ANALYTICAL STUDY OF THE MECHANICS OF SCOUR FOR THREE-DIMENSIONAL JET

by

Yuichi Iwagaki

George L. Smith

Maurice L. Albertson

Prepared for

U. S. Bureau of Public Roads

under contract CPR 11-5504

(Presented at

ASCE Hydraulics Conference

Atlanta, Georgia

August 20, 1958)

Colorado State University Research Foundation

Fort Collins, Colorado

ANALYTICAL STUDY OF THE MECHANICS OF SCOUR FOR THREE-DIMENSIONAL JET

by

Yuichi Iwagaki

George L. Smith

Maurice L. Albertson

Prepared for

U. S. Bureau of Public Roads

under contract CPR 11-5504

(Presented at

ASCE Hydraulics Conference

Atlanta, Georgia

August 20, 1958)

Colorado State University Research Foundation

Fort Collins, Colorado

ABSTRACT

A theory of scour for a three-dimensional jet impinging on an erodible bed is developed and applied to the analysis of experimental data obtained in the Hydraulic Laboratory, Colorado State University.

SYNOPSIS

Using the polar coordinate system the equation of continuity of mass sediment transport has been derived, and the phenomenon of scour is then described mathematically by the continuity equation. The relationship between the shape of the scour hole and its variation with time is assumed for different conditions of scour and deposition. Expressions for distribution of sediment transport along the bed are derived for each condition.

The impingement of a three-dimensional jet on a normal boundary is analyzed by making the assumption that the Bernoulli Equation is valid in the neighborhood of the stagnation point.. The ideal jet flow, with uniform velocity distribution and without diffusion, is considered first, followed by real jet flow with non-uniform velocity distribution and vertical diffusion. For conditions of submerged and non-submerged outlets, expressions for the horizontal velocity and shear distribution along the boundary are developed using Bernoulli's theorem and the boundary layer theory.

The variation of the depth of scour is determined for two conditions of outlets by assuming a law of open channel flow for sediment transportation and by using the previously determined shear distributions and the continuity equation. In particular, the variation of scour depth with respect to time and the final depth of scour are described theoretically in terms of dimensionless parameters. It is then shown that the development of the scour hole with respect to time follows the power and logarithmic laws for the submerged and non-submerged outlets respectively before the final state is reached.

The influence of the angle of the jet is analyzed in the same manner as in the case of a vertical jet. In this case, expressions for the variations of the depth of scour with respect to time and the final depth of scour are also developed.

Results of the theoretical analysis are applied to the experimental data for the cases of vertical and horizontal non-submerged outlets obtained by tests conducted and being conducted in the Hydraulics Laboratory of Colorado State University. Three regimes of scour, maximum and minimum jet deflections defined by H. Rouse (1), and a final state are determined for the case of the horizontal non-submerged outlet.

INTRODUCTION

There is no essential difference between two- and three-dimensional cases of the scour phenomena due to jets. This is proven in the analysis of scour by a two-dimensional jet (2). The existence of the regimes of maximum and minimum jet deflections (2) and a final state of scour (3) will be expected also for the case of the three-dimensional jet. The problem lies only in the extension of the mathematical treatment to the cylindrical coordinate system. Since experiments of scour for the three-dimensional jet have been conducted (4) and are now being continued (5), the developed theory will be verified by application to experimental data.

EQUATION OF CONTINUITY OF MASS SEDIMENT TRANSPORT

Theoretical Development of the Continuity Equation

To determine the equation of continuity of mass sediment transport, the quantity of sediment transported through the infinitesimal element ABCD, Fig. 1, defined by arcs $\widehat{AB}$ and $\widehat{CD}$ and radial lines $\overline{AC}$ and $\overline{BD}$ may be considered.

The quantity of sediment transported through arcs $\widehat{AB}$ and $\widehat{CD}$ per unit of time is given by

$$\widehat{AB} : \quad q_s r\, d\psi$$
$$\widehat{CD} : \quad q_s r\, d\psi + \frac{\partial}{\partial r}(q_s r\, d\psi)\, dr$$

in which q_s is the mass rate of sediment transport per unit width from $Z = 0$ to $Z = \infty$.

The quantity of sediment scoured by the jet per unit of time can be expressed by

$$-\frac{\partial}{\partial t}(r\, d\psi\, dr\, z)(1-\lambda)$$

in which λ is the porosity of the sediment.

The equation of continuity of mass sediment transport in terms of cylindrical coordinates (see Fig. 1) is, therefore, expressed by

$$(1-\lambda)\, r \frac{\partial z}{\partial t} + \frac{\partial}{\partial r}(q_s r) = 0 \tag{1}$$

Conditions of Application of the Continuity Equation

The relation between the shape of the scour hole and the distribution of the scoured sediment will now be considered for the following

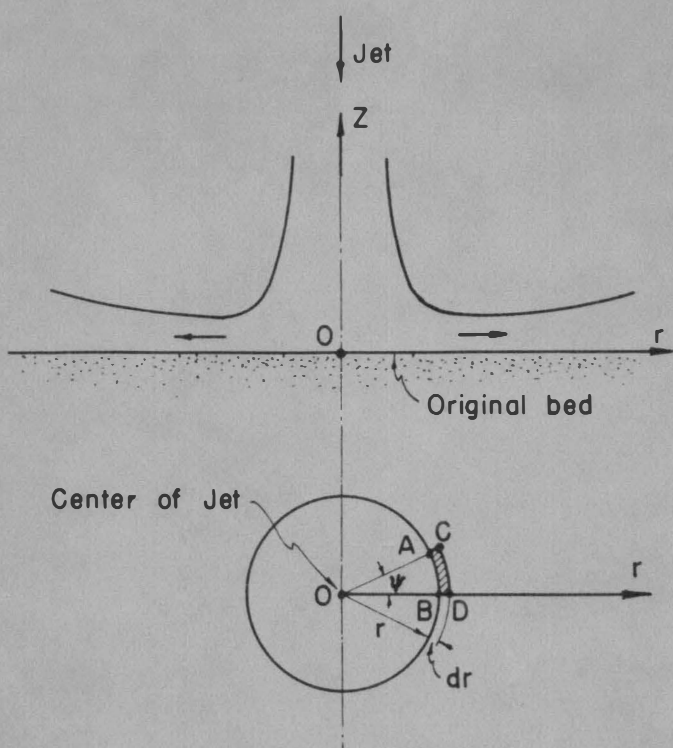

Fig. 1 Cylindrical coordinates for theoretical development of the continuity equation

conditions:

a. The condition of $\partial Z/\partial t = 0$ (No scour)

For this condition Eq 1 becomes

$$\frac{dq_s}{dr} + q_s/r = 0 \tag{2}$$

Integrating Eq 2 gives

$$q_s = \frac{C_1}{r} \tag{3}$$

in which C_1, is the integral constant.

b. The condition of $\partial Z/\partial t = -C(t)$, in which $C(t) > 0$

(Uniform scour)

From Eq 1, the continuity equation for this condition is

$$\frac{d(q_s r)}{dr} = (1-\lambda) C r \tag{4}$$

Integrating Eq 4 gives

$$q_s = \frac{(1-\lambda) C r}{2} + \frac{C_2}{r} \tag{5}$$

in which C_2 is the integral constant.

c. The condition of $\partial Z/\partial t = C(t)$, in which $C(t) > 0$

(Uniform deposition).

The solution for this condition is obtained by changing the sign of C in Eq 5, or

$$q_s = \frac{C_3}{r} - \frac{(1-\lambda) C r}{2} \tag{6}$$

in which C_3 is the integral constant.

d. The condition of $\partial Z=\partial t = -(A_1-B_1 r)$, in which A_1 and B_1 are functions of t and positive. (Linear scour)

From Eq 1

$$\frac{d(q_s \tau)}{d\tau} = (1-\lambda)(A_1 - B_1\tau)\tau \tag{7}$$

The solution of Eq 7 is

$$q_s = (1-\lambda)\left(\frac{A_1}{2} - \frac{B_1\tau}{3}\right)\tau + \frac{C_4}{\tau} \tag{8}$$

in which C_4 is the integral constant.

e. The condition of $\partial Z/\partial t = B_1\tau - A_1$, in which A_1 and B_1 are funtions of t and positive. (Linear deposition)

The equation for this condition is the same as for condition (d) defined by Eq 8.

f. The condition in which conditions defined by sections b, d, and e are combined as follows:

$\partial Z/\partial t = -C(t)$ for $\tau < \tau_0$ (Uniform scour)

$\partial Z/\partial t = -(A_1 - B_1 r)$ for $\tau_0 < \tau < A_1/B_1$ (Linear scour)

$\partial Z/\partial t = B_1 r - A_1$ for $\tau > A_1/B_1$ (Linear deposition)

The relations given q_s in this case are obtained by combining the equations determined for the conditions of sections b, d, and e; that is

$$\begin{aligned} q_s &= \frac{(1-\lambda)C\tau}{2} + \frac{C_2}{\tau}, \quad \text{for } \tau < \tau_0 \\ q_s &= (1-\lambda)\left(\frac{A_1}{2} - \frac{B_1\tau}{3}\right)\tau + \frac{C_4}{\tau}, \quad \text{for } \tau_0 < \tau < \frac{A_1}{B_1} \\ q_s &= (1-\lambda)\left(\frac{A_1}{2} - \frac{B_1\tau}{3}\right)\tau + \frac{C_4}{\tau}; \quad \text{for } \tau > \frac{A_1}{B_1} \end{aligned} \tag{9}$$

The distribution of sediment transport for conditions a through f are illustrated schematically in Figs. 2, 3, 4, 5 and 6. The direction of sediment movement is in the direction of the arrow on each figure.

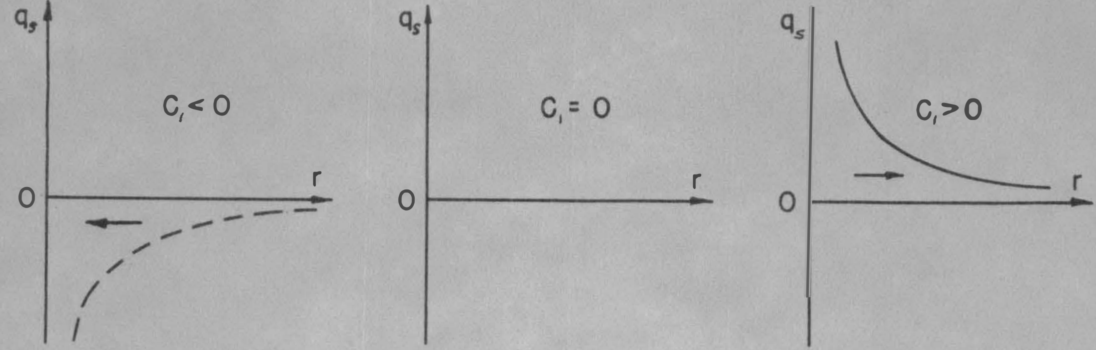

Fig. 2 q_s-distributions in the case of no scour (Eq. 3)

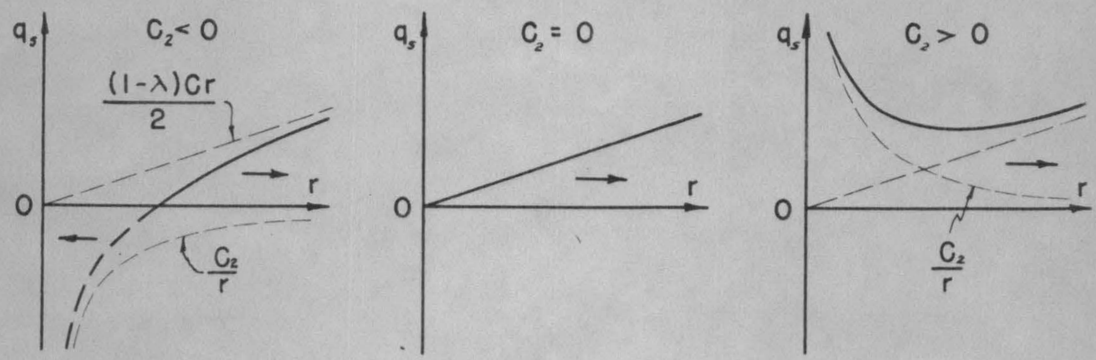

Fig. 3 q_s-distributions in the case of uniform scour (Eq. 5)

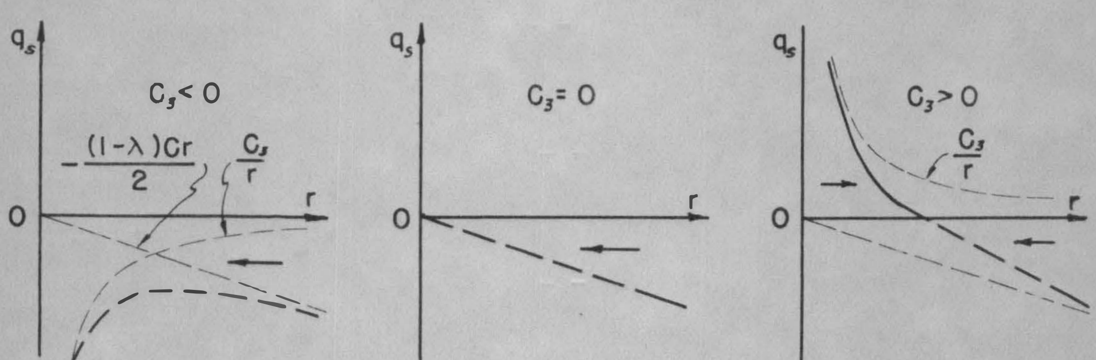

Fig. 4 q_s- distributions in the case of uniform deposition (Eq. 6)

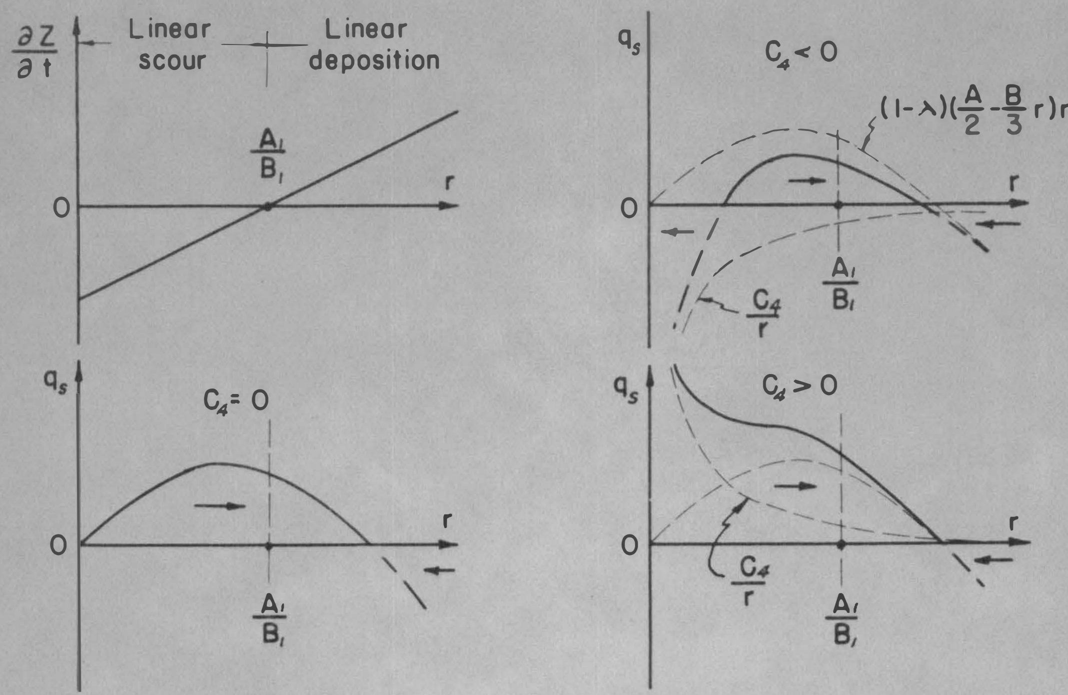

Fig. 5 q_s-and $\partial Z/\partial t$-distributions in the cases of linear scour and linear deposition (Eq. 8)

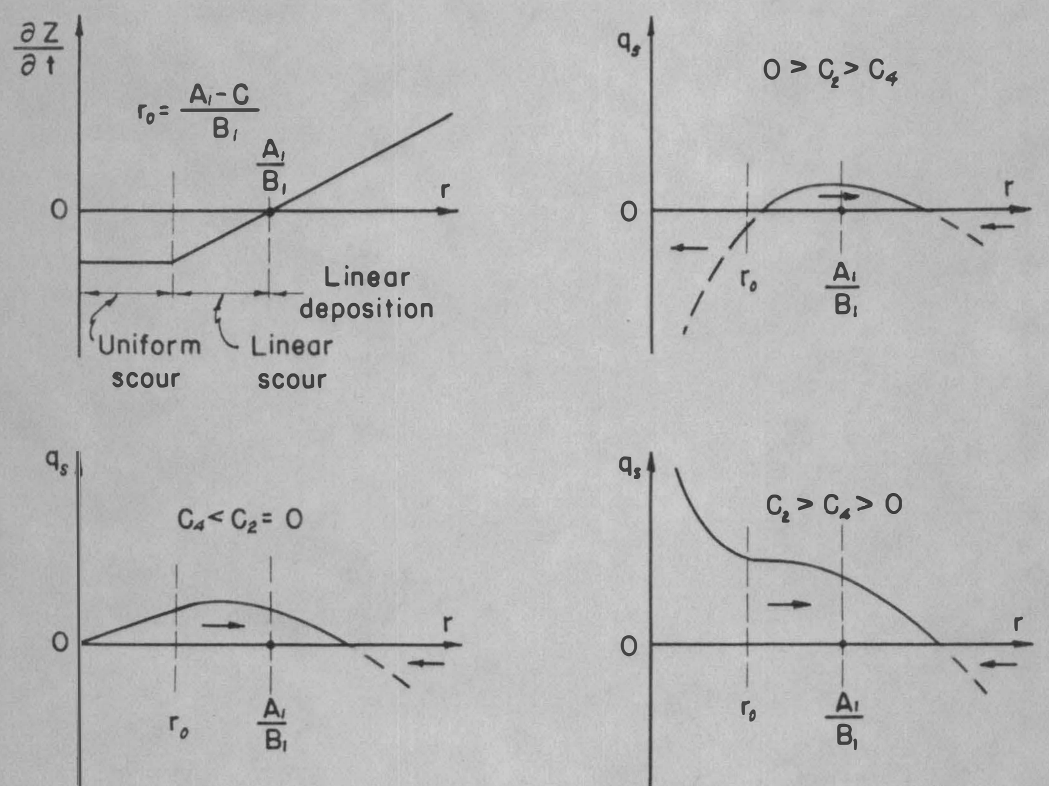

Fig. 6 q_s-and $\partial Z/\partial t$-distributions in the combination of uniform and linear scours and linear deposition

The foregoing mathematical analysis has considered both the physically possible and the physically impossible cases of scour, deposition, and scour and deposition. Therefore, it is essential that the physically possible cases, which have a bearing on the analysis of scour as presented in this report, be identified.

First, since q_s is a function of C_n/r, the origin must be excluded as a zone of consideration; for, if the constant of integration has a finite value $C_n > 0$, then C_n/r has a value of infinity at the origin. However, it is possible to circumvent this discrepancy by assuming that in the vicinity of the stagnation point ($r = 0$) the constant C_n changes rapidly from 0 at $r = 0$ to C_n at $r = r_o$. This assumption is valid since the constant of integration denotes the initial sediment discharge carried into the zone under consideration by the incoming flow. Due to the finite width of the jet, the discharge cannot be introduced at a single point. Furthermore, since the origin is a point of stagnation, the sediment discharge at the origin must be zero since the velocity is zero.

Second, since the constant of integration denotes the initial sediment discharge, it can never become negative as long as the jet is impinging on the boundary. Therefore, all of the foregoing cases in which the direction of sediment movement is toward the origin, $C_n \leq 0$, are not relevant to the analysis of this report.

Lastly, the sediment discharge cannot change sign, that is, as shown in Fig. 4 for the case $C_3 > 0$, only that quantity of sediment contained by the flow can be deposited. Also, if the initial discharge is zero, no sediment can be deposited unless sediment has been scoured from the region near the origin.

IMPINGEMENT OF A THREE-DIMENSIONAL JET ON A NORMAL BOUNDARY

Ideal Fluid:

The impingement of a three-dimensional jet on a non-erodible bed covered by a tailwater of depth b is considered for the case of ideal flow, that is, flow without diffusion or boundary shear (see Fig. 7).

For this case, Bernoulli's theorem applied between points M and N in Fig. 7 gives:

$$\frac{W_o^2}{2g} + Z_M + [b - Z_M] = \frac{U_o^2}{2g} + Z_N + (b - Z_N)$$

in which Wo is the velocity of jet at a distance Z_M from the boundary and Uo the velocity in the deflected jet at a point N located a great distance from the stagnation point O and a distance Z_N from the boundary. Bernoulli's equation shows that

$$Wo = Uo \tag{10}$$

Furthermore, from the continuity equation of water mass flux

$$\frac{\pi}{4} d^2 W_o = 2\pi r h U_o$$

and Eq 10, the following relation for the thickness of the deflected jet h is derived:

$$\frac{h}{d} = \frac{1}{8}\frac{d}{r} \tag{11}$$

The relationship expressed by Eq 11 is valid only in the region where the streamlines are parallel to the boundary.

The flow within the stagnation region can be closely approximated by three-dimensional, axisymmetrical, potential flow with stagnation. For this flow, the vertical and horizontal velocity components W and U can be expressed, according to Schlichting (6), by

$$U = ar \tag{12}$$

$$W = -2a\,Z \tag{13}$$

in which a is a constant. Equations 12 and 13 satisfy the equation of continuity

$$\frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial Z} = 0 \tag{14}$$

For ideal flow, Eq 10 expresses the velocity distribution along the boundary for the case where the streamlines are essentially parallel to the bed; whereas, Eq 12 expresses the velocity distribution near the stagnation point. Furthermore, by assuming a smooth transition between the two cases, the velocity distribution along the boundary would be as shown in Fig. 8.

Real Flow: Non-submerged Outlet:

For real flow, the analysis of the flow conditions is more complex. The jet impinging upon a normal, plane bed covered by a tailwater of depth b results in the development of two types of flow phenomena. One is the development of a boundary layer along the bed through boundary shear, and the other is the jet diffusion, or more specifically, the development of kinetic energy turbulence through interaction between the jet and the tailwater

For an analysis of the development of the boundary layer, use will be made of the fundamental ideas regarding boundary layer development on a rotating disk put forth by Schlichting and Truckenbrodt (7,8). In their study the assumption was made that the velocity distribution in the deflected jet is uniform

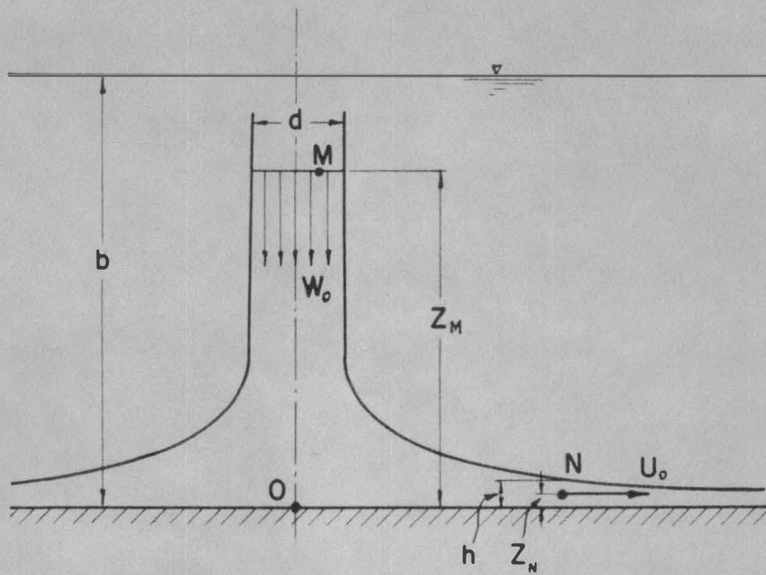

Fig. 7 Schematic drawing of impingement of the jet in ideal fluid

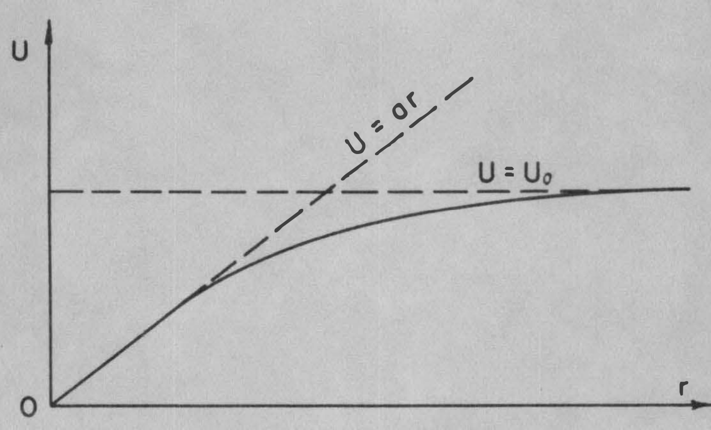

Fig. 8 Distribution of component velocity U along the bottom.

outside of the boundary layer. On the basis of this assumption, they found that the thickness of the boundary layer δ and the shear velocity U_* in the neighborhood of the stagnation point may be expressed as follows:

For laminar boundary layer,

$$\sqrt{\frac{a}{\nu}}\,\delta = \text{constant} \tag{15}$$

$$\frac{U_*^2}{\sqrt{\nu_a}\,U} = 1.312 \tag{16}$$

and for turbulent boundary layer and smooth boundary,

$$\frac{\delta}{r} = 0.086\left(\frac{a r^2}{\nu}\right)^{-\frac{1}{5}} \tag{17}$$

$$\frac{U_*^2}{U^2} = 0.0414\left(\frac{a r^2}{\nu}\right)^{-\frac{1}{5}} \tag{18}$$

That is, U_*^2 is proportional to r in the laminar boundary layer and to $r^{8/5}$ in the turbulent boundary layer.

For an analysis of jet diffusion for the case of a pipe outlet at a fixed distance above the water surface, use will be made of the experiment investigation by Homma (9), who found that the velocity distribution along the center-line of jet can be expressed as follows:

$$\frac{W_m}{W_o} = 1.24\, e^{-0.109\left(\frac{b-Z}{d}\right)} \quad \text{for } Re < 2.5 \times 10^4 \tag{19}$$

$$\frac{W_m}{W_o} = 1.24\, e^{-0.137\left(\frac{b-Z}{d}\right)} \quad \text{for } Re > 3 \times 10^4 \tag{20}$$

in which W_m is the maximum velocity at the center of the diffused jet, d the diameter of jet as shown in Fig. 9, and Re the Reynolds number of jet flow.

Since an expression for the velocity distribution across the jet in the case of a non-submerged outlet, has not been developed; the following general expression, as assumed for the two-dimensional case (2), is introduced herein:

$$\frac{W}{W_m} = \phi_1\left(\frac{b}{d}, \frac{r}{b}, \frac{b-Z}{d}\right) \tag{21}$$

in which W is the vertical velocity component whose distribution across the normal section of any particular zone of flow can be approximated by the Gaussian normal probability function (see Fig. 9).

At the deflecting boundary Z = 0 and the variables defining the mean flow pattern are given by the relationship

$$\frac{W_b}{W_{bm}} = \phi_2\left(\frac{b}{d}; \frac{r}{b}\right) = \phi_0 \tag{22}$$

in which W_{bm} is the maximum--centerline-- velocity and W_b is the previously defined vertical velocity component.

In order to approximate the velocity distribution along the deflecting boundary, use will be made of the results of the experimental study by Homma (9). On the basis of the experimental results, the pressure distribution on the boundary in the proximity of the stagnation point is given by

$$p = \frac{\rho}{2} W_b^2 + \rho g b \tag{22a}$$

and for the stagnation point is given by

$$p_0 = \frac{\rho}{2} W_{bm}^2 + \rho g b \tag{22b}$$

If it is assumed that the vertical centerline of the jet and the upper surface of the boundary layer form a streamline along which Bernoulli's equation is valid, then the following relationship is obtained

$$\frac{\rho}{2}(U^2 + W^2) + \rho g Z = p_0 - p \tag{23}$$

Along the r-axis at the upper surface of the boundary layer Z = $\delta \approx 0$ and $W \approx 0$; thus, the horizontal velocity component U_b above the boundary layer is given by

$$\frac{\rho}{2} U_b^2 = p_0 - p \tag{24}$$

Substituting the expressions for p and p_o given by Eqs 22a and 22b in Eq 24 obtain

$$\frac{U_b}{W_{bm}} = \left\{ 1 - \left(\frac{W_b}{W_{bm}} \right)^2 \right\}^{1/2} \tag{25a}$$

Substituting Eq 22 into Eq 25a gives

$$\frac{U_b}{W_{bm}} = \left(1 - \phi_2^{\,2} \right)^{1/2} \tag{25b}$$

The relation given by Eq 25b is valid only in the proximity of the stagnation point, because the pressure rapidly approaches the constant pressure ρgb as the flow moves away from the point of stagnation. For a constant pressure ρgb, Eq 22a gives $W_b = 0$ and Eq 25a gives $U_b = W_{bm}$ = const., which, of course, does not agree with actual flow conditions.

An approximation of the boundary layer and the shear velocity along the boundary can be made, however, but use of Eq 25b and the momentum equation given by Truckenbrodt (8), which is as follows:

$$r U_*^2 = \frac{d}{dr} \left\{ r \delta U_b^2 \int_0^1 \frac{u}{U_b} \left(1 - \frac{u}{U_b} \right) ds \right\} + r \delta U_b \frac{dU_b}{dr} \int_0^1 \left(1 - \frac{u}{U_b} \right) ds \tag{26}$$

in which $s = Z/\delta$ and u is the horizontal velocity component in the boundary layer of thickness δ as shown in Fig. 10

Assume that for a smooth boundary the velocity profile in the boundary layer can be expressed by

$$\frac{u}{U_b} = f\left(\frac{z}{\delta} \right) = s^{m'} \tag{27}$$

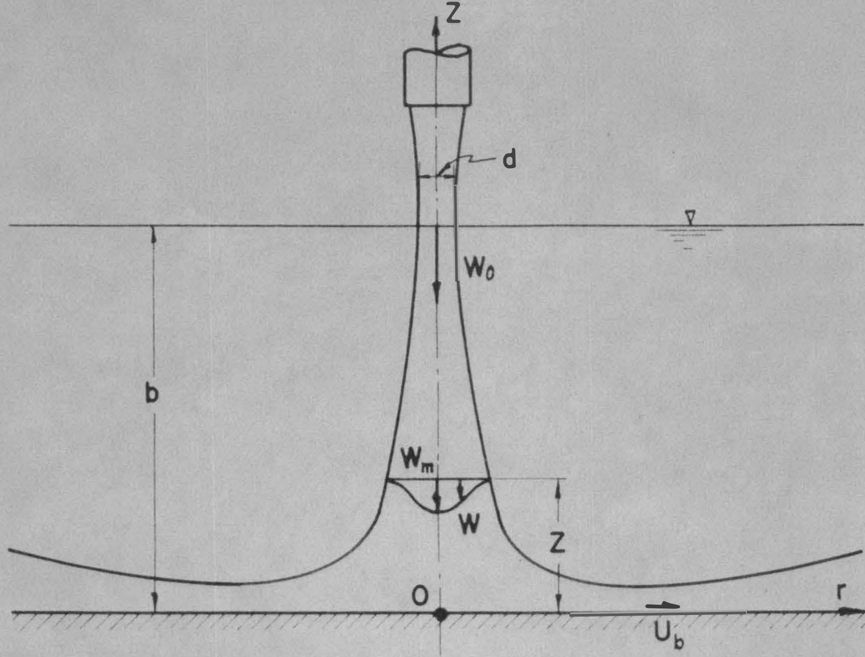

Fig. 9 Schematic drawing of impingement of the jet issuing from a non-submerged outlet on a plane bed

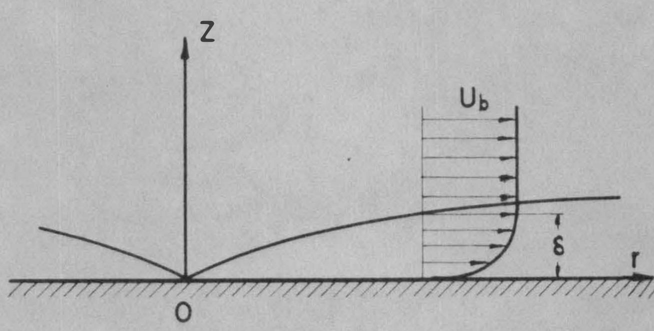

Fig 10 Boundary layer along the bottom

and the shear velocity by

$$\frac{U_*^2}{U_b^2} = \frac{k}{\left(\frac{U_b \delta}{\nu}\right)^n} \qquad (28)$$

in which

$$n = \frac{2m'}{1+m'} \; ; \text{ and } \; n = \frac{1}{4} \qquad (29);\ (30)$$

when k = 0.0225 and m′ = 1/7.

Substitution Eqs 27 and 28 into Eq 26 gives

$$\frac{d}{d\tau}(A\tau\delta U_b^2) + B\tau\delta U_b \frac{dU_b}{d\tau} = \tau \frac{U_b^2}{\left(\frac{U_b\delta}{\nu}\right)^n} \qquad (31)$$

in which

$$A = \int_0^1 s^{m'}(1-s^{m'})\,ds \qquad (32)$$

$$B = \int_0^1 (1-s^{m'})\,ds \qquad (33)$$

Letting

$$\frac{\delta}{\tau} = \delta_* \qquad (34)$$

Eq 31 becomes

$$2A\delta_*^{n+1} + A\tau\delta_*^n \frac{d\delta_*}{d\tau} + \delta_*^{n+1} L\left(A + \frac{B}{2}\right) = \frac{k}{\left(\frac{U_b\tau}{\nu}\right)^n} \qquad (35)$$

in which

$$L = \frac{\tau}{U_b^2}\,\frac{dU_b^2}{d\tau} \qquad (36)$$

Eq 35 can be solved only for the condition that L is constant, that is, when $U_b = C_5 r^{m''}$, in which C_5 and m″ are constants. Therefore, assuming L constant, the solution of Eq 35 may be written as

$$\delta_* = \left(\frac{U_b\tau}{\nu}\right)^{-\frac{n}{1+n}} \beta' \qquad (37)$$

The value of β' is obtained by substituting Eq 37 into Eq 35, or

$$\beta' = \left[\frac{K}{2A - A\left(\frac{n}{n+1}\right) + L\left\{A + \frac{B}{2} - \frac{A}{2}\left(\frac{n}{n+1}\right)\right\}} \right]^{\frac{1}{n+1}} \tag{38}$$

Furthermore, substitution of Eq 37 into Eq 28 gives

$$\frac{U_*^2}{U_b^2} = \frac{k}{\left(\frac{U_b r}{\nu}\right)^n \delta_*^n} = k\left(\frac{U_b r}{\nu}\right)^{-\frac{n}{n+1}} \beta'^{-n} \tag{39}$$

For $n = 1/4$, Eqs 37 and 39 can be written

$$\frac{\delta}{r} = \delta_* = \left(\frac{U_b r}{\nu}\right)^{-\frac{1}{5}} \beta' \tag{40}$$

$$\frac{U_*^2}{U_b^2} = k\, \beta'^{-\frac{1}{4}} \left(\frac{U_b r}{\nu}\right)^{-\frac{1}{5}} \tag{41}$$

Substituting Eq 25b into Eq 41, the following expression for the shear velocity distribution can be obtained:

$$\frac{U_*^2}{W_{bm}^2} = k\, \beta'^{-\frac{1}{4}} \left(\frac{W_{bm}\, b}{\nu}\right)^{-\frac{1}{5}} \left(\frac{r}{b}\right)^{-\frac{1}{5}} \left(1 - \phi_o^2\right)^{\frac{9}{10}} \tag{42}$$

<u>Real Fluid: Jet Issuing From a Submerged Outlet.</u>

Albertson and others (10) determined expressions for the longitudinal distribution of the velocity Wm as well as the transverse distribution of the vertical velocity W along the centerline of a diffusing jet. These expressions adopted herein are

$$\frac{W_m}{W_o} = 6.2\, \frac{d}{b - z} \tag{43}$$

$$\frac{W}{W_m} = e^{-76 \frac{r^2}{(b - z)^2}} \tag{44}$$

in which b is the distance from the outlet to the bottom, and d and Wo are the diameter and the velocity of jet at the outlet respectively as shown in Fig. 11. In particular, along the r-axis where $z = 0$,

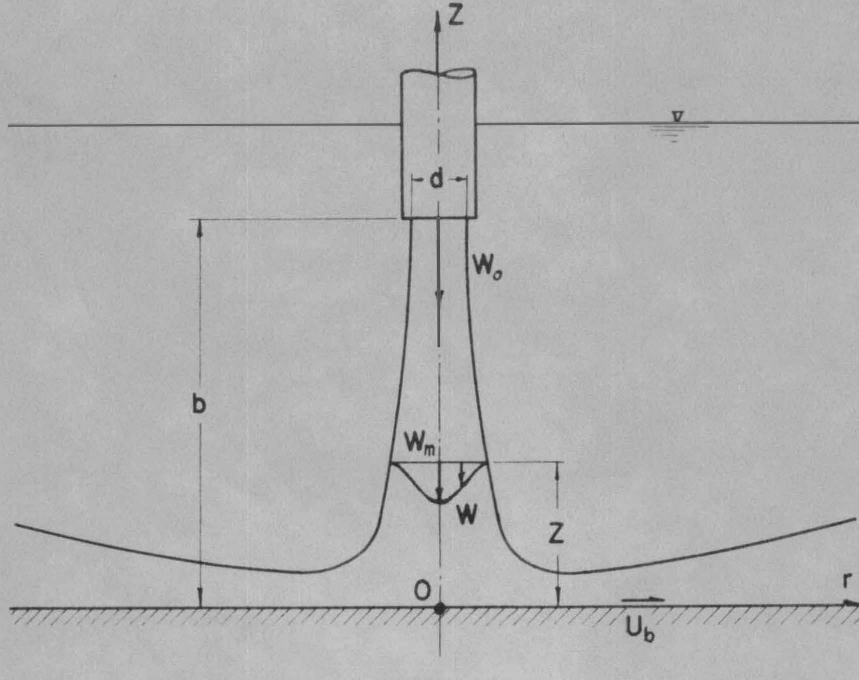

Fig. 11 Schematic drawing of impingement of the jet issuing from a submerged outlet on a plane bed

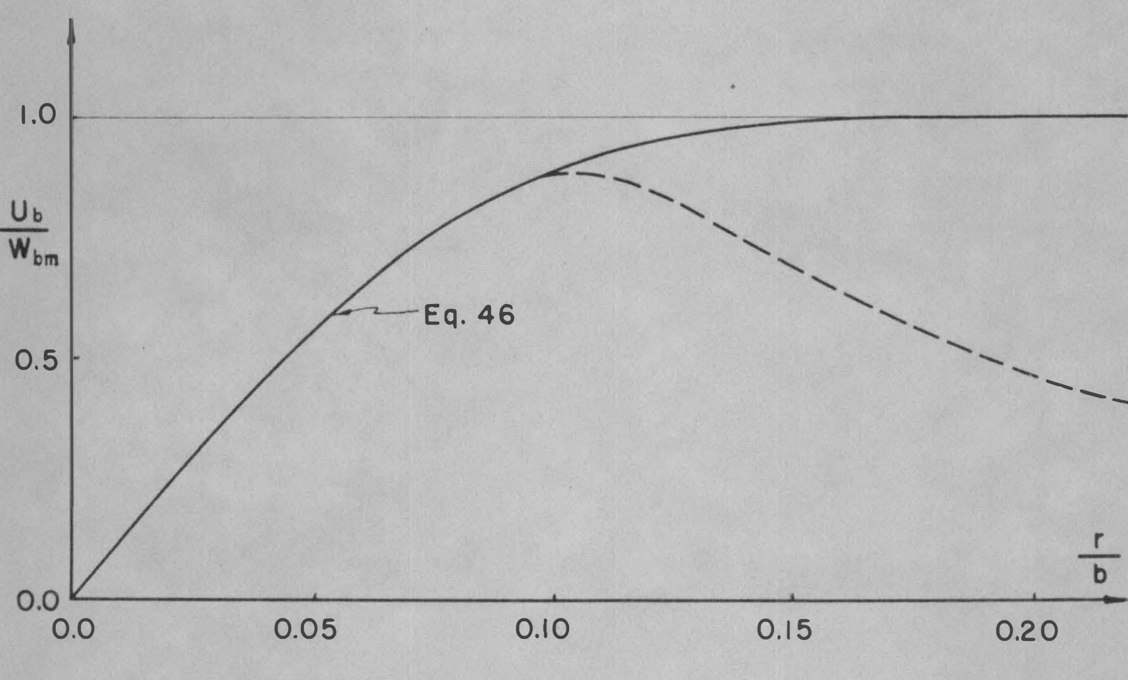

Fig. 12 Horizontal velocity distribution along the plane bed

Eq 44 becomes

$$\frac{W_b}{W_{bm}} = e^{-76\left(\frac{r}{b}\right)^2} \tag{45}$$

Using a similar approach to the one used for the case of a non-submerged outlet, the horizontal velocity distribution near the bottom corresponding to Eq 25b can be derived, or

$$\frac{U_b}{W_{bm}} = \left[1 - e^{-152\left(\frac{r}{b}\right)^2}\right]^{\frac{1}{2}} \tag{46}$$

The horizontal velocity distribution given by Eq 46 is shown by the full line in Fig. 12 and is assumed to be correct in the proximity of the stagnation point. The dotted line indicates a more probable velocity distribution when the lateral diffusion is taken into account.

The method for deriving the boundary layer thickness and shear velocity used in the case of a non-submerged outlet is also valid in this case. Similar results are found as follows:

$$\frac{\delta}{r} = \delta_* = \left(\frac{U_b r}{\nu}\right)^{-\frac{1}{5}} \beta' \tag{47}$$

$$\frac{U_*^2}{W_{bm}^2} = k\beta'^{-\frac{1}{4}}\left(\frac{W_{bm} b}{\nu}\right)^{-\frac{1}{5}}\left(\frac{r}{b}\right)^{-\frac{1}{5}}\left(1 - e^{-152\left(\frac{r}{b}\right)^2}\right)^{\frac{9}{10}} \tag{48}$$

in which β' is expressed by Eq 38. However, L in Eq 38 defined by Eq 36 can be calculated by using Eq 46, or

$$L = \frac{r}{U_b^2}\frac{dU_b^2}{dr} = 304\left(\frac{r}{b}\right)^2 e^{-152\left(\frac{r}{b}\right)^2}\left(1 - e^{-152\left(\frac{r}{b}\right)^2}\right)^{-1} \tag{49}$$

The variation of L with r/b is given in Fig. 13, which shows that, in the vicinity of the stagnation point, L is approximately constant and equal to 2; whereas for values of $\frac{r}{b} > 0.2$, L tends rapidly to zero. Eqs 47 and 48 were derived under the assumption L = constant, in the neighborhood of the stagnation point at least this assumption seems valid.

Fig. 14 represents the shear velocity distribution given by Eq 48. In the computation of $K\beta'^{-\frac{1}{4}}$ in Eq 48, it has been assumed that $m' = 1/7$ (therefore, $A = 7/72$, $B = 1/8$ and n= 1=4 from Eqs 32, 33 and 30) and $k = 0.0225$.

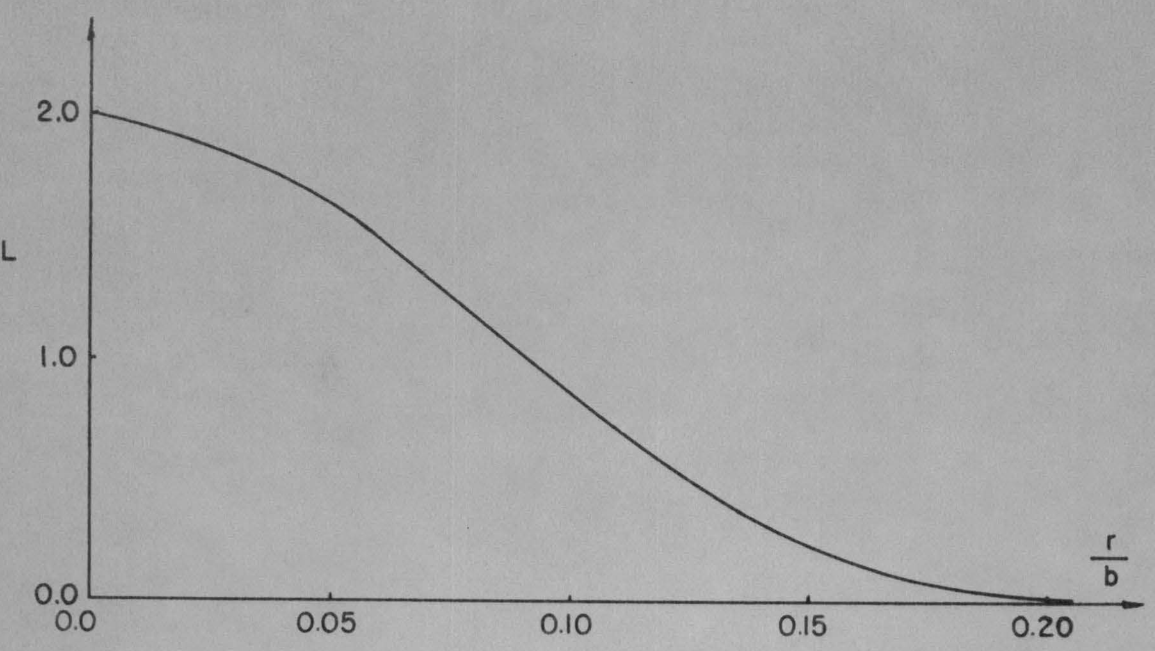

Fig. 13 Variation of L with r/b

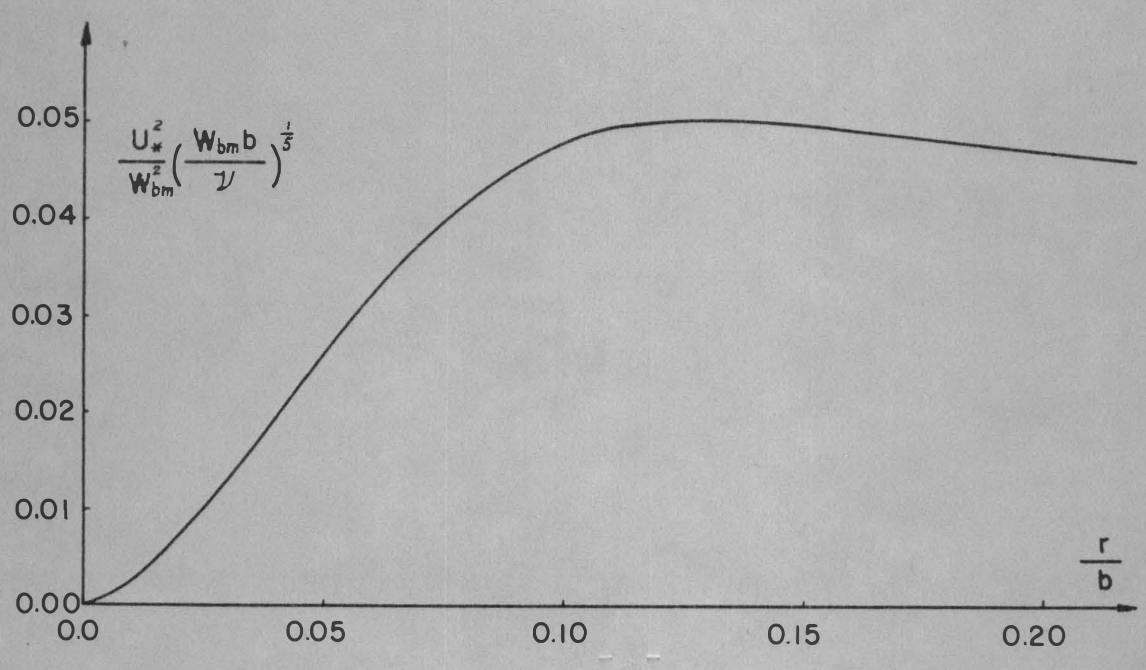

Fig. 14 Shear velocity distribution along the plane bed

IMPINGEMENT OF A THREE-DIMENSIONAL VERTICAL JET ON AN ERODIBLE BED

Rate of Sediment Transport q_s

In order to solve Eq 1, it is necessary to know the relationship between q_s and r. Similarly to the case of a two-dimensional jet (1), the expression for sediment transport in open channels mainly as bed-load is:

$$\frac{q_s}{U_* d_s} = A_o \left[\frac{U_*^2 - U_{*c}^2}{V_s^2} \right]^m \tag{50}$$

and

$$V_s^2 = \left(\frac{\sigma}{\rho} - 1 \right) g \, d_s \tag{51}$$

In Eqs 50 and 51, U_{*c} is the critical shear velocity, $\bar{\sigma}$ and d_s the density and the mean diameter of sediment particules respectively, and A_o and m are constants which are, in general, the function of U_*^2/Vs^2. When $U_* \gg U_{*c}$, Eq 50 simplifies to

$$\frac{q_s}{U_* d_s} = A_o \left[\frac{U_*}{V_s} \right]^{2m} \tag{52}$$

and then applied, as will be shown, in the calculation of the depth of scour. When U_* is of the same order of magnitude as U_{*c}, the value of m increases and the value of Ao decreases with decreasing U_*/Vs.

Non-Submerged Outlet

Substituting Eq 42 into Eq 52 gives

$$\frac{q_s}{W_{bm}\, d_s} = A_o\left(K\beta'^{-\frac{1}{4}}\right)^{\frac{2m+1}{2}}\left(\frac{W_{bm}}{V_s}\right)^{2m}\left(\frac{W_{bm}\, r}{\nu}\right)^{-\frac{2m+1}{10}}\left(1-\phi_o^2\right)^{\frac{9(2m+1)}{20}} \tag{53}$$

The following quantities are now defined

$$M = A_o\left(K\beta'^{-\frac{1}{4}}\right)^{\frac{2m+1}{2}}\left(\frac{1.24\, W_o b}{\nu}\,\frac{r}{b}\right)^{-\frac{2m+1}{10}}\left(1-\phi_o^2\right)^{\frac{9(2m+1)}{20}} \tag{54}$$

$$M' = \frac{dM}{d\left(\frac{r}{b}\right)} \tag{55}$$

$$\eta = \left(\frac{0.137}{1-\lambda}\right)\left(\frac{18m+9}{10}\right) e^{-0.137\left(\frac{18m+9}{10}\right)\frac{b}{d}}\left(\frac{1.24\, W_o}{V_s}\right)^{2m}\left\{\frac{M}{\left(\frac{r}{b}\right)} + M'\right\}\frac{d_s}{d}\,\frac{1.24\, W_o t}{b} \tag{56}$$

Then Eq 53 and the derivative of q_s with respect to (r/b) are written:

$$\frac{q_s}{W_{bm}\, d_s} = \left(\frac{W_{bm}}{V_s}\right)^{2m}\left(\frac{W_{bm}}{1.24\, W_o}\right)^{-\frac{2m+1}{10}} M \tag{57}$$

$$\frac{1}{W_{bm}\, d_s}\,\frac{\partial q_s}{\partial (r/b)} = \left(\frac{W_{bm}}{V_s}\right)^{2m}\left(\frac{W_{bm}}{1.24\, W_o}\right)^{-\frac{2m+1}{10}} M' \tag{58}$$

Substituting Eqs 57 and 58 into Eq 1 gives

$$\frac{\partial z}{\partial t} + \left(\frac{W_{bm}}{1.24\, W_o}\right)^{\frac{18m+9}{10}}\frac{d_s}{1-\lambda}(1.24\, W_o)\left(\frac{1.24\, W_o}{V_s}\right)^{2m}\left(\frac{M}{r}+\frac{M'}{b}\right) = 0 \tag{59}$$

Since W_{bm}/W_o depends only on the depth of z, the integration of Eq 59 gives the depth of scour for the case in which $U_* >> U_{*c}$ and is written

$$\int_0^z \left(\frac{W_{bm}}{1.24\, W_o}\right)^{-\frac{18m+9}{10}} dz + \frac{10\, d\, e^{0.137\left(\frac{18+9}{10}\right)\frac{b}{d}}}{0.137\,(18m+9)}\,\eta = 0 \tag{60}$$

In fact, r/b in M and M′ defined by Eqs 54 and 55 should be written r/(b - Z) since the bed is erodible. For this case, however, the integration of Eq 59 is impossible; therefore, the study is limited to the case in which $b \gg Z$, that is, when r/(b - Z) can be written as r/b.

Substituting Wm=Wo given by Eq 20 into W_{bm}/W_o in Eq 60 and integrating the first term yields

$$1 - e^{-0.137\left(\frac{18+9}{10}\right)\frac{Z}{d}} + \eta = 0 \qquad (61)$$

Putting $Z_s = -Z$, Eq 61 becomes

$$\frac{Z_s}{d} = \frac{10}{0.137(18m+9)} \ln(\eta + 1) \qquad (62)$$

in which η is a function of the following dimensionless parameters as seen from Eqs 56, 54, 55, and 22 (noting that m and Ao are both functions of $U_*^2/(\frac{\sigma}{\rho}-1)g d_s$):

$$\frac{W_o^2}{(\frac{\sigma}{\rho}-1)g d_s}, \quad \frac{r}{b}, \quad \frac{d_s}{d} \quad \frac{W_o t}{b}, \quad \frac{W_o b}{\nu} \quad \text{and} \quad \frac{b}{d}$$

The final depth of scour $Z_{s\infty}$ will be attained when $U_* = U_{*c}$. From Eq 42, this equality is written

$$\left(\frac{U_{*c}}{1.24 W_o}\right)^2 = K\beta^{1-\frac{1}{4}}\left(\frac{W_{bm}}{1.24 W_o}\right)^{\frac{9}{5}}\left(\frac{1.24 W_o b}{\nu}\right)^{-\frac{1}{5}}\left(\frac{r}{b}\right)^{-\frac{1}{5}}(1-\phi_o^2)^{\frac{9}{10}} \qquad (63)$$

Furthermore, replacing Wbm/1.24Wo by Wm/1.24Wo and using Eq 20, Eq 63 becomes

$$e^{0.137)\frac{9}{5}\left(\frac{b+Z_{s\infty}}{d}\right)} = K\beta^{1-\frac{1}{4}}\left(\frac{1.24 W_o}{U_{*c}}\right)^2\left(\frac{1.24 W_o b}{\nu}\right)^{-\frac{1}{5}}\left(\frac{r}{b}\right)^{-\frac{1}{5}}(1-\phi_o^2)^{\frac{9}{10}}$$

or

$$\frac{b + z_{s\infty}}{d} = \frac{5}{9 \times 0.137} \ln \left\{ k \beta^{-\frac{1}{4}} \left(\frac{1.24 W_o}{U_{*c}} \right)^2 \left(\frac{1.24 W_o b}{\nu} \right)^{-\frac{1}{5}} \left(\frac{r}{b} \right)^{-\frac{1}{5}} \left(1 - \phi_o^2 \right)^{\frac{9}{10}} \right. \tag{64}$$

From Eqs 64 and 22, it can be seen that $(b + z_{s\infty})/d$ is a function of W_o/U_*, W_ob/ν, r/b, and b/d.

Submerged Outlet

The following quantities are defined in a manner similar to that for the case of the non-submerged outlet.

$$N = A_o \left(k \beta^{-\frac{1}{4}} \right)^{\frac{2m+1}{2}} \left(\frac{6.2 W_o b}{\nu} \frac{r}{b} \right)^{-\frac{2m+1}{10}} \left(1 - e^{-152 \left(\frac{r}{b} \right)^2} \right)^{\frac{9(2m+1)}{20}} \tag{65}$$

$$N' = dN/d\left(\frac{r}{b}\right) \tag{66}$$

$$\xi = \frac{18m+9}{10} \frac{1}{1-\lambda} \left(\frac{6.2 W_o}{V_s} \right)^{2m} \left\{ \frac{N}{(r/b)} + N' \right\} \frac{d_s}{d} \frac{6.2 W_o t}{b} \tag{67}$$

Then Eq 1 can be written in the same manner as in the case of the non-submerged outlet.

$$\frac{\partial z}{\partial t} + \left(\frac{W_{bm}}{6.2 W_o} \right)^{\frac{18m+9}{10}} \frac{d_s}{1-\lambda} (6.2 W_o) \left(\frac{6.2 W_o}{V_s} \right)^{2m} \left(\frac{N}{r} + \frac{N'}{b} \right) = 0 \tag{68}$$

The integration of Eq. 68 gives, with ξ defined by Eq. 67,

$$\int_0^z \left(\frac{W_{bm}}{6.2 W_o} \right)^{-\frac{18m+9}{10}} dz + \frac{10 d}{18m+19} \xi = 0 \tag{69}$$

Substituting the value of Wm/Wo given by Eq. 43 into Wbm/Wo in Eq. 69 and integrating, the first term gives

$$\left(\frac{b}{d}\right)^{\frac{18m+19}{10}} - \left(\frac{b-Z}{d}\right)^{\frac{18m+19}{10}} + \xi = 0 \tag{70}$$

Finally, letting $Z_s = -Z$, the depth of scour in this case is given by the following relation:

$$\frac{b+Z_s}{d} = \left\{ \xi + \left(\frac{b}{d}\right)^{\frac{18m+19}{10}} \right\}^{\frac{10}{18m+19}} \tag{71}$$

in which ξ is a function of the same dimensionless parameters as those for η. However, it should be noted that the form of Eq 71 is different from that of Eq 62 which gives the depth of scour for the case of a non-submerged outlet.

An expression for the final depth of scour $Z_{s\infty}$ can be obtained by putting $U_* = U_{*c}$ in Eq 48 and replacing Wbm by Wm in Eq 43, or

$$\frac{b+Z_{s\infty}}{d} = \left(K\beta'^{-\frac{1}{4}}\right)^{\frac{5}{9}} \left(\frac{6.2\,W_0}{U_{*c}}\right)^{\frac{10}{9}} \left(\frac{6.2\,W_0\,b}{\nu}\right)^{-\frac{1}{9}} \left(\frac{r}{b}\right)^{-\frac{1}{9}} \left(1-e^{-152\left(\frac{r}{b}\right)^2}\right)^{\frac{1}{2}} \tag{72}$$

in which $(b + Z_{s\infty})/d$ is a function of W_0/U_{*c}, $W_0 b/\nu$, and r/b. however, a better approximation for $Z_{s\infty}$ is obtained if the exponent r/b in the last bracket is replaced by $r/(b + Z_{s\infty})$, especially when $Z_{s\infty}$ and b are of the same order of magnitude.

IMPINGEMENT OF A THREE-DIMENSIONAL INCLINED JET ON AN ERODIBLE BED

Jet Issuing from a Non-Submerged Outlet

The Z'-axis and r'-axis are taken along the centerline of the jet and perpendicularly to the Z'-axis respectively as shown in Fig. 15. On the vertical plane containing the centerline of the jet, the relations between the system of coordinates (r', Z') and (r, Z) are

$$r' = r \sin\theta + Z \cos\theta \tag{73}$$

$$Z' = -r \cos\theta + Z \sin\theta \tag{74}$$

Since $\overline{AH} = b' - Z'$ and $\overline{HM} = r'$ in Fig. 15, an expression similar to Eq 19 or 20 for the maximum velocity of the jet W_m' along the centerline can be written as follows:

$$\frac{W_m'}{W_o'} = \alpha\, e^{-\beta\left(\frac{b'-Z'}{d}\right)} \tag{75a}$$

According to Homma's experiments (19), the values of α and β are 1.24 and 0.024 respectively for the case in which $\theta = 60°$ and $Re \doteqdot 25{,}000$.

The velocity distribution in the direction parallel to or', i.e. along the cross-section $\overline{HM}$, is assumed, similarly to Eq 21, as

$$\frac{W'}{W_m'} = \Phi\left(\frac{b'}{d}, \frac{r'}{b'}, \frac{b'-Z'}{d}\right) \tag{76a}$$

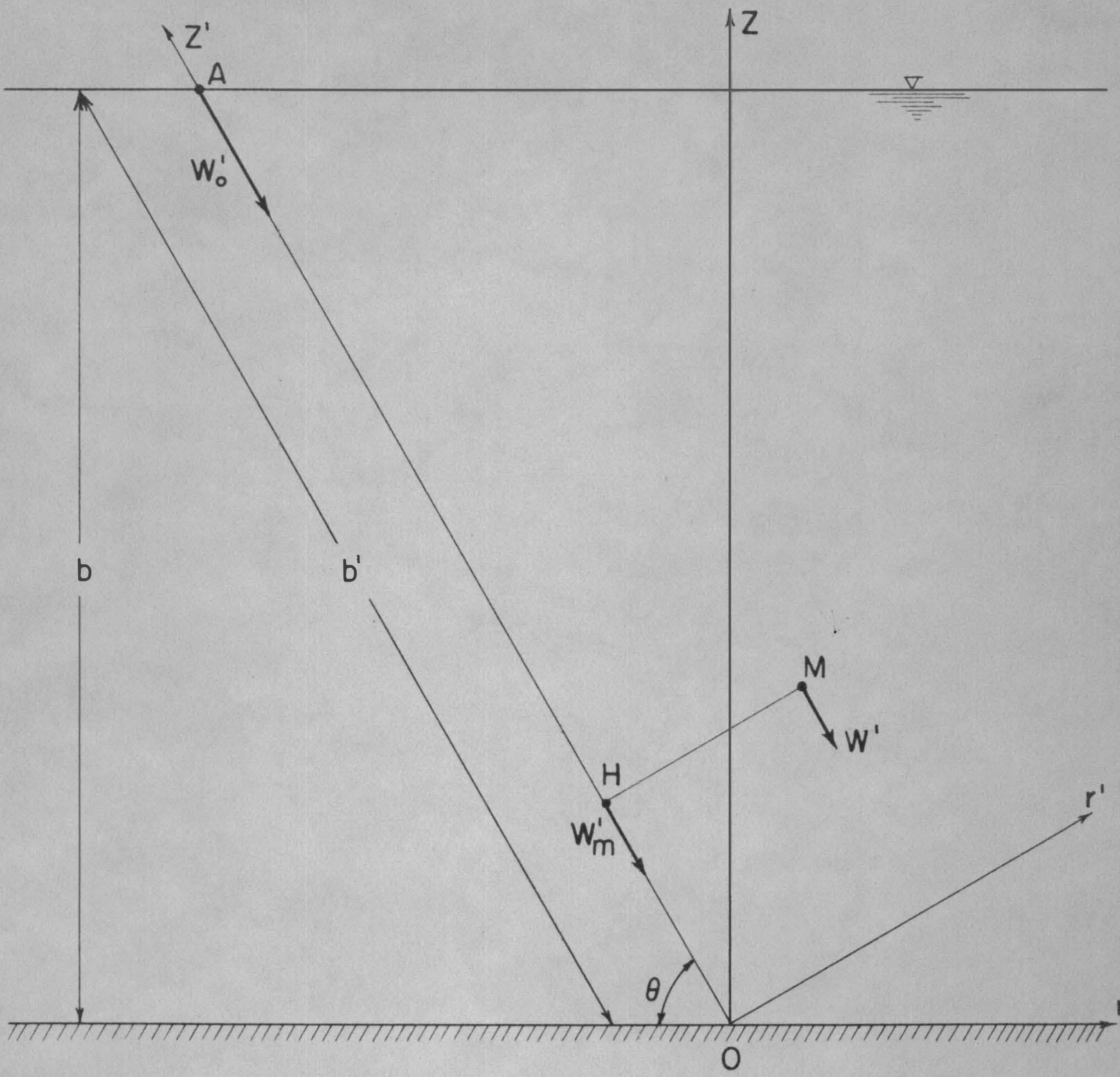

Fig. 15 Coordinate systems for an inclined jet

Expressions for Eqs 75a and 76b using the relations 73 and 74 as functions of r, Z, and O are as follows:

For any point on the vertical plans

$$\frac{W_m'}{W_o'} = \alpha \exp\left\{-\beta\left(\frac{b'}{d} + \frac{r}{d}\cos\Theta - \frac{Z}{d}\sin\Theta\right)\right\} \tag{75b}$$

$$\frac{W'}{W_M'} = \Phi\left\{\frac{b'}{d}; \left(\frac{r}{b'}\sin\Theta + \frac{Z}{b'}\cos\Theta\right); \left(\frac{b'}{d} + \frac{r}{d}\cos\Theta - \frac{Z}{d}\sin\Theta\right)\right\} \tag{76b}$$

For a point along the axis or (Z = 0)

$$\frac{W_{bM}'}{W_o'} = \alpha \exp\left\{-\beta\left(\frac{b'}{d} + \frac{r}{d}\cos\Theta\right)\right\} \tag{75c}$$

$$\frac{W_b'}{W_{bM}'} = \Phi\left\{\frac{b'}{d}; \frac{r}{b'}\sin\Theta; \left(\frac{b'}{d} + \frac{r}{d}\cos\Theta\right)\right\} \tag{76c}$$

For the point O (Z = 0 and r = 0)

$$\frac{W_{bMo}'}{W_o'} = \alpha \exp\left(-\frac{\beta b'}{d}\right) \tag{76d}$$

As in the case of a vertical jet, the horizontal velocity U_b along the r-axis can be expressed by

$$\frac{U_b}{W_{bMo}'\sin\Theta} = \left(1 - \Phi_o^2\right)^{1/2} \tag{77}$$

in which

$$\Phi_o = \frac{W_b'}{W_{bMo}'} \tag{78}$$

By combining Eqs 75c, 76c, and 76d, Φ_o becomes

$$\Phi_o = \frac{W_b'}{W_{bMo}'} = \frac{W_b'}{W_{bM}'}\,\frac{W_{bM}'}{W_o'}\,\frac{W_o'}{W_{bMo}'} \tag{79}$$

$$= \Phi\left\{\frac{b'}{d}; \frac{r}{b'}\sin\Theta; \left(\frac{b'}{d} + \frac{r}{d}\cos\Theta\right)\right\} \exp\left(-\frac{\beta r}{d}\cos\Theta\right)$$

For the shear distribution along the bed, the approach given for the case of a vertical jet is still valid if L in Eq 36 is assumed constant; therefore, it can be shown that

$$\frac{U_*^2}{U_b^2} = K\beta'^{-n}\left(\frac{U_b r}{\nu}\right)^{-\frac{n}{1+n}} \tag{39}$$

Introducing Eq 77 into Eq 39 and taking n as 1/4 gives

$$\frac{U_*^2}{(Wb'_{mo}\sin\theta)^2} = K\beta'^{-1/4}\left(\frac{Wb'_{mo}\sin\theta\, r}{\nu}\right)^{-\frac{1}{5}}\left(1-\Phi_o^2\right)^{\frac{9}{10}} \tag{80}$$

Since Eq 52 is an expression for the rate of sediment transport for the case in which $U_* \gg U_{*c}$, Eq 80 may be substituted into Eq 52 to give

$$\frac{q_s}{\alpha W_o' \sin\theta\, d_s} = \left(\frac{Wb'_{mo}}{\alpha W_o'}\right)^{\frac{9(2m+1)}{10}} P \tag{81}$$

in which

$$P = A_o\left(K\beta'^{-1/4}\right)^{\frac{2m+1}{2}}\left(\frac{\alpha W_o' \sin\theta}{1/5}\right)^{2m}\left(\frac{\alpha W_o' \sin\theta\, b}{\nu}\,\frac{r}{b}\right)^{-\frac{2m+1}{10}}\left(1-\Phi^2\right)^{\frac{9(2m+1)}{20}} \tag{82}$$

Let

$$p' = dP/d\left(\frac{r}{b}\right) \tag{83}$$

The derivative of qs with respect to r/b is

$$\frac{1}{\alpha W_o' \sin\theta\, d_s}\,\frac{\partial q_s}{\partial (r/b)} = \left(\frac{Wb'_{mo}}{\alpha W_o'}\right)^{\frac{9(2m+1)}{10}} P' \tag{84}$$

Substituting Eqs 81 and 84 into Eq 1 gives

$$\frac{\partial z}{\partial t} + \left(\frac{Wb'_{mo}}{\alpha W_o'}\right)^{\frac{9(2m+1)}{10}} \frac{\alpha W_o' \sin\theta\, d_s}{(1-\lambda)\, b}\left\{P' + \frac{P}{(r/b)}\right\} = 0 \tag{85}$$

For the case of an erodible bed, $W_b{}'_{mo}/W_o{}'$ depends upon Z and is equal to Wm/Wo given by Eq 75a.

Since Eqs 73 and 74 can be written

$$r = r' \sin \Theta - Z' \cos \Theta \tag{86}$$

$$Z = r' \cos \Theta + Z'' \sin \Theta \tag{87a}$$

along the Z′-axis, that is $r' = 0$, Eq 87a becomes

$$Z = Z'' \sin \Theta \tag{87b}$$

Therefore, Eq 75a is written for any point along the Z′-axis as follows:

$$\frac{W_m{}'}{W_o{}'} = \alpha\, e^{-\beta\left(\frac{b}{d} - \frac{Z}{d \sin \Theta}\right)} \tag{75d}$$

Substitution of Eq 75d into Eq 85 yields

$$\int_0^Z e^{\frac{9(2m+1)}{10}\beta\left(\frac{b'}{d} - \frac{Z}{d \sin \Theta}\right)} dZ + \frac{d\, d_s}{1-\lambda}\left\{p' + \frac{p}{(\gamma/g)}\right\}\frac{W_o' \sin \Theta t}{b} = 0 \tag{88}$$

Moreover, integrating Eq 88 gives

$$1 - e^{-\frac{9(2m+1)}{10}\frac{\beta Z}{d \sin \Theta}} + \eta' = 0 \tag{89}$$

in which

$$\eta' = \frac{\alpha\beta}{1-\lambda} \cdot \frac{9(2m+1)}{10}\, e^{-\beta\frac{9(2m+1)}{10}\frac{b'}{d}}\left\{p' + \frac{p}{(\gamma/g)}\right\}\frac{d_s}{d}\,\frac{W_o' t}{b} \tag{90}$$

Denoting the actual depth of scour by Zs = −Z, Eq 89 becomes

$$\frac{Z_s}{d \sin \Theta} = \frac{10}{9(2m+1)\beta} \ln(\eta' + 1) \tag{91}$$

From Eqs 79, 82, 83, and 90, it can be seen that η' depends upon the following parameters:

$$\frac{d_s}{d} \quad \frac{W_o' t}{b} \ , \quad \frac{(W_o' \sin\Theta)^2}{(\frac{\sigma}{\rho}-1)\, g\, d_s} \ , \quad \Phi_o\left(\frac{r}{b}, \frac{b}{d\sin\Theta}, \Theta, \beta\right)$$

$$\frac{W_o' \sin\Theta\, b}{\nu} \ , \quad \frac{r}{b} \ ; \quad \frac{b}{d\sin\Theta} \ ; \ \text{and} \ \beta$$

The final value of the scour depth $Z_{s\infty}$ is attained when $U_* = U_{*c}$. From this equality, Eq 80 yields

$$\frac{U_{*c}^2}{(\alpha W_o' \sin\Theta)^2} = K\beta'^{-\frac{1}{4}}\left(\frac{W_{bmo}'}{\alpha W_o'}\right)^{\frac{9}{5}}\left(\frac{\alpha W_o' \sin\Theta\, r}{\nu}\right)^{-\frac{1}{5}}\left(1 - \Phi_o^2\right)^{\frac{9}{10}} \qquad (92)$$

Replacing $W_{bmo}'/\alpha W_o'$ by $W_m/\alpha W_o'$ and using Eq 75d, Eq 92 becomes, for the critical value of the scour depth ($Z_{s\infty} = -Z_{\infty}$),

$$e^{\frac{9}{5}\beta \frac{Z_{s\infty}+b}{d\sin\Theta}} = K\beta'^{-\frac{1}{4}}\left(\frac{\alpha W_o' \sin\Theta}{U_{*c}}\right)^2\left(\frac{\alpha W_o' \sin\Theta\, r}{\nu}\right)^{-\frac{1}{5}}\left(1 - \Phi_o^2\right)^{\frac{9}{10}}$$

or

$$\frac{Z_{s\infty}+b}{d\sin\Theta} = \frac{5}{9\beta}\ \ln\left\{K\beta'^{-\frac{1}{4}}\left(\frac{\alpha W_o' \sin\Theta}{U_{*c}}\right)^2\left(\frac{\alpha W_o' \sin\Theta\, r}{\nu}\right)^{-\frac{1}{5}}\left(1 - \Phi_o^2\right)^{\frac{9}{10}}\right\} \qquad (93)$$

From Eqs 93 and 79, it can be seen that $(Z_{s\infty} + b) / (d \sin\Theta)$ is a function of

$$\frac{W_o' \sin\Theta}{U_{*c}} \ , \quad \Phi_o\left(\frac{r}{b}, \frac{b}{d\sin\Theta}, \Theta, \beta\right) \quad \frac{W_o' \sin\Theta\, b}{\nu} \ , \quad \frac{r}{b} \ \text{and} \ \beta$$

<u>Jet Issuing from a Submerged Outlet</u>

In this case, the same expressions for the longitudinal distribution of the velocity Wm' along the centerline of the jet and the transverse distribution of the velocity W' as given by

Eqs 43 and 44 for the vertical jet are also employed

$$\frac{W_m'}{W_o'} = 6.2 \frac{d}{b' - Z'} \tag{94a}$$

$$\frac{W'}{W_m'} = e^{-76 \frac{Z'^2}{(b'-Z')^2}} \tag{95a}$$

Using the relations between both coordinates (r', Z') and (r, Z) Eqs 73 and 74, expressions for the velocity distributions are given for the following cases:

For a point along the axis or ($Z = 0$)

$$\frac{W_{bm}'}{W_o'} = 6.2 \frac{d}{b' + \eta \cos\theta} \tag{94b}$$

$$\frac{W_b'}{W_{bm}'} = \exp\left[-76 \left(\frac{\eta \sin\theta}{b' + \eta\cos\theta}\right)^2\right] \tag{95b}$$

For the point 0 ($Z = 0$ and $r = 0$)

$$\frac{W_{bmo}'}{W_o'} = 6.2 \frac{d}{b'} \tag{94c}$$

Although Eq 77 is still valid, in this case, the value of Φ_o becomes, by combining Eqs 95b, 94b, and 94c, as follows:

$$\Phi_o = \frac{W_b'}{W_{bmo}'} = \frac{W_b'}{W_{bm}'}\,\frac{W_{bm}'}{W_o'}\,\frac{W_o'}{W_{bmo}'} = \left(1 + \frac{\eta}{2b}\sin 2\theta\right)^{-1} \exp\left[-76\left\{\frac{(\eta/b)\sin^2\theta}{1 + (\eta/2b)\sin 2\theta}\right\}^2\right]$$

For the shear velocity distribution, Eq 80 can be applied as given. Furthermore, Eqs 81, 82, 83, 84, and 85 can also be used by changing P and P′ to Q and Q′ respectively and letting $\alpha = 6.2$; that is,

$$\frac{q_s}{6.2\, W_o' \sin\theta\, d_s} = \left(\frac{W b'_{mo}}{6.2\, W_o'}\right)^{\frac{9(2m+1)}{10}} Q \quad (97)$$

in which

$$Q = A_o (K\beta'^{-\frac{1}{4}})^{\frac{2m+1}{2}} \left(\frac{6.2 W_o' \sin\theta}{VS}\right)^{2m} \left(\frac{6.2\, W_o' \sin\theta\, b}{\nu} \frac{r}{b}\right)^{-\frac{2m+1}{10}} \left(1 - \bar{\Phi}_o^2\right)^{\frac{9(2m+1)}{20}} \quad (98)$$

$$Q' = dQ/d\left(\frac{r}{b}\right)$$

$$\frac{1}{6.2\, W_o' \sin\theta\, d_s} \frac{\partial q_s}{\partial (r/b)} = \left(\frac{W b'_{mo}}{6.2\, W_o'}\right)^{\frac{9(2m+1)}{10}} Q' \quad (100)$$

$$\frac{\partial Z}{\partial t} + \left(\frac{W b'_{mo}}{6.2\, W_o'}\right)^{\frac{9(2m+1)}{10}} \frac{6.2\, W_o' \sin\theta\, d_s}{(1-\lambda)\, b} \left\{ Q' + \frac{Q}{(r/b)} \right\} = 0 \quad (101)$$

It can be shown that $W'_{b_{m_o}}/W_o'$ for an erodible bed is equal to W_m'/W_o' given by Eq 94a. Therefore, substituting Eq 94a into Eq 101 and using Eq 87a yields

$$\left\{\frac{b' - (Z/\sin\theta)}{d}\right\}^{\frac{9(2m+1)}{10}} dZ + \frac{6.2\, d_s}{1-\lambda} \left\{ Q' + \frac{Q}{(r/b)} \right\} \frac{W_o' \sin\theta}{b}\, dt = 0 \quad (102)$$

Integration of Eq 102, with the initial condition $Z = 0$ at $t = 0$, gives

$$\left(\frac{b'}{d}\right)^{\frac{18m+19}{10}} - \left\{\frac{b' - (Z/\sin\theta)}{d}\right\}^{\frac{18m+19}{10}} + \xi' = 0 \quad (103)$$

in which

$$\xi' = \frac{18m+19}{10} \frac{6.2}{1-\lambda} \left\{ Q' + \frac{Q}{(r/b)} \right\} \frac{d_s}{d} \frac{W_o' t}{b} \tag{104}$$

Letting Zs = -Z, Eq 103 becomes

$$\frac{b + Z_s}{d \sin\Theta} = \left\{ \xi' + \left(\frac{b}{d \sin\Theta} \right)^{\frac{18m+19}{10}} \right\}^{\frac{10}{18m+19}} \tag{105}$$

in which ξ' depends, since m and Ao are functions of $U_*^2/(\frac{\sigma}{\rho} - 1)gds$, upon the following parameters:

$$\frac{d_s}{d} \frac{W_o' t}{b} ; \frac{(W_o' \sin\Theta)^2}{(\frac{\sigma}{\rho} - 1) g d_s} ; \Phi_o\left(\frac{r}{b}, \Theta\right) ; \frac{W_o' \sin\Theta b}{\nu} ; \frac{r}{b}$$

and $\frac{b}{d \sin\Theta}$

An expression for the final depth of scour Zs_∞ can be obtained by letting $U_* = U_{*c}$ in Eq 80 and using Eq 94a as the value of $Wb'mo/Wo'$, or

$$\frac{b + Z_{s\infty}}{d \sin\Theta} = \left(K \beta'^{-\frac{1}{4}} \right)^{\frac{5}{9}} \left(\frac{6.2 W_o' \sin\Theta}{U_{*c}} \right)^{\frac{10}{9}} \left(\frac{6.2 W_o' \sin\Theta r}{\nu} \right)^{-\frac{1}{9}} \left(1 - \Phi_o^2 \right)^{\frac{1}{2}} \tag{106}$$

Therefore, it can be shown that $(b + Zs_\infty)/(d \sin\Theta)$ depends upon the following parameters

$$\frac{W_o' \sin\Theta}{U_{*c}} ; \frac{W_o' \sin\Theta b}{\nu} ; \frac{r}{b} \text{ and } \Phi_o\left(\frac{r}{b}, \Theta\right)$$

These parameters are similar to those for the case of a non-submerged outlet except for the fact that $(b + Zs_\infty)/(d \sin\Theta)$ is a function of a power law while for the non-submerged outlet, as given by Eq 93, it is a function of the logarithmic law.

EXPERIMENTAL VERIFICATIONS

Vertical Jet with a Non-Submerged Outlet

In the experimental study by Doddiah (11), the characteristic depth of scour was adopted as the distance from the original bed to the point of intersection of the side slopes of the crosssection instead of the measured maximum depth of scour at the stagnation point as shown in Fig. 16. This expression for the depth of scour is similar to the one used by Rouse (1) for the case of a two-dimensional jet. The equation for the depth of scour for the case of the vertical jet with a non-submerged outlet is expressed by Eq 62, or

$$\frac{Z_s}{d} = \frac{10}{0.137(18m+9)} \ln(\eta + 1)$$

Furthermore, if the characteristic depth of scour Z_s is assumed to be equivalent to that for a certain value of r/b which contains the dimensionless parameter η and if the square root of the crosssectional area of the jet at the water surface $\sqrt{A_j}$ is used instead of d, then Eq 62, for values of $\eta \gg 1$, can be written as

$$\frac{Z_s}{\sqrt{A_j}} = Y \left\{ \log\left(\frac{d_s}{\sqrt{A_j}} \frac{W_o t}{b} \right) + \log Z \right\} \tag{107}$$

with

$$Y = \frac{10}{0.137(18m+9)} \sqrt{\frac{4}{\pi}} \tag{108}$$

and

$$Z = \frac{1.24}{(1-\lambda)Y} \sqrt{\frac{\pi}{4}} \exp\left(-\sqrt{\frac{\pi}{4}} \frac{1}{Y} \frac{b}{\sqrt{A_j}} \right) \left(\frac{1.24 W_o}{V_s} \right)^{2m} \left\{ \frac{M_o}{(\eta/b)_o} + M_o' \right\} \tag{109}$$

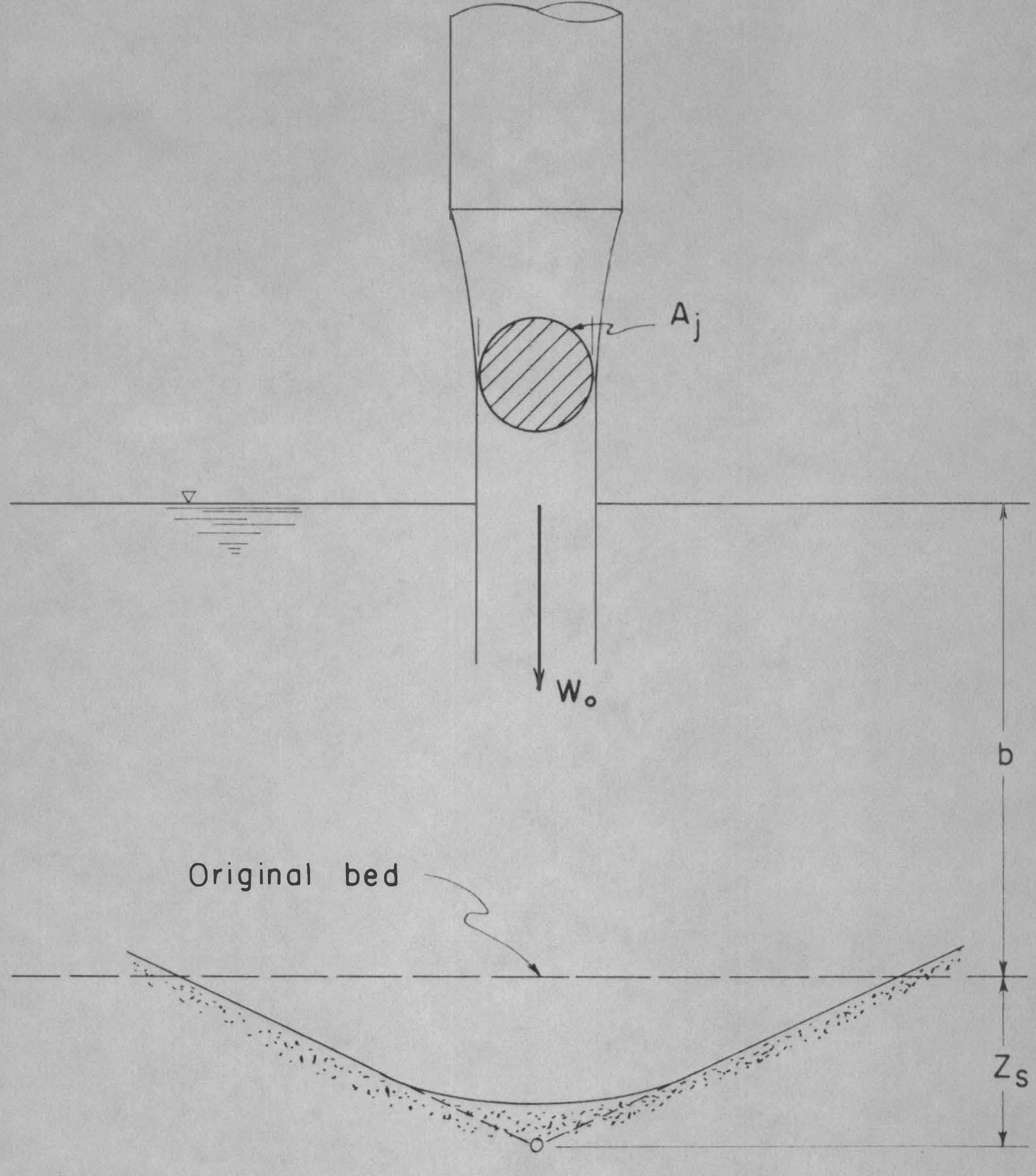

Fig. 16 Schematic drawing for a characteristic depth of scour in the case of vertical jet issuing from a non-submerged outlet

in which Mo and Mo′ are the values of M and M′ corresponding to a certain value of r/b denoted by $(r/b)_0$.

Since m and Ao are contained in the parameters Mo and Mo′, and are functions of U_*^2/Vs^2, it is evident that both Y and Z are functions of the following parameters.

$$\frac{Wo^2}{\left(\frac{\sigma}{\rho}-1\right) g d_s}; \quad \frac{b}{\sqrt{A_j}}; \text{ and } \quad \frac{Wo\, b}{\nu} \qquad \left(\text{note } Vs^2 = \left(\frac{\sigma}{\rho}-1\right) g d_s\right)$$

Furthermore, it is to be noted that the exponent of Wob/ν is considerably smaller than that of Wo^2/Vs^2; therefore, b/d_s should be introduced instead of Wob/ν when the bed is regarded as a rough boundary. Thus, for a rough boundary, the effect of Wob/ν on the development of the scour hole will be insignificant.

From Eq 107, it is evident that the slope of the straight line obtained by the plotting of $Zs/\sqrt{A_j}$ against $\log\left(\frac{d_s}{\sqrt{A_j}} \frac{W_o t}{b}\right)$ (see Fig. 17) represents the values of Y and of $Zs/\sqrt{A_j}$ at the point where $\left(\frac{d_s}{\sqrt{A_j}} \frac{W_o t}{b}\right) = 1$ which are equal to that of Y log Z. In the plots of Fig. 17, the computed values of $\sqrt{A_j}$ and Wo, were used.

Fig. 18, giving the relationship between Y and $\frac{Wo^2}{\left(\frac{\sigma}{\rho}-1\right) g d_s}$ shows that the effect of the parameter $\frac{b}{\sqrt{A_j}}$ is very small although there is considerable scattering of data, especially for the case when b= 16 in. The solid line in Fig. 18 represents the following equation:

$$Y = 0.00484 \left\{ \frac{Wo^2}{\left(\frac{\sigma}{\rho}-1\right) g d_s} \right\}^{0.757} \qquad (110)$$

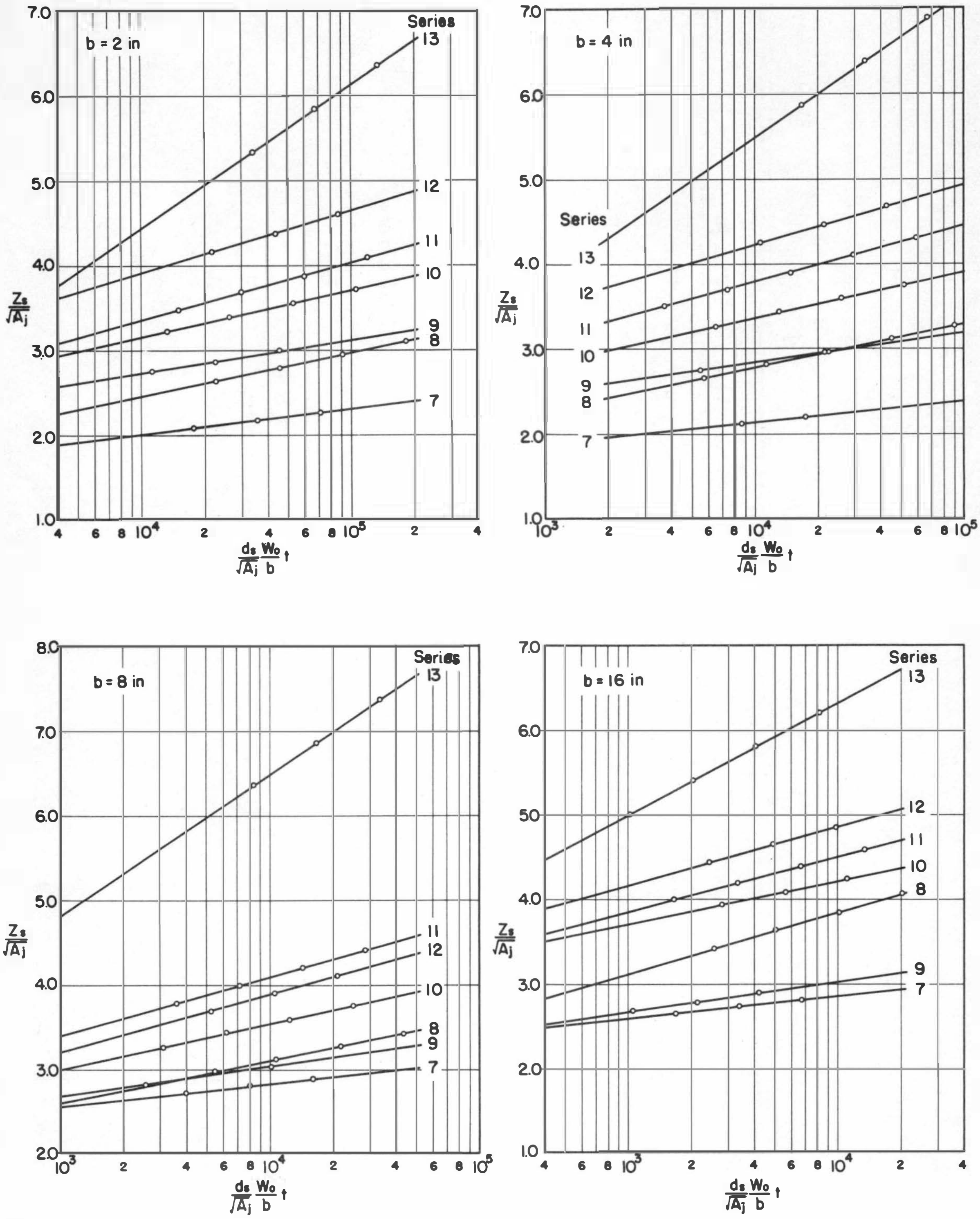

Fig. 17 Variations of $\frac{Z_s}{\sqrt{A_j}}$ with $\frac{d_s}{\sqrt{A_j}}\frac{W_o}{b}t$ for the vertical jet issuing from a non-submerged outlet

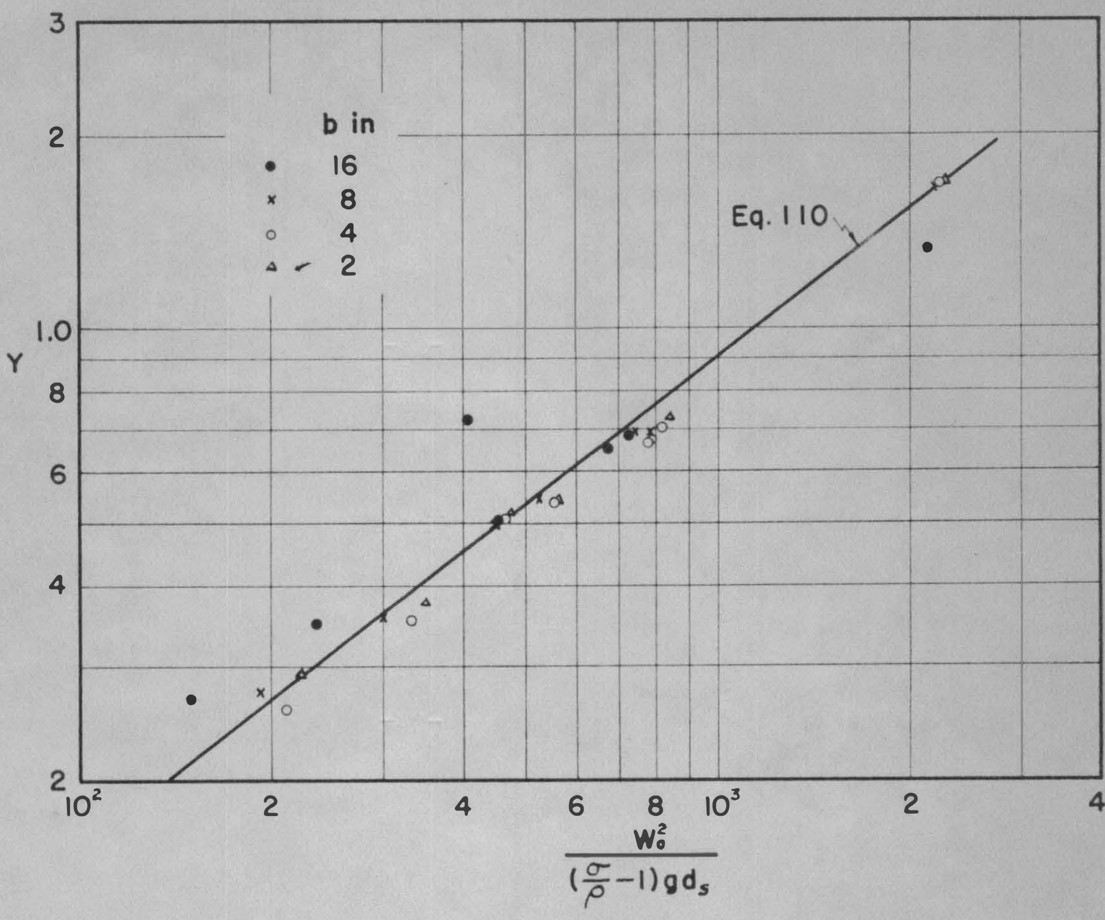

Fig. 18 Relationship between Y and $\frac{W_o^2}{(\frac{\sigma}{\rho}-1)gd_s}$

Fig. 19, which gives the relationships between Z and $\frac{b}{\sqrt{A_j}}$, shows that Z increases with increasing $\frac{b}{\sqrt{A_j}}$ except for the case of Series 8. This trend is valid, however, only for the region where $\frac{b}{\sqrt{A_j}}$ is less than a certain value; that is, after $\frac{b}{\sqrt{A_j}}$ reaches a certain value, Z will decrease with increase of $\frac{b}{\sqrt{A_j}}$. Fig. 20 shows the effect of $\frac{W_o^2}{(\frac{\sigma}{\rho}-1)g\,d_s}$ on the magnitude of Z, in which $-2\log\frac{b}{\sqrt{A_j}}$ is added to log Z to eliminate the effect of $\frac{b}{\sqrt{A_j}}$. The solid line in Fig. 20 represents the following equation:

$$\log Z - 2\log\frac{b}{\sqrt{A_j}} = 16 - 5.36\log\left\{\frac{W_o^2}{(\frac{\sigma}{\rho}-1)g\,d_s}\right\} \qquad (111)$$

The data plotted on Fig. 20 are for the sediment size 6.25 mm for Series 7 and 8 and for the sediment size 4.11 mm for Series 9, 10, 11, 12, and 13. The effect of sediment size cannot be checked satisfactorily by Figs. 18 and 19, since these figures represent only the effect of the velocity of the jet W_o. The result of the analysis of Rouse's data for the two-dimensional case in the previous paper (1) suggests that $\frac{W_o^2}{\omega_m^2}$ should be taken rather than $\frac{W_o^2}{(\frac{\sigma}{\rho}-1)g\,d_s}$. The influence of sediment size can be established only by further experiments using a wider range of sediment size.

Inclined Jet and Non-Submerged Outlet

In the analysis of the experimental study by Smith (12), the characteristic depth of scour was again taken as the distance from the original bed to the point of intersection of the side slopes of the scoured basin. (See Fig. 21).

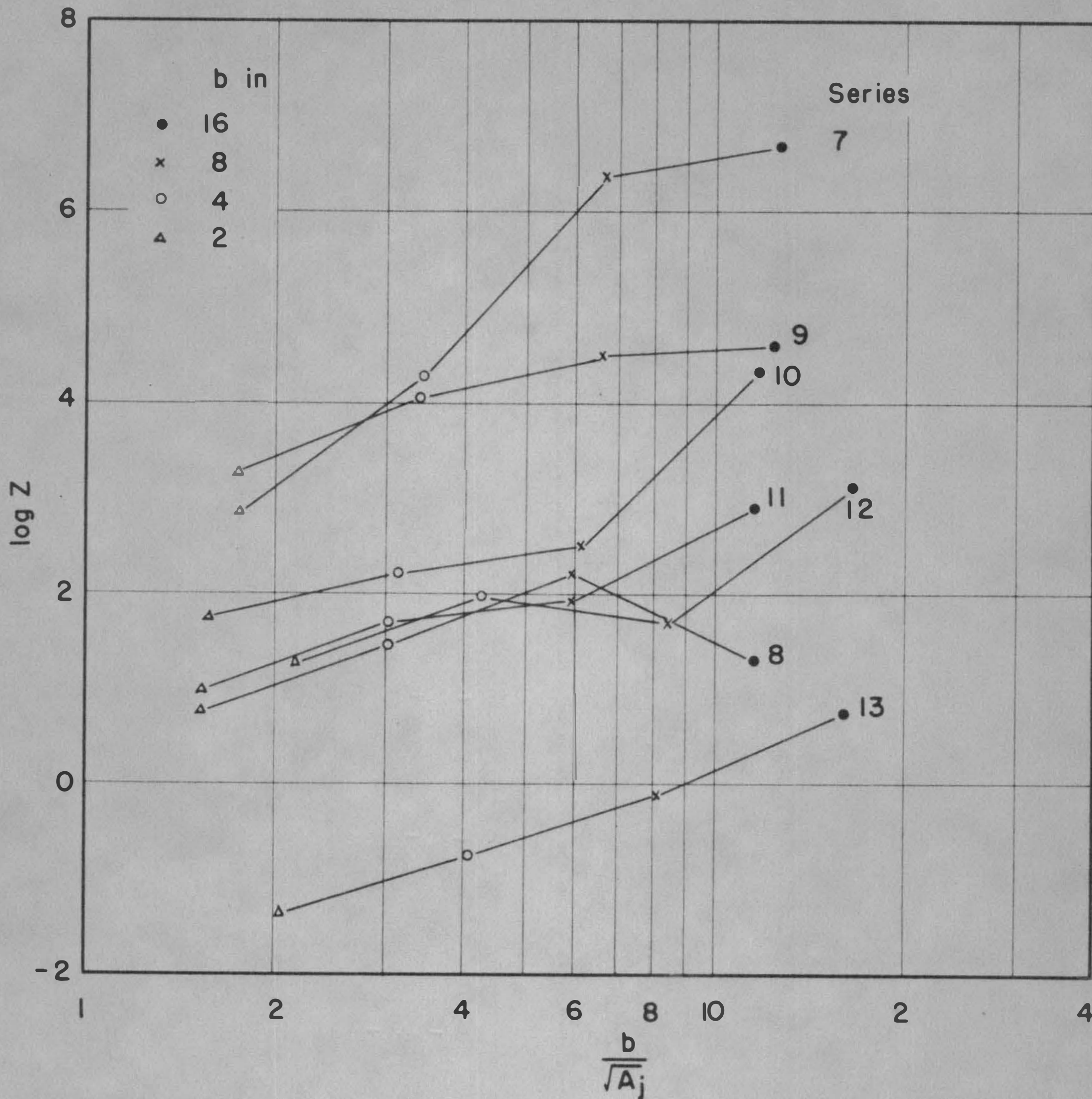

Fig. 19 Relationship between Z and $\frac{b}{\sqrt{A_j}}$

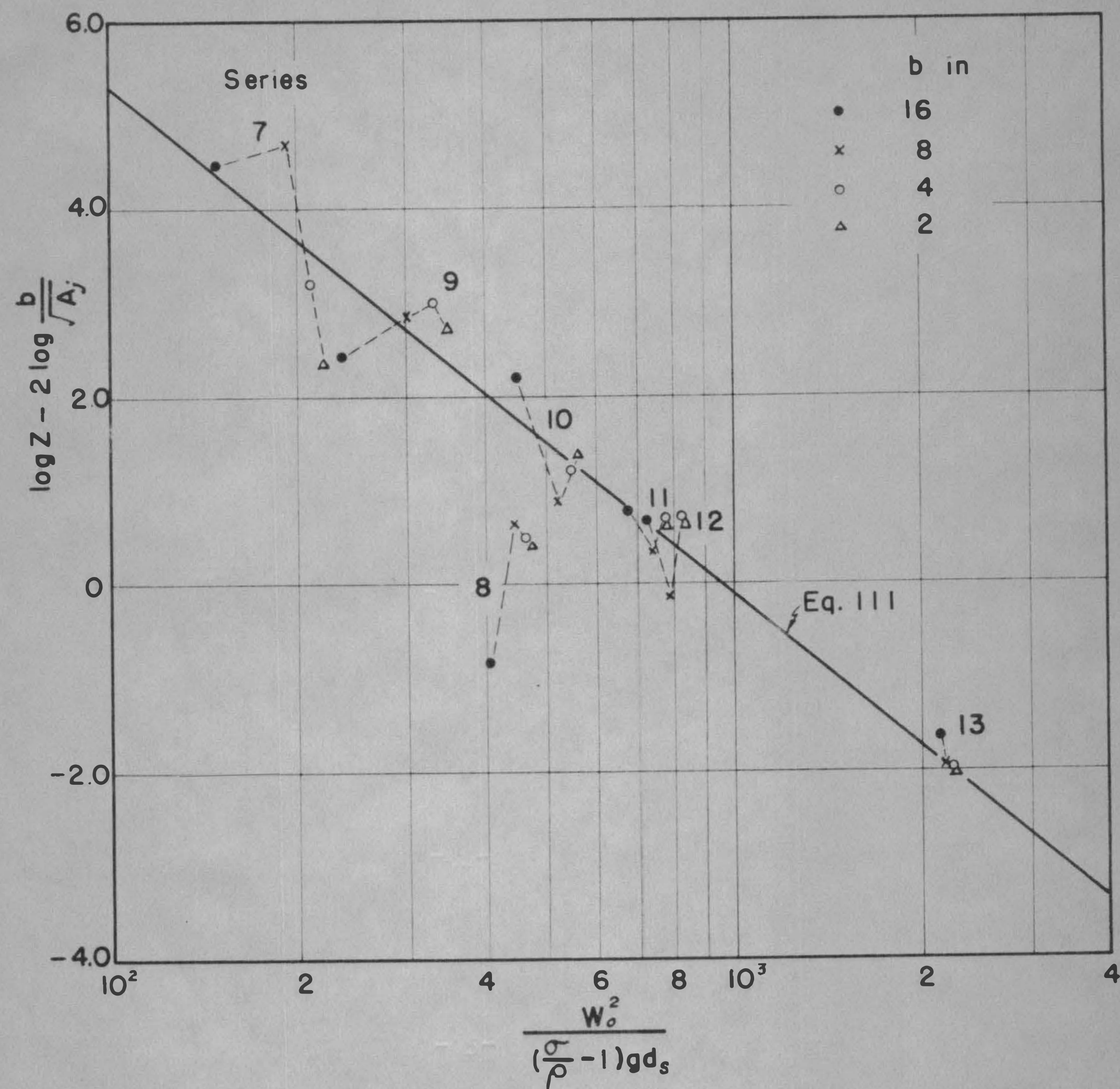

Fig. 20 Relationship between $\log Z - 2\log \frac{b}{\sqrt{A_j}}$ and $\frac{W_o^2}{(\frac{\sigma}{\rho}-1)gd_s}$

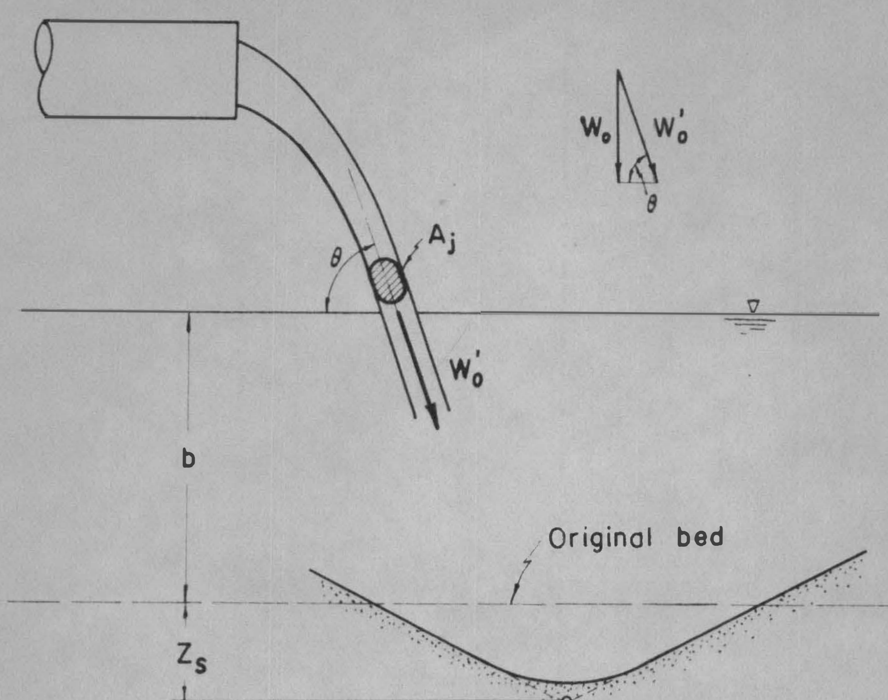

Fig. 21 Schematic drawing for a characteristic depth of scour in the case of the inclined jet issuing from a non-submerged outlet

The equation for the depth of scour for the case of the inclined jet issuing from a non-submerged outlet is expressed by Eq 91, or

$$\frac{Z_s}{d \sin\theta} = \frac{10}{9(2m+1)\beta} \ln(\eta' + 1)$$

Substituting $\sqrt{A_j}$ for d, Eq 91 may be written when $\eta' \gg 1$, as

$$\frac{Z_s}{\sqrt{A_j} \sin\theta} = Y' \left\{ \log\left(\frac{d_s}{\sqrt{A_j} \sin\theta} \frac{W_o t}{b} \right) + \log Z' \right\} \qquad (112)$$

with

$$Y' = \frac{10}{9(2m+1)\beta} \sqrt{\frac{4}{\pi}} \qquad (113)$$

and

$$Z' = \frac{\alpha}{(1-\lambda)Y'} \sqrt{\frac{\pi}{4}} \exp\left(-\sqrt{\frac{\pi}{4}} \frac{1}{Y'} \frac{b}{\sqrt{A_j} \sin\theta} \right) \left\{ P_o' + \frac{P_o}{(r/b)_o} \right\} \qquad (114)$$

in which Po and Po' are the values of P and P' corresponding to a certain value of r/b denoted by $(r/b)_o$. Furthermore, it is obvious that both Y' and Z' are functions of

$$\frac{W_o^2}{(\frac{\sigma}{\rho} - 1) g d_s} \; ; \; \frac{b}{\sqrt{A_j} \sin\theta} \; ; \; \theta \text{ and } \frac{W_o b}{\nu}$$

Fig. 22 shows, in dimensionless terms, the variation of depth of scour with time, and based on Eq 112. This figure shows that there are apparently three regimes of the scour development: (a) maximum jet deflection, (b) minimum jet deflection, which are identified by Rouse (2), and (c) a final condition of scour.

Significantly, the regimes shift from maximum jet deflection (curves with flatter slopes) to minimum jet deflection at approximately the same time as determined by Rouse (1) for the two-dimensional case.

By the foregoing procedure, values of Y′ and Z′ in Eq 112 are obtained from Fig. 22 for each regime. However, the range of $\frac{W_o^2}{(\frac{\rho_s}{\rho}-1)g d_s}$ in the experimental data is so limited that its effect on Y′ and Z′ cannot be determined. Among the parameters influencing Y′ and Z′, the most effective is the angle of impingement of the jet Θ as can be seen in Figs. 23 and 24. There is, however, a considerable scatter of the data on these figures which is due to other undefined factors as well as experimental errors.

It is evident that there exists opposite trends in Y′ and Z′ for Θ between maximum and minimum jet deflections. Moreover, it seems apparent that the regime of maximum jet deflection disappears at $\Theta = 61°$. When $\Theta < 61°$, there exists only the regimes of minimum jet deflection. On these figures have been plotted the values of Y′ and Z′ for $\Theta = 90°$ estimated from Doddiah's data. It can been seen that Doddiah's data belongs to the regime of maximum jet deflection.

The spread of three points at $\sin\Theta = 1.0$ in Fig. 23 is caused by different values of $\frac{W_o^2}{(\frac{\rho_s}{\rho}-1)g d_s}$ as a third variable. The three dotted lines in Fig. 23 show the variation of $\frac{W_o^2}{(\frac{\rho_s}{\rho}-1)g d_s}$ with $\frac{b}{\sqrt{A_j}}$, of which the various ranges of variation are indicated by heavy lines between the same symbols of the line of $\sin\Theta = 1.0$

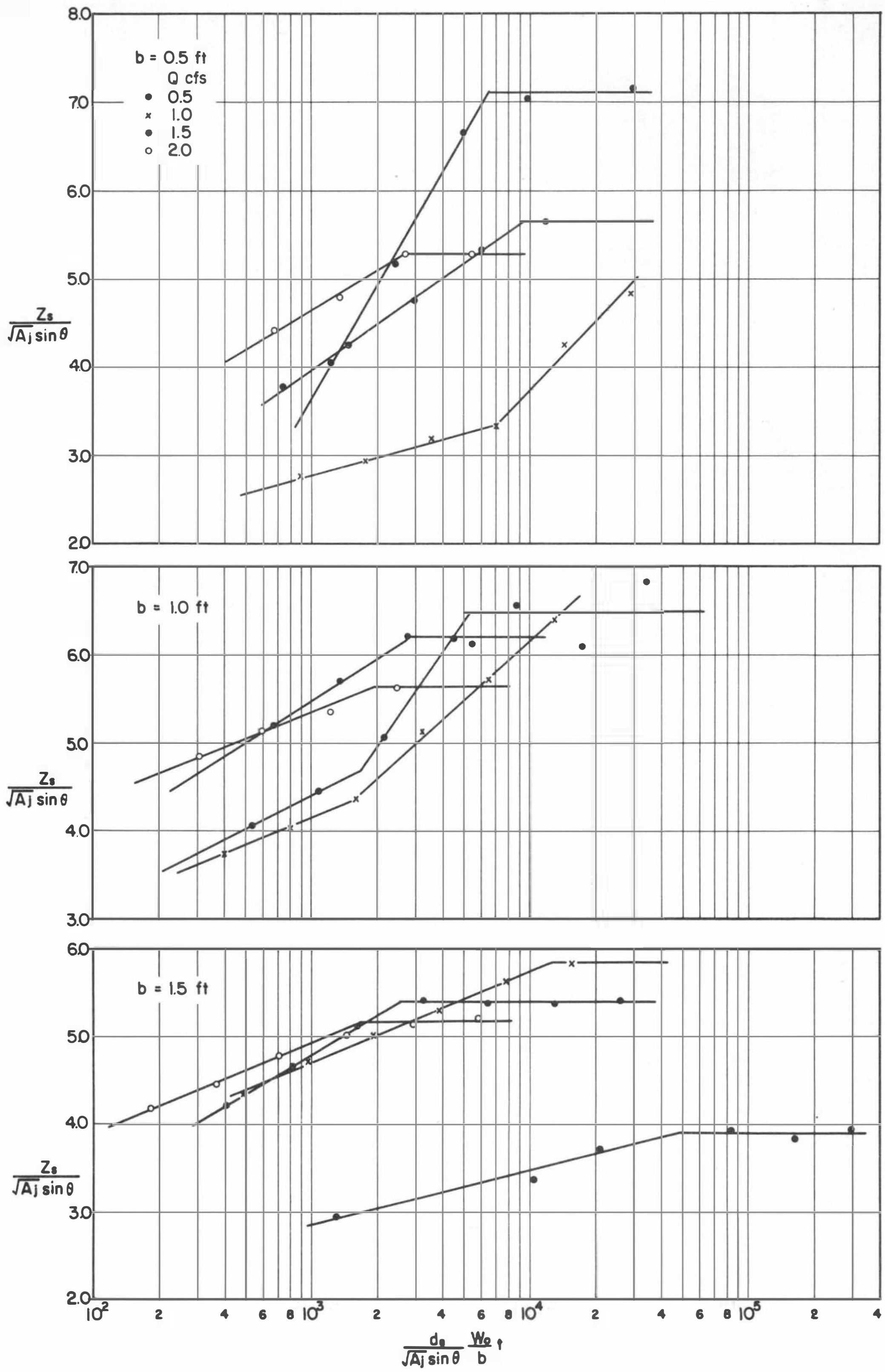

Fig. 22 Variations of $\frac{Z_s}{\sqrt{A_j}\sin\theta}$ with $\frac{d_s}{\sqrt{A_j}\sin\theta}\frac{W_o}{b}t$ for the inclined jet issuing from a non-submerged outlet

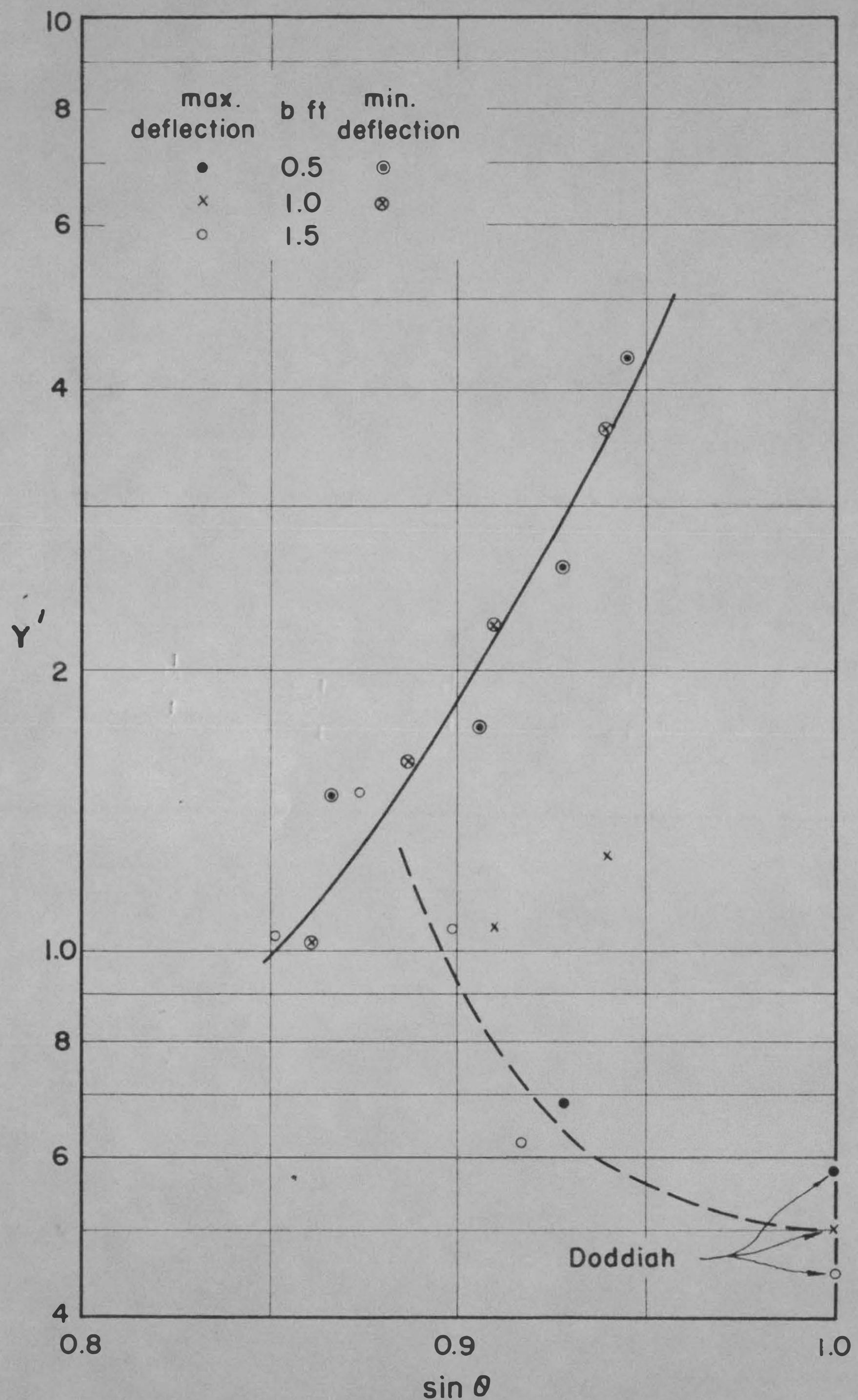

Fig. 23 Effects of the angle of jet θ on Y'

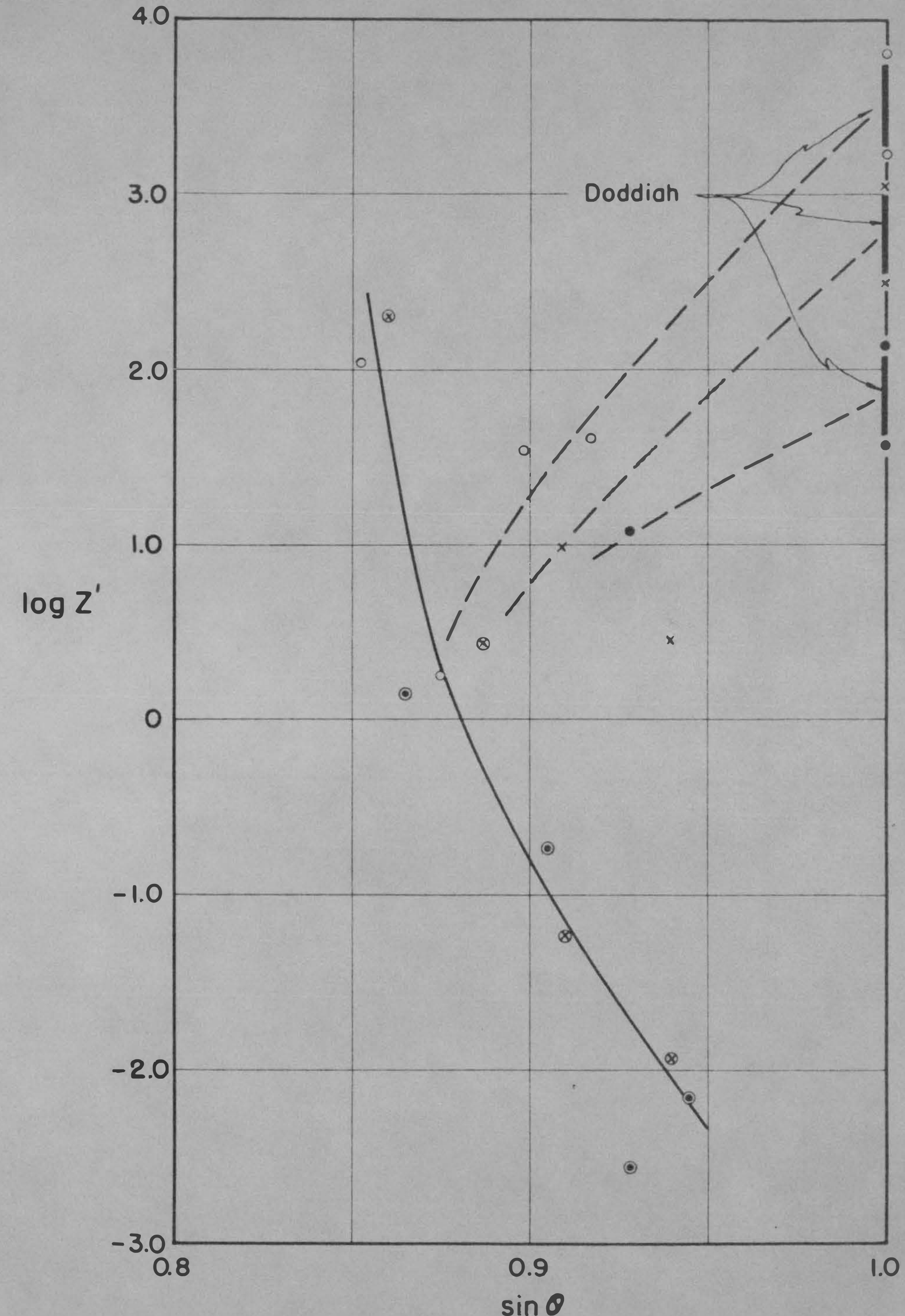

Fig. 24 Effects of the angle of jet θ on Z'

The value of β, which is contained in the parameter Y' of Eq 113, seems to be a function of θ on the basis of experimental results by Homma (9). Evidence of this is shown by the values of β in Eq 75a where β = 0.024 for $\theta = 60^{\circ}$ and Re $\doteqdot$ 25,000 and β = 0.137 for $\theta = 90^{\circ}$ and Re $>$ 30,000 (See Fig.20). This fact is an explanation for the increase in the value of Y' with a decrease in the value of θ for maximum jet deflection. However, the variation of Y' for minimum jet deflection, which shows the opposite trend to that in the case of maximum jet deflection, cannot be explained by the variation of β with θ. The reason for this will depend upon the eventual explanation of the difference between the fluid mechanics of maximum and minimum jet deflection phenomena. Also, there is need for detailed studies of the effect of θ on the value of β and the characteristics of the exponent m in scour phenomena.

The relationship of the final depth of scour for the case of the inclined jet issuing from a non-submerged outlet is expressed by Eq 93. Since it is expected that β is a function of θ and the influence of $\frac{W_0 b}{\nu}$ is small, $\frac{Z_{s\infty}+b}{d \sin\theta}$ depends on $\frac{W_0}{U_{*c}}$, $\frac{b}{d \sin\theta}$ and θ for a given r/b. Replacing d by $\sqrt{A_j}$ and considering $\frac{U_{*c}^2}{(\frac{\sigma}{\rho}-1) g d_s}$ as approximately constant, $\frac{Z_{s\infty}+b}{\sqrt{A_j} \sin\theta}$ at the point of the maximum depth of scour is a function of $\frac{W_0^2}{(\frac{\sigma}{\rho}-1) g d_s}$, $\frac{b}{\sqrt{A_j} \sin\theta}$ and θ. Furthermore, the parameter $\frac{b}{\sqrt{A_j} \sin\theta}$ might be omitted, because Φ_0 for the case of a submerged outlet does not include this parameter as may be seen in Eq. 96. Thus, $\frac{Z_{s\infty}+b}{\sqrt{A_j} \sin\theta}$ is a function only of the parameters $\dfrac{W_0^2}{(\frac{\rho_s}{\rho}-1)g\,d_s}$ and Θ .

Fig. 25 shows a plot of the data for the final depth of scour obtained from Fig. 22. It is evident from this figure that there are two points of discrepancy:

1. For a certain value of Θ , the smaller of the velocity ? of the jet, the greater the depth of scour.

2. For a given velocity of jet, the smaller the angle of jet, the smaller the depth of scour.

Furthermore, the value of β decreases as the depth of scour increases with a decrease in the angle of jet.

The influence of Θ on the final depth of scour, as well as the inconsistencies in Y'' and Z' in the case of minimum jet deflection, needs further investigation.

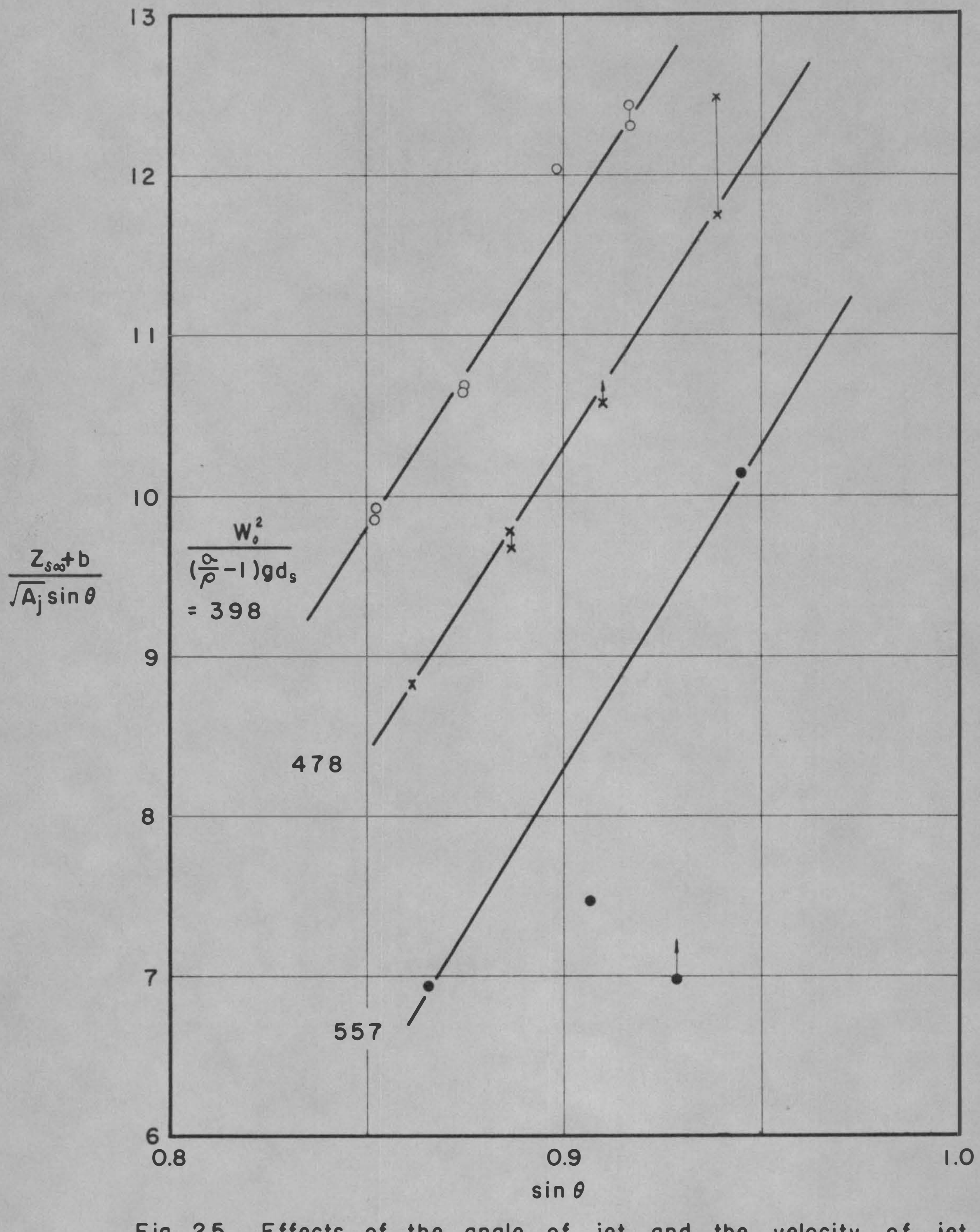

Fig. 25 Effects of the angle of jet and the velocity of jet on the final depth of scour

SUMMARY AND CONCLUSION

Atheoretical analysis has been attempted herein for the mechanics of scour for three-dimensional jets issuing from submerged and non-submerged outlets. In developing the theory, some assumptions have been introduced and some effects have been neglected, such as:

1. It has been assumed that the Bernoulli's theorem is valid in the neighborhood of the stagnation point.

2. The value of L in Eq 36 has been assumed constant in solving the boundary layer equation.

3. It has been assumed that the depth of scour is small compared with the tail water depth in integrating the continuity equation of mass sediment transport.

4. The lateral diffusion along the bed, after impingement of the jet on the bed, has been neglected.

5. The bed has been treated as a hydraulically smooth boundary in obtaining the shear distribution from the boundary layer equation.

6. The equation of sediment transport for open channel flow has been applied, that is, it has been assumed that the rate of sediment transport depends only upon the shear velocity and the size and specific weight of sediment.

This theoretical investigation with the experimental verifications for the two cases of vertical and inclined jets from non-submerged outlets indicates the following:

1. The rate of scour by a jet issuing in or into still water from a submerged or non-submerged outlet is governed by the characteristics of jet diffusion. In the case of a submerged outlet, the depth of scour varies with the power law with respect to time, and in the case of a non-submerged outlet, the variation of the scour depth follows the logarithmic law with respect to time.

2. The dimensionless parameters defining the depth of scour with respect to time, are

$$\frac{Z_s}{d \sin \Theta} \quad \text{and} \quad \frac{d_s}{d \sin \Theta} \quad \frac{W_o t}{b} \quad .$$

3. The effect of sediment characteristics can be expressed by the parameter

$$\frac{W_o^2}{(\frac{\sigma}{\rho} - 1) g d_s} \quad ;$$

however, its validity has not been established by experimental data because of the narrow range of sediment sizes used in the experiments.

4. The tail water depth is introduced by the dimensionless parameter

$$\frac{b}{d \sin \Theta}$$

5. The angle of impingement of the jet Θ indicates a significant influence on the depth of scour in the case of the non-submerged outlet.

6. The dimensionless form for the final depth of scour should be of the form

$$\frac{Z_{s\infty} + b}{d \sin \Theta}$$

which is expressed by power and logarithmic equations with other parameters for the cases of submerged and non-submerged outlets respectively.

Applications of the theoretical results to the analyses of experimental data lead to the following conclusions:

1. In the case of the non-submerged outlet, the variation of the depth of scour with time follows the logarithmic law as found empirically before.

2. The influence of b/d on the value of Y in the case of a vertical jet and of

$$\frac{b}{d \sin \theta}$$

on Y′ in the case of an inclined jet seems to be negligible.

3. There exists three regimes of scour in the case of the three-dimensional jet:

   a) maximum jet deflection

   b) minimum jet deflection

   c) the final condition

   The first two were identified by Rouse for the case of a two-dimensional jet.

4. The regime of maximum jet deflection disappears when the angle θ of jet impingement on the tail water reaches approximately 61°. When θ is less than 61°, the regime is minimum jet deflection only.

ACKNOWLEDGMENTS

This research was performed in the Hydraulic Laboratory of Colorado State University, sponsored by the U. S. Bureau of Public Roads. The permission of Mr. Carl Izzard, Chief of the Division of Hydraulic Research, U. S. Bureau of Public Roads, to present this study is gratefully acknowledged. The writers also wish to thank Mr. Lucien Duckstein, who gave valuable advice and assistance in developing the theory; Mr. Eric Plate for his criticism and assistance in clarification of the basic theory and of the concept of sediment transport for different conditions of scour and deposition; and Dr. A. R. Chamberlain, Acting Dean and Chief, Engineering Research, Colorado State University, for his encouragement in preparation of this report.

REFERENCES

1. Rouse, Hunter. "Criteria for Similarity in the Transportation of Sediment." Proc. Hyd. Conf., Univ. of Iowa Studies in Engineering. Bulletin 20, 1940. pp. 33-49.

2. Iwagaki, Yuichi; Smith, George L.; Duckstain, Lucien and Albertson, Maurice L. "Analytical Study of the Mechanics of Scour for Two-Dimensional Jet." Unpublished paper.

3. Laursen, Emmett M. "Observations on the Nature of Scour." Proc. Fifth Hyd. Conf. Bulletin 34. State Univ. of Iowa, Studies of Engineering. 1953. pp. 179-197.

4. Doddiah, Doddiah; Albertson, Maurice L; and Thomas, Robert. "Scour from Jets." Proc. of Minnesota International Hydraulics Convention. Sept. 1953. pp. 161-169.

5. Smith, George L. "Scour and Energy Dissipation below Culvert Outlets." Dept. of Civ. Eng., Colorado A and M College. April 1957. pp. 1-122.

6. Schlichting, H. "Boundary Layer Theory." McGraw-Hill Book Co., Inc. New York, 1955. p. 73.

7. Schlichting, H. and Truckenbrodt, E. "Die Strömung an einer angeblasenen rotierenden Scheibe." ZAMM Band 32. Heft 4/5. April/Mai, 1952. S. 97-111.

8. Truckenbrodt, E. "Die trubulent Strömung an einer angeblasenen rotierenden Scheibe." ZAMM, Band 34. Heft 4/5. April/Mai 1954. S. 150-162.

9. Homma, M. "An Experimental Study on Water Fall." Proc. of Minnesota International Hydraulic Convention. Sept. 1953. pp. 477-481.

10. Albertson, M. L.; Dia, Y. B., Jensen, R. A. and Rouse H. "Diffusion of Submerged Jets." Trans. ASCE. Vol. 115, 1950. pp. 639-697.

11. Doddiah, Doddiah. "Comparison of Scour Caused by Hollow and Solid Jets of Water." Unpublished Master's thesis. Colorado A and M College. 1950. 156 pages.

12. Smith, George L. "An Analysis of Scour below Culvert Outlets." Unpublished Master's thesis. Colorado State University, 1957. 152 pages.

13. Iwagaki, Yuichi and Tsuchiya, Yoshito. "Some Experiments on Water-Drop Erosims. Trans. Japan Society of Civil Engineers, No. 35. 1956. (in Japanese).

LIST OF SYMBOLS

Symbols	Definition
a	Coefficient of proportionality between U and r or W and Z
A	Ratio of momentum thickness of boundary layer to thickness of boundary layer δ
A_o	Constant in sediment transport equation
A_1	Function of time having positive value
A_j	Cross-sectional area of jet at water surface
b	Tail water depth or height of outlet from bed
b′	$b/\sin\theta$
B	Ratio of displacement thickness of boundary layer to thickness of boundary layer
B_1	Function of time having positive value
C	Function of time having positive value
C_1, C_2, C_3, C_4	Integral constants
d	Diameter of jet at outlet or water surface
d_s	Mean diameter of sediment particle
g	Acceleration of gravity
h	Thickness of deflected jet in case of ideal fluid
k	Constant, 0.0225 for smooth boundary and $m' = 1/7$
L	Coefficient in boundary layer equation
m	Exponent in sediment transport equation
m′	Exponent in power-law velocity distribution
M	Function defined by Eq 54
M_o	M corresponding to a certain value of (r/b)

Symbols	Definition
M'	Derivative of M with respect to (r/b)
M'_o	M' corresponding to a certain value of (r/b)
n	$2m'/(1+m')$
N	Function defined by Eq 65
N'	Derivative of N with respect to (r/b)
p	Pressure intensity at a point
p_o	Pressure intensity at stagnation point
P	Function defined by Eq 82
P_o	P corresponding to a certain value of (r/b)
P'	Derivative of P with respect to (r/b)
P'_o	P' corresponding to a certain value of (r/b)
q_s	Mass rate of sediment transport per unit width
Q	Function defined by Eq 98
Q''	Derivative of Q with respect to (r/b)
r	Radial coordinate parallel to bottom
r'	Radial coordinate perpendicular to inclined jet
r_o	Radius of uniform scour
Re	Reynolds number
s	z/δ
t	Time
U	Horizontal component of velocity at a point
U_o	Horizontal velocity of deflected jet at a point N in Fig. 7
U_b	Horizontal component of velocity along r-axis
U_*	Shear velocity
U_{*c}	Critical shear velocity

Symbol	Definition
V_s	$\left[\left\{(\gamma_s/\gamma)-1\right\} g\, d_s\right]^{1/2}$
W	Vertical component of velocity at a point
Wo	Vertical component of jet velocity at outlet or water surface
Wm	Maximum velocity at center of jet
W_b	Value of W at Z = b
Wbm	Value of Wm at Z = b
W′	Velocity of inclined jet at a point
Wo′	Velocity of inclined jet at outlet or water surface
Wm′	Maximum velocity of jet at center of inclined jet
Wb′	Value of W′ at Z = b
Wb′m	Value of Wm′ corresponding at point along r-axis
Wb′mo	Value of Wb′m at Z = b and r = o
Y	Function defined by Eq 108
Y′	Function defined by Eq 113
Z	Vertical coordinate perpendicular to bottom
Z_M, Z_N	Vertical distances from r-axis to points M and N in Fig. 7
Z_s	Depth of scour
$Z_{s\infty}$	Final depth of scour
Z′	Parallel to inclined jet
Z	Function defined by Eq 109
Z′	Function defined by Eq 114
α	Constant in equation of maximum velocity of inclined jet
β	Coefficient in exponent in equation of maximum velocity of inclined jet

Symbol	Definition
β'	Coefficient in equation of boundary layer thickness
δ	Thickness of boundary layer
δ_*	δ/r
η	Function defined by Eq 56
η'	Function defined by Eq 90
θ	Angle of jet at water surface
λ	Porosity of sediment
ν	Kinematic viscosity of water
ξ	Function defined by Eq 67
ξ'	Function defined by Eq 104
ρ	Density of water
σ	Density of sediment
ϕ	Function expressed by Eq 21
ϕ_o	ϕ at $Z = b$
Φ	Function representing velocity distribution of inclined jet
Φ_o	Φ at $Z = b$
ψ	Angular Coordinate

DISCUSSION

E.M. Laursen[1] Eq 1 is correct. The sections on conditions of application of the continuity equation and ideal fluid seem to be principally for academic interest. The essence of the study begins with the section on real flow: non-submerged outlet.

The purpose of the paper is commendable--to obtain a transport relation as a function of r and Z_s and the initial conditions of Wo, d and Z. The attempt is valiant and the general idea of the required steps reasonable. Individual steps, however, are more or less questionable. In order of appearance, they are as follows:

1. As demonstrated by Corrsin in his discussion of "Diffusion of Submerged Jets" by Albertson (10), a simple similarity criteria $W/W_m = f\ (r/b)$ where b is any typical width of the jet, and the assumption of constant pressure throughout the field is sufficient to demonstrate linear spread of the jet and that $W_m/W_o \propto 1/X$ where X is the distance from the free surface. On this basis, Eqs 21 and 22 seem to be at variance with Eq 23.

2. Application of Bernoulli's theorem to obtain the horizontal or radial velocity distribution in Eq 27 is a questionable procedure. It is sufficient for obtaining an approximation of the velocity distribution near the stagnation point, but as the jet continues to spread by diffusion and is affected by boundary layer development becomes less and less applicable.

3. The determination of shear velocity is only approximate but is better than the previous assumptions that enter into the relationship.

4. Eq 41 is a sediment transport equation that upon examination seems questionable as to its validity. For example, dropping the critical term by making m a variable exponent is only proper if m becomes infinite when the shear becomes critical. By disregarding critical shear and/or the fact that m becomes infinite the scour seems to proceed to infinity at infinite time. More important,

[1] Associate Professor, Department of Civil Engineering, Michigan State University, East Lansing, Michigan.

the influence of some critical term is completely overlooked. At low rates of scour the shear will be almost the critical shear and the latter should not be ignored.

L. K. Duckstein , Y. Iwagaki , and G. L. Smith .--The comments by Mr. Laursen were greatly appreciated.

Mr. Laursen, in reference to the discussion by Corrsin, points out that Eqs 21 and 22 seem to be at variance with Eq 23.

In his discussion, Corrsin actually proves that a similarity hypothesis of the type of Eq 23 necessarily leads to a vertical velocity distribution where the velocity relationship $\overline{W}_m / \overline{W}_o$ is proportional to (b - z / d).

It should be noted, however, that Eqs 21 and 22 are experimental expressions, which have been verified by experimental data, that define a jet issuing from an non-submerged outlet; whereas, Eq 23 is an experimental expression for a jet issuing from a submerged outlet.

Since, in this analytical study of scour, we are considering jet flow from non-submerged outlets; a more rigorous approach, as implied by Mr. Laursen, would be to replace Eq. 23 by a general relationship of the type

$$\frac{\overline{W}}{\overline{W}_M} = \Phi\left(\frac{r}{b}, \frac{b-z}{d}, \frac{b}{d}\right)$$

Although the function Φ will have to be determined by experiment, Eq 23 may be considered as an order of magnitude of the function Φ.

A basic assumption in this study was that the Bernoulli equation could be applied in the neighborhood of the stagnation point to determine the velocity distribution. As a first approximation, it seems to give a representative velocity distribution. It is recognized, however, that the application of the Bernoulli equation, as the jet continues to spread by diffusion in the radial direction and is affected by boundary layer development, can be substantiated only by experiment.

An important point of this study is that it is valid only at the initial stage of the scouring phenomenon. Therefore, to better understand the scour phenomena, Eq 41 must be applied instead of Eq 42, and $(b - Z)$ must replace b. The solution of Eq 43 then requires numerical integration. However, the limiting value of the scour hole depth can be obtained by making q_s equal to zero. Furthermore, by substituting Eq 41 into Eq 43 an approximate value of the time t when the depth is 99 per cent of its final value can be determined.