1} - l_1)!} \frac{s_{k_2}!}{(s_{k_2} - l_2)!} \cdots \frac{s_{k_q}!}{\left(s_{k_q} - l_q\right)!}$$

$C_{\mathcal{P}_k}$ can also be written as follows:

$$C_{\mathcal{P}_k} = \left(\frac{1}{m_1!} \frac{1}{m_2!} \cdots \frac{1}{m_f!} \prod_{P_i, k_j} \frac{1}{r_{ij}!} \right) \left(\frac{s_0!}{(s_0 - l_1)!} \frac{s_1!}{(s_1 - l_1)!} \frac{s_2!}{(s_2 - l_2)!} \cdots \frac{s_n!}{(s_n - l_n)!} \right) \tag{4.17}$$

□

The reason for organizing $C_{\mathcal{P}_k}$ as in (4.17) will become clear in the proof of theorem (4.3.1).

Lemma 4.3.2. *With the definitions and notations above, corollary (4.2.1) can be rewritten as :*

$$\begin{gathered} \int_{\overline{M}_{0,(\frac{1}{q})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n} = \\ \sum_{\mathcal{P}_k \in \mathfrak{Q}} (-1)^{n+|\mathcal{P}|} C_{\mathcal{P}_k} \int_{\overline{M}_{0,(n-m)}} \psi_{P_1}^{k_{P_1}} \ldots \psi_{P_m}^{k_{P_m}} \psi_{S_1}^{k_{S_1}} \ldots \ldots \psi_{S_c}^{k_{S_c}} \end{gathered} \tag{4.18}$$

where $C_{\mathcal{P}_k}$ is the number of $\mathcal{P}$-graphs that map to this $\mathcal{P}_k$-graph, and $\mathfrak{Q}$ is the set of all $\mathcal{P}_k$-graphs. And, k_{P_i} is as defined in (4.10).

Proof. This version of the closed formula for $\int_{\overline{M}_{0,(\frac{1}{q})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n}$ is just a reorganization of (4.9) using $\mathcal{P}_k$-graphs instead of $\mathcal{P}$-graphs. As the map
$\{\mathcal{P}\text{-}graphs\} \to \{\mathcal{P}_k\text{-}graphs\}$ is a surjection, we get all the terms in (4.9). □

Now, for a general $\mathcal{P}_k$-graph corresponding to the following $\mathcal{P}$-graph with m number of forks shown below, we define a $\mathcal{P}_k$-operator $\mathcal{D}_{\mathcal{P}_k}$ which appears in $\hat{\mathcal{L}}$, a $\mathcal{P}_k$-term that appears as a summand in (3.15), a t-monomial $T_{\mathcal{P}_k}$ which appears in $F(\mathbf{t})$ as follows.

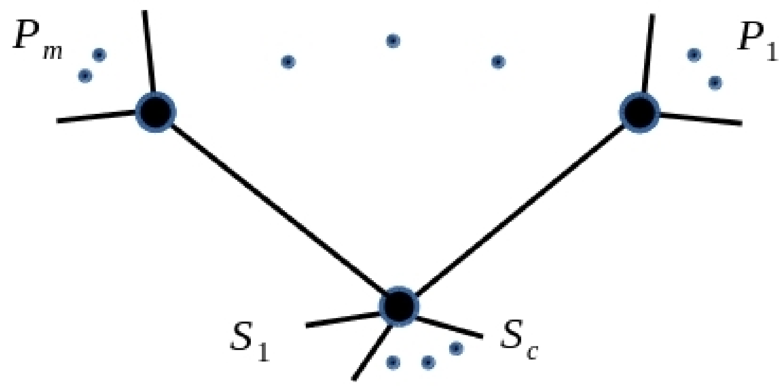


Figure 4.6: $\mathcal{P}$-graph with m forks

Definition 4.3.3. *A $\mathcal{P}_k$-operator $\mathcal{D}_{\mathcal{P}_k}$ (which appears in $\hat{\mathcal{L}}$), corresponding to a $\mathcal{P}$-graph with m number of forks shown in figure (4.7) is defined as follows:*

$$\mathcal{D}_{\mathcal{P}_k} := t_{P_1} t_{P_2} \dots t_{P_m} \frac{\partial}{\partial t_{k_{P_1}}} \dots \frac{\partial}{\partial t_{k_{P_m}}} \tag{4.19}$$

where,

$$t_{P_i} := t_{k_{P_{i,1}}} t_{k_{P_{i,2}}} \dots t_{k_{P_{i,z_i}}} \tag{4.20}$$

$\mathcal{D}_{\mathcal{P}_k}$ can also be written as follows:

$$\mathcal{D}_{\mathcal{P}_k} = \left(\prod_{i=0}^{n} t_i^{l_i} \right) \frac{\partial}{\partial t_{k_{P_1}}} \dots \frac{\partial}{\partial t_{k_{P_m}}} \tag{4.21}$$

Definition 4.3.4. *A $\mathcal{P}_k$-term that appears as a summand in (3.15), corresponding to a $\mathcal{P}$-graph with m number of forks shown in figure (4.7) is defined as follows :*

$$(-1)^{n+|\mathcal{P}|} C_{\mathcal{P}_k} \int_{\overline{M}_{0,(n-m)}} \psi_{P_1}^{K_{P_1}} \dots \psi_{P_m}^{K_{P_m}} \psi_{S_1}^{k_{S_1}} \dots \dots \psi_{S_c}^{k_{S_c}} \tag{4.22}$$

Definition 4.3.5. *A t-monomial $T_{\mathcal{P}_k}$ corresponding to a $\mathcal{P}$-graph with m number of forks shown in figure (4.7) is constructed as follows:*

1. *To every $P_i \in \mathcal{P}$, and every $S_i \in \mathcal{P}$, associate a $t_{k_{P_i}}$ and $t_{k_{S_i}}$ where,*

$$t_{k_{P_i}} := t_{k_{P_{i,1}}+k_{P_{i,2}}+\ldots+k_{P_{i,z_i}}-z_i+1}$$

   And form the product :

$$\left(t_{k_{P_1}} t_{k_{P_2}} \ldots t_{k_{P_m}}\right)\left(t_{k_{S_1}} t_{k_{S_2}} \ldots t_{k_{S_c}}\right)$$

2. *Collecting all t_i's in the expression above, and dividing by $\mathbf{s}!$, as defined for terms in Witten-potential (4.12), we get the $T_{\mathcal{P}_k}$ that appears in $F(\mathbf{t})$ associated with the above $\mathcal{P}_k$-graph :*

$$\frac{\mathbf{t}^{\mathbf{s}}}{\mathbf{s}!} =: T_{\mathcal{P}_k} \tag{4.23}$$

Let

$$t_{k_{P_1}} t_{k_{P_2}} \ldots t_{k_{P_m}} = \prod_{i=0}^{n} t_i^{a_i}$$

And as

$$t_{k_{S_1}} t_{k_{S_2}} \ldots t_{k_{S_c}} = \prod_{i=0}^{n} t_i^{s_i - l_i}$$

we can rewrite $T_{\mathcal{P}_k}$ as follows:

$$T_{\mathcal{P}_k} = \left(\prod_{i=0}^{n} \frac{t_i^{s_i-l_i}}{(s_i-l_i)!}\right)\left(\prod_{P_i \in \mathcal{P}_k} \frac{t_i^{a_i}}{(s_i-l_i+1)\ldots(s_i-l_i+a_i)}\right) \tag{4.24}$$

Lemma 4.3.3. *With $\mathcal{D}_{\mathcal{P}_k}$, $\hat{\mathcal{L}}$, and $A_{\mathcal{P}_k}$ as defined, the coefficient of $\mathcal{D}_{\mathcal{P}_k}$ in $\hat{\mathcal{L}}$ is $(-1)^{\sum|P_i|-m} A_{\mathcal{P}_k}$.*

Proof. This follows directly by looking at the multinomial coefficient of $\mathcal{D}_{\mathcal{P}_k}$ in $\hat{\mathcal{L}}$, which is

$$(-1)^{\sum|P_i|-m} A_{\mathcal{P}_k} \frac{1}{m!}\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\right)\left(\prod_{P_i,k_j}\frac{1}{z_i!}\frac{z_i!}{r_{ij}!}\right)$$

$$= (-1)^{\sum|P_i|-m} A_{\mathcal{P}_k}\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)$$

$$= (-1)^{\sum |P_i| - m} A_{\mathcal{P}_k} A_{\mathcal{P}_k}$$

$\square$

Lemma 4.3.4. *With $\mathcal{D}_{\mathcal{P}_k}$, $T_{\mathcal{P}_k}$, and $S_{\mathcal{P}_k}$ as defined, and $T := \frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!}$,*

$$\mathcal{D}_{\mathcal{P}_k} (T_{\mathcal{P}_k}) = S_{\mathcal{P}_k} T$$

Proof. Applying $\mathcal{D}_{\mathcal{P}_k}$ as defined in (4.21) to $T_{\mathcal{P}_k}$ as defined in (4.24), we get :

$$\begin{aligned}
\mathcal{D}_{\mathcal{P}_k} (T_{\mathcal{P}_k}) &= t_{P_1} t_{P_2} \ldots t_{P_m} \frac{\partial}{\partial t_{K_{P_1}}} \cdots \frac{\partial}{\partial t_{K_{P_m}}} (T_{\mathcal{P}_k}) \\
&= \left(\prod_{i=0}^{n} t_i^{l_i} \right) \frac{\partial}{\partial t_{K_{P_1}}} \cdots \frac{\partial}{\partial t_{K_{P_m}}} \left(\prod_{i=0}^{n} \frac{t_i^{s_i - l_i}}{(s_i - l_i)!} \right) \left(\prod_{P_i \in \mathcal{P}_k} \frac{t_i^{a_i}}{(s_i - l_i + 1) \ldots (s_i - l_i + a_i)} \right) \\
&= \left(\prod_{i=0}^{n} t_i^{l_i} \right) \left(\prod_{i=0}^{n} \left(\frac{\partial}{\partial t_i} \right)^{a_i} \right) \left(\prod_{i=0}^{n} \frac{t_i^{s_i - l_i}}{(s_i - l_i)!} \right) \left(\prod_{P_i \in \mathcal{P}_k} \frac{t_i^{a_i}}{(s_i - l_i + 1) \ldots (s_i - l_i + a_i)} \right) \\
&= \left(\prod_{i=0}^{n} \frac{s_i}{(s_i - l_i)!} \right) \left(\prod_{i=0}^{n} \frac{t_i^{s_i}}{s_i!} \right) \\
&= S_{\mathcal{P}_k} T
\end{aligned}$$

$\square$

Strategy of proof of theorem (4.3.1): we will show that for a general t-monomial $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!}$, its coefficients in $G(\mathbf{t})$ and $\hat{\mathcal{L}}(F(\mathbf{t})) - \frac{t_0^3}{3!}$ are equal. And that both equal $\int_{\overline{M}_{0,(\frac{1}{q})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n}$. To show that, we show a bijection between $\mathcal{P}_k$ graphs and the terms (4.22) which are the summands in the formula (4.18). To show this bijection, we pick a $\mathcal{P}_k$ graph, and find the term (4.20) in $\hat{\mathcal{L}}$ (with some multiplicity); then we find the term $T_{\mathcal{P}_k}$ in $F(\mathbf{t})$, such that the term (4.20) when applied to $T_{\mathcal{P}_k}$ gives the t-monomial $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!}$ with coefficient the intersection number defined in (4.22). As that exactly is the summand in (4.18) corresponding to the chosen $\mathcal{P}_k$-graph, this proves the theorem in one direction. In the other direction, we pick a term (4.22) which is a summand in the coefficient of t-monomial $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_l{}^{s_l}}{s_0! s_1! \ldots s_l!}$, and show that this maps to the same $\mathcal{P}_k$-graph.

Proof. (of theorem (4.3.1))

Consider a general term in $G(\mathbf{t})$ with the corresponding t-monomial $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_n{}^{s_n}}{s_0! s_1! \ldots s_n!}$, with coefficient $\int_{\overline{M}_{0,(\frac{1}{q})^n}} \hat{\psi}_1^{k_1} \hat{\psi}_2^{k_2} \ldots \hat{\psi}_n^{k_n}$. Now, we will show that we get all the terms of formula (4.18) in $\hat{\mathcal{L}}(F(\mathbf{t}))$ and that each term corresponds to a $\mathcal{P}_k$-graph.

Now consider a $\mathcal{P}_k$-graph with m number of forks. Let this $\mathcal{P}_k$-graph corresponding to the following $\mathcal{P}$-graph with m number of forks shown below.

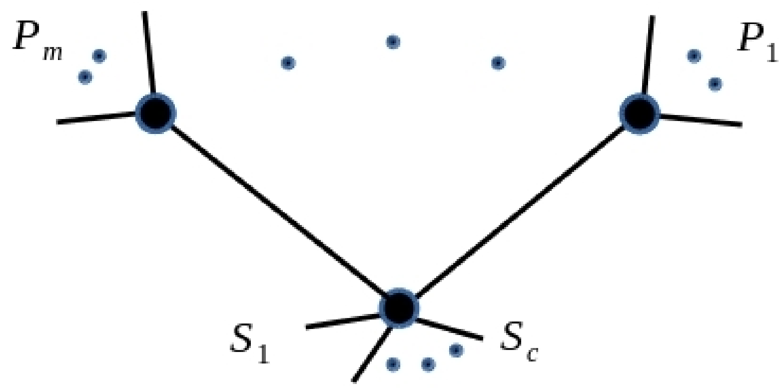


Figure 4.7: $\mathcal{P}$-graph with m forks

Then the corresponding operator (4.20) is

$$\begin{aligned} \mathcal{D}_{\mathcal{P}_k} &= t_{P_1} t_{P_2} \ldots t_{P_m} \frac{\partial}{\partial t_{k_{P_1}}} \cdots \frac{\partial}{\partial t_{k_{P_m}}} \\ &= \left(\prod_{i=0}^{n} t_i^{l_i} \right) \left(\prod_{i=0}^{n} \left(\frac{\partial}{\partial t_i} \right)^{a_i} \right) \end{aligned}$$

The coefficient of this term in $\hat{\mathcal{L}}$ by Lemma (4.3.3) as a summand in : $\frac{\mathcal{L}^m}{m!}$: is :

$$\begin{aligned} &(-1)^{\sum |P_i| - m} A_{\mathcal{P}_k} \\ &= (-1)^{\sum |P_i| - m} \frac{1}{m_1!} \frac{1}{m_2!} \cdots \frac{1}{m_f!} \prod_{P_i, k_j} \frac{1}{r_{ij}!} \end{aligned}$$

The corresponding term in $F(\mathbf{t})$ is

$$T_{\mathcal{P}_k} = \left(\prod_{i=0}^{n} \frac{t_i^{s_i-l_i}}{(s_i-l_i)!}\right)\left(\prod_{P_i\in\mathcal{P}_k} \frac{t_i^{a_i}}{(s_i-l_i+1)\dots(s_i-l_i+a_i)}\right)$$

So, in $\hat{\mathcal{L}}(F(\mathbf{t}))$, the corresponding term is

$$(-1)^{\sum|P_i|-m}A_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)\mathcal{D}_{\mathcal{P}_k}(T_{\mathcal{P}_k})$$

where $\langle\tau^{\mathbf{s}}\rangle$ is the appropriate ψ-monomial on $\overline{M}_{0,\left(\frac{1}{q}\right)^{|\mathcal{P}|}}$ that appears as coefficient of $T_{\mathcal{P}_k}$ in $F(\mathbf{t})$. Observe that the coefficient of $D_{\mathcal{P}_k}$ in : $\frac{\mathcal{L}^m}{m!}$: is exactly $(-1)^{\sum|P_i|-m}\frac{1}{|Aut(\hat{\mathcal{P}}_k)|}$ as claimed earlier. Now by Lemma (4.3.4),

$$\begin{aligned}
&(-1)^{\sum|P_i|-m}A_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)\mathcal{D}_{\mathcal{P}_k}(T_{\mathcal{P}_k})\\
&=(-1)^{\sum|P_i|-m}A_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)S_{\mathcal{P}_k}T\\
&=(-1)^{\sum|P_i|-m}C_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)T\\
&=(-1)^{\sum|P_i|-m}C_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)T\\
&=(-1)^{n-|S|-|F|}C_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)T\\
&=(-1)^{n-|\mathcal{P}|}C_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)T\\
&=(-1)^{n+|\mathcal{P}|}C_{\mathcal{P}_k}\left(\langle\tau^{\mathbf{s}}\rangle\right)T
\end{aligned}$$

which term in $\hat{\mathcal{L}}(F(\mathbf{t}))$ has as coefficient of T exactly the term (4.22) which is the summand in (4.18) corresponding to the chosen $\mathcal{P}_k$-graph.

So, one direction is proved.

To show bijection in the other direction, we pick a summand in the coefficient of $\frac{t_0{}^{s_0}t_1{}^{s_1}\dots t_n{}^{s_n}}{s_0!s_1!\dots s_n!} = T$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ that comes from a summand in : $\frac{1}{m!}\mathcal{L}^m$: $(F(\mathbf{t}))$. Let this summand be :

$$(-1)^{\sum z_i-m}\left(\frac{1}{m_1!}\frac{1}{m_2!}\dots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)\left(\langle\tau^{\mathbf{s}}\rangle\right)$$

Also, let the corresponding summand in : $\frac{1}{m!}\mathcal{L}^m$: be

$$(-1)^{\sum z_i - m}\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)D$$

and let it act on $\tilde{T}$,

such that

$$\begin{aligned}
&(-1)^{\sum z_i - m}\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,D(\tilde{T})\\
&= (-1)^{\sum z_i - m}\left(\frac{1}{m_1!}\frac{1}{m_2!}\cdots\frac{1}{m_f!}\prod_{P_i,k_j}\frac{1}{r_{ij}!}\right)(\langle\tau^{\mathbf{s}}\rangle)\,S.T
\end{aligned}$$

Then, this implies that

$$\begin{aligned}
D &= \mathcal{D}_{\mathcal{P}_k}\\
&= t_{P_1}t_{P_2}\ldots t_{P_m}\frac{\partial}{\partial t_{K_{P_1}}}\cdots\frac{\partial}{\partial t_{K_{P_m}}}\\
&= \left(\prod_{i=0}^{n} t_i^{l_i}\right)\left(\prod_{i=0}^{n}\left(\frac{\partial}{\partial t_i}\right)^{a_i}\right)
\end{aligned}$$

and

$$\tilde{T} = T_{\mathcal{P}_k}$$

and

$$S = S_{\mathcal{P}_k}$$

as defined earlier. $\langle\tau^{\mathbf{s}}\rangle$ is uniquely determined by $T_{\mathcal{P}_k}$, and $T_{\mathcal{P}_k}$ is uniquely determined by the following relation from lemma (4.3.4).

$$\mathcal{D}_{\mathcal{P}_k}(T_{\mathcal{P}_k}) = S_{\mathcal{P}_k}T$$

As this $\mathcal{D}_{\mathcal{P}_k}$ is in bijection with the $\mathcal{P}_k$-graph as in the figure by construction, we get the term in (4.18) corresponding to this $\mathcal{P}_k$-graph as the chosen summand in $\hat{\mathcal{L}}(F(\mathbf{t}))$ as

$$(-1)^{n+|\mathcal{P}|} C_{\mathcal{P}_k} \int_{\overline{M}_{0,(n-m)}} \psi_{P_1}^{K_{P_1}} \ldots \psi_{P_m}^{K_{P_m}} \psi_{S_1}^{k_{S_1}} \ldots \ldots \psi_{S_c}^{k_{S_c}}$$

So the coefficient of $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_n{}^{s_n}}{s_0! s_1! \ldots s_n!}$ in $\hat{\mathcal{L}}(F(\mathbf{t}))$ equals the coefficient of $\frac{t_0{}^{s_0} t_1{}^{s_1} \ldots t_n{}^{s_n}}{s_0! s_1! \ldots s_n!}$ in $G(\mathbf{t})$.

$\square$

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