THESIS

THE GROUP EXTENSIONS PROBLEM AND ITS RESOLUTION IN COHOMOLOGY FOR THE CASE OF AN ELEMENTARY ABELIAN NORMAL SUB-GROUP

Submitted by

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ABSTRACT

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The Jordan-Hölder theorem gives a way to deconstruct a group into smaller groups, The converse problem is the construction of group extensions, that is to construct a group G from two groups Q and K where $K \leq G$ and $G/K \cong Q$. Extension theory allows us to construct groups from smaller order groups. The extension problem then is to construct all extensions G, up to suitable equivalence, for given groups K and Q. This talk will explore the extension problem by first constructing extensions as cartesian products and examining the connections to group cohomology.

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I would like to thank my loving wife for her patience and support, my family for never giving up on me no matter how much dumb stuff I did, and my friends for anchoring me to reality during the rigors of graduate school. Finally I thank my advisor, for his saintly patience in the face of my at times profound hardheadedness.

DEDICATION

I would like to dedicate my masters thesis to the memory of my grandfather Wilfred Adams whose dazzling intelligence was matched only by the love he had for his family, and to the memory of my friend and brother Luke Monsma whose lust for life is an example I will carry with me to the end of my days. I miss you both.

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Chapter 1

Introduction

In many, if not all, of the scientific disciplines, the task of classification is of great importance. Take for instance the effort of the taxonomist in categorizing the world's flora and fauna, or the work of Mendeleev in organizing the periodic table of elements. In mathematics the drive to classify is also a prime mover. Whether for the sake of the thing or for reasons of usefulness, mathematicians seek to classify continuous functions by their smoothness, topological spaces for their separability, and graphs by their sub-graphs. This paper examines a classification of groups by their normal subgroups and factor groups.

A group is an algebraic structure that describes the symmetry inherent to an object. Some groups describe concrete symmetries like those of a regular polygon or polytope, while others describe more abstract symmetries like relations among the roots of a polynomial over an algebraically closed field. Concretely, a group is often defined to be set closed under an associative binary operation so that there is an identity element, and that each element has an inverse. Yet, we may think of a group as a collection of symmetries of a certain concrete or abstract object.

The most fundamental property of a group is its order, or the cardinality of its underlying set. A group may be of finite order, like the symmetries of a square (the dihedral group of order 8) or infinite as in the integers under addition. In the finite case, a good way to begin a classification of groups is by their order though this classification is a coarse one.

In order to refine the classification, one would like to classify the groups of a given order. One way to make this classification is to separate the simple groups from the groups with a normal subgroup. That a composition series exists for every group is a good motivation for making this distinction because a classification of simple groups will provide a classification of all other groups. The better part of twentieth century finite group theory was dedicated to a classification theorem for finite simple groups, an effort which found its success in about 1980.

Attention must now be paid to classifying groups which have a normal subgroup. Normal subgroups are special. From a group G with normal subgroup K, the group $Q = G/K$ can be formed. In this case, G is called an extension of Q by K. Groups with a normal subgroup can thus be classified by considering the groups of which it is an extension. Thus a question of importance is: given a two groups can we calculate all of their extensions? In the case of an elementary abelian normal subgroup, this question, treated below, is solved by way of group cohomology.

Chapter 2

Preliminaries

2.1 Free Groups and Group Presentations

Given a set Γ, It is possible to construct the *free group*, on Γ (in this construction, Γ is generally called the set of *generators* of the free group). The free group is the set of words made from the letters (elements) of Γ and their formal inverses. The multiplication is concatenation of words with a reduction process whereby xx^{-1} and $x^{-1}x$ are replaced by the empty word. Free groups allow the concept of a group presentation to be formalized, and eventually, it allows us to reduce the number of variables when calculating the cohomology in aid of solving the extension problem.

Free Groups

Definition 1. *A group F is free on a set Γ, denoted $F = \langle \Gamma \rangle$, if every map $\Gamma \to H$ where H is a group, can be extended to a homomorphism $F \to H$. Γ is called a basis of F and an element of Γ is called a generator of F*

This is the definition of a free object in a category specialized to the category of groups and while this definition fits the general case it is useless in practice. Thus it is expedient to construct a group satisfying the free property for the purposes of calculation.

Definition 2. *Given a set Ω, the alphabet A associated to Ω is the set of symbols $A = \Omega \cup \{\omega^{-1} : \omega \in \Omega\}$*

In plain language, the alphabet associated with a set is that set united with a set of symbols to represent the inverse of each element.

Definition 3. *The set A^* of Words on an alphabet A is the set of all finite sequences whose terms (letters) come from A, including the empty sequence.*

We now define an equivalence relation $\sim$ on the set of words A^* by declaring two words to be equivalent if they differ by the term xx^{-1} or $x^{-1}x$ for any x in A.

Lemma 1. *The set $(A^*/\sim, \circ)$ where $\circ$ maps the equivalence classes of two words to the equivalence class containing their concatenation is the Free Object in the category of groups. [1]*

This way representation of the free group as equivalence classes of words under concatenation gives us a way to perform concrete calculation in the group instead of talking about it in broad theoretical terms. Armed with this construction, we give a few properties of free groups.

Theorem 2.1.1. *Given any set Γ there exists a Free group with generators Γ.*

Proof: Consider The group $\Gamma^*/\sim$ Which is a free Group under $\circ$

We call the free group with basis Γ is denoted $F(\Gamma)$.

Theorem 2.1.2. *Two Free groups are isomorphic iff their bases have the same cardinality.*

Theorem 2.1.3. *(Nielsen-Schreier) The subgroups of free groups are themselves free groups.*

Lemma 2. *Every group is the quotient of a free group.*

Proof: Let G be a group and define the set $\Gamma = \{x_g\}_{g \in G}$. There is a clear bijection between G and Γ and so the map from Γ to G can be extended to a homomorphism, which is indeed an epimorphism as the map between G and Γ is surjective. Call the kernel of this map K. Then by the first isomorphism theorem, $G \cong F(\Gamma)/K$. $\square$

Group Presentations In applications, it is useful to write groups in a compact form that captures the key structures that uniquely identify a group. The presentation allows us to write a group in terms of generators and relations among generators and hence gives a compact way to write and represent a group.

Definition 4. *Let Γ be a set and Δ be a set of words on Γ. A group G has generators Γ and relators Δ if G is the quotient of $F(\Gamma)/R$ where R is the normal subgroup of F generated by Δ. This is captured by the pair $\langle\Gamma|\Delta\rangle$. The group G is called finitely generated if $|\Gamma| < \infty$.*

Since every group is the quotient of a free group (Lemma 2), every group has a presentation.

Definition 5. *A finitely generated group is called finitely presented if* $G \cong \langle \Gamma | \Delta \rangle$ *and* $|\Delta| < \infty$

Here we stop to give a few examples of finitely presented groups and their presentations:

- The Klein 4 Group is finitely presented and has presentation $\langle a, b | a^2, b^2, (ab)^2 \rangle$.
- The Dihedral group of order $2n$ is finitely presented with presentation $\langle r, s | s^2, r^n, rsrs \rangle$.
- The Symmetric S_3 on three points has presentation $\langle a, b, c | a^2, b^2, c^3, abc \rangle$.
- The cyclic group of order n, C_n has presentation $\langle a | a^n \rangle$.

Exact sequences

Definition 6. *Let* G*,* K*, and* Q *be groups with maps* $\iota : K \hookrightarrow G$ *and* $\pi : G \twoheadrightarrow Q$*. Let* 1 *denote the trivial group. A short exact sequence is*

$$1 \to K \hookrightarrow G \twoheadrightarrow Q \to 1$$

where $im(\iota) = ker(\pi)$ *or equivalently, if* $G/K \cong Q$.

Chapter 3

Group extensions

Given two groups, it is reasonable to ask if their cartesian product can be endowed with a product that defines a group. The first example of such a construction is the direct product, that is the cartesian product where the multiplication is defined point-wise. The obvious question is whether there is another way to form a group out of the cartesian product. A slightly more complicated construction is the semi-direct product. Given two groups and a homomorphism ϕ from one group to the automorphism group of the other, we can define a multiplication in the Cartesian product by

$$(k_1, h_1)(k_2, h_2) = (k_1\phi(h)(k_2), h_1, h_2).$$

All direct products are semi-direct products (simply consider the map to the automorphisms to be trivial). Yet, not all semi-direct products are direct products. As an example, consider D_8 which has a normal subgroup of order 4 and an element of order 2, yet $C_4 \times C_2$ is abelian whereas D_8 is not.

The question remains, is the semi-direct product the most general multiplication which can be defined in the cartesian product. To answer it we generalize the semi-direct product. Let H and K be groups, and G a group such that $K \rtimes_\phi H = G$. Note that $K \lhd G$ and that $G/K = H$. The proofs of these facts are standard fare for a first course in algebra. These properties define a short exact sequence

$$0 \to K \to G \to H \to 0$$

So a good generalization is captured in the following definition.

Definition 7. *Given two groups H and K, a group G is called an extension of H by K if the sequence*

$$0 \to K \to G \to H \to 0$$

is short exact.

It is clear that all semi-direct products are extensions, yet there are groups that are extensions but are not semi-direct products. As an example, consider Q_8 as an extension of V_4 by C_2 where $|\langle -1 \rangle| = 2$ and $Q_8/\langle -1 \rangle \cong V_4$. Yet V_4 is not a subgroup of Q_8 and so Q_8 is not the semi-direct product of C_2 and V_4 .

The notion of group extension leads to the extension problem. Given two groups H and K, what groups G, up to isomorphism, are extensions of H by K? The solution then is to determine groups as extensions of other groups. Determining a group is not a well defined task, indeed it could be that determining a group is to write down a multiplication table for a group, or perhaps it means to determine the isomorphism classes of such groups G. The problem of constructing multiplication tables of extensions is well understood and indeed solved by way of group cohomology in the case where K is abelian.

Equivalence of extensions

Definition 8. *Two extensions are said to be equivalent if there exists an isomorphism $\phi : G \to G'$ such that the following diagram commutes:*

$$\begin{tikzcd}
 & & G \arrow[dr, two heads] \arrow[dd, dashed, leftrightarrow] & & \\
0 \arrow[r] & K \arrow[ur, hook] \arrow[dr, hook] & & H \arrow[r] & 0 \\
 & & G' \arrow[ur, two heads] & &
\end{tikzcd}$$

Lemma 3. *The above definition induces an equivalence relation on the set of extensions.*

Proof: Reflexivity and symmetry are clear by the properties of isomorphism. Transitivity is demonstrated by the following diagram.

$$\begin{tikzcd}
 & & G_1 \arrow[d, leftrightarrow] \arrow[dr, two heads] & & \\
0 \arrow[r] & K \arrow[ur, hook] \arrow[r, hook] \arrow[dr, hook] & G_2 \arrow[r, two heads] & H \arrow[r] & 1 \\
 & & G_2 \arrow[u, leftrightarrow] \arrow[ur, two heads] & &
\end{tikzcd}$$

Definition 9. *$Ext(H, K)$ is the set of equivalence classes of extensions of H by K*

Note that equivalent groups are isomorphic but that isomorphic groups are not necessarily equivalent.

Arithmetic in the extension:

Definition 10. *A transversal of a (normal) subgroup K of a group G is a set of representatives of cosets of K in G. That is, a set containing a single element of every coset.*

A transversal T can be interpreted as a function $\tau : Q \to G$ such that $\tau(Q) = T$. These functions are also called transversals. As a convention, we use 1_G to represent the coset K (equivalently $\tau(1_K) = 1_G$). Thus we construct the extension in terms of its normal sub-group and factor group as follows.

Lemma 4. *Let G be a group with a normal sub-group K, and let T be a transversal of K in G, Then*

$$G = \cup_{t \in T} Kt$$

Proof: The reverse inclusion is clear as all cosets of K are subsets of G. So it suffices to show that $G \subset \cup_{t \in T} Kt$ Let $g \in G$, then because the cosets of K partition G, there is some $t \in T$ such that $g \in Kt$. Thus g is in the union of the cosets, proving the lemma. [1] $\square$

Since G is a union of K cosets, we have that each element of G can be expressed uniquely up to choice of transversal as a pair in the Cartesian product $H \times K$ as follows. Let $g \in G$, then by the previous lemma there is a unique $t \in T$ such that $g \in Kt$, and thus there is a unique $k \in K$ such that $g = kt$. Further, there is an $h \in H$ such that $\tau(h) = t$. We can then write $g = k\tau(h)$ and thus make the following identification

$$g = kt = k\tau(h) := (k, h)$$

which allows us to represent elements of the extension as pairs from the Cartesian product of the normal subgroup and the factor group.

With a way to describe the set G, we need only to describe the multiplication in G using elements of the Cartesian product. Let g_1 and $g_2 \in G$, such that $g_1 = (k_1, h_1)$, $g_2 = (k_2, h_2)$ where k_1 , $k_2 \in K$ and h_1 , $h_2 \in H$ such that t_1 , $t_2 \in T$ with $\tau(h_1) = t_1$, $\tau(h_2) = t_2$. Then

$$\begin{aligned}
g_1g_2 &= (k_1, h_1)(k_2, h_2) \\
&= k_1\tau(h_1)k_2\tau(h_2) \\
&= k_1t_1k_2t_2 \\
&= k_1t_1k_2t_1^{-1}t_1t_2 \\
&= k_1k_2^{t_1^{-1}}t_1t_2 \\
&= k_1k_2^{t^{-1}}\tau(h_1)\tau(h_2) \\
&= k_1k_2^{t^{-1}}\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1}\tau(h_1h_2) \\
&= (k_1k_2^{t^{-1}}\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1})(\tau(h_1h_2))
\end{aligned}$$

Note that $k_1k_2^{t^{-1}} \in K$ because K is a normal subgroup. Thus for each $h \in H$ there is a unique automorphism of K and hence a map

$$\begin{aligned}
\theta : H &\rightarrow Aut(K) \\
h &\mapsto (k \mapsto \tau(h)k\tau(h)^{-1})
\end{aligned}$$

If K is an abelian group, then this map is a homomorphism, as it is with the semi-direct product (i.e. $(k^{\tau(h_1)})^{\tau(h_2)} = k^{\tau(h_1)\tau(h_2)}$. We choose to consider K to be abelian from here on. Further $\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1} \in K$ and $\tau(h_1h_2) = t$ for some $t \in T$. Thus

$$\begin{aligned} g_1g_2 &= (k_1k_2^{t^{-1}}\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1})(\tau(h_1h_2)) \\ &= (k_1k_2^{t^{-1}}\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1}, h_1h_2). \end{aligned}$$

We collect these calculations into the following definition.

Definition 11. *Let K and H be groups and $\theta : H \to Aut(K)$ be a homomorphism . $K \times_\theta H$ is the set $K \times H$ under the operation given:*

$$(k, x)(l, y) = (k \cdot l^x, xy)$$

While this seems to be enough to specify the group multiplication, there is a problem lurking in the elements of G. Note that while we know that $\tau(h_1)\tau(h_2)$ lies in $\tau(h_1h_2)K$ thus $\tau(h_1h_2) = k$ yet we don't know the value of k without using the multiplication in G.

Considering the problem of multiplication without access to G, we have that

$$k_1\tau(h_1)k_2\tau(h_2) = k_1k_2^{\tau(h_1)^{-1}}\tau(h_1) \cdot \tau(h_2)k_1k_2^{\tau(h_1)^{-1}}\tau(h_1h_2)f(x, y)$$

For some $f(x, y) \in K$.

Definition 12. *A factor set for G is a function $f : H \times H \to K$ such that for some transversal τ ,*

$$\tau(x)\tau(y) = f(x, y)\tau(xy)$$

There are a two things to note here. First, the factor set f depends directly on the choice of τ . Second, the factor set f is a measure of how different τ is from a homomorphism. Since the particular extension of H by K is determined by the choice of transversal τ, and since τ

determines the factor set f, finding the groups extensions of H by K is equivalent, up to a choice of τ, to finding the factor sets f . Understanding factor sets is integral to understanding extensions and so we take some time to prove some of their properties.

At this point, we adopt the convention that K will be written additively and operations in H will be written multiplicatively. Thus the action of H on K is written multiplicatively instead of as a power (that is, $\theta(x)(y) = xy$)

Definition 13. *$Z^2_\theta(H,K)$ is the set of all factor sets. Note, some authors call factor sets, co-factors and we will use these interchangeably.*

Lemma 5. *Let K be abelian and let $\theta : H \to Aut(K)$. A function $f : H \times H \to K$ is a factor set if and only if it satisfies the following equations: For all $x, y, z \in H$*

$$f(1, y) = 1 = f(x, 1)$$

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0$$

Proof: We assumed that $\tau(1) = 0$. So the following equality holds

$$0 + \tau(y) = \tau(1) + \tau(y) = f(1, y) + \tau(1y)$$

So $f(1, y) = 0$.

Further,

$$\tau(x) + 0 = \tau(x) + \tau(1) = f(x, 1) + \tau(x1)$$

so $f(x, 1) = 0$ proving the first identity.

For identity two, let $x, y, z \in H$ and consider the following equalities.

$$(\tau(x)+\tau(y))+\tau(z)=\tau(x)+(\tau(y)+\tau(z))$$
$$f(x,y)+\tau(xy)+\tau(z)=\tau(x)+f(y,z)+\tau(yz)$$
$$f(x,y)+f(xy,z)+\tau(xyz)=f(y,z)+\tau(xyz)+f(x,yz)$$

which verifies identity 2 proving necessity.

For sufficiency, suppose that $f : H \times H \to K$ is a function satisfying the above relations. We now construct a group G, an extension of H by K realizing θ with transversal τ, such that f is the associated factor set.

Let G be the group with set $K \times H$ with operation:

$$(k,x)+(l,y)=(k+xl+f(x,y),xy)$$

The proof that G is a group, in particular that it is associative, is an exercise in bookkeeping and will not be proven here. Note that G is an extension as it fits the format given above where $f(x,y)=\tau(x)\tau(y)\tau(xy)^{-1}$. □

These factor sets give us the ability to calculate in extensions of H by K without using the multiplication in G.

Theorem 3.0.1. *$Z^2_\theta(H,K)$ is an abelian group under point-wise addition*

Proof: Let $f,g \in Z^2_\theta(H,K)$ and $x,y,z \in H$ then,

$$\begin{aligned}
&x(f+g)(y,z)-(f+g)(xy,z)+(f+g)(x,yz)-(f+g)(x,y)\\
=&xf(y,z)+xg(y,z)-(f(xy,z)+g(xy,z))+f(x,yz)+g(x,yz)+f(x,y)+g(x,y)\\
=&(xf(y,z)-f(xy,z)+f(x,yz)+f(x,y))+(xg(y,z)-g(xy,z)+g(xy,z)+g(x,y))\\
=&0
\end{aligned}$$

As f and g are both factor sets. Thus $f+g \in Z^2_\theta(H,K)$, giving closure. Associativity and commutativity are inherited from K. Note that f has an inverse as

$$0 = -xf(y,z) + f(xy,z) - f(x,yz) + f(x,y)$$

Thus $Z^2_\theta(H,K)$ is an abelian group under point-wise addition. $\square$

The following lemma describes how a change of transversal affects the associated factor set.

Lemma 6. *Let G be an extensions of H by K realizing θ and let τ, τ' be two choices of transversals with associated factor sets f and f' respectively. There exists a function $e : H \to K$ with $e(1) = 0$ such that*

$$f'(x,y) - f(x,y) = xe(y) - e(xy) + e(x)$$

Proof : Note that for any x, $\tau(x)$ and $\tau'(x)$ lie in the same coset of K. So there exists an $e(x) \in K$ such that $\tau'(x) = \tau(x) + e(x)$. Note that since $\tau(1) = \tau'(1) = 0$ we have $e(1) = 0$. Now consider the following identity

$$\begin{aligned}
\tau'(x) + \tau'(y) &= e(x) + \tau(x) + e(y) + \tau(y) && (3.1)\\
&= e(x) + xe(y) + \tau(x) + \tau(y) && (3.2)\\
&= e(x) + xe(y) + f(x,y) + \tau(xy) && (3.3)\\
&= e(x) + xe(y) + f(x,y) - e(xy) + \tau'(xy) && (3.4)
\end{aligned}$$

in which line 2 holds because G realizes θ. These equalities imply

$$\tau'(x) + \tau'(y) - \tau(xy) = f'(x,y) = e(x) + xe(y) + f(x,y) - e(xy)$$

as desired. $\square$

The preceding lemma demonstrates that two co-factors differ by some function $g(x,y) = xe(y) - e(xy) + e(x)$. We formalize this in the following definition,

Definition 14. *A Coboundary is a function $g : H \times H \to K$ such that*

$$g : (x, y) \mapsto xe(y) - e(xy) + e(x)$$

for some $e : H \to K$ with $e(1) = 0$. The set of all Coboundaries is denoted by $B^2_\theta(H, K)$

Lemma 7. *Two extensions are equivalent if their associated factor sets differ by a co-boundary.*

Proof: Since G and G' are isomorphic, identify G' with its isomorphic image G. Then by Lemma 5, we have a co-boundary g such that $f' - f = g$ $\square$

Lemma 8. *B^2_θ is an abelian group and $B^2_\theta \leq Z^2_\theta$*

Proof: For closure, let $f, g \in B^2_\theta$ and $x, y \in H$ There exist functions d and e from H to K such that $f(x, y) = xd(y) - d(xy) + d(x)$ and $g(x, y) = xe(y) - e(xy) + e(y)$ and so

$$f+g(x, y) = xd(y)-d(xy)+d(x)+xe(y)-e(xy)+e(y) = x(d+e)(y)-(d+e)(xy)+(d+e)(x).$$

Hence, $f + g \in B^2_\theta$. To show that co-boundaries are a subgroup of the co-factors, let $g \in B^2_\theta$ and let $x, y, z \in H$. Then

$$\begin{aligned}
&xg(y, z) - g(xy, z) + g(x, yz) - g(x, y) \\
=&x(ye(z) - e(yz) + e(y)) - (xye(z) - e(xyz) + e(xy)) \\
+&xe(yz) - e(xyz) + e(x)) - (xe(y) - e(xy) + e(x)) \\
=&0
\end{aligned}$$

and so $g \in Z^2_\theta$ as desired. $\square$

In the next chapter, we borrow some tools from homological algebra to build up a theory which allows us to calculate all extensions of H by K by understanding the factor sets for $H \times H$ and K. First we make a brief digression to discuss semi-direct products.

Every semi-direct product is an extension of its two factors (i.e $G = K \rtimes H$ then G an extension of H by K) The next lemma gives a way of detecting whether or not an extension is a semi-direct product.

Lemma 9. *A group extension is G of H by K is a semi-direct product iff there exists a transversal $\tau : H \to G$ that is not just a set map, but also a homomorphism.*

Proof: It is clear that every semi-direct product is an extension. Let τ be a transversal that is a homomorphism. Then

$$\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1} = \tau(h_1h_2h_2^{-1}h_1^{-1}) = \tau(1)$$

and hence

$$(k_1, h_1)(k_2, h_2) = (k_1k_2^{\tau(h_1^{-1})}\tau(h_1)\tau(h_2)\tau(h_1h_2)^{-1}, h_1h_2) = (k_1k_2^{\tau(h_1^{-1})}, h_1h_2)$$

which is the multiplication of the semi-direct product. $\square$

Chapter 4

Solving the problem with group Cohomology

4.1 Group Cohomology

4.1.1 Why study Group Cohomology

The extensions of H by K are determined by the co-factors $f : H \times H \to K$. These co-factors arose as functions such that $\tau(x)\tau(y) = \tau(xy)f(x, y)$, that is, the co-factors measure how different τ is from a homomorphism. Co-homology does something similar. Given a co-chain complex as defined below, the cohomology groups associated to the co-chain complex are a measure of the inexactness of the co-chain complex. Thus, If we can define a chain complex that captures the co-factors, we will be able to count the co-factors using cohomology. Before developing the specific theory of group cohomology, we first step back to consider cohomology in a wider context.

4.1.2 Co-Chain Complexes and General Cohomology

Definition 15. *Let $\{A^i\}_{i\in\mathbb{N}}$ be a sequence of abelian H-modules and let $\{\partial^i | \partial^i : A^i \to A^{i+1}\}$ be an associated collection of homomorphisms such that $\partial^{n+1} \circ \partial^n = 0$ for all n. We call (A^*, ∂^*) a co-chain complex.*

$$\ldots \xrightarrow{\partial^{i-2}} A^{i-1} \xrightarrow{\partial^{i-1}} A^i \xrightarrow{\partial^i} A^{i+1} \xrightarrow{\partial^{i+1}} \ldots$$

A complex is called exact if $im(\partial^{i-1}) \leq ker(\partial^i)$

These abelian modules may be algebraic objects assigned to a topological space, or a differentiable manifold, or indeed to a pair of groups.

Note that $\partial^{n+1} \circ \partial^n = 0$ implies that $image(\partial^n) \subset ker(\partial^{n+1})$ which makes the following well defined.

Definition 16. *The i^{th} Cohomology group associated to the chain complex (A^*, ∂^*) is $H^i = image(\partial^i)/\ker(\partial^{i+1})$*

4.1.3 Specializing to group cohomology

We now move from this general setting to develop a more specific cohomology theory of groups. We first construct certain abelian groups from the normal sub-group and factor group of which we hope to find the extensions then build the second cohomology group which we will prove, contains the data needed to determine the extensions of H by K.

Lemma 10. *Let H and K be groups and let F^i be the set of all functions from the i fold product of H to K. F^i is an abelian group under pointwise operations.*

Proof: Because the operations are defined point-wise, the set inherits the abelian group structure from K

Lemma 11. *The map*

$$\partial_i : F_i \to F_{i+1}$$

$$f(x_0 \ldots x_i) \mapsto x_0 f(x_1, \ldots, x_i) + \sum_{j=1}^{i} (-1)^j f(x_0, \ldots, x_{j-1}x_j, \ldots x_i) + (-1)^{i+1} f(x_0, \ldots, x_{i-1})$$

has the property that $\partial_{i+1} \circ \partial_i = 0$.

Proof: Let $f \in F^i$ and $x_1, \ldots, x_{i+1} \in H$. It follows that

$$
\begin{aligned}
&\partial^{i+1}\partial^i(f)\\
&= x_0\partial^i(f)(x_1\dots x_{i+1}) + \sum_{j=1}^{i+1}(-1)^j\partial^i f(x_0\dots x_{j-1}x_j\dots x_{i+1}) + (-1)^{i+2}\partial^i f(x_0\dots x_i)\\
&= x_0(x_1 f(x_2\dots x_{i+1}) + \sum_{j=2}^{i+1}(-1)^j f(x_1\dots x_{j-1}x_j\dots x_{i+1}) + (-1)^{i+1}f(x_1\dots x_i)\\
&+ \sum_{k=1}^{i+1}(-1)^k[x_0 f(x_1,\dots,x_{k-1}x_k,\dots x_{i+1}) + \sum_{j=1}^{k-1}(-1)^j f(x_1\dots x_{j-1}x_j,\dots x_{k-1}x_k,\dots,x_{i+1})\\
&+ \sum_{j=k+1}^{i+1}(-1)^j f(x_1\dots x_{k-1}x_k,\dots,x_{j-1}x_j,\dots x_{i+1}) + (-1)^{i+1}f(x_1,\dots,x_{k-1}x_k,\dots x_i)]\\
&+ (-1)^{i+2}(x_0 f(x_1,\dots,x_i) + \sum_{j=1}^{k}(-1)^j f(x_0,\dots,x_{j-1}x_j,\dots,x_i) + (-1)^i f(x_0\dots x_{i-1}))\\
&= x_0x_1 f(x_2\dots x_{i+1}) + \sum_{j=2}^{i+1} x_0 f(x_1,\dots,x_{j-1}x_j,\dots,x_{i+1}) + (-1)^{i+1}x_0 f(x_1,\dots,x_i)\\
&+ (-1)x_0x_1 f(x_2,\dots,x_{i+1}) + \sum_{k=2}^{i+1}(-1)^k x_0 f(x_1,\dots,x_{k-1}x_k,\dots x_{i+1})\\
&+ \sum_{k=1}^{i+1}\sum_{j=1}^{k-1}(-1)^{k+j} f(x_1\dots x_{j-1}x_j,\dots,x_{k-1}x_k,\dots x_{i+1})\\
&+ \sum_{k=1}^{i+1}\sum_{j=k+1}^{i+1}(-1)^{k+j} f(x_1,\dots x_{k-1}x_k,\dots x_{j-1}x_j,\dots x_{i+1})\\
&+ \sum_{k=1}^{i}(-1)^{k+i+1} f(x_0,\dots,x_{k-1}x_k,\dots,x_i) + (-1)f(x_0,\dots x_{i-1})\\
&+ (-1)^{i+2}x_0 f(x_1,\dots x_i) + \sum_{j=1}^{k}(-1)^{j+i+2} f(x_0,\dots,x_{j-1}x_j,\dots,x_i) + f(x_0,\dots,x_{i-1})\\
&= 0
\end{aligned}
$$

which proves the theorem.

Taken together, the last two lemmata give us that the pair (F^*, ∂^*) is a cochain complex and so can be studied with cohomology. But the reason we might study this cohomology group is not immediately clear.

Lemma 12. *$image(\partial_1) = Z^2_\theta(H, K)$ and $ker(\partial_2) = B^2_\theta(H, K)$*

Theorem 4.1.1. *There exists a bijection between $Ext(H, K)$ and the second cohomology group $H^2_\theta(H, K)$*

Proof : Let $B^2_\theta(H, K) = B$ and let $[G_f]$ be the extension realizing the factor set f . Define

$$\phi : H^2_\theta(H, K) \to Ext(H, K)$$

$$(f + B) \mapsto [G_f]$$

ϕ is well defined, as if f and f' lie in the same coset, then by Lemma 8 their associated extensions are equivalent. For injectivity, note that if $f + B$ and $f' + B$ map to to equivalent extensions then they differ by a coboundary and so are the same element in the quotient set. For surjectivity, suppose that G represents some equivalence class of extensions. Then there exists a cofactor f of G. So $\phi(f + B) = [G]$ as desired. □

The second cohomology group is in bijection with the set of co-factors and so determines the number of extensions of K by H. In the case where K is elementary abelian, cohomology turns difficult and complicated group theory into linear algebra over a finite field, standard operations for a computer algebra system.

4.1.4 An example

We stop here to make concrete the calculation of group extensions by considering the cohomology group associated to the groups C_2 and V_4. That is, we want to calculate the groups G of order eight with normal subgroup C_2 such that $G/C_2 \simeq V_4$. As an example consider $D_8 = \langle r, s | r^4, s^2, rsrs \rangle$ with normal subgroup $\langle r^2 \rangle$ of order two. Note that by the fourth isomorphism theorem, $D_8/C_2 \simeq V_4$ so D_8 is such an extension. We seek here to enumerate all such extensions using cohomology. Since the second cohomology group associated to C_2 and V_4 is in bijection with the Extensions we want, we then seek to calculate it by calculating the image and kernel of $\partial^2 : V_4 \times V_4 \to C_2$ by looking for restrictions imposed on functions $f : V_4 \times V_4 \to C_2$

by the identities from chapter two. Namely we require a cofactor f to be such that

$$f(1,y) = f(x,1) = 0 \tag{4.1}$$

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = 0. \tag{4.2}$$

Note that because C_2 is a field, the set of all functions into C_2 is a vector space and so we can leverage the tools of linear algebra to help solve this problem.

We make a choice of coordinates for $V_4 = \langle a, b | a^2, b^2, (ab)^2, abab \rangle = \{1, a, b, ab\}$. Equation (5) tells us that we need not worry about pairs of the form $(1, x)$ of $(x, 1)$. Hence, under the assumption that $f(a, b) = 0$ (an assumption we can make because co-factors are determined by a choice of coset representatives), a sensible choice of coordinates for $V_4 \times V_4$ is :

$$\begin{aligned} f(a,a) &:= v_1 & f(a,ab) &:= v_2 \\ f(b,a) &:= v_3 & f(b,b) &:= v_4 \\ f(b,ab) &:= v_5 & f(ab,a) &:= v_6 \\ f(ab,b) &:= v_7 & f(ab,ab) &:= v_8 \end{aligned}$$

Now consider that the action of V_4 on C_2 is trivial since C_2 has trivial automorphism group, and as (written additively) $-1 = 1$ so in this particular example we have,

$$xf(y,z) - f(xy,z) + f(x,yz) - f(x,y) = f(y,z) + f(xy,z) + f(x,yz) + f(x,y).$$

We check the condition for all triples x, y, z in V_4. I give here the first few calculations to demonstrate the process. for $x = a$, $y = b$, and $z = ab$ we have that

$$f(b,ab) + f(ab,ab) + f(a,bab) + f(a,b) = v_5 + v_8 + v_1 = 0$$

for $x = b$, $y = a$ $z = ab$ we have

$$f(a, ab) + f(ba, ab) + f(b, aab) + f(b, a) = v_2 + v_8 + v_4 + v_3 = 0$$

using similar processes, we have

$$\begin{aligned} v_1 + v_4 + v_5 + v_6 = 0, &\;\; v_1 + v_2 + v_4 + v_7 = 0, \\ v_4 + v_6 + v_8 = 0, &\;\; v_3 + v_1 + v_8 + v_7 = 0, \\ v_1 + v_2 = 0 & \end{aligned}$$

Hence the co-factors are the functions spanned by the right null space of the following matrix over $\mathbb{Z}/2\mathbb{Z}$

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

Using GAP, We find that A has nullspace space of dimension three.

Next, we want to find a basis for the space spanned by the set of all the 2 co-boundaries so that we can calculate the quotient space.

Recall that the 2 co-boundaries are the functions $g : Q \times Q \to K$ such that $g(x, y) = xe(y) - e(xy) + e(x)$ for $e : Q \to K$, and hence in this example we are looking for the functions $g : V_4 \times V_4 \to C_2$ with the property above (note that since the action is trivial, we can ignore it) . To find these, consider the functions $e : V_4 \to C_2$. Since $\tau(1) = 1$ we have $e(1) = 0$ a good basis for the space of all functions from V_4 to C_2 is functions e_a, e_b , and e_{ab} that map their corresponding element to 1 and the rest to 0. Recall our assumption that all transversals τ have the property, $\tau(a)\tau(b) = \tau(ab)$. This implies that any function $e : V_4 \to C_2$ such that for transversals τ and τ', $\tau'(x) = \tau(x)e(x)$ as,

$$\begin{aligned}\tau'(ab) &= \tau'(a)\tau'(b)\\ \tau(ab)e(ab) &= \tau(a)e(a)\tau(b)e(b)\\ e(ab) &= e(a)e(b).\end{aligned}$$

Hence, $\partial e_{ab} = 0$. Further, one can check that these assumptions imply that $\partial e_a = \partial e_b = 0$. Thus the co-boundaries are a dimension 0 vector space.

Since the Co-factors are of dimension 3 and the co-boundaries are of dimension 0, over $\mathbb{Z}/2\mathbb{Z}$, the second Co-homology group is of order 8. Thus there are 8 extensions of V_4 by C_2 . Yet there are only 5 groups of order eight up to isomorphism. Surely C_8 is not such an extension as the normal sub-group and factor group are both cyclic. Note here that the three non-identity elements in V_4 are all have the same order, and so can be switched one for another. Thus, in the extension D_8, the identity coset can be assigned to any of the three elements and so there are three non-equivalent copies of D_8 among the extensions. A similar argument holds for $C_4 \times C_2$. combine with C_2^3 and Q_8 and that makes eight total extensions.

4.2 Conclusions

The example at the end of the last chapter highlights some difficulties in computing extensions with cohomology. First of all, to run over all of the triples in the group to calculate the cohomology. This process is computationally expensive, at best $O(n^3)$. This algorithm only calculates the cohomology group, but what one would really like is the isomorphism classes of the extensions. The ad-hoc arguments presented at the end of the example used to determine isomorphism classes does not translate the groups of order even slightly larger than explored here. One most rely on isomorphism tests. Thus the problem is expensive to solve in general and future work in the area should be concerned with reducing the cost of calculation.

Bibliography

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