PRESSURE DISTRIBUTION DURING STEADY FLOW
IN UNSATURATED SANDS

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ABSTRACT

A differential equation is derived which describes the pressure
distribution during steady flow in a porous material occupied by two
immiscible fluids such as air and water. It is assumed that Darcy's
equation applies simultaneously to the wetting and the non-wetting
phase. Each phase is assumed to be continuous, and therefore, any
isolated portions of either phase must be regarded as part of the
porous matrix. The equation may be applied to fluids flowing in
any direction with respect to each other or in any direction with
respect to the earth's gravitational field. In order to solve the equa-
tion, it is necessary to know the relationship between the pressure
discontinuity across interfaces between the phases and the conduct-
ivity of the flowing phase or phases. The nature of this function and
a method of obtaining it are discussed briefly.

Experiments were conducted using a hydrocarbon liquid as the
wetting fluid, air as the non-wetting fluid, and long columns of sand
as porous media. Several cases were investigated, and results of
two are presented: (1) Downward flow through a uniform sand, and
(2) Downward flow through a sand into another sand of finer texture.
Good agreement between experimental data and theory was obtained
for all cases.

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Introduction

Steady flow in unsaturated soils is a phenomenon that rarely if ever exists in nature. Nevertheless, an anlysis of this type of flow leads to the development of certain principles that can be applied qualitatively to situations of practical interest. Such situations include the drainage of soils and evaporation of water from the surface of soils in contact with a water table.

In 1945 Childs (1) reported a study of steady downward flow through long columns of uniform soil. He concluded that (provided the column is sufficiently long) the moisture content and suction are constant over a considerable length of the column and, as a consequence, the potential gradient is the gravitational gradient. Childs and George (2) have also pointed out that the existence of an invariant suction over a considerable portion of a long column during steady downward flow can be the basis of a convenient method of measuring the permeability of unsaturated soils.

The analysis presented here provides an explanation for these and other phenomena associated with the steady flow of fluids through porous solids.

Analysis

It is assumed that Darcy's equation applies simultaneously to a wetting and a non-wetting phase as was first suggested by Muskat and Mere's (7). Any isolated mass of fluid is regarded as a part of the porous matrix and not as a part of either fluid phase. The forces in each phase are assumed to constitute separately a conservative potential field in a homogeneous and isotropic matrix.

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The capillary pressure p_c is defined as

$$p_c = p_{nw} - p_w \qquad (1)$$

so that

$$\frac{\partial p_c}{\partial r} = \frac{\partial p_{nw}}{\partial r} - \frac{\partial p_w}{\partial r} \qquad (2)$$

Where the subscripts nw and w refer to the non-wetting and wetting phases, respectively, and r is any direction in which it is desired to find the variation of capillary pressure. For the range of p_c investigated in this study, the term p_c is identical with "suction" as the latter term is usually employed.

For a situation in which the only body force acting on the fluid is gravity, Darcy's equation may be written as

$$q = -\frac{K_e}{\eta} \left(\frac{\partial p}{\partial r} + \rho g \sin \Theta \right) \qquad (3)$$

in which q is the component of volume flow per unit of area in the direction of r, K_e is the effective permeability, η is the fluid viscosity, ρ is the fluid density, and g is the force per unit mass due to gravity, and Θ is the angle of r with the horizontal.

Solving equation 3 for $\frac{p}{r}$ and substituting its equivalent into equation 2 gives

$$\frac{\partial p_c}{\partial r} = \left(\frac{q\eta}{K_e} + \rho g \sin \Theta \right)_w - \left(\frac{q\eta}{K_e} + \rho g \sin \Theta \right)_{nw} \qquad (4)$$

For the sake of brevity, this is written as

$$\frac{\partial p_c}{\partial r} = \left[\Delta(\rho g) \right] \sin \Theta - \Delta \left(\frac{q\eta}{K_e} \right) \qquad (5)$$

Equation 5 can be written as an ordinary differential equation provided the following conditions (in addition to those previously mentioned) exist:

1. The system is at steady state

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2. Thermal equilibrium exists

3. The flow rate q does not vary in space for either phase.

4. The properties of the matrix do not vary in space.

With the foregoing condition satisfied, equation 5 becomes

$$\frac{dp_c}{dr} = \left[\Delta(\rho g)\right] \sin \Theta - \Delta\left(\frac{q n}{K_e}\right) \qquad (6)$$

If it is assumed further that the flow is in one direction only (and there-fore its component in any other direction is zero), it is permissible to drop the requirement that the matrix be isotropic. In this case, K_e may be regarded as the effective permeability in the direction r. The term "effective" is used here to distinguish the permeability of a par-ticular fluid in the presence of another fluid from the permeability K that exists when only one fluid occupies the pores of the matrix.

An insight into the significance of equation 6 can be gained simply by inspection. If there is no flow of either phase, the equation merely describes the rate of change of p_c in the direction r resulting from a change in elevation. On the other hand, if there is no change in ele-vation in the direction r (as for horizontal flow), the equation describes only the difference in the rate of pressure loss of the two phases result-ing from flow. In the general case, the combined effect of both factors is described. With proper consideration of signs, the equation should be applicable to situations in which two fluids are flowing in the same, oblique, or even opposite directions.

Equation 6 can be written in terms of dimensionless variables by dividing both sides by $(\Delta \rho g) \sin \Theta$, and scaling p_c by dividing it by a characteristic capillary pressure p_d. A description of a para-meter p_d and the method of its determination are given in the discus-sion of experimental procedures. The term K_e may be replaced by its equivalent $K_r K$, where K_r is the relative permeability for the particular phase under consideration and K is the permeability for a saturation of unity, and the term $r \sin \Theta$ by its equivalent, the elevation z.

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For the case of a static non-wetting phase, equation 6 may be written in terms of the scaled variables as

$$\frac{dP.}{dZ.} = 1 + \frac{Q.}{K_{rw}} \tag{7}$$

in which $P.$ is p_c/p_d, $Z.$ is $\dfrac{z\,(\Delta\rho g)}{p_d}$, and $Q.$ is $\Delta\left(\dfrac{q\eta}{K_e(\Delta\rho g)\sin\Theta}\right)$.

The designation of scaled variables by dots after the letters follows the precedent set by Miller and Miller (5). The scaled variables permit the behavior of laboratory models to be compared with the behavior of more extensive systems in the field. The values of scaled variables do not depend on the units employed, provided that a consistent set of units is used.

Equation 7 can be solved for particular values of $Q.$ provided the functional relationship

$$K_{rw} = f(P.) \tag{8}$$

is known or assumed. The method of determining K_{rw} as a function of $P.$ is described in the section dealing with experimental procedures. For the present purposes it is sufficient to assume that K_{rw} is a continuous function of $P.$ and furthermore that K_{rw} decreases with increasing $P.$. The analysis that follows is applicable only to the cycle of increasing $P.$ so that the problem of hysteresis is avoided.

For downward flow, the value of $Q.$ is negative. When the matrix is completely saturated and the downward flow is steady, K_{rw} has a value of one, and $Q.$ is a constant. The solution of equation 7 for the latter case is

$$P. = (1 - |Q.|)\,Z. + c \tag{9}$$

which is a linear equation. By defining $Z.$ as zero where $P.$ is zero, the constant c can be eliminated. For very small flow rates

$$P. \approx Z. \tag{10}$$

which can be written as an exact solution for the static case. As $P.$ is increased, K_{rw} is decreased and will eventually reach the same value as $|Q.|$. When $K_{rw} = |Q.|$ for steady downward flow in a uniform matrix, the solution of equation 7 is

$$\frac{dP.}{dZ.} = 1 - \frac{|Q.|}{K_{rw}} = 0 \tag{11}$$

or

$$P. = \text{a constant.} \tag{12}$$

Theoretically $P.$ could not become greater than this constant because a smaller value of K_{rw} would result in a negative value of $dP./dZ.$ and thus cause $P.$ to decrease. If the value of $P.$ is large over a substantial portion of the column, the value of $Q.$ will be small because of the high resistance of the column to flow.

A long vertical column of uniform sand is considered which is assumed to be partly saturated with a wetting liquid, the non-wetting phase being air. The wetting liquid is flowing downward at a steady rate. The value of $P.$ is substantially greater than unity for most of the column. At the bottom of the column $P.$ is zero and $Z.$ is defined as zero at this point. At the top of the column, $P.$ is some arbitrary constant, say 2.0. Without knowing anything more about K_{rw} as a function of $P.$ than has already been assumed, it is possible to deduce the following:

1. Near the bottom of the column, $P.$ will vary linearly with $Z.$ and probably $P. \approx Z.$.

2. Near the top of the column, provided the column is sufficiently long, $P.$ will be invariant with $Z.$.

3. The portion of the column between the region where $P. \approx Z.$ and that where $P.$ is invariant will consist of a transition zone in which $P.$ will approach a limiting value asymptotically.

4. The value of K_{rw} near the top of the column will be the value of $|Q.|$.

By assuming a particular functional relationship for equation 8 it is possible to predict in detail the curve representing P. vs Z. . It is also possible to predict the form of P. vs Z. when the column undergoes a change in texture, a change in slope, or when Q. changes abruptly as a result of a source or sink for the flow at some point within the column. Before the latter situations are discussed, however, the experimental procedures and results are explained in order to justify the assumption of a particular kind of relationship between K_{rw} and P. .

Experimental Procedures

In all of the laboratory tests of equation 7 , the non-wetting phase was air and was stationary. The wetting phase was a hydrocarbon oil called Soltrol[1], and the matrix was sand packed into a Lucite tube which ranged in length up to five feet. The tube consisted of a large number of short sections separated by tensiometer rings. No attempt was made to seal the joints in the assembled column, because it was desired that air (at atmospheric pressure) be in contact with the column at as many points as possible in order to hasten equilibrium.

Before each experimental run, the entire column was vacuum-saturated with Soltrol. This was accomplished by clamping the column into a metal tray made especially for the column, surrounding the column with Soltrol, covering the container with a lid which sealed the system, and then evacuating the entire assembly. When the column was returned to atmospheric pressure, it appeared to be completely saturated with Soltrol. The metal tray without the lid served as a support for mounting the column in its position for the experimental run. In cases where the run involved a change in slope, the container was built to accomodate column of this shape.

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[1] Phillips Cone Test Fluid, produced by Phillips Petroleum Co., Special Products Division, Bartlesville, Oklahoma.

The use of Soltrol as the wetting phase permitted values of Z. about twice as great as would be possible with water because the smaller surface tension of Soltrol results in smaller values of p_d for particular sands. Soltrol also has a much smaller vapor pressure than water and as a consequence, the problem of evaporation from joints in the column was greatly diminished. Because of the more consistent wetting properties of Soltrol, pressure-sensing manometers consisting of vertical capillary tubes were less subject to error due to variations in capillarity than is the case with water. The sands which have small values of p_d also permitted much larger scaled elevations than would have been possible with fine soils.

The tensiometers consisted of annular strips of Porvic[2] cemented over grooves machined into the inside wall of the Lucite tube, an arrangement that permitted the manometers to equilibrate in much less time than would be possible with any ceramic tensiometer commonly employed. Because the Porvic strips were flush with the inside wall of the tube, the tensiometers did not reduce the cross-sectional area of the column or interfere with the flow in any way.

Disks made from the same porous plastic were placed at the top and bottom of the column, and were connected to constant-head siphons to provide the means whereby the capillary pressure within the column was controlled. In all cases, the wetting phase was maintained at a pressure less than atmospheric so that there was no tendency for the Soltrol to run out the joints of the Lucite tube. A typical experimental arrangement is shown in figure 1.

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[2] Supplied by Pritchett and Gold and E.P.S., Co., Dagenham Dock, Essex England.
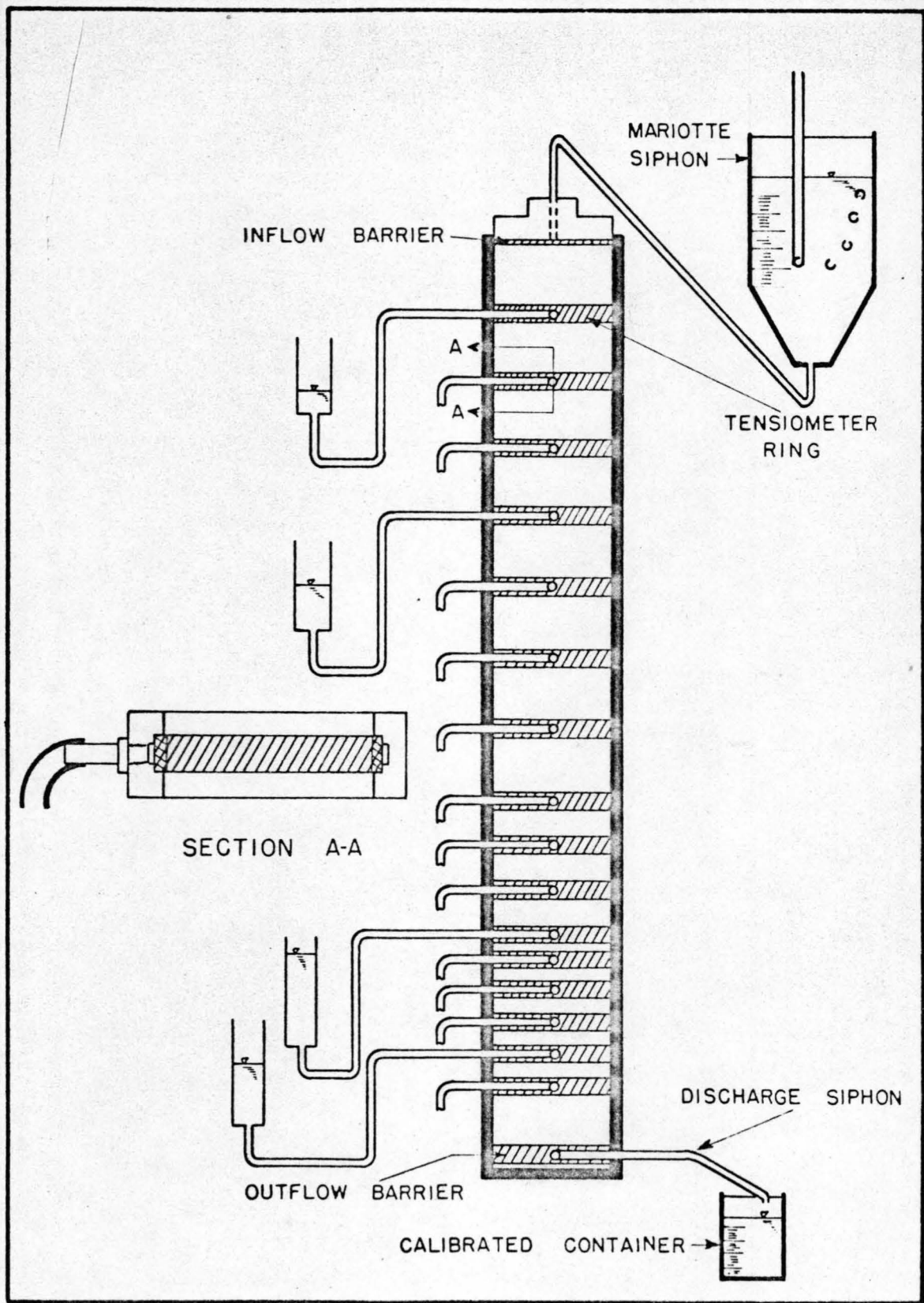
FIGURE *I* — SCHEMATIC DIAGRAM OF EXPERIMENTAL APPARATUS

All tests were on the drainage cycle; i.e., each successive run (with a particular setup) was conducted with a smaller saturation than the preceding run. In each case, the first run was conducted with the entire column at a p_c sufficiently small to avoid desaturating the columns in order to obtain K_w for the sand or sands packed into the tube.

The curves of K_{rw} vs p_c were determined for the sands in the assembled column. The technique for doing this was a modification of Richards (8) original controlled-pressure method. It was similar in some respects to Childs' and George's (3) long-column method. Downward flow was established under a hydraulic gradient of unity and with an increased value of p_c and smaller value of Q. for each succeeding run. The p_c at the extreme lower end of the column was kept sufficiently small to insure complete saturation in that region. The rates of flow into and out of the column were measured and when these rates were the same, the system was assumed to be at steady state. At steady state, the average was determined over that portion of the column in which p_c was constant except for deviations caused by variations in packing. The average p_c was determined by averaging the readings of the manometers attached to the tensiometers.

When K_{rw} had been determined over the desired range of p_c, a graph of ln K_{rw} vs ln p_c was made as shown in figure 2. In every case the curve was linear over most of the range of p_c, the exception being in the range of very small values of p_c. An extrapolation of the linear portion of the curve to the abscissa representing $K_{rw} = 1.0$ was used to define the parameter p_d as shown in figure 2.

Having determined the value of the scaling factor p_d, a graph of P. vs Z. was made for the entire column for each run.
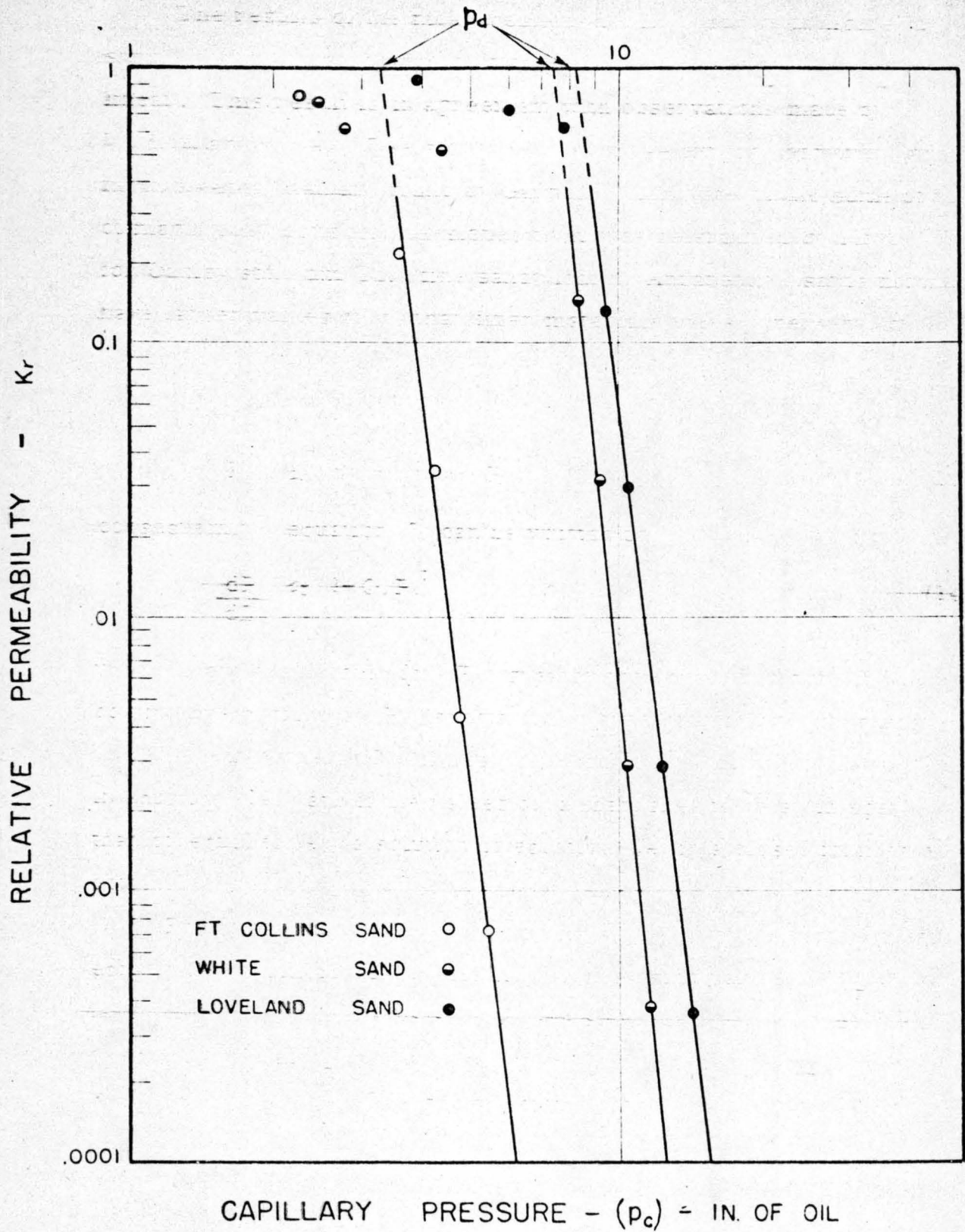
FIGURE 2 -   Relative Permeability As A Function Of
              Capillary Pressure For Three Sands

Results

The results of the measurements of K_{rw} shown in figure 2 show that for the sands investigated, the curves of K_{rw} vs p_c are linear. This result is in agreement with observations made by W. R. Gardner (4). The slopes (-n) of these curves, however, are much steeper than any found by Gardner. They are of the same order of magnitude, however, as slopes of curves determined by Moore (6) for Oakley sand and Yolo fine sandy loam. Apparently, sands normally have larger values of n than finer materials unless, perhaps, if the finer materials are well aggregated.

It is apparent that for values of P. greater than one,

$$K_{rw} \approx \left(p_c / p_{d}\right)^n \approx P.^n \tag{13}$$

consequently, equation 7 can be written as

$$\frac{dP.}{dZ.} \approx 1 + Q.P.^n \tag{14}$$

which is applicable only for P. greater than one. If a part of the column has a value of P. substantially greater than one, it is known that $P. \approx Z.$ where P. is less than one. The variables are separable in equation 14, and if the values of n happened to be an integer less than or equal to 4, a solution of equation 14 in a closed form is easily obtained.

For large values of n , however, it is more convenient to solve equation 7 numerically by plotting slopes using values of K_{rw} vs P. taken directly from graphs such as are shown in figure 2. Solutions of equation 7 along with data which substantiate the theory are shown in figures 3 and 4. The smooth curves drawn on the figures were computed for the measured values of Q. using equation 7 and graphs similar to those in figure 2. The smooth curves are not to be construed as best-fit curves through the data.
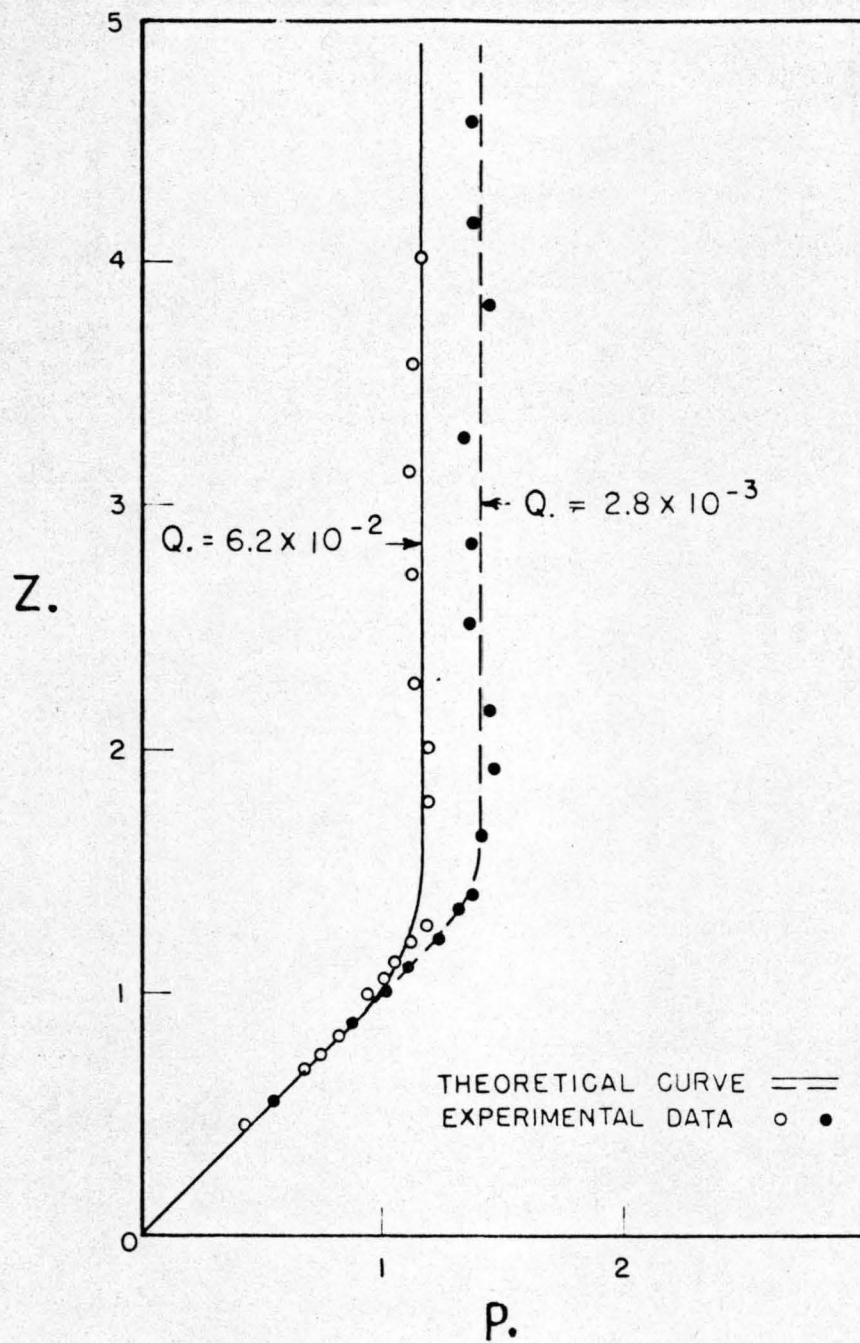
FIGURE 3 – Scaled Capillary Pressure P. As A Function Of Scaled Elevation Z. For Downward Flow Through A Uniform Sand
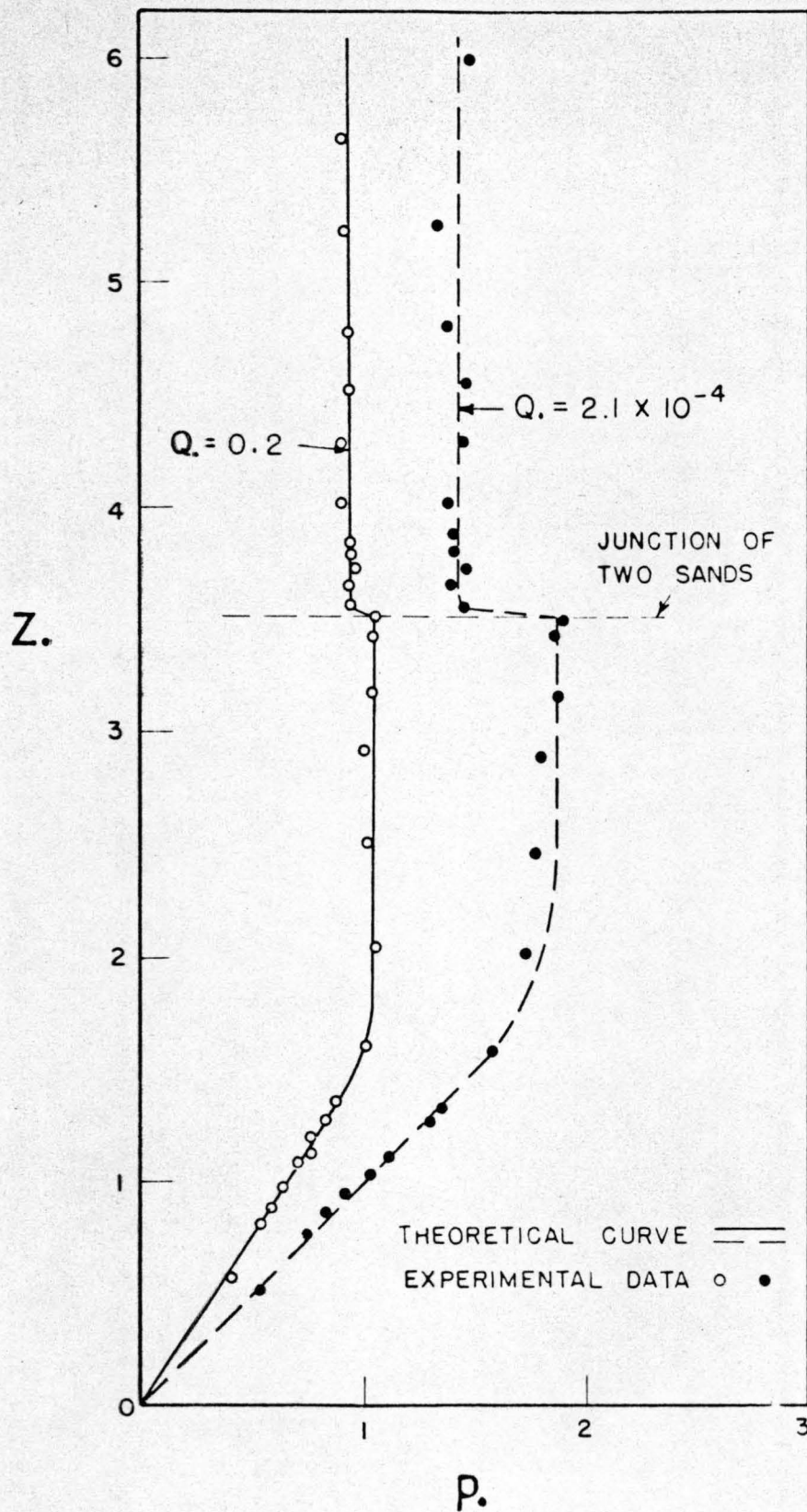
FIGURE 4 — Scaled Capillary Pressure P. As A Function Of Scaled Elevation Z. For Downward Flow Through A Sand Into Another Sand Of Finer Texture

Figure 3 shows the results for steady downward flow through a column of uniform sand. Figure 4 shows the results for steady downward flow through a sand into another sand having a slightly finer texture. In plotting the graph for figure 4, the p_d of the lower sand was used as a scaling parameter for both sands in order to avoid a discontinuity in scales at the juncture of the two sands.

Many similar curves were obtained for other values of $|Q.|$ and for other situations such as those that involved changes in slope, sources and sinks within the column, and flow from a finer to a coarser-textured sand. By coarser-textured, in this case, is meant a sand having a smaller p_d. The results for virtually every test (which have all been reported by Scott (9)) were in as good agreement with equation 7 as those shown in figures 3 and 4.

The results shown in figure 4 demonstrate the interesting phenomenon that when steady flow is downward through a long column of unsaturated sand, the effective permeability tends to reach the same value in each stratum provided the stratum are sufficiently thick. At the bottom of a coarse-textured stratum underlain by a finer-textured stratum, however, there will develop a region of very low saturation and low permeability. The reason for this is that the relationship between p_c and Z. must be continuous regardless of abrupt changes in texture. If this were not true, there would be an infinite pressure gradient at the juncture of two constrasting sands. Since this is a physical impossibility, it is obvious that the p_c at the juncture must be the same for both sands and, as a consequence, the sand having the coarser texture will have a smaller saturation and permeability. This generalization would, of course, not apply if both sands were still completely saturated because in this case the sand with the smaller p_d would usually have the higher permeability.

The zone of low effective permeability at the bottom of coarse-textured strata during downward flow undoubtedly accounts for the abnormally low suctions in and above such strata for long periods following rains or irrigations. This can also explain why the moisture equivalent is a poor approximation of field capacity for coarse-textured soils.

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