Proceedings of the

Symposium on

Transient Ground Water Hydraulics

JULY 25-27, 1963

CIVIL ENGINEERING SECTION

COLORADO STATE UNIVERSITY

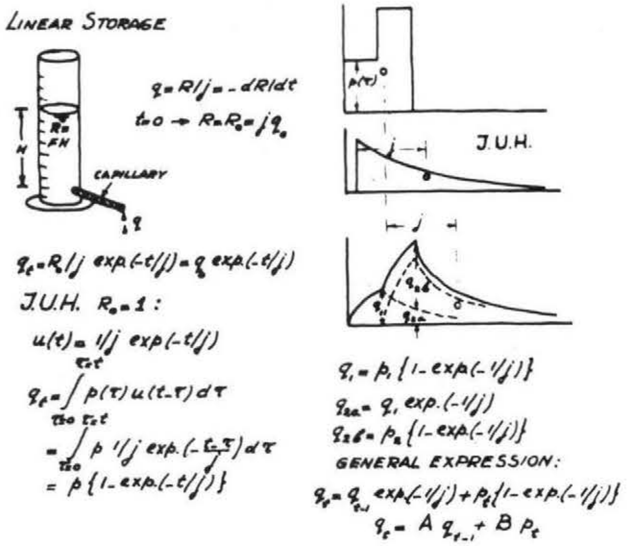
FORT COLLINS COLORADO

CER63DEM-MWB70

PROCEEDINGS ON THE
SYMPOSIUM ON TRANSIENT GROUND WATER HYDRAULICS
JULY, 1963

ERRATA

Page 9: Figure 9 should be replaced with figure below



Page 18: Left column, SCHEIDEGGER is misspelled twice.

Page 58: In figure 10, the dashed line (f) should be horizontal at the well.

Page 60: In figure 11, the part of the circuit connecting Ri with A is missing.

Page 87: Under Comments, the eq. is $L = \dfrac{9\alpha t}{2\left(\dfrac{H}{Z_c} - 1\right)}$

Page 111: First equation should read:

$$h(x,z,t) = h_o \frac{N_o(\alpha\rho)}{N_o(\alpha)} \sin\left[\frac{2\pi t}{t_o} + \phi_o(\alpha\rho) - \phi_o(\alpha)\right] - h_o A_1(\rho,\tau,\tau_o)$$

Page 113: Second Boundary Condition (IA) is

$$h(0,t) = h_s(0) + \delta h_1 = h_1 + \delta h_1$$

Page 116: On the right figure, the "datum plane" should be the base of the aquifer.
Second Boundary Condition is $h(x,y,o) = h_o$

Page 119: The title of Case A-20 should be: Fluctuations of water levels in response to flood waves.

Page 153: On the graph, the ordinate is $\frac{S}{\frac{Q}{2\pi KD}}$ and the third variable is $\frac{\alpha t}{b^2}$ (0.02 to 0.40 and ∞).

Page 155: Under II, Average Drawdown, the first equation is

$$\overline{s} = \frac{Q}{4\pi Kb} \left[W(u) + \overline{f} \left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{l'}{b}, \frac{d'}{b} \right) \right]$$

Under (a) Small values of time

$$t < \frac{(2b - l - l')^2 \, S_s}{20K}$$

Page 157: On first drawing, the thickness of the aquifer is

$$b = b_o \exp\left(-2 \frac{x - x_o}{a}\right)$$

Page 160: Under Symbols, $r_o = (a/2) \ln(a/10b_o)$

Page 163: The last equation should be

$$f(R, m, r_w) = 2 \ln(R/m) \ln(m/r_w) + \ln(m/2r_w) \ln(R/r_w)$$

Page 166: Under Solutions: The drawdown in the caisson of a collector well etc.

$$s_c = \frac{Q}{2\pi Kl} \left\{ 0.5 \frac{\left(\frac{Z_i}{a}\right)^2 \left(\frac{l}{a}\right)}{\left[1 - \left(\frac{l}{2a}\right)^2\right]^2} + \ln \frac{2\left(1 + \frac{2Z_i}{r_w}\right)}{1 + \sqrt{1 + \left(\frac{2Z_i}{l}\right)^2}} \right\}$$

provided $l > 10 r_w$, $t \geq a^2 S_s/K$ and $Z_i \leq \frac{a}{2}$.

Page 199: Under Solutions: 2.(a) Expressions for short times,

For $t_o < 0.09/\nu\alpha^2$

Page 200: Under Solutions: 3.(b) Expressions for long times,

$$q_s = Q \exp\left(-\frac{\nu t}{B^2}\right) \left\{ 1 - \exp\left[x_o \left(\frac{1}{\beta} - \alpha + \frac{1}{2\alpha B^2} \right) \right] \right\}$$

Page 202: Under Symbols, $u' = r'^2/4\nu t$, and

$$W(u, w) = \int_u^\infty \frac{1}{y} \exp\left(-y - \frac{w^2}{4y}\right) dy$$

PROCEEDINGS OF THE SYMPOSIUM ON TRANSIENT GROUND WATER HYDRAULICS

held at

Colorado State University
Fort Collins, Colorado
July 25-27, 1963

edited by

D. E. L. Maasland and Morton W. Bittinger

December 1963

CER63DEM-MWB70

PREFACE

The non-equilibrium equation, introduced by C. V. Theis in 1935, sparked the beginning of the capability to adequately describe the dynamics of a ground-water system mathematically. In the ensuing years a wealth of literature has been developed to meet many geohydrologic conditions. This literature is scattered throughout many scientific journals, bulletins, and reports. Unsteady state, or transient, ground-water hydraulics has developed into an important but somewhat uncoordinated field of endeavor. Several disciplines, including soil physics, soil engineering, drainage engineering, ground-water hydrology, and petroleum engineering have produced significant contributions.

Because of the scattered literature, and the rare opportunity for individuals of various disciplines working in the field of porous media hydraulics to meet to discuss their common interests and problems, it was decided to hold the "Symposium on Transient Ground-Water Hydraulics." The Symposium was planned by a committee composed of R. H. Brooks, Agricultural Research Service, R. E. Glover, U. S. Bureau of Reclamation, R. W. Stallman, U. S. Geological Survey, and the undersigned. The panel discussion approach was decided upon in order to stimulate the exchange of ideas. Those contacted to serve on panels and act as moderators responded enthusiastically. The attendance of over 150 individuals and the active discussions during the Symposium further assured the planning committee and the sponsors that their efforts served a worthwhile purpose.

The first three one-half day sessions of the Symposium (July 25-26) were devoted to mathematical developments. An admittedly rather arbitrary differentiation was made for these sessions in order to limit the extent of each. The first session dealt with cases generally described by a rectangular coordinate system, whereas,the second session included cases described by a cylindrical coordinate system. Discussion of leaky aquifer conditions was reserved for the third one-half day. The fourth and final session (July 27) was devoted to discussions of the use of models, analogs and computers in the field of transient ground-water hydraulics.

All sessions of the Symposium were recorded on tape. Those participating in the Symposium kindly reviewed the transcribed material and furnished illustrations for the Proceedings. The first four parts of the Proceedings include the discussions held during the four one-half day sessions. These are followed by a talk presented the evening of July 25 by Dr. H. K. Van Poollen of the Marathon Oil Company. Appendicies A, B, and C include summaries of mathematical developments, largely prepared by the participants. These served as reference material during the Symposium. Two papers pertinent to the subject, submitted after the Symposium, are included in the Proceedings as Appendices D and E. Appendices F and G are lists of selected references. The first, on ground-water models, was prepared by A. I. Johnson of the U. S. Geological Survey. The second, Appendix G, lists mainly publications in which transient ground-water hydraulic equations are developed.

Many individuals and agencies contributed to the success of the Symposium. Thanks are due to the planning committee and the participants, as well as to the agencies, institutions, and firms which were represented. Colorado State University personnel helping with local arrangements included R. A. Longenbaugh, M. M. Skinner, George Palos, Ali Eshett, E. Bruce Jones, and Wayne Stafford.

Financial help to defray the cost of preparing the Proceedings was supplied by a grant from the National Science Foundation. The sponsoring institution and the editors gratefully acknowledge this support.

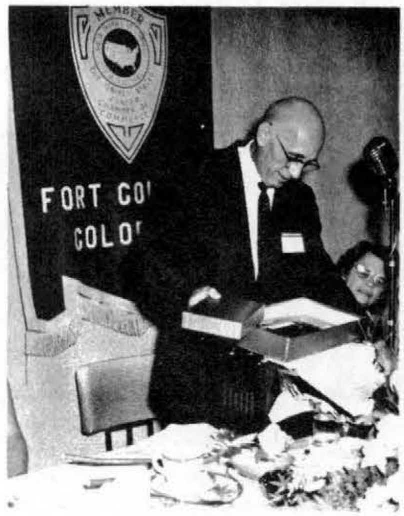
D. E. L. Maasland

Morton W. Bittinger
Editors

SYMPOSIUM BANQUET, JULY 26, 1963

Dr. and Mrs. C. V. Theis and Stan Lohman.

Mrs. Theis receives a gift from Master of Ceremonies Lohman -



- then C. V. receives tools no ground-water hydrologist should be without, a willow stick and an electronic black box with an abundance of dials, buttons, flashing lights and buzzers. Finally, the principal gift from the U.S.G.S. colleagues, a desk pen set inscribed with the following:

To Dr. Charles V. Theis
for his many contributions to the science of ground-water hydrology
from admiring colleagues
Fort Collins, Colorado, July 26, 1963

It is in the same spirit that these Proceedings
of the Symposium of Transient Ground Water Hydraulics
are dedicated to Dr. Theis.

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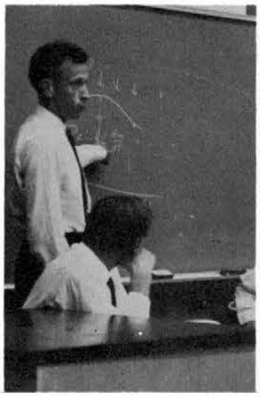
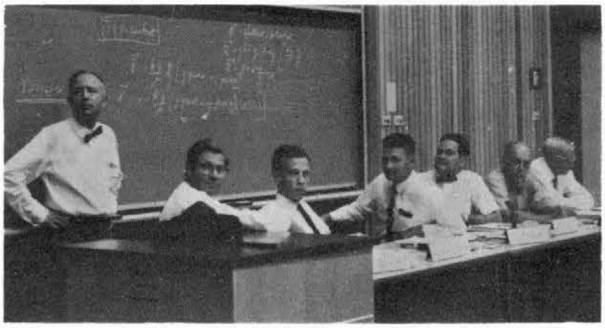
TABLE OF CONTENTS



Panel Session 1: Left to right: Dr. Zane Spiegel, Dr. Jan van Schilfgaarde, Dr. A. T. Corey (moderator), Mr. John G. Ferris, Mr. R. E. Glover, Mr. D. A. Kraijenhoff van de Leur and Dr. R. J. M. De Wiest.

Not shown, Mr. R. W. Nelson, shown below between Dr. van Schilfgaarde and Dr. Corey.



(Photographs by George Palos)

SESSION 1

MATHEMATICAL DEVELOPMENTS IN TRANSIENT GROUND WATER HYDRAULICS USING A RECTANGULAR COORDINATE SYSTEM

MR. BITTINGER: I would like to introduce Dr. M. L. Albertson to give you a word of welcome. He has made the printed program obsolete by changing jobs. He is now Director of the Office of International Programs here at CSU. Dr. Albertson is also a Professor of Civil Engineering with a great deal of experience and interest in hydraulics and ground water.

DR. ALBERTSON: It is a real pleasure for me to represent the University, and to speak not only for the University but also for myself in welcoming you to this meeting. It is very seldom that you get as distinguished a group of individuals in the field of ground water together, as we have here.

Actually we have here a very large proportion of the individuals who have been making the major contributions in recent years, and even dating back a couple or three decades in the field of ground water hydrology and hydraulics, especially when you are dealing with the unsteady state aspect of this. It is a field that has been of great interest to us here at CSU. As an institution we have been chartered with responsibility of doing research and having the programs necessary, to serve the State of Colorado in particular and the region and even beyond that, in general,with respect to ground water. Consequently, we have quite a history here of interest in ground water problems. As you well know, Mort Bittinger has been taking leadership in recent years with respect to ground water problems and the applied aspects of it but we can go back to the turn of the century and even before that with respect to studies that have been carried on here with L. G. Carpenter and then in more recent years, Ralph Parshall and Carl Rowher and Bill Code, all men that I am sure you are all familiar with and also D. F. Peterson and Art Corey, who is on the panel up here and also Bob Longenbaugh who is going to help to keep things moving during this conference.

You are all aware of the vast quantities of water that are available underground and the potential this holds for the future. Some of us have struggled with the problems of evaporation with surface storage and yet I think we have not taken full advantage of the opportunities that are available to us in underground storage, where the problem of evaporation is of little or no consequence. The problem that has been hard to get our fingers on is the unsteady state situation in underground flow. Underground storage and the whole concept of underground dynamic storage reservoirs is something that we are only beginning to have the analysis for. We should be able to develop the kind of management of these reservoirs and water supplies that we have to do if we really carry out a proper conservation and utilization of our water resources. This certainly accentuates the need for a conference as this where those individuals who are right at the forefront of the knowledge can exchange ideas with each other and others of you that are interested in this type of thing can take part in this sort of exchange. I think it is interesting that you are also going to have a special session on models, both mathematical and physical models. There has been progress in recent years here that opens up completely new possibilities for the solution of problems.

Again I want to welcome all of you to CSU. I hope that this meeting proves beneficial to all of you as you anticipated. Thank you.

MR. BITTINGER: I would like to turn the meeting over to Art Corey, Professor of Agricultural Engineering at Colorado State University, who will be moderator this afternoon. I should mention that Professor Harr, who we originally had on the program was not able to come. Also, Professor Hantush,listed as a panel member has had to change his plans and will not be able to attend.

DR. COREY: First, I would like to introduce the members of our panel for this afternoon. Dr. Spiegel is a Water Resource Engineer from the New Mexico State Engineer Office, Santa Fe. Next to him is Professor Jan van Schilfgaarde from the Agricultural Engineering Department at North Carolina State College. Then we have one of our former students, Mr. R. W. Nelson. He is working for the General Electric Corporation in the Hanford Laboratories at Richland, Washington. On my left is Mr. John G. Ferris. He works for the Ground Water Branch of the U. S. Geological Survey at Tucson, Arizona. Next to him is Mr. Robert E. Glover, U. S. Bureau of Reclamation and on his left is Professor Kraijenhoff van de Leur, visiting scientist from Wageningen in the Netherlands. On the extreme left is Professor DeWiest. He is now teaching at a NSF Summer Institute on Hydrology at Colorado State University, but his normal base is Princeton University.

The subject of this particular panel discussion is, of course, unsteady flow. We are going to talk this afternoon, however, about such unsteady flow ground water problems as we would normally describe using a rectangular coordinate system. We will reserve for tomorrow's panel meeting discussion of all those unsteady problems that we would describe either with a polar or a cylindrical coordinate system. I am going to ask the several panel members here to tell us about the work that is being done in this particular area in their organization or institution, and I am going to start with Dr. Zane Spiegel.

DR. SPIEGEL: The State Engineer Office at Santa Fe, New Mexico has been involved quite closely in using non-steady solutions for determination of the effect of changing or adding appropriations of water by wells on the other wells in the region or on streams and drains. There are other problems which come up, such as new surface water development in one part of the basin that releases water, or may release water in the future for additional supplementary surface water irrigation in other parts of the basin. This means that the recharge to a fairly large area will be changed and this increase in recharge caused by a new surface water development will increase the supplies for the existing developments downstream by increasing the base flow, or the drain flow resulting from the surface water projects. The rectangular aquifer solutions presented in the Appendix represent some of the work done on this phase. The nonsteady applications come by the method of superposition of steady state solutions given by Jacob many years ago and by other problems presented in this collection. The steady state solution is also quite useful in itself because we can measure the water levels and draw contours for existing conditions and use the steady state solutions to verify the coefficients of the aquifer we need to use in the nonsteady state in order to predict what will happen when we change the present conditions.

DR. COREY: Professor van Schilfgaarde, would you like to tell us what you are doing in North Carolina?

DR. VAN SCHILFGAARDE: Maybe I can tell you a little bit about what we are doing. I am going to do this in chronological order. For the last couple of years we have done some work on the theory of transient saturated flow and some of you may not be familiar with these references. One, with senior-author Herman Bouwer, has been submitted to ASAE but is not out yet. It deals with a very simple little gimmick for adapting the steady state solutions to a transient solution of a drainage problem. Everything I am talking about just now has to do with agricultural drainage.

Let us look at the problem which Hooghoudt, for example, and Donnan and many others have considered in the past, of steady rainfall being removed by tile drains. We may modify this--put this in a nonsteady state--by considering that at a given time the rainfall rate P is stopped. The water table starts to drop, and, at least instantaneously, the discharge rate will stay the same. Well, if you do that you arrive at a simple differential equation stating that the equivalent drainage rate P is proportional to the rate of change of the height of the water table. If f is the porosity and C is a factor which has to do with the change of the shape of the water table, then we can write

$$P = -fC\frac{dm}{dt}$$

We can integrate this equation if we have a relationship between m and P, and dozens of these relationships are available in literature. We can take Hooghoudt's original equation, for example, putting in the ellipse equation. We can take a graphic solution such as the one presented by Ernst and Boumans some years ago, obtained from relaxation solutions. We can do this analytically or we can do it graphically. Graphically, you may plot the relationship between precipitation and the height of the water table for different spacings and then we can obtain an average precipitation rate $\overline{P}$ by assigning a specified rate of drawdown, $\Delta m/\Delta t$.

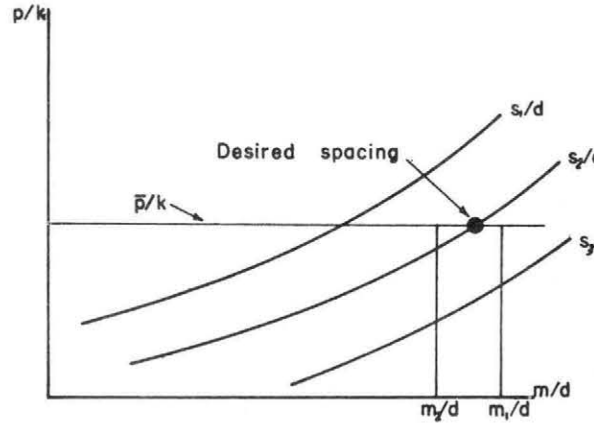


Figure 1

Let us say we want to drop the water table from a certain level to another certain level in a given time increment. This also fixes for us an average precipitation rate. After you have plotted (figure 1) the dimensionless plot of P/K versus m/d for different values of S/d, by simply reading the center point off this graph you can get the spacing which is required to get the prescribed rate of drawdown of the water table, given the specified depth of the impervious layer and the specified conductivity of the soil, and the porosity. The thing that is appealing to us is the fact that it is extremely simple and as such it can be adapted for field use.

As another development, we take a problem that Mr. Glover worked out a number of years ago, and which I noticed in the Appendix A, where he considered the rate of drop of the water table by looking at the differential equation:

$$f\left(\frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x}\left(yk\frac{\partial y}{\partial x}\right)$$

This is the standard heat-flow equation which Mr. Glover integrated by assuming that y was constant. He also solved it for the case where the value of d (depth of impervious layer below drain axis) was zero. Now, the only modification that I have made is that I have integrated this equation where y is a variable. This gives us a solution to the falling water table problem, which is valid for any relative depth of the impervious layer, except for the convergence correction, which has to be brought into any one of these problems. This can be done using Hooghoudt's tables or the graphic equivalent of those. I want to point out however, that Mr. Brooks recently published a similar paper using somewhat different initial conditions, and his solution and mine are not identical; as far as I know they are both correct if you accept the assumption on which they are based. I'll make a sketch (figure 2) of the initial conditions used by Mr. Brooks and myself.

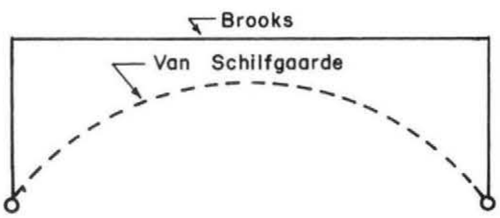


Figure 2

We are working right now on something which falls also within the scope of this topic, i.e., on a different method for measuring the saturated conductivity in the field using very large samples. We have finished the theoretical work and the laboratory work and are quite happy with it. A paper submitted to ASAE describes this work. Our field work tends to bog down during this first season. This is a classical example of difficulties in the field, but I think we can overcome them.

I want to say a few words about what we would like to do in the future. In the first place, let me put this in a philosophical framework. I am interested in agricultural drainage primarily. I feel that logically, the first question to be asked in agricultural drainage is, what are the requirements of the crop. The second question that comes up is what is the physics of moisture and air movement through the soil. And the third question is, in an engineering sense, how can we combine what we know about the botanical aspects and the physical situation to do a little better job of engineering design. Solutions such as published by Mr. Kraijenhoff, Mr. Brooks, Mr. Glover and others are examples of design tools. But there is one thing that, at least in humid regions, we need very badly. That is, we do not have to follow the water table down after one irrigation, or after a series of irrigations at regular intervals. We are concerned with what will happen when we have erratic, variable rainfall during the growing season. So it seems to me that the next step that we would like to take is to take the type of mathematical solution I described, or a solution such as that of Kraijenhoff, and use it in combination with a probability statement of the rainfall distribution over the season, and from that determine the distribution of the water table height over the season, on a probability basis. This is what our thinking is for our next line of attack, but I must admit that I have done some talking about it and not much more.

DR. COREY: Our next speaker is Mr. Bill Nelson.

MR. NELSON: Many of the fundamental principles that all of us are concerned with find their application in a wide variety of practical situations. We have had discussions on drainage, water supply and replenishment. The general area of work and interest at Hanford is that of the disposal of industrial effluents to the ground. Of course, in particular, our concern is that of radioactive wastes. I should also emphasize here that the radioactive materials that I will be talking about are low-level wastes. Since this area of application is a little different from the others, I have prepared slides that I would like to use as background, pointing out the type of questions that are asked of one who is going to analyze flow systems in order to help predict how nuclides will move through the soil. Often the questions center around the time involved in travel to potable supplies in order to allow decay. The first slide is a schematic diagram (figure 3) to illustrate a typical problem requiring analysis. The figure shows a pond; at some distance further downstream the natural ground water flow is into a surface water river. The classical equipotential lines and streamlines are shown. The streamlines in particular become a very vital part of the solution that is needed. In other words, the distribution of arrival time is dependent on two things (1) the length of the flow path, and (2), the velocity, or more precisely the flow time which is an integral of the velocity along the various flow paths. Consider the next slide (figure 4). The shortest streamline is from the pond directly towards the river. On this graph the shortest arrival is represented by $q/Q = 0$. The material first reaches the river 11 days after the waste was put into the pond. At 20 days, approximately 15 percent ($q/Q = 0.15$) of the original material has arrived. A convenient way to think about the curve is: At zero time all of the water which instantaneously left the pond was colored blue; then by the end of 20 days about 15 percent of the blue water will have arrived at the river, entered the river, and been swept away. Similarly, at 97 days 50 percent of the water which left the pond at zero time has entered the river.

The curve in figure 4 is the result needed from the flow system analysis for contamination analysis. All of the concern about the heterogeneity of soil permeability, the boundary conditions, and the other flow effects combine to give a curve of this type (figure 4). It is paramount then that the streamlines are obtained in order to get the all-important time distribution.

In the next slide (figure 5) is shown a comparison of mathematical formulations for transient flow. Both the classical equations for transient flow and

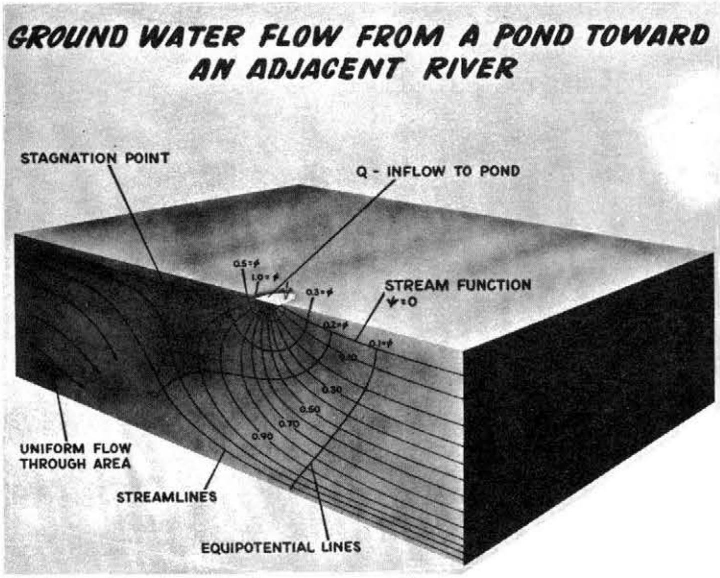


Figure 3

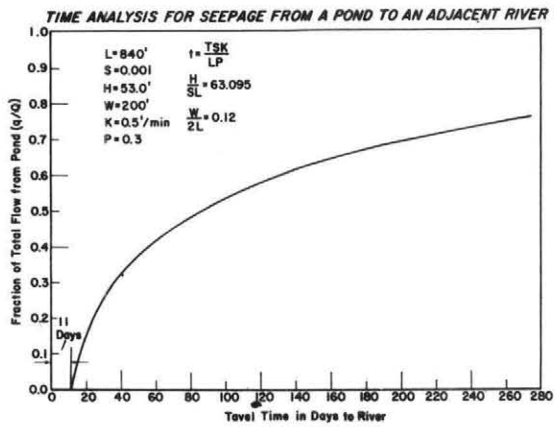


Figure 4

MATHEMATICAL MODELS FOR TRANSIENT SATURATED FLOW IN HOMOGENEOUS SOIL

DUPUIT-FORCHHEIMER APPROXIMATE FORMULATION

$$K\left[\frac{\partial\left(h\frac{\partial h}{\partial x}\right)}{\partial x} + \frac{\partial\left(h\frac{\partial h}{\partial y}\right)}{\partial y}\right] = S\frac{\partial h}{\partial t} \quad (1)$$

Where

K is permeability
S is the storage coeficient,
h is the water table elevation. above some datum
x & y are the two horizontal space coordinates, and
t is time.

CLASSICAL POTENTIAL FORMULATION

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad (2)$$

Where

$\phi = p/\gamma + z$ is the hydraulic potential
p/γ is the pressure head
z is the verticle coordinate oriented parallel with the gravitational vector.

The time variable, t, enters the solution through time dependent boundary conditions.

Figure 5

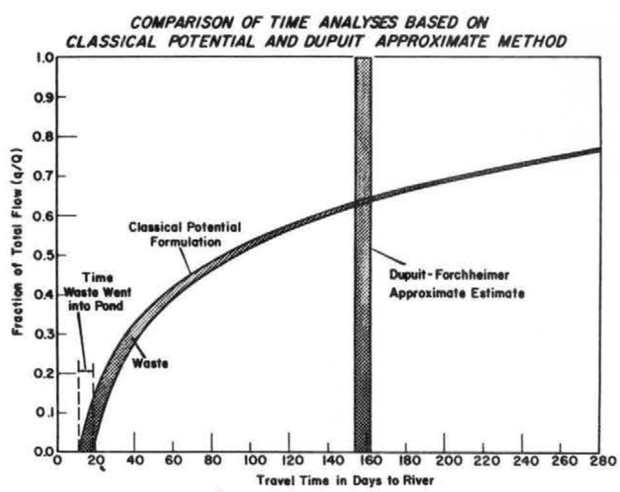


Figure 6

the Dupuit-Forchheimer approximate equations are shown. We notice in the Dupuit-Forchheimer expressions the transient enters the flow system as the partial derivative of h with respect to t , or in other words, time dependence enters through the partial differential equation. In contrast, in the classical potential analysis, time enters through the boundary conditions being time dependent.

Often in unconfined systems, the water table shape is a function of time, and this is the manner in which time dependence enters the analysis. When analyzing contamination we use the classical equations on the right. The next slide (figure 6) indicates the reason why we used the exact formulation. The slide (figure 6) compares the two formulations of the solution for the pond in the original schematic diagram. A waste solution was discharged into the pond for a week. That is a longer period than one usually finds in practice; yet it is used in order to illustrate the effect shown. From the figure for the exact equations, at 40 days from the time the waste entered the pond, the vertical difference between these two curves will be the amount of activity that is being bled into the river. The advantage for disposal lies in the flow system spreading and diluting effect as well as allowing time for radioactive decay. In contrast, if the Dupuit-Forchheimer approximate method is used there is a single pulse, and that pulse predicts a peak which reaches 1 at 160 days. At first this might appear to be a very conservative estimate; in other words, the predicted activity from the Dupuit-Forchheimer assumptions may be high. If this were truly the case, the Dupuit-Forchheimer results could be used, but the next slide (figure 7) contrasts the results and shows the error.

COMPARISON OF ACTIVITY ESTIMATES BASED ON DUPUIT AND CLASSICAL METHODS OF FLOW ANALYSIS

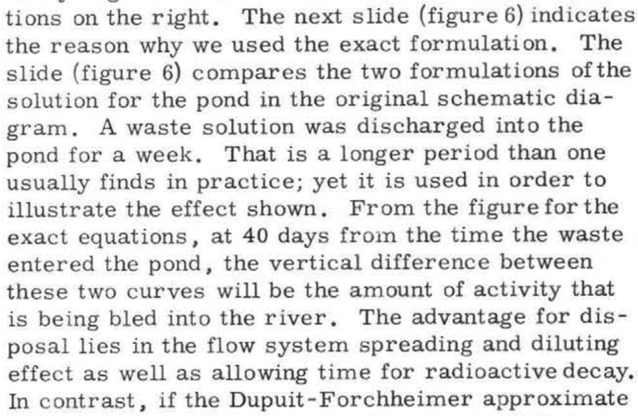


Figure 7

The Dupuit-Forchheimer method does not offer the conservative estimate that we would like. Two isotopes, P^{32} and Cr^{51} are shown. Notice that the classical analysis gives a peak early in both cases and then it drops off. However, in contrast, the predictions utilizing the Dupuit-Forchheimer assumptions give very low peaks and they arrive much too late.

The reason for this is that the approximate theory predicts some 140 days in which decay occurs, whereas, in the actual case and in the classical analysis it has really decayed only 11 days. Perhaps this indicates some of the reasons why, as far as disposal analysis, the classical formulations rather than those with the Dupuit-Forchheimer are required. This leads us then to the work summarized in Appendix A. A great deal of our work has been associated with taking care of heterogeneity of soil permeability and being able to measure the permeability distribution, in place, for natural flow systems. This is a major part of the work we are engaged in at the present time. The work on stream and path functions for flow in heterogeneous porous media is part of the overall measurement research effort.

Stream functions have been available in classical hydrodynamics for many years, particularly the singularly elegant methods of complex variables and Stokes stream function for axisymmetrical problems. More recently, in 1957, Yih presented stream functions for inviscid hydrodynamics in terms of three-dimensional stream and path functions. These stream functions can be carried across directly for flow in a homogeneous porous medium, so long as it is saturated flow. As one moves to a heterogeneous medium, changes are required; the permeability falls out of the equations for streamlines. However, it has to be reintroduced in terms of a boundary condition. Considering transient path functions, which are covered here, the permeability distribution itself is inside the equation. The three classical characteristics of a stream function are found: (1) The paths of fluid flow are determined by setting the adjunct functions, f, g and η equal to constants; (2) the velocity components are described in terms of these three stream function groups, and (3) through very careful handling of the boundary condition the material distribution is obtained.

There is a very worthwile field yet to be examined through studying the path functions and their relationship to the potential function for flow in heterogeneous media. A few results found to date may be of interest. We have been able to show that the kinetic energy distribution is not a conservative system, i.e., the curl of the velocity is not zero. The practical implication of this result is that homogeneous systems are very, very special cases in nature. Very special in the sense that for the amount of energy that one has in the flow system, you get more flow in the homogeneous system than you can in any heterogeneous flow system than can exist. We are just getting into some of these phases, but I think there are important things and implications that need further study.

Let me mention briefly some of the other work that we have underway. We have done quite a bit of work on partially-saturated flow by developing computer programs to take care of solutions through numerical means. We have one program called "Steady Darcian Flow" which has exceeded our original expectations for usefulness in solving steady, partially-saturated flow problems and their reduced forms. It will also solve heterogeneous saturated flow systems and homogeneous saturated flow systems, since they are all reduced forms of the general partially-saturated case. The program can solve for up to 8000 grid points very nicely.

We have another computer program which can solve one-dimensional, transient, partially-saturated flow problems very nicely. When one uses the program for two-dimensions and axisymmetrical cases, instabilities develop. There is a great deal of development left to be done in the higher dimensionality problems. A visiting summer professor from the Mathematics Department of Washington State University has been studying the instability characteristics. It is hoped that before too long we will be in a position to solve multiple dimensional, partially-saturated, transient problems. As for work in the future, we are working toward measurement in place of the permeability distribution. In the session on models and computers I will talk about a series of programs and steps designed to accomplish the measurement and ultimately enable us to build an electrical analog model of the Hanford ground water system.

DR. COREY: Mr. Ferris, it is your turn to tell us what is going on in the Ground Water Branch of the USGS.

MR. FERRIS: Although Slichter's[1] treatise of 1898 did not include the nonsteady state, it is of interest to note that he presented solutions for a number of boundary-value problems which involved the flow to trenches or drains. Further, he set forth at this early date a lucid exposition of the role of potential theory as applied to ground-water flow systems and the methods of conformal mapping for resolution of complex boundary problems. In subsequent work[2] he developed field methodology and instrumentation for measuring ground-water velocity with the aid initially of fluorescein dye as the tracer, but later the use of an electrolyte when he developed his under-flow conductivity meter. In a study of filtration through sands of the Fort Caswell area, Stearns[3] measured rates of underflow from upstream trenches to positions downstream using dye, electrolytes, and biologic tracers.

A mathematical model for the movement of water toward a plane sink was developed by Theis[4] in his study of the influence of drains on the water resources of the Middle Rio Grande Valley, New Mexico. In a study of the ground-water resources of the High Plains, Theis[5] developed a piezometric parabola model for a flow system which is recharged at a steady rate over a region bounded by parallel sinks. With the aid of this model he made the first quantitative determinations of recharge to the High Plains. The piezometric parabola model for system analysis was treated later in more detail by Jacob[6] who developed the form of the recession equation which describes the decay of the piezometric profile when recharge ceases. He illustrated the applicability of the profile recession model to correlation with long-term records of rainfall on Long Island and with a high degree of success developed the functional relation between rainfall trends and ground-water stages from the turn of the century to date. In the closing statement of his report Jacob pointed out the applicability of these methods to problems of land drainage and to the analysis of the base flow component of the stream flow.

(1) Slichter, C. S., 1898, Theoretical investigations of the motion of ground waters: U. S. Geol. Survey 19th Ann. Rept., pt. 2, p. 295-384.

(2) Slichter, C. S., 1905, Field measurements of the rate of movement of underground waters: U. S. Geol. Survey Water-Supply Paper 153.

(3) Stearns, N. D., 1927, The geology and ground-water hydrology of the experimental area of the United States Public Health Service at Fort Caswell, N. C.: U. S. Pub. Health Service, Hygienic Lab. Bul. 147, p. 137-168.

(4) Theis, C. V., 1938, Ground water in the middle Rio Grande Valley, N. Mex.: Nat. Resources Comm. Regional Planning, Part 6, Upper Rio Grande, vol. 1, part 2, Ground-Water Resources, p. 277-285.

(5) Theis, C. V., 1935, Amount of ground-water recharge in the southern High Plains: Am. Geophys. Union Trans., 18th Ann. Meeting, pt. II, p. 564-568.

(6) Jacob, C. E., 1945, Correlation of ground-water levels and precipitation on Long Island, N. Y.: New York Dept. Conserv., Water Power and Control Comm. Bul. GW-14.

The work of Cooper[7], Glover[8], and Henry[9] developed mathematical models for flow systems which involve a fresh water-salt water interfacial boundary. Their work has contributed much to clarifying certain aspects of this complex problem with its moving boundary plus diffusion-dispersion forces. A paper[10] on the subject of the cyclic fluctuation of streams illustrated a method for determining the hydraulic diffusivity of the ground-water reservoir and information on its locus of submarine discharge. A recent report by Cooper and Rorabaugh[11] summarizes their progress in developing mathematical models to evaluate bank storage and the diffusivity characteristic of the ground water reservoir in its modulation of cyclic pulses from the stream. They have applied these models to determine the ground-water portion of the stream flow hydrograph.

DR. COREY: Now, we are going to hear from Mr. Glover of the U. S. Bureau of Reclamation.

MR. GLOVER: Irrigation constitutes a major part of the work at the Bureau of Reclamation. Experience indicates that irrigation can not ordinarily be practiced without becoming involved with ground water. The first type of involvement to be met is often that of the need for drainage. When irrigation is practiced without provision for drainage, the water table is apt to rise until the land is waterlogged and the productivity of the area brought to an end. It is natural, then, that the Bureau will become concerned with the problem of drainage, with open drains, tile drains, or parallel drains. This involvement comes from many sources. A canal leaks and contributes to the ground water, and that may have effect upon the ground water levels, so that a line-source function becomes useful for investigating the effect of canal seepage on ground water. The development of parallel drains was attacked first by a first approximation procedure, which involved a restriction that the drainable depth would be small compared with the depth of the aquifer below the drains. In order to get some idea of the effect of having the drains near the bottom of the aquifer another development was made as explained in Appendix A. It has been brought to my attention by Marinus Maasland that this problem was also attacked by Boussinesq in 1904. I hope you people's French will enable you to pursue this elegant treatment somewhat better than mine does. He did treat this particular case and in an elegant way. In the case where drains are placed along the slope and there is a gradient transverse to the drains it then becomes of some importance to investigate what happens due to this transverse slope. This case has been investigated and there is a summary in Appendix A. It also became certain that we would have to deal with a layered aquifer and this has been given some treatment. The problem of the return flow is of importance. Other factors had to be treated also. These cases came up and were handled individually, in the beginning, and the solutions, recommendations, etc . were placed in informal memos which took care of the problem at the time. Later a number of these were collected together and assembled into Technical Memorandum 657. This will soon be followed by a Technical Monograph in which the material will be organized in a somewhat better fashion and will be supplemented by some machine-computed tables which will facilitate computations for cases of transient ground water movement.

DR. COREY: Mr. Kraijenhoff will you report on your work?

MR. KRAIJENHOFF: At the State Agricultural University at Wageningen, the Netherlands, we have done work that is more or less similar to the work Professor van Schilfgaarde has presented here. That is, we have studied the unsteady flow to drains as a consequence of a certain succession of application rates of seepage into the saturated zone. I say seepage into the saturated zone because I am very much convinced of the importance of the unsaturated zone which effectively transforms the rainfall distribution diagram into a time diagram of the inflow into the saturated zone.

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(7) Cooper, H. H., 1959, A hypothesis concerning the dynamic balance of fresh water and salt water in a coastal aquifer: Jour. of Geophysical Research, vol. 64, no. 4, p. 461-467.

(8) Glover, R. E., 1959, The pattern of fresh-water flow in a coastal aquifer: Journal of Geophysical Research, vol. 64, no.4, p. 457-459.

(9) Henry, Harold R., 1961, Salt intrusion into coastal aquifers: International Association of Scientific Hydrology, Commission of Subterranean Waters, publication no. 52, p. 478-487.

(10) Ferris, John G., 1951, Cyclic fluctuations of water level as a basis for determining aquifer transmissibility: Union Geodesique et Geophysique Internationale, Association Internationale d'Hydrologie Scientifique Assemblee Generale de Bruxelles--Tome II, p. 148-155.

(11) Cooper, H. H., and Rorabaugh, M. I., 1963, Changes in ground-water movement and bank storage caused by flood waves in surface streams: U. S. Geol. Survey Prof. Paper 475-B, Article 53, p. B192-B195.

We started with this inflow into the saturated zone and used an equation as given by Mr. Glover for a recession period after an instantaneous addition of water to a drainage situation typified by two parallel channels with a constant level (figure 8).

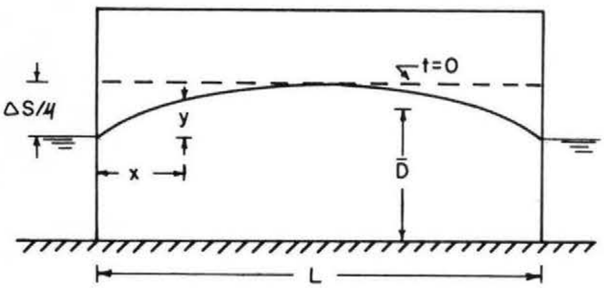


Figure 8

This two-dimensional Dupuit-Forchheimer model was used for an instantaneous application to the saturated zone of ΔS per unit surface at zero time. This inflow causes the water table to rise instantaneously over $\Delta S/\mu$, where μ stands for the active porosity.

The shape of the water table after zero time is given by:

$$y_t = \frac{\Delta S}{\mu} \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \exp(-n^2 t/j) \sin \frac{n\pi x}{L}$$

The reservoir coefficient j, which incorporates all hydrologic properties of the drainage situation is given by

$$j = \frac{1}{\pi^2} \frac{\mu L^2}{K \overline{D}}$$

where L is the distance between drains, μ the active porosity, which equals the real porosity minus the enclosed air in the drainable zone, K the permeability and $\overline{D}$ the equivalent mean depth of horizontal flow according to the Dupuit assumption. This is called equivalent depth because radial resistance and horizontal components of flow above the water table are to be accounted for.

By application of Darey's Law we obtain for q_t:

$$q_t = 2 K \overline{D} \left.\frac{\partial y}{\partial x}\right|_{x=0} = 8 \frac{K \overline{D}}{\mu L} \Delta S \sum_{n=1,3,5,\ldots}^{\infty} \exp(-n^2 t/j)$$

We introduce now the instantaneous unit-hydrograph, which is the reaction of outflow to an instantaneous inflow of a unit volume into the system. Therefore, substituting unity for ΔSL we obtain:

$$u(t) = \frac{8}{\pi^2} \frac{1}{j} \sum_{n=1,3,5,\ldots}^{\infty} \exp(-n^2 t/j)$$

This expression for the instantaneous unit hydrograph is then used in a convolution integral in order to find the outflow rate as caused by a steady inflow at a rate p into the saturated zone, starting at zero time. Thus we find:

$$q_t = \frac{8}{\pi^2} p \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \left[1 - \exp(-n^2 t/j)\right] =$$

$$= \frac{8}{\pi^2} p \left\{1 - \exp(-t/j)\right\} + \frac{1}{9} \frac{8}{\pi^2} p \left\{1 - \exp(-9t/j)\right\} +$$

$$+ \ldots = q_t^* + q_t^{**}$$

For the computation of the outflow rate q_t a simple tabular computation was developed. First, we approximate the time distribution of inflow by a block diagram of equal time intervals and during each interval the value of the inflow rate p is considered a constant. We then write the series in its separate terms q_t^*, q_t^{**}, etc. We observe that these terms are similar to the outflow rate of a linear storage system (figure 9) and therefore successive values of each term can be easily computed.

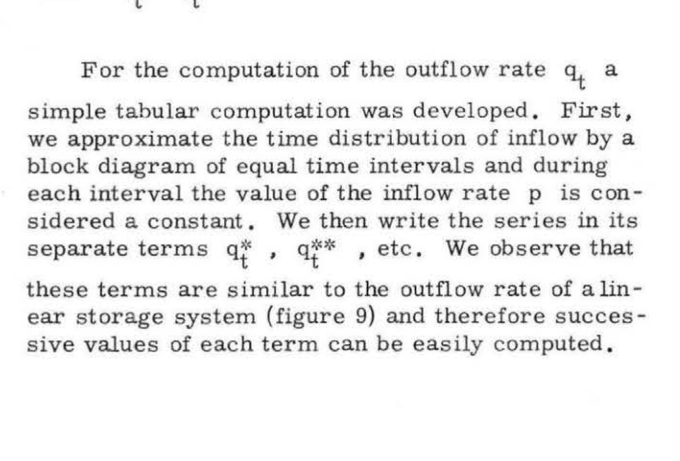


Figure 9

Fortunately the number of terms to be computed can be restricted. This is shown in figure 10. The left half pictures the growth of q/p where p is the rate of addition to the saturated zone, so that q/p is a dimensionless expression for the outflow rate.

On the horizontal axis you find t/j which is a dimensionless expression for time. Here j is again the reservoir coefficient expressed in the same time units as the interval t during which the inflow rate p is considered to be a constant.

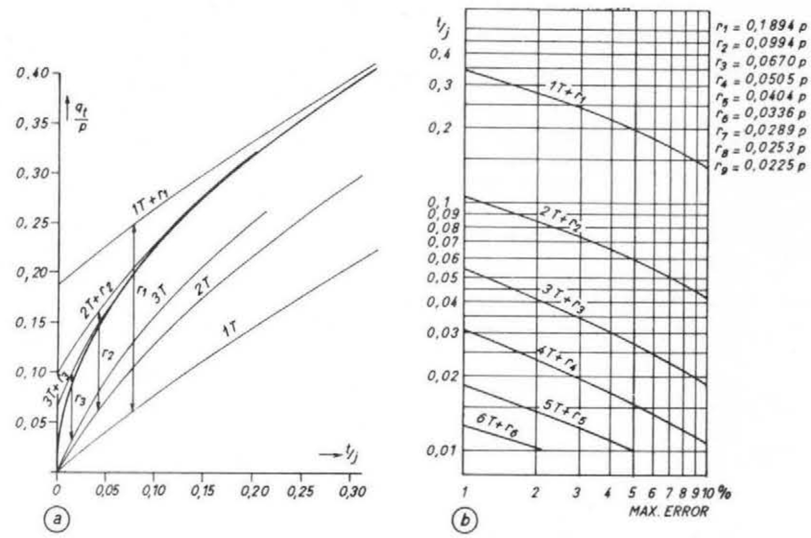


Figure 10

Table 1. Simultaneous Computation of q_t and y_t .

$j = 16{,}7$ intervals: $\quad \exp.\left(-\frac{1}{j}\right) = \exp.(-0{,}06) = 0{,}9418 \quad \exp.\left(-\frac{9}{j}\right) = \exp.(-0{,}54) = 0{,}5827$

$\mu = 0{,}10 : \frac{j}{\mu} = 167$ int. $\quad \frac{8}{\pi^2}\left\{1-\exp.\left(-\frac{1}{j}\right)\right\} = \frac{8}{\pi^2} . 0{,}0582 = 0{,}0472 \quad \frac{8}{\pi^2}\cdot\frac{1}{9}\left\{1-\exp.\left(-\frac{9}{j}\right)\right\} = \frac{8}{\pi^2}\cdot\frac{1}{9} . 0{,}4173 = 0{,}0376$

t intervals	1		2		3		4		5		6		7	
$p_t \frac{mm}{interval}$	5		20		10		5		0		2		0	
$0{,}9418\ q_{t-1}^*$			0,222		1,098		1,479		1,615		1,520		1,520	
$0{,}0472\ p_t$	0,236		0,944		0,472		0,236				0,094			
q_t^*		0,236		1,166		1,570		1,715		1,615		1,614		1,520
$0{,}5827\ q_{t-1}^{**}$			0,110		0,502		0,512		0,408		0,238		0,182	
$0{,}0376\ p_t$	0,188		0,752		0,376		0,188				0,075			
q_t^{**}		0,188		0,862		0,878		0,700		0,408		0,313		0,182
$r_2 = 0{,}0994\ p_t$		0,497		1,988		0,994		0,497				0,199		
$q_t \frac{mm}{interval}$		0,921		4,016		3,442		2,912		2,023		2,126		1,702
$\frac{\pi}{2} q_t^* = \frac{\mu}{j} y_t^*$		0,371		1,832		2,466		2,696		2,540		2,543		2,400
$-\frac{1}{3}\frac{\pi}{2} q_t^{**} = -\frac{\mu}{j} y_t^{**}$		—0,099		—0,452		—0,460		—0,367		—0,214		—0,163		—0,095
$\frac{\mu}{j}(y_t^* - y_t^{**})$		0,272		1,380		2,006		2,329		2,326		2,380		2,305
$\frac{\mu}{j} r_2 = 0{,}00761\ p_t$		0,038		0,152		0,076		0,038				0,015		
Σ		0,310		1,532		2,082		2,367		2,326		2,395		2,305
$y_t = 167\ \Sigma$ mm		51,7		255		348		395		388		399		384

One can approximate the real solution of q/p by just taking the first term of the series, which is the line marked 1T, and adding a constant factor times the rate of inflow p. The resulting line marked $1T + r_1$ only approximates the real value after a relatively long time. We could, however, take two terms and the appropriate rest term to obtain a quicker approximation and finally the figure shows that the line which represents three terms and the appropriate rest term very soon merges into the real line.

It is this ratio t/j, or $1/j$ if the duration of one interval is the time unit, which determines how many separate terms of the series should be computed in order to obtain a chosen accuracy. This is shown on the right half of the figure.

Table 1 shows this computation in tabular form. We can leave the lower half of this table out of consideration because it applies to the water table elevations which can be computed in a similar way.

To start any computation we must first know the reservoir coefficient which typifies the drainage situation. In this example j equals 16 intervals so $1/j = 0.06$ and the exponential functions can be found in tables. The preceding diagram (figure 10) shows that at this value of $1/j$ only two terms should be computed separately in order to keep the errors well below 1 percent. It follows that we must first find the four constants as computed above the table and these constants are used for determining successive values of the first term q_t^* and the second term q_t^{**}. Then we compute the rest term r_2 and add them all together and thus find the successive values of the outflow rate q_t.

The reservoir coefficient j for a certain situation follows from the recession curve when there is no inflow into the saturated zone (figure 11). These recession curves must merge into straight lines if we plot the log of the outflow rates against time and then their slope determines the reservoir coefficient. There are other ways of determining the reservoir coefficient: If the hydrologic properties of the drainage situation are known, $j = 1/\pi^2 . \mu L^2/k\overline{D}$ can be computed. One can also try to find the time lag between the centers of area of the inflow diagram and the resulting wave of outflow.

Figure 12 shows the result of an experiment with a granular model. We could run a series of "rainfall" rates and gage the successive outflow rates. By repeating the wave twice we got an idea of the accuracy.

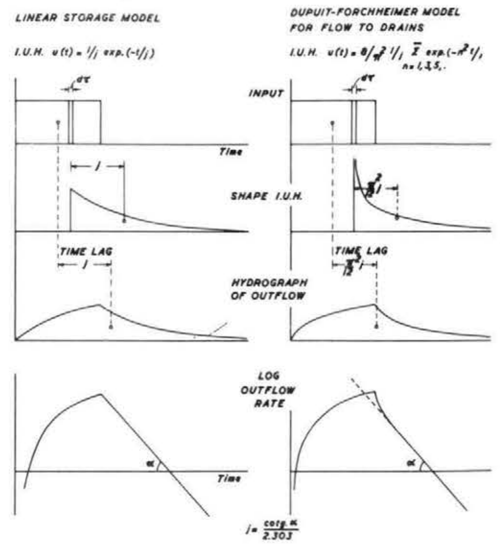


Figure 11

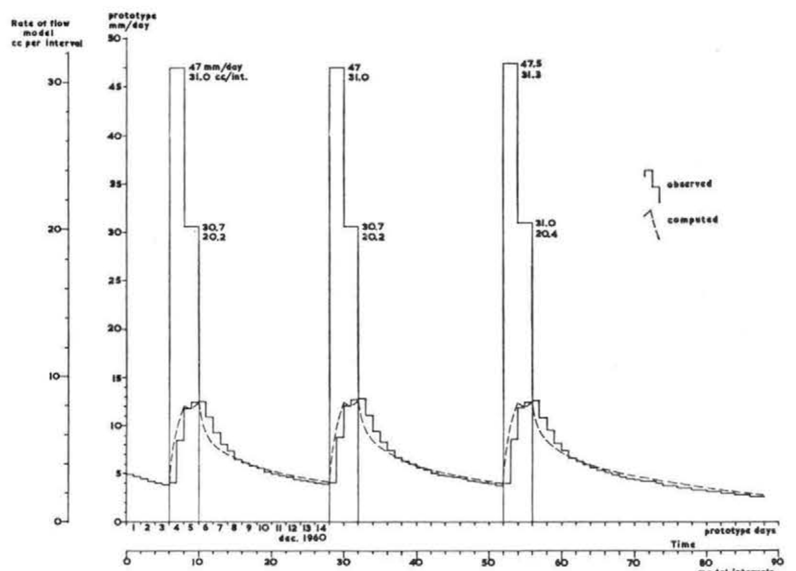


Figure 12

There is a good agreement of observed and computed hydrographs of outflow. The latter have been obtained by the computation method which I have presented here. Figure 13 shows how the reservoir coefficient was derived from the log q_t plottings of the recession curves.

We did not find the same agreement for the hydrographs of ground water level (figure 14) and I think that we must ascribe that to the influence of the unsaturated zone which has been completely disregarded in the computation. There is not only a transformation and delay of the downward flow of moisture in the unsaturated zone but there is also the fact that the rising water table swallows moisture and the falling water table leaves moisture behind creating some kind of Doppler effect which amplifies the vertical motion of the water table.

We are investigating the influence of the unsaturated zone and I hope to show you some of the results when we will be discussing model research later on in this symposium.

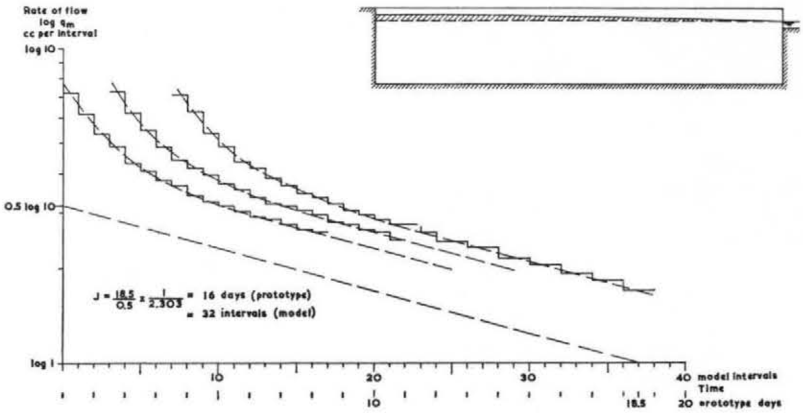


Figure 13

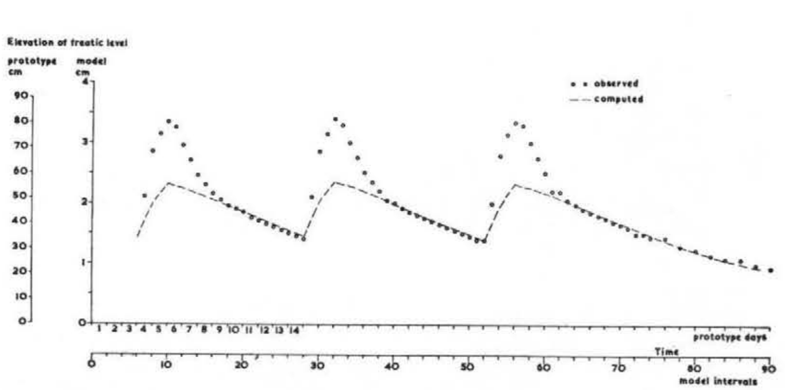


Figure 14

DR. COREY: Now we are going to hear from Dr. Roger De Wiest from Princeton University.

DR. DE WIEST: Thank you, Dr. Corey. Gentlemen, two years ago I came here as an uninvited observer to the Conference on Water Resources. I found since then that Colorado State University is a very hospitable place and I am very glad to be here and I can say the efforts that have been made here to further the progress in water resources in general and in hydrology and ground water hydrology in particular are very significant. Now I consider myself a neophyte in this field of ground water since I have been working only four years in it. I was not so fortunate as to have instructors like C. V. Theis or C. E. Jacob, or other distinguished teachers in the audience, I had to teach myself. My own work started really as a Doctoral Thesis at Stanford. The idea was to study the unsteady state flow that takes place when a reservoir is filled. This is a practical problem, as described in a recent article in the French Journal "La Houille Blanche," about the filling of a reservoir in Grevelingen (Holland). The mathematics underlying the solution were borrowed from an article by N. Curle in the Proceedings of the Royal Society of London (Series A), Vol. 235, 1956, in which the unsteady flow of a jet through an orifice was investigated. In our case the flow through an embankment or a levee with horizontal underdrain was examined. The mathematics involved were much more complicated as it turned out to be than those of the jet problem. The essential idea used in the analysis was the consideration of the unsteady flow as a time-dependent perturbation of the final steady flow. The unsteady potential function was expanded in a power series of $e^{-\lambda t}$, of the form

$$\phi(x,y,t) = \phi_o(x,y) + \phi_1(x,y)\, e^{-\lambda t} + O\left(e^{-2\lambda t}\right)$$

in which $\phi_o(x,y)$ was the known steady-state potential, $\phi_1(x,y)$ was the perturbation potential, and

$$O\left(e^{-2\lambda t}\right) = \phi_2(x,y)\, e^{-2\lambda t} + \phi_3(x,y)\, e^{-3\lambda t} + \ldots$$

Each of the terms $\phi_n(x,y)\, e^{-n\lambda t}$ can be thought of as a perturbation term of its precursor in the series, and the approach of my study was limited to the computation of the first perturbation term $\phi_1(x,y)\, e^{-\lambda t}$.

It was shown that ϕ, satisfied Laplace's equation $\nabla^2 \phi_1 = 0$ in a dimensionless hodograph plane. The free-boundary condition was linear but complicated, especially in the case of rapid level rises. This boundary condition contained the eigenvalue λ, which was found by the solution of a determinantal equation. This equation, by coincidence, was similar to one developed by the English astronomer, Hill, in his Lunar Theory. A paper extracted from my Doctoral Dissertation was published in the Journal of Fluid Mechanics, May 1960, vol. 8, p. 1-9.

At Princeton University, I completed the above analysis by means of model tests with a Hele-Shaw apparatus. In particular we tested whether it would be worthwhile to go through the more complicated analysis resulting from the boundary condition for rapid water rise behind the dam, or whether we could limit our efforts to examining slow water rises, as a first approximation. The experiments confirmed rather well the hypothesis that for practical purposes the study of slow water rises was sufficient. Another interesting feature was revealed as a result of the construction of the Hele-Shaw model. Previous experimentors with this kind of model reported very good agreement between model tests and analytical computations in which Dupuit's simplifying assumptions about the free surface were made. It was found that this agreement was due to incorrect scaling of the model. A paper on these investigations was made available in the transactions of the American Society of Civil Engineers, vol. 127, 1962, part I, p. 1045-1089.

Another paper that deals with a problem in rectangular coordinates, on aquifers intersected by streams, will appear in the Journal of the Hydraulics Division of the ASCE, November 1963. In this paper we had to estimate analytically the yield of a bank storage project, in the vicinity of Princeton, New Jersey, as predicted by a consultant hydrologist. The mathematical tool used in this paper was made available to engineers in 1956, when Bernard Friedman published his book "Principles and Techniques in Applied Mathematics." (John Wiley and Son, New York, N. Y.). It shows us how to construct the Green's function for some differential equations and boundary conditions, and how to expand the delta-function in its Fourier series, by integration of the Green's function in the complex plane.

Finally, a few words could be said about the English translation which I made from the Russian classic "The Theory of Ground-Water Movement," by P. Ya. Polubarinova-Kochina. (Princeton University Press, 1962, 613 pp.). The book contains a vast amount of material that has become classic and sometimes may be found in a better form in the works of Morris Muskat. Nevertheless it has some very original and elegant methods which make it worthwhile as a tool for researchers and for advanced course work. Among these methods I would like to quote the so-called method of the "small parameter," as used also, I believe, in a paper by Roy Brooks. Would Mr. Brooks like to comment on this method?

DR. COREY: Roy, would you like to say something about that?

MR. BROOKS: This method of small parameters which as was indicated by Roger De Wiest is found in Madame Polubarinova-Kochina's book, is also found in Advances in Applied Mechanics, Vol. IV, 1956, p. 281, called the Poincare-Lighthill-Kuo method. Also, Mr. Glover suggested the method of Emile Picard in "Memoire sur la theorie des equations aux derives partielles et la methode des approximations successives." This is from a French Journal published in 1890. Picard discusses a method similar to the method of small parameters discussed by Polubarinova-Kochina. These references are in the paper, "Unsteady Flow of Ground Water into Drain Tile," by R. H. Brooks, ASCE Journal of Irrigation and Drainage, vol. 87, no. IR 2, June 1961.

DR. COREY: I would like members from the audience to ask questions or make comments.

MR. BROOKS: I am not sure to whom this question should be directed on the panel. I would like to talk about the parallel drain problem which consists of a system of horizontal parallel drains above an impermeable boundary. Glover first solved this problem, by assuming that the drainable depth was quite large with respect to the distance of the tile above the barrier.

If we make an inspectional analysis of the nonlinear partial differential equation which makes no such assumption, we find that a dimensionless constant appears from the analysis which I call H_o/D, where H_o is the initial height of the water table above the drain, and D is related to the depth of the drain above the barrier. Because of the fact that we are using the Dupuit-Forchheimer horizontal flow assumptions, this is the only geometry factor that appears in the equation. In another paper published by Hammad, a solution was developed using the La Place equation. Hammad's solution involved only the geometry factor L/D , the ratio of the length of the spacing to the depth of the tile above the barrier. This factor L/D is also an important parameter in a general solution as well as H_o/D. Because of the Dupuit-Forchheimer assumptions, L/D obviously does not appear in the approximations by Brooks, Glover and van Schilfgaarde. Obviously, if the drains are close together the L/D parameter is very important. Should we come up with some computer solutions or some exact two-dimensional solutions which would indicate when the factor L/D is important in these Dupuit-Forchheimer solutions? Are we satisfied with the equations that we presently use in design with their assumptions and limitations, or do we need some exact solutions to indicate the useful limits of the easy to use approximate solutions?

Maybe Bill Nelson or Jan van Schilfgaarde would like to comment on this.

DR. VAN SCHILFGAARDE: To begin with, I would say that the Dupuit-Forchheimer theory as it stands will never give a solution that will take into account this convergence effect into the drain that you referred to. From a pragmatic viewpoint, we can circumvent this problem for the time being by the use of an equivalent depth as was proposed by Hooghoudt in 1946. It was used in the same form by Professor Kraijenhoff in his paper in "De Ingenieur" and was used by Herman Bouwer and myself in a couple of papers we have written. This is not an answer to Brooks' question, because theoretically we still have not accounted for convergence. For those of you who have not seen this correction, let me just mention the origin of Houghoudt's development. The original idea was essentially this: If we have a drain which is close to an impervious layer so that this ratio L/d is large, then the Dupuit-Forchheimer assumption is relatively good. The ellipse equation for steady state as given by Hooghoudt, or the Donnan equation, which is essentially the same thing, is based on this kind of situation. Now, if we go to the other extreme, when the impervious layer is way down, in that case the horizontal flow assumption is very poor, but a good approximation can be made if we assume an aquiclude through the center of the drain, which is the other extreme. In this case we get radial flow into the drain with only half of a circle acting as a sink. In intermediate cases we get some mixture of the two, and Hooghoudt in his original work combined these two in a rather ingenious fashion, which I do not want to describe now, and calculated the so called true relationship between Q and m . m is the height of the water table at the midpoint above the drain axis, Q is the discharge, or the precipitation rate, whichever you want to use. And then, he went back to the ellipse equation, which is valid when the Dupuit assumption holds, and said, if we calculated a correction factor for the ellipse equation always used the ellipse equation with this correction factor, we would get the right answer, whether the ellipse equation per se holds or not. This correction factor was obtained, then, in terms of a substitution d_e of an equivalent depth which had to be used in the place of the depth d in the equation, and d , you will remember, is the depth to the impervious layer. This resulted in a tabulation of some 50 or 75 pages in metric units and was published in Dutch, and not very useful to an American audience. In my recent paper in the ASCE Proceedings, I have taken these tables of Hooghoudt for one special case, a 5-inch diameter drain, and plotted them in English units so that you get a series of curves of d_e plotted against d , spacing as the parameter on the curves.

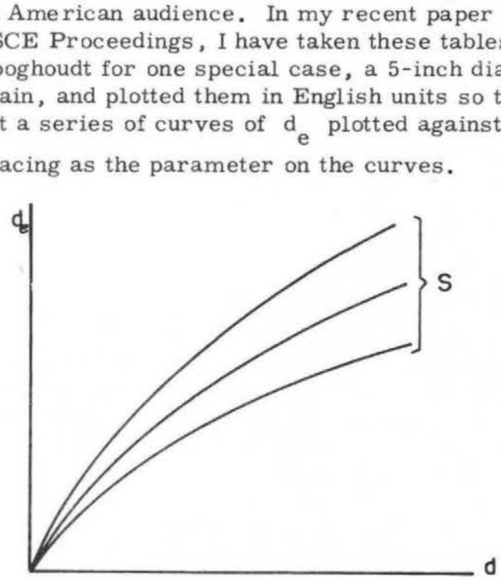


Figure 15

So for any given depth at a given spacing the equivalent depth that has to be plugged into the equation, can be read off directly. This method of correcting for the convergence is a practical way out of the difficulty that Brooks mentioned, but it is not a very sophisticated way. And the only method, in the conventional sense, is to avoid the Dupuit-Forchheimer Theory.

DR. COREY: Would you like to say something about that, Mr. Nelson?

MR. NELSON: The Dupuit-Forchheimer equations are not used very often in our work, and we are moving even further away from them. We are going in this direction because, for our particular application, these approximations do not give results that are of value and use. Perhaps I did not make it clear that the Dupuit-Forchheimer assumptions, when used to predict concentrations of wastes, do two things. First, they predict that the arrival of waste will be a good deal later than it is in actuality; and secondly, they predict that when it does arrive, the activities are far lower than the true condition. From an engineering point of view we must be on the conservative side. For this reason, our research group is moving away from the Dupuit-Forchheimer-type analysis to the classical potential analysis where transients enter as the time dependents of a boundary condition. For me to say that this is the way one should go in drainage design or ground water supply would be extremely unwise, since these applications represent a different set of circumstances.

In analyzing heterogeneous systems, the number of analytical solutions we can find is extremely limited; this naturally forces the use of computer and model solutions. If you are going to use such methods you will do better, in our experience to use classical methods of physics. The next step is to look at how the time dependence enters the boundary value problem. In the majority of the field situations further analysis shows the upper boundary, or water table, should be analyzed as a partially-saturated system. I would expect ultimately to solve partially-saturated flow systems with heterogeneous media. When this is done there is no longer any worry about the arbitrary boundary, the water table, which is more a figment of our imagination--arising from analytical complexities--than a reality in nature.

DR. COREY: Thank you, Mr. Nelson. Mr. Glover, would you like to add something to this?

MR. GLOVER: Mr. Brooks' question concerned applications in drainage. We have drains as indicated in figure 16.

We call the drainable depth H the distance between drains L and the depth below the drains D. The question as I understand it was this; whether in a complete solution we would not have to have a parameter depending upon L/D , another one on H/D . I think the answer would be yes.

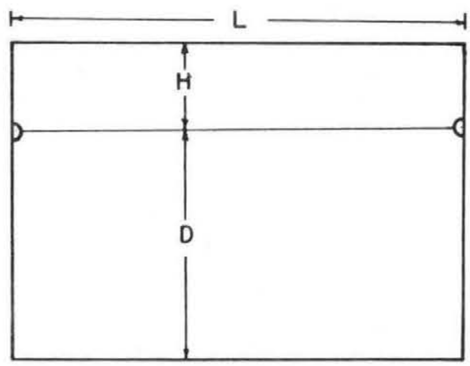


Figure 16

I would like to discuss here a development recently used at the Bureau, which is a partial solution. Mr. Moody of the Bureau has developed a solution which represents a flow coming along a strip which then goes out of a circular drain. The convergence loss can be separated out from that due to the flow which would pass through the strip if this convergence were not present.

In the American Concrete Institute Journal, in an article by Tuthill, Glover, Spencer and Bierce, there was a heat flow case described that concerned the winter curing of concrete, by the use of insulation applied to the forms. There is a very similar mathematical situation in the drainage case and in the heat case. In the heat case there is a certain amount of heat generated between the insulated forms. It raises the temperature and produces a gradient with respect to the exterior. Here we have a resistance to the flow of heat caused by the insulation. In the present case the resistance to flow of water, which takes the place of heat, is due to the convergence. By a modification of the formulas, which were originally used for the flow of heat, we can use the charts which are presented in the Journal to solve the case of the drain, taking into account the effect of convergence. It happens, that the notations are almost identical. The term which takes care of the emissivity in the heat flow case is given in Moody's development by the expression

$$E = \frac{\pi K}{\pi a + 2 D \ln\left(\frac{D}{\pi a}\right)}$$

where a is the radius of the drain, D is the saturated depth below the drain and K is the permeability of the material. If this value of E is introduced into the charts, then it is possible to take account of the local drain resistance by using this previous development. It is a partial solution to the question you just asked.

MR. BEAN: I would like to speak to Mr. Nelson's statement that he is much interested in future work in the unsaturated zone. I would just like to let him know, and the others of you too, that we in California have actually two programs in progress that are concerned with this zone. One is a cooperative program with Ivan Johnson of the U. S. Geological

Survey, who is here today. This program has to do mainly with unsaturated permeability in the vadose zone. The other is a cooperative program with the Agricultural Research Service on artificial recharge. Bill Bianchi, a representative of ARS in that program, is also here. In the ARS program we are concerned with infiltration of water and its downward movement in the vadose zone. This zone is not a shallow zone at all in much of our State. We have several hundred feet to the water table in parts of the San Joaquin Valley. So we have a deep zone to consider, and it takes a significantly long period of time for water to move through; so we have quite a problem.

MR. NELSON: Near the separations plants at Hanford the depth to the water table is in the order of 200 feet. For this reason the partially-saturated zone is of vital interest to us. Mr. Reisenauer of our group has collaborated with some or your people regarding our partially-saturated flow studies.

MR. MOODY: I would be interested in hearing a little bit more from Mr. Nelson with regard to his computer program. He spoke of being able to handle up to 8000 grid points. Do I understand correctly, is this a nonlinear solution and is this a relaxation method?

MR. NELSON: Yes, it is the nonlinear equation where the permeability is a function of the ground water potential. I don't want to become bogged down in belaboring semantics; however, everybody seems to have their own name for a particular type of iterative solution technique. Relaxation, which Southwell introduced, sought out and reduced the nodes of greatest change in potential. Southwell's relaxation is rather inefficient computerwise, since the search is time-consuming. We use the classical Gauss-Seidel method which sequentially improves the potential estimate at the nodes. We further use over-relaxation to obtain the solution rapidly.

What we do is set up a grid work covering the problem. We replace the partial differential equation with the finite difference equivalent and seek the potential at a given set of node points. This gives a set of simultaneous equations for solution rather than solving the partial nonlinear equation. However, the simultaneous equations are not linear since the partial differential equation is nonlinear. Three conditions have to be satisfied simultaneously at each node, namely: (1) the finite difference equation, (2) the relationship between capillary pressure and the potential, and (3) the relationship between capillary pressure and relative capillary conductivity.

Computational instabilities occur in many problems but experience has provided a series of methods which enable overcoming the instabilities. They are controlled by using an average of capillary conductivity as the estimate at the central node point instead of the true one. There is a rational basis for analyzing instability for the steady flow problems. You say that there exists an electrical network which itself has characteristics analogous to the nonlinear equations being used. Then go to the theory for stability in electrical networks, which has been rather well worked out, to get the general criteria. Utilizing these criteria (when combined with moving through the improvement procedure in an optimum way) we have been able to handle any instabilities that have arisen for steady, partially-saturated flow systems. In summary, we can solve one, two, axisymmetrical and three-dimensional problems, although the cube root of 8000 is only 20 which indicates restrictions in three dimensional problems.

DR. BOUWER: I am very much interested in this. A few years ago I published a paper showing how solutions of unsaturated phases may be obtained with a resistance network analog. Are you using any new principles or is your technique basically the same as the network analog procedure?

MR. NELSON: There is very little difference in the methods. We are solving the node equations digitally and you are using an analog. You iterate with the resistance values in the network in order to satisfy the three conditions mentioned earlier. The latter iteration can be eliminated by approximating the capillary conductivity as a step function. For our soils there is difficulty in justifying the step function approximation.

DR. SPIEGEL: I would like to add another practical note to what Mr. van Schilfgaarde said. First, we must realize that we approach this problem of ground water with at least three different viewpoints. Perhaps we could call one the agricultural viewpoint. This viewpoint considers fairly small drain spacing. Second, the geological viewpoint is a very broad-scaled one, and third, we have been talking about a geochemical viewpoint which requires that we consider first and foremost the heterogeneities in aquifers. For the first viewpoints we don't have to consider the micro-heterogeneity very carefully, but we do have to consider macro-heterogeneity. I think we should differentiate the two relative scales used in the geologic and the agricultural aspects of hydrology. Let me illustrate. In the agricultural viewpoint the ratio between vertical and horizontal scales is relatively large. However, for most geological problems we can relax our equations and boundary conditions quite a bit because most aquifers are actually very thin in comparison with their extent.

Thus, in most aquifers the Dupuit-Forchheimer assumptions certainly can be shown to hold everywhere. And furthermore you can generalize the Dupuit-Forchheimer assumptions for the case of partial penetration and it will still hold in most of the aquifer whether the aquifer is penetrated to the bottom, to two-thirds or half, or even the drain problem with almost zero penetration. Another geological point of interest is that streams don't usually fully penetrate an aquifer. The stream may penetrate only a shallow portion of the aquifer and so near the stream convergence of flow lines toward the stream might be expected to give some trouble if we analyze the problem using theoretical equations based on the Dupuit-Forchheimer approximation.

However, if we realize that most streams, because of the Pleistocene geological history of the stream, more nearly approximate the fully penetrated situation, because the streams actually run on a gravel fill, which if it doesn't penetrate fully to the base of the aquifer may penetrate half way. Therefore, we should keep in mind not only the limitations of a physical system in using our mathematical methods or analogs, but we should always keep in mind some of the general relations that we can make, to relax the assumptions and reduce the complexity of the equations.

DR. VAN SCHILFGAARDE: I would like to make two comments. In the first place, I am very much in agreement with what my partner here just said. I also want to say in relation to Brooks' earlier question that we've got to be a little careful as to the type of degree of sophistication that we are interested in. From an agricultural viewpoint we are interested in studying in detail what actually happens. We are kidding ourselves if we use a transient, saturated solution or any kind of a saturated solution, and as soon as we are using anything different, then the Dupuit-Forchheimer theory of course, is out the window. This comes back then to what Bill Nelson was saying earlier. To the extent that we can make use of the Dupuit-Forchheimer solution it is a matter of arriving at a first approximation of what might happen in the field. In that case the type of correction shown here by Mr. Glover, or as I indicated earlier, is plenty good enough. We are fooling ourselves if we ask for higher accuracy. Probably the biggest problem here is to identify or to measure the characteristics of the soil or the aquifer in geological terms. If you can only measure the conductivity within 500 percent, and we can get an approximate theory that is within 10 percent, then we should be least concerned with the theory. The difference between geological and agricultural interpretation is fairly obvious if it is presented as it was just now, but it can also get us into some serious trouble, because of our different backgrounds as we tackle some of these problems. One problem which is a rather simple example of this type of thing, but is not a simple or a small problem, involves the present work on the Arkansas River. The U. S. Geological Survey did a beautiful job in a rather sophisticated manner of predicting the annual changes in the water table height through the basin as a result of proposed structures. It was a beautiful piece of work assuming the type of thing we are talking about here in an unconfined aquifer, but now agriculturalists must take the data from the USGS and interpret them in terms of what the proposed program does to the plant; and if we are talking about the plant, we are interested in what is happening in the top 4 feet. And if the geologist tells you that on the average the water table will change from 10 ft below the surface to 4 feet below the surface on an annual basis, we still know nothing about what it will do to the plant that the farmer wants to grow commercially. This is a good example of an instance where geological techniques can give very good answers to geological questions, but cannot give answers that are useful in answering questions of interest to agriculture.

DR. COREY: Do we have a question from the audience?

MR. PAPADOPULOS: Tcharnyi in 1951 and Hantush recently have independently proved the validity of the Dupuit-Forchheimer well-discharge formula. They report that although the Dupuit-Forchheimer theory might give erroneous results for the head in the vicinity of the well, it gives accurate results for the discharge of the well and the head at large distances from the well. Isn't this also true for drain problems?

DR. VAN SCHILFGAARDE: Yes, the same thing is true for drain problems. As a matter of fact, Kirkham published a paper in 1958 (Am. Geophys. Union Trans. v. 39: p. 892-908) on this particular problem and showed under what circumstances the Dupuit-Forchheimer theory will predict the proper shape of the water table and when it will not.

MR. PAPADOPULOS: Does the Dupuit-Forchheimer theory give an incorrect answer for the position of the water table at the midpoint between drains?

DR. VAN SCHILFGAARDE: No, I could be wrong, but as an off-hand answer, I do not believe the Dupuit-Forchheimer theory gives an incorrect answer for the position of the phreatic surface at the midpoint between drains if we consider y as a variable in the differential equation as was done by Brooks and by myself. It probably will result in an error in the vicinity of the drain but since the critical point for agricultural drainage is the midpoint, we don't care how far we are off close to the drain.

DR. COREY: I think we should allow some time for discussion of the direction we should go in future research.

DR. DE WIEST: My point in the field of future research concerns a valid expression of Darcy's Law in the zone of dispersion. I've been lecturing on ground water here at the NSF Hydrology Institute and I have pointed out that in a dispersed medium, under the heading single phase-flow, we use an expression of Darcy's Law which has been questioned by Professor Scheidegger particularly in the case of unsteady flow. The question arises is this expression that we now have and that we now use for Darcy's Law, is it valid in steady flow? I could derive the expression on the blackboard if you are interested.

DR. VAN SCHILFGAARDE: Will you define the region of dispersion?

DR. DE WIEST: Yes, let me illustrate this by the problem of salt water intrusion, in which, considering as a first approximation that salt water and fresh water are immiscible, we first tried to locate the position of the interface between salt water and fresh water. This of course, was the first step, made by K. Hubbert, in analogy with the interface between a hydrocarbon and its water environment in the rock strata. Recently we have had a number of interesting studies of diffusion and dispersion in porous media, by researchers at the University of California in Berkeley and Davis, at MIT, and at the Technion in Haifa. A very important contribution, in my opinion at least, has been made by the late N. Lusczynski in his paper published by the Journal of Geophysical Research, Dec. 1961, pp. 4247-4256. The results of this paper are extremely interesting, provided the expression used for Darcy's law is still valid in the region of dispersion, i.e. the region where fresh water and salt water mix. This paper shows that we can obtain a three-dimensional flow picture in the zone of dispersion by making some observations in test wells and by applying some rather elementary vector calculus. Let me write Darcy's equation as it is used in the paper:

$$\vec{v} = -\frac{K}{g}\ \mathrm{grad}\ \phi^* \qquad K = \frac{k\rho g}{\mu}$$

ϕ^* = Hubbert potential

$\phi^* = gh = g(z + \frac{1}{g}\int_{p_o}^{p} \frac{dp}{\rho(p)})$ If ρ = constant then

$$\phi^* = gz + \frac{p}{\rho} + \text{constant}$$

$$\vec{v} = -\frac{kg}{\mu}\frac{\rho}{g}\left[g\ \mathrm{grad}\ z + \frac{1}{\rho}\mathrm{grad}\ p\right]$$

$$= -\frac{kg}{\mu}\quad \rho\ \mathrm{grad}\ z + \frac{1}{g}\ \mathrm{grad}\ p\Big]$$

This is the equation that we derive from the force potential ϕ^* and we know that this is a good equation in the salt water zone and in the fresh water zone. We use this equation also in the zone of dispersion, where there is no force potential ϕ^* because ρ is a function of position there. The validity of the equation, at least for unsteady state flow, has been questioned by Scheidigger in his book "The physics of flow through porous media," 1960, The MacMillan Co., New York, N. Y., pp. 256-258. My question is the following. Is this a valid expression of Darcy's Law in the zone of dispersion?

MR. JACOB: Yes, it is. Now, for unsteady flow the variation of density in space and in time enters the equation of flow through the continuity equation. I am on the defensive, but I am in agreement with Scheidigger. Nobody has yet solved satisfactorily an exact three-dimensional equation for unsteady flow in a compressible aquifer. From an engineering standpoint this is not necessary. The storage coefficient, or storativity, is empirically determined and reflects the variation of fluid density with pressure and also of porosity with fluid pressure as closely as we are capable of measuring them.

DR. DE WIEST: I have come to the conclusion that we should do research about the validity of this formula either starting from the basic principles or as a second approach that we should try to come up with an experiment in which we could then prove or disapprove the validity of the formula. May I ask for the opinion of Mr. Nelson?

MR. NELSON: I have considered this in some detail in connection with the transient stream functions. There are some interesting complications which show up in the functions when the density varies. The complications do not show up with the steady cases. Presently I do not know how to interpret this observation. It certainly deserves careful study.

If we ask how valid Darcy's Law is I think we need to differentiate between clays, which may have high surface charge densities and the essentially inert sands and gravels. There is little doubt in my mind that as we go further in very careful studies that in clays alterations in Darcy's Law probably will be needed. The very complicated nature of water along with the work on the double layer points in this direction. There have been some good papers recently published that have pointed out such alterations. The information is not all in, and I certainly discourage panic or suggestions to throw away what is presently available.

MR. JACOB: Perhaps you all know of the Muskat-Hubbert discussions. Muskat was charged with an awkward formulation of the equations of flow in compressible systems. The petroleum engineers have thus inherited imprecise equations using either presized theories of compressible flow in three dimensions.

DR. COREY: Whether or not this equation will be valid for a steady situation depends on how the difference in density arises and how the density gradients are oriented with respect to the body forces that are acting on the fluid. You are thinking about a system in which there is some kind of relationship between the density and the body forces. The question that Dr. De Wiest orginally posed was a situation in which the density gradient had nothing whatever to do with the body forces that are acting. In other words they were not due to the weight of the fluid. Now this particular problem was presented to me once as a consultant. The problem concerned a huge brine aquifer, extending over many square miles, in fact hundreds of square miles, in a huge basin. The density of that fluid varied from point to point in a heterogeneous manner. They had measurements, of the bottom hole pressure at points within this aquifer and they were trying to relate the movement of fluid in the aquifer to the equal potential lines which they could plot. But there was no correlation between the movement of the fluid and the equipotential lines and in fact I think you can show that there is no reason why there should be in that situation. There is no potential in that problem. You cannot define it and that is the basic problem. I think the theory is this. You can define a potential provided that the density gradients that exist are in the same direction as the resultant of all body forces. But if they are not then there is no way that you can define one. And so it becomes useless and not meaningful to talk about the gradient of any kind of potential if you can not define any. You can perhaps combine a velocity potential with an energy potential and get an answer, I think there are techniques to get an answer.

MR. JACOB: You don't think this equation is invalid in a variable density situation?

DR. COREY: It depends on how the density variation occurs. In general I would have to say yes there would be cases where it would be invalid.

MR. JACOB: Irrespective of how the density varies, the gradient of the density contributes no driving force and should be omitted from Darcy's law, should it not?

DR. COREY: The particular case that was presented to me, was a steady flow case and there was no way to correlate the flow with the potential gradient because I suppose that there is no such thing as a potential in that case.

DR. DE WIEST: There is of course no potential here and still we use an equation which is exactly the same, at least in letter form as the one derived for potential. The introduction of "environmental potential" by Lusczynski to study the flow in the zone of dispersion was ingenious and realistic as the potential indeed turns out to be multivalued and therefore not compatible with the meaning of the classical potential concept.

DR. COREY: I think Dr. De Wiest has pointed out a very interesting problem that deserves attention. It seems that most of the people here have been interested in making assumptions and getting solutions for the problems that have been presented to them. My own field of endeavor has been investigating assumptions and seeing whether or not they are valid. I find that as long as the system is a Hele-Shaw model or a sand model, assumptions that have been made are very good, but if they are soils in the field or rock formations that occur in nature then assumptions are not so good. Working in the laboratory it is rather difficult to determine to what extent the fact that some particular assumption is not valid is going to effect the results that we get from our solutions as they apply to field situations. It seems to me that we can argue forever about the relative merits of two different ways to determine what tile spacing ought to be, but if we don't get out into the field and make some measurements and see what has happened in places where tiles have eventually been installed we will never be able to resolve the questions we are always asking. I wonder how many are actually doing research along the lines of investigating what is the performance of installations in the field and if you don't think that some additional work along those lines would be very useful?

DR. DE WIEST: I think I would like to ask Mr. Walton to say something here on that subject. I think that Mr. Walton is well qualified because he is very familiar with well drilling and he is as you know the editor of the Ground Water Journal of the National Water Well Association.

MR. WALTON: Case histories of heavy ground-water development in nine areas in Illinois have been studied by the Illinois State Water Survey. These case histories suggest that it is often possible to evaluate ground-water resources with available analytical expressions by devising approximate methods of analysis based on idealized models of aquifer situations. By checking the performance of wells and aquifers computed with transient ground-water formulas against records of past pumpage and water levels, the validity of analytical expressions when applied with professional judgement to field conditions has been established.

MR. TAPP: I am from the Bureau of Reclamation, Office of Chief Engineer, Denver. I am glad Dr. Corey brought up this subject. I have been working in the Office of Drainage and Groundwater Engineering for the past 12 years. I am closely associated with Mr. Glover, Mr. Moody, Mr. Dumm, and Mr. Winger. We are doing exactly the thing that you mentioned. Messrs. Glover and Moody started in about 1951 or 1952 developing transient drain-spacing equations. The Bureau used them because we had to have practical drainage answers to give to our administrative officers regarding overall drainage requirements and costs for planning reports and for planning and designing drainage systems on operating projects. Over the years, the original equations have been improved. We have gathered field measurements from drainage installations over the world and used them to check the equations against tank experiments. We find, in almost every instance, that there is good agreement of water-table behavior predicted by the equations and the field and tank data. Our data come from Canada, from Australia, and from projects in the United States. We believe that, for practical field work, we should not stop at the development of suitable equations. Such equations should be checked against field data. We would appreciate receiving field data relative to drain operation from any of you.

MR. MYERS: I would like to point out also that we in the Agricultural Research Service are making a number of these field checks that you mentioned, Dr. Corey, and we find, as Mr. Tapp mentioned, that this is an extremely important facet of verification of equations and spacing formulas that would be used by design engineers in the field. I believe it is particularly important that we work with non-homogeneous soils in analog and mathematical studies for drainage solutions. These techniques can be applied to actual non-homogeneous soils, with drainage installations in the field, to help us perfect prediction equations. As you know, most theoretical studies deal with idealized situations that seldon occur in the field.

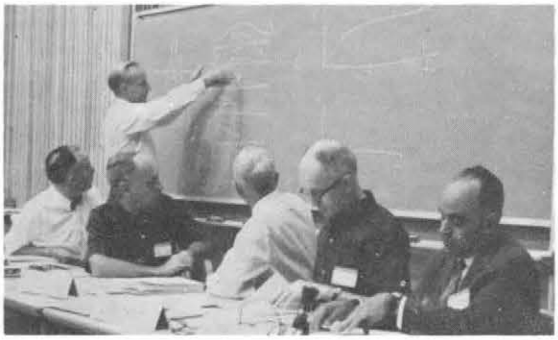
MR. CROMWELL: I would like to know in regard to what the last two gentlemen have said if they have found some way of eliminating these 500 percent variations in permeability measurements which Dr. van Schilfgaarde talked about?

MR. TAPP: Frankly, we have not. I came here with the intention of reminding you that much must be done to improve present techniques of collecting field data. We have not, by any means, perfected our techniques for determining such things as permeability. I think that some of the talent here could be directed profitably toward improving the field methods of obtaining the constants needed to solve the drain-spacing equations. I feel very strongly about it.

Another thing about which I feel very strongly is that we do not have satisfactory tools for sampling a deep, cohesionless aquifer. Just before I came here, our Information Retrieval Service sent me a notice of a new tool developed in Holland to get samples from deep aquifers. If it will retrieve samples from depth, as stated, the Bureau will use it.

MR. NELSON: As practical engineers we must ask ourselves how much effort and how much money are we willing to spend in order to get the type of numbers that are required. I suggest that this may be particularly important in connection with drainage. There has been a tendency to restrict the development of permeability measurement methods. All of the methods that we are finding to handle heterogeneity perhaps cannot be justified on an economic basis for agricultural drainage. More elaborate methods can be justified and are required in the disposal field.

DR. COREY: This brings us to the end of this session. I would like to thank all of you present.



Panel Session 2: Left to right: Dr. Herman Bouwer, Dr. R. J. M. De Wiest, Mr. John G. Ferris, Mr. R. E. Glover, Mr. C. E. Jacob, Dr. A. I. Kashef, Dr. C. V. Theis, Mr. W. T. Moody and Dr. Verne H. Scott (moderator).

(Photographs by George Palos)

SESSION 2

MATHEMATICAL DEVELOPMENTS IN TRANSIENT GROUND WATER HYDRAULICS USING A CYLINDRICAL COORDINATE SYSTEM

DR. SCOTT: I would like to make a few introductory comments. First, in connection with the remark that Dr. Albertson made in his introduction concerning the importance of this meeting. It is a significant milestone, since it appears that ground water is coming into its own. In the past only a few people have been involved in ground water work. Now, however, there are an increasing number of people being involved. I believe there is a growing interest on behalf of the lay people and this is coming on as part of the concern for and understanding of ground water problems which is resulting in increasing support for ground water work. I think this is encouraging. Now, I know that those of us who are involved in research think this support is slow in coming, but this is something we all have to endure. It is encouraging to note that there are so many people who are interested in ground water and its importance. We found out yesterday in connection with the disposal of radioactive wastes and also in the disposal of irrigation water that the whole system of ground water has become increasingly important.

We find a growing number of ground water activities in the professional societies: American Society of Civil Engineers, American Society of Agricultural Engineers, American Geophysical Union, American Waterworks Association, National Water Well Association. All of these are devoting increasing amounts of time in their technical sessions to ground water. This is coming about because of the growing number of people in this field of work.

In the line of coming meetings some of you are aware of the session devoted to ground water in the ASCE Hydraulics Division Meeting in Pennsylvania in August. The IUGG has some interesting papers on ground water in its sessions the latter part of August in Berkeley. From September 30th to October 3rd in San Francisco, the National Water Well Association will have technical sessions on ground water. The American Society of Agricultural Engineers will also have a session in Chicago at their winter meeting, so there are certainly enough meetings taking place and I think this is an encouraging sign.

Also, in publications we find an increasing number of outlets for information on ground water. I understand that the American Geophysical Union is adapting a new format which will include hydrology, and of course ground water will be an important part. There is a new publication, The Journal of Hydrology, which has come out recently. In ground water, we have a new publication called the "Ground Water Journal," which is making a significant mark in the area of publications since it is devoted exclusively to ground water problems. Bill Walton, with the Illinois Water Survey, is doing an excellent job in editing that Journal. So I think all these things point encouragingly to the importance of ground water and the developments that are being made. We learned from the discussions yesterday that there are a multitude of problems and as we go through this day, I am sure that there will be further complexities called to our attention.

This morning's session will deal with problems and solutions of radial-flow ground water systems and we have a knowledgable panel of experts including: Dr. Herman Bouwer, Research Agricultural Engineer, Agricultural Research Service, Water Conservation Laboratory, Tempe, Arizona. A native of Holland, he received his education over there and at Cornell; Dr. DeWiest who is on the faculty at Princeton, and an authority in this field, having many years of experience; Mr. John Ferris, Research Engineer, Ground Water Branch, U. S. G. S. He is also on the faculty of the University of Arizona in their program of hydrology. He has also had a lot of ground water experience in a number of states - Michigan, Indiana, and New York, as well as in Arizona; Robert Glover, who has had a number of years experience with the Bureau of Reclamation and the U. S. G. S. in a variety of problems, not only in ground water but in engineering mechanics; C. E. Jacob, who had many years of experience with the U. S. G. S. and on the faculty at Utah University. He is now a ground water consultant in California; Dr. Kashef, who comes from Egypt. He was on the faculty at Cairo University, American University in Lebanon, and more recently in the Civil Engineering Department, North Carolina State College at Raleigh; Dr. Theis, whom I know all of you know - considered one of the Deans, if not the Dean, of modern ground water analytical methods, and we are certainly pleased to have him participate; William Moody, Chief of the Technical Engineering Branch, U. S. B. R., Denver, having had a number of years of experience in a variety of engineering activities with the Bureau.

I would like to point out that the success of this program depends on the participation of not only the panel, but of all of you, and I hope that you will engage in discussion, questions and debate, and contribute ideas and information you may have. Certainly the success of this meeting will depend on an exchange of information.

We will hear from Dr. Bouwer first.

DR. BOUWER: Thank you Verne, I would like to take up where we left off yesterday, namely on the importance of hydraulic conductivity measurements. Of course the hydraulic conductivity is a critical factor when it comes to predicting rates of flow in porous media. We can divide the methods for measuring hydraulic conductivity into two groups. For the first group, we are dealing below the water table and there we can use the various pumped-well techniques. The second group is the less fortunate case where we want to know the saturated hydraulic conductivity but where we do not have a water table. Several methods have been developed to measure conductivity in the absence of a water table. One of the activities of my institution has been the development of another field method for measuring this saturated hydraulic conductivity in dry holes. This method is called the double-tube method. We have two concentric tubes in an auger hole (figure 1). We fill these tubes with water to create in a few hours a zone of positive pressures in the soil below the hole. For purposes of simplicity we will call this the zone of saturation, although we may always have some entrapped air, of course.

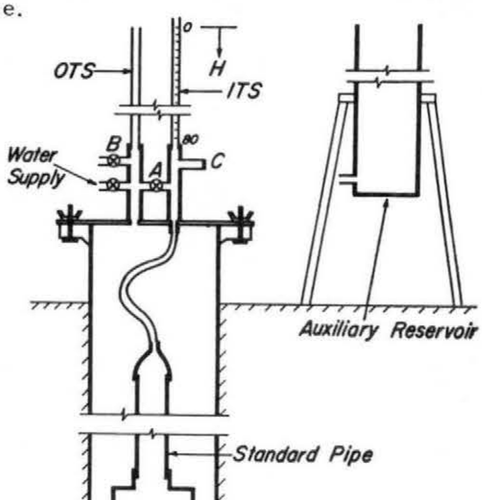


Figure 1—Sketch of double-tube installation with top plate and standpipes (ITS for inner-tube standpipe and OTS for outer-tube standpipe).

If the two levels in the tube are the same, the outflow from the inner tube is due to infiltration alone. If we let the water level in the inner tube be a distance H below that in the outer tube, the original outflow due to infiltration is diminished by an inflow component Q_H due to the water level difference H .

The field measurements are aimed at evaluating the inflow component Q_H as a function of water level difference H . We do that by having standpipes on both tubes and by carrying out various falling water-level measurements in the standpipe on the inner tube. The hydraulic conductivity calculation utilizes a dimensionless factor which describes the flow system below the augerhole due to different water levels in the concentric tubes. These factors have been evaluated with a resistance network analog and they have been presented in the form of a graph.

The field procedure with the double tube method takes a few hours in most soils and we think of the method as a rather useful tool in measuring conductivity for, say, recharge systems, disposal systems, drainage aspects of newly irrigated projects, seepage from projected canals, etc. The method has been field tested and is now being used or tested by several groups in this country. Our current program consists of efforts to use the double tube principle to measure both vertical and horizontal conductivity in case of anisotropic soils.

The resistance network analog at our Laboratory has a capacity of 736 nodes. The resistors are constructed as calibrated, variable resistance, plug-in units. Thus, desired resistance values can be directly dialed on the analog, which saves a lot of time in setting up and solving flow systems. With the plug-in units, flow systems can be quickly assembled and disassembled and the units can be used over and over again. In addition to potential problems, flow problems involving free boundary development, steady unsaturated flow, or moving water tables can be analyzed. Use of variable resistors also permits solution of axially symmetrical systems.

We have used the analog, among others, in an analysis of ground water mounds in connection with artificial recharge. If the distance between the original water table and the impermeable layer is quite small compared with the width of the recharge system, then the streamlines are essentially horizontal. However in a flow system where we have a mound above an unconfined aquifer of large or even infinite vertical extent, the stream lines are much more vertical. This shows the limitation of using the Dupuit-Forchheimer theory in flow systems where we do have vertical components. The limitations of the Dupuit-Forchheimer theory are discussed in Appendix E. Because the Dupuit-Forchheimer theory forms the foundation of so many of our theoretical analyses, it might be of interest to again examine its validity, particularly in connection with theoretical analyses of recharge systems. Since the results of our electrical resistance studies are free from major simplifying assumptions, comparison will give us some idea of the limitations of the Dupuit-Forchheimer theory.

With ground water recharge, we have a flow system whereby the stream lines originate and terminate at a water table. We might call such a system an "upper-region" flow system, because only the upper region of the aquifer contributes actively to the flow. A similar flow system occurs in artificial drainage, where the stream lines originate at the water table and terminate at the parallel drains. This also might be called an upper region flow. Pavlowsky has called the upper and lower part of such flow systems active and passive, respectively. We might also define these as flowing regions and stagnant regions. Now what is the consequence of this type flow system in applying the Dupuit-Forchheimer assumption? If we start with a shallow impermeable layer, we limit the extent of the zone of active flow by the presence of this impermeable boundary. If we lower the impermeable boundary, the region of active flow can further expand until we finally reach a level whereby the active flow region has fully developed. Further lowering of the impermeable boundary will only add more passive region to the flow system without affecting the active region any more. The total flow will then also remain essentially unaffected.

If, for the case of drainage, we plot the flow Q against the depth D of impermeable material below the drains, we see that first we get a linear increase in Q as we increase D . Soon, however, the rate of increase starts to diminish and finally becomes zero when increasing D does not affect Q any more. In case of drainage, the point whereby Q is no longer affected by increases in D occurs when D has reached about 2/10 spacing of the drains. How does a similar plot look for a recharge system? From the analog results, although not directly applicable to this case, I could at least postulate that the following relations would exist for a mound above a water table of an unconfined aquifer.

Let W define the width of the mound or the width of the percolation zone for two-dimensional mounds, $2R_c$ the diameter of a circular mound, and D the distance of the impermeable layer below the original water table. If we plot the total flow Q below the mound against D, we see again that Q first increases linearly with D . The rate of increase in Q soon diminishes and the point where Q is no longer affected by increasing D is reached when D has become equal to approximately $1.5\ R_c$ for the circular mounds, and equal to W for the two-dimensional mounds. How would these curves look if we would apply the Dupuit-Forchheimer assumptions to relate Q and D ? They would be straight lines and they would continue to be straight up to infinity. The vertical difference between the curve and the straight line is the error we would incur if we apply the Dupuit-Forchheimer assumptions to the type flow systems where we have active and passive regions. Unless D is relatively small, this error can be quite large. The philosophy underlying the indiscriminate application of the Dupuit-Forchheimer theory to systems of this type could well be the expediency of getting solutions, rather than the validity of the concept. Thank you.

DR. SCOTT: Very good, Herman. Next will be Roger DeWiest.

DR. DE WIEST: My work in radial flow has been limited. The one paper I have considered in radial flow is included in the session on the theory of leaky aquifers. The other two are for eccentric or non-radial conditions. I may add some comments to Herman's expose'. He pointed out the work of Pavlowsky. In the book by Polubarinova-Kochina we have some considerations on this problem. Actually, when we talk about the Dupuit-Forchheimer assumptions, we are starting from Boussinesq's equation for the free surface and it is pointed out under which conditions this equation is derived and which are the limitations. Then, there are various ways of linearization, one way being the Dupuit-Forchheimer way of attack. But there are other ways of linearizing the free surface equation, and it might be worthwhile to take a look at the work done by some of these Russian authors as summarized in the book.

MR. FERRIS: There follows a brief resume of the principal work that has been done on radial flow regimes by the U. S. Geological Survey and some of the work in progress. Omitted from the present discussion are those radial flow models which will be covered this afternoon under the subject of flow in leaky aquifers. Although he limited his attention to the steady state, Slichter,[1] in 1898, developed in some detail several mathematical models for radial flow systems and identified the analogy between fluid permeation and other fields of potential flow. In 1931 Wenzel[2] made an intensive study near Grand Island, Nebraska, to evaluate the applicability of the Thiem equation under field conditions and to evaluate the use of a method suggested by Meinzer for determining specific yield.

The first mathematical analysis of non-steady flow was Theis'[3] contribution in 1935 of the non-equilibrium formula. In clarifying the analogy between heat conduction and fluid permeation he pointed the way to a wealth of mathematical developments which could be translated to ground water hydrology.

(1) Slichter, C. S., 1898, Theoretical investigations of the motion of ground waters: U. S. Geol. Survey 19th Ann. Rept., pt. 2, p. 295-384.

(2) Wenzel, L. K., 1936, The Thiem method for determining permeability of water-bearing materials and its application to the determination of specific yield, results of investigations in the Platte River Valley, Nebr.: U. S. Geol. Survey Water-Supply Paper 679-A, p. 1-57.

(3) Theis, C. V., 1935, The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage: Am. Geophys. Union Trans., 16th Ann. Meeting, pt. II, p. 522.

There followed in 1940 Jacob's [4] work on the elastic artesian aquifer wherein he developed Theis equation from fluid mechanics and thereby elucidated the physical significance of the storage coefficient. Jacob also clarified the nature of Thiem's equation and identified its relation to the storage coefficient. He also developed several very useful asymptotic solutions of the Theis equation and with Cooper [5] published an exposition of the applicability of these asymptote models to field problems. Jacob contributed the first mathematical analysis for radial flow to a well of constant drawdown and with Lohman [6] described the first field application of the model.

In Theis' [7] study of the effect of a well on a nearby stream he introduced the image-well method as an aid to the solution of boundary-value problems. Within a few years applications of the image-well method were extended over a wide range of field problems which involved geo-hydrologic boundaries of diverse form. A study by Guyton [8] in the Houston area, in 1941, provided the first quantative field tests of an artesian aquifer of great areal extent. These tests marked also the first application of Theis' equation to a regional complex of many individual well stations which pumped simultaneously at widely different rates and operated on diverse schedules. In 1945, Jacob analyzed the problem of flow toward a partially penetrating well and pointed out the effects of flow convergence on head distribution near the pumped well. In 1947 [9] he developed the mathematical models and a method of step-drawdown tests for evaluating entrance losses in wells. Amplification and extension of the method for evaluating entrance losses was made by Rorabaugh [10]

Recent studies reported by Stallman[11] on flow toward a well in a water-table aquifer identify the relative magnitude of certain energy losses that were not included in Theis' analysis of radial flow. Following Boulton's development, Stallman prepared a series of type curves which identify head losses by vertical flow components that accompany cross-bed movement of streamlines as they flow toward a well in a water-table aquifer. Use of this series of type curves requires a knowledge of the space coordinates of the point of head measurement relative to the point of withdrawal. Using the field data for Wenzel's test at Grand Island, Nebraska, Stallman found that after accounting for the vertical flow components there remained significant departures of the field data from the Boulton type curves and thus established that other physical aspects of the problem yet to be resolved are of considerable importance.

Work in progress on the physics of drainage and unsaturated flow as related to the problem of flow toward a well in a water table aquifer was reported by Smith[12] . Studies in progress, by Stallman[13] on the use of earth temperature as an index of water movement in the unsaturated phase will contribute to studies of the free surface problem in the water-table aquifer.

MR. SCOTT: We are glad to have you mention recent work and I hope others will take advantage of this opportunity to convey any information on work that has been completed, or will be completed in the near future so that we will all have a chance to know

(4) Jacob, C. E., 1940, On the flow of water in an elastic artesian aquifer: Am. Geophys. Union Trans., 21st Ann. Meeting, p. 574-586.

(5) Cooper, H. H., Jr., and Jacob, C. E., 1946, A generalized graphical method for evaluating formation constants and summarizing well-field history: Am. Geophys. Union Trans., vol. 27, p. 526-534.

(6) Jacob, C. E., and Lohman, S. W., 1952, Nonsteady flow to a well of constant drawdown in an extensive aquifer: Am. Geophys. Union Trans., vol. 33, no. 4, p. 559-569.

(7) Theis, C. V., 1941, The effect of a well on the flow of a nearby stream: Am. Geophys. Union Trans., 22nd Ann. Meeting, pt. 3, p. 734-738.

(8) Guyton, W. V., 1941, Applications of coefficients of transmissibility and storage to regional problems in the Houston district, Texas: Am. Geophys. Union Trans., 21st Ann. Meeting, pt. 3, p. 756-772.

(9) Jacob, C. E., 1947, Drawdown test to determine effective radius of artesian well: Am. Soc. Civil Eng. Trans., vol. 112, p. 1049.

(10) Rorabaugh, M. I., 1953, Graphical and theoretical analysis of step drawdown test of artesian well: Am. Soc. Civil Eng. Proc., vol. 79, Separate no. 362.

(11) Stallman, R. W., 1960, Notes on the use of temperature data for computing ground-water velocity: U.S. Geol. Survey open-file report, 17 p.

(12) Smith, W. O., 1961, Mechanism of gravity drainage and its relation to specific yield of uniform sands: U. S. Geol. Survey Prof. Paper 402-A.

(13) Stallman, R. W., 1961, Boulton's integral for pumping-test analysis: U. S. Geol. Survey Prof. Paper 424-C, p. 24-29.

about it. Mr. Glover, Will be next.

MR. GLOVER: Inasmuch as Mr. Moody and I are in the same organization, I would be pleased if Mr. Moody will take care of this when his turn comes.

MR. SCOTT: Alright. We will call on Mr. C. E. Jacob next.

MR. JACOB: It is very stimulating to be here, and I congratulate CSU on the Symposium. I think it has been long needed. Being a consultant, you do not have much time to do research, if you behave as most consulting firms do. You may have your own handbook or may use one you used twenty years ago. You do not have much time for reflection and study of the literature as you do when you are employed by someone who can backstop you. However, it has its advantages; I gained an appreciation for engineering, which I did not have when I worked for the Government, and which, I am sure, most Civil Service people do not have. In other words, what is engineering?

I think, rather than talk about details, I would like to talk about keeping your feet on the ground, keeping a little perspective about water. A lot of money is spent for research on water now. It is true we are trying to get more money, but more than ten-times as much money is now spent on ground water as was spent when I joined the U.S. Geological Survey about 25 years ago. Water can be got for a fraction of a cent a barrel, some of the best water in some of the poorest places. Petroleum is worth about $2.00 a barrel. We heard last night a very excellent lecture on the difference between the science and the art of petroleum engineering. There is a tremendous gap. There will always be a tremendous gap in water, and it will last much longer. And I think that we in this business should increase our efficiency. I think a great deal of irrelevant data are collected - field data, collected in groundwater research. And I think that there should be a great deal of sharpening up of thinking and of techniques, and particularly of resources that are now available for research, to greatly sharpen what we are doing.

Now, we talked about being in the ballpark; someone from the Netherlands brought it up. Lots of times we talk about someone being "way out in left field." Is that so bad after all? You have a left-fielder, don't you? Of course, if he is out there swinging a bat, then something is wrong. We should try to do the batting at home plate. Now, a great deal of engineering work is just getting yourself inside the ballpark. Very often in consulting practice you are asked to give an answer next Thursday. Would it be possible to put a large industrial plant here? How long will ground water last? What will it cost us? What are the objections? What are the legal angles? These problems require answers. You do not have time to send it back to your district or your region or have it edited, you have to have the answer now. So a great deal of rule-of-thumb work is done in engineering practice, it is valid, and it is admissable. It can be done.

Now, a little bit about the hydraulics of wells. We had a recital of the history of ground-water hydraulics, and now we are talking about radial flow. But you cannot separate radial flow from what we called yesterday "uni-directional flow". In fact, we were not talking about uni-directional flow yesterday. We were talking mostly about unconfined flow in a two-dimensional system in a vertical plane, which is not a circuit. That is, it is not a uni-directional ground-water circuit. So you cannot unscramble these things. Yesterday we talked mostly about unconfined flow because we realize that in confined flow, where the geometry is constant, it is very easy to describe the flow by the mathematics of heat conduction, which is an old science that goes back to the early 19th century. So we have borrowed a great many solutions and so has the petroleum industry. And we have interchange with each other, unfortunately not as much as there should be.

There is still a lot of work being done, and a great renewed interest in heat conduction today. Some of the greatest applied mathematicians are working today on heat conduction and diffusion because suddenly they become interested in this because of rocket engines and reactors, neutron diffusion, etc. So, attention the last ten years has been focused very sharply by mathematicians upon these problems and unfortunately we do not have - except for a few outstanding examples that you see here - we do not have people in applied mathematics field working on this as there should be. I think that every university should have close correlation between the civil engineering and geology departments and the people in applied mathematics so that the best talent is put to bear on these things. Now, educational institutions and some government bureaus can afford to do this, consulting engineering firms cannot.

Still there is a great deal of work that needs to be done, and is being done because of the invention of digital computers. Very high-speed memory recovery and computational facilities, have revolutionized the computation industry. I think this has had a bad effect on engineering offices. A lot of engineers have forgotten there is such a thing as a desk calculator. Now, when our clients have been able to afford it - we have programmed problems on digital and analog computation machines. But most of our work is done in numerical integration. We have tricks, known and unknown, which enable a person, by certain techniques, very rapidly with a desk calculator to do numerical integration for whatever configuration you may have, whether it is uni-directional or radial or three-dimensional or whatever it may be.

In the hydraulics of wells a great number of mixed boundary value problems are solved. For example, if you have a well that has large internal storage, let us say a "silo" such as a missile silo, or such as an excavation that has radial flow, you have an internal boundary condition of the third kind, which involves not only the gradient, but the rate of change of head on that boundary, and these are related to the rate at which you take the water out of that excavation. In fact, you may have to analyze it while the excavation is being constructed. These problems can be solved very nicely on desk calculators using numerical techniques.

As to the hydraulics of wells, there are unlimited combinations. Of course we have specific examples. Then there are attempts made which are logical, I think, to get a generalized theory, for example of the gravity well. This has never been successfully done. I don't think it will ever be successfully done in closed form until some mathematicians set forth with new ways of handling boundary value problems with moving boundaries. Now, great advances are being made in hydrodynamics, which ground water people are not aware of and are not able to capitalize on.

Now, just a little about the hydraulics of wells, because we had something to do with the earlier thinking about this. If you statistically study a great number of wells in similar geological environments with similar design, you find out a very interesting thing which every well driller should know, and that is, wells differ more than the transmissivity of the aquifer differs from place to place. This is because of the difficulty of developing the well and uniformily cleaning up the bore of the well and opening up the formation. Now, we have statistically studied hundreds of wells in the world, for example in West Pakistan, in the coastal plain of Formosa and in the coastal plain of Israel, and other places where we have great groups of identically designed wells. At one place in Venezuela (Calabozo) we had one hundred pressure-relief wells on an earth dam. These can be studied by statistical approaches. Just think of a simple relationship, that the drawdown in a well is some constant of proportionality A times the discharge Q over the transmissivity T plus some other constant C times Q^2. Thus:

$$s_w = \frac{A(r_w, t)\,Q}{T} + CQ^2$$

Now, this is where pipe hydraulics was in 1850 when Darcy was making filter beds and discovered his well-known law of flow. This leads to a specific-capacity diagram. The specific capacity, of course, is a function of time. So if you were to plot drawdown versus discharge of the well, you get a parabola for each time: for each successive time you get a lower parabola. This is a specific-capacity diagram. Most well drillers still think that there is just one line on this diagram. This is a handicap. This is the difference between the art and the science. Most pump salesmen, and a lot of civil engineers also, still think there is only one straight line, and this is not true.

Now, if you want to you can compare a great number of wells in the same geologic formation, that is, in similar rock. The transmissivity, of course, has been separated out for that purpose. We can plot this in a very interesting way, for example drawdown divided by the square of the discharge, giving an equation:

$$\frac{s_w}{Q^2} = C + \frac{A(r_w, t)}{TQ}$$

and if I wanted to take the variation of that ratio with $(1/TQ)$ for a uniform period of time, we would have something like in Figure 2, if the A-factor is the same for all the wells. However, you know that this

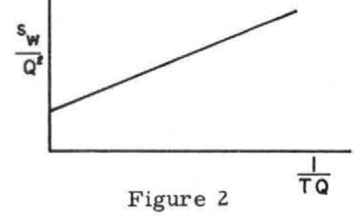


Figure 2

A factor is not only a function of time but also of what we call the "effective radius" of the well r_w.

I can make this artificial division of the resistance formula. When this is done statistically with a great number of wells, the points are very scattered, which indicates to you the thing I pointed out, that the wells vary much more (from well to well) than the formation does. Always this is true. I have never seen a well conducted drilling campaign - where there have been very rigid specifications and very close control on well development and pumping and testing - I have never seen a place yet where the drillers have been able to drill wells that are as uniform as the formation.

DR. SCOTT: Thank you Mr. Jacob, we appreciate your philosophical views on the problems of wells.

DR. KASHEF: I am going to discuss a summary of two publications which were made some time ago. One of them was published in 1952. When I was at Purdue, a group of people including myself were interested in solving some of the problems of ground water flow by the numerical methods or the finite difference methods. At that time only the steady state problems had been solved by the relaxation methods. The well-known problems were those of Southwell and his associates in Britain. They solved the problem of the seepage through a dam with a free surface and then Van Deemter in the Netherlands solved the problem of tile drains and in 1950 Yang at Harvard solved the problem of seepage through earth dams. These are all two-dimensional problems and they involve what is called "the line of seepage", and are more complex than straight-forward two-dimensional problems with known boundaries.

In 1952 we published the solution of the steady-state flow and the nonsteady-state flow of both artesian and water table wells. We have in the Appendix

B, only the summary that concerns the nonsteady-state flow. The finite difference equation was based on the general mathematical solution that governs the flow towards the well. Both problems of the artesian and water table wells were solved. In the water table wells, the Dupuit-Forchheimer assumptions again were assumed, and we compared these results with Theis' equations. They compared very well except in the very vicinity of the well bore, and we attributed that, to the fact that in the Theis solution, a line source or line sink was assumed and in our solution it was the actual well diameter that was considered. Naturally, as in all numerical methods, some difficulties are involved in applying these methods. The main drawbacks were, in the first place, they are very laborious and you cannot reach a certain drawdown curve at a certain time until you work out all the previous time intervals. In the second place, to start this solution we have to make an assumption. So if you will refer back to the figure in Appendix B (B 12) the aquifer was divided into concentric shells -- it starts with W, 1, 2, 3, etc., and the general shells L, O and R. Now to start the solution we have to make an assumption, that the head in the first shell, W , was assumed to be constant during the first time interval. So if these shells are chosen to be of a very small thickness, or very small base area, and the intervals were selected too small, then this assumption would not be severe. That was the only assumption made here. But we were confident that the results are fairly correct.

Now, in order to improve this, in 1961, I thought about including the finite difference method in a graphical solution. That was published in the International Conference of Soil Mechanics (Paris, 1961). I made a semi-graphical solution based on the finite difference method. If you can graph the drawdown for a certain time interval, for a transient case, you can determine any other drawdown after any other time interval. This is done simply by shortening or delongating the radii and measuring the corresponding drawdowns as shown in the original paper. This is proved in this paper mathematically. I have here the drawdown after six hours and then I constructed the drawdown after twelve hours, eighteen hours and twenty-four hours by this simple construction on the graph. So it is very simple to plot the drawdown curve once and for all, and then you can determine all other drawdowns by making this simple projection.

This, of course, was not done just for the sake of one well, but it was going to be extended for a group of wells - wells near a stream and other complex problems. I am working in this area now and I hope I can finish it in the near future. I am also considering the study of the coefficient of storage which is not so well accepted to be considered as a constant. The solution may be corollated to the principle of effective pressures in the soil mechanics field. I hope to succeed in determining the variation with pressures and to arrive at a simple solution.

DR. SCOTT: May we hear from you next, Dr. Theis?

DR. THEIS: We have been talking about applications of equations and new types of equations, and I thought I would go back to the time we developed the first transient flow equation. The original transient case equation was developed from a geological standpoint. We have not talked here very much about the geology to which these equations must be applied. In the days when this equation was being worked out, which was before publication of Muskat's book or King Hubbert's paper on ground-water flow, I was faced with the proposition of how to think about pumping from the High Plains, particularly from the Llano Estacado, or Staked Plain of Texas. Here, very evidently, we could not think of a small withdrawal of water from a water body,which had several times the capacity of Lake Mead,in equilibrium terms, which had been the practice up to that time. So I was driven to find a new way to think about the problem and, of course, as any of you who have read the paper will know, I ended up with a mathematician friend who helped me in my quest to find the new way. Mr. Jacob was not on the Geological Survey, or at least I did not know him, at that time. But when we first proposed this, the main adverse comment about it in the Geological Survey was that everyone knows aquifers are not homogeneous, which irritated me at the time because the only thing I could say was that we might hope that there may be some sort of statistical homogeneity sufficient for the problem. Well, I did not know I was going to open up a Pandora's box of new equations in such numbers that it would be necessary to have a conference like this to try to bring some order into them. But the criticism that was made originally is still valid, it seems to me. Everyone knows that an aquifer is not homogeneous, so I began saying fifteen or twenty years ago that I would spend the first half of my career arguing that there was enough homogeneity that such equations could be used in solving problems about an aquifer and spending the last half of it warning against putting too much faith in this homogeneity.

One of the groundwater phenomena which has only lately been studied and that emphasizes the fact that aquifers are not homogeneous is that of dispersion on the field scale. The study of these phenomena has been made particularly necessary by the use of input wells for waste disposal and the consequent necessity for knowing the course of a possible contaminant; but the knowledge gained has application also to the interpretation of data concerned with discharging wells.

The first studies of these phenomona, notably at Harvard and Berkeley, were made in the laboratory on homogeneous materials and resulted in showing a rather surprising amount of dispersion. However, it soon became apparent that the amount of dispersion varied only with the square root of the distance traveled. Therefore, such dispersion loses importance when extrapolated to field dimensions.

However, when experiments in the field were performed at Berkeley and elsewhere the dispersion observed was orders of magnitude greater than that predicted from studies on homogeneous materials in the laboratory. In an experiment at Hanford about one hundred pounds of fluorescein was placed in an unused well over a period of about one day. Laboratory experience indicates that the angle of spread of a tracer would be about 6 degrees, but the dye at Hanford was observed in two unused wells about two miles down-gradient making an angle of about 30 degrees with the input well and was observed in both wells over a period extending from about sixty days to about two-hundred fifty days after input, peaking at about one-hundred ten days. The angle of spread was probably much greater than the 30 degrees observed; however, no other wells were available for sampling. The aquifer tested was unusual in that it consisted of gravelly glacial outwash with an exceedingly high permeability.

Such dispersion must be due to non-homogeneity in the aquifer - to the many different velocities that may be characteristic of various available flow tubes through lenses of various permeabilities and to the refraction of flow lines in passing from lens to lens. The course of ground-water is much more complicated than the simple flow lines we are compelled to assume assume for most mathematical treatment.

I have sometimes referred to the "ghastly" assumption of homogeneity that we must make in at least most quantitative ground-water studies. This is perhaps an overstatement because the assumption has proved very fruitful and resulted in increasing greatly our knowledge of the hydraulics of aquifers. Bill Nelson has indicated how complex and costly is a program to actually take into account movement through a heterogeneous medium. However, the more knowledge we gain of the actual movement of ground-water the more we are reminded that we are generally using greatly simplified mathematical or other models as a criteria to judge the actual flow and our only safeguard is in very complete data on the actual system, both areally and in time.

DR. SCOTT: Thank you, Dr. Theis. Next, we have Mr. Moody.

MR. MOODY: Following as I do, after these illustrious gentlemen, I feel like anything that I will say will be anti-climactic.

Mr. Glover, you will remember, passed the buck to me, as it were. If you will look on the list of papers that are included here, you will see that he is not a buck-passer as far as it comes to putting out work. Many of these solutions have his name on them. He pointed out earlier the primary interest of the Reclamation Bureau is in the field of irrigation. There are two aspects of that field in areas where we use solutions of these types; one in the field of drainage and the second in the production of water from underground aquifers. The linear cases we discussed yesterday have, for the most part, applications to the area of drainage. The radial cases which we are discussing today can be used to apply both to production and drainage applications. All I intend to do here is to discuss a group of the papers containing solutions which are included in Appendix B.

The first paper is a parallel to the work of Jacob and Lohman in some respects. It is a solution to the flowing artesian well. However, the chart which accompanies this paper does enable us to determine the drawdown at any given radius from the well at any time. The table which also accompanies the solution is a function very similar to the one which Jacob and Lohman gave, in fact, if you take into account the fact that the parameter in this case is directly related to the square root of their parameter, you can check some of these values directly. And they do check out to three significant figures for every case in which the argument parameters are comparable.

The second paper here is again a parallel of Dr. Theis's solution in which we have found it useful on occasion to use a parameter which is the square root of his. This makes a slightly more compact table, a copy of which is included along with the solution.

Our third contribution is found useful in estimating the effect where we have a condition of distributed pumping over a large area. We wish to estimate the effect of that generalized pumping at the center of the area. This solution, including a dimensionless curve, provides a means of making such an estimate.

A fourth contribution represents a similar situation in which again we are interested in the minimum drawdown which may be accomplished by pumping, this time by a small group, such as square array, of wells. In general the chart shows minimum drawdown at the center of arrays of $4n^2$ wells - say 4, 16, 64, 100 - on up to an infinite number. In this respect these last two cases are similar except that the latter one can be used for small, finite numbers of wells, whereas the other one is more applicable to a general distributed pumping over an entire area.

The next case is one which is derived in order to evaluate the portion of a well's production which might be coming from a bounding river, the river in the neighborhood. This solution, again is by Mr. Glover, and has a chart in dimensionless parameters which gives the results.

We have one more case. This case gives the drawdown within a circular region when the water table is maintained constant at a given radius. That one is a more recent development. Thank you.

DR. SCOTT: Mr. Glover, would you like to make any comments?

MR. GLOVER: The chart used to determine a piecewise depletion of a stream by an adjacent well, was not prepared by me, but was due to Mr. Florey of the Bureau staff.

DR. SCOTT: Are there any other comments, Mr.

Glover?

MR. GLOVER: Your question concerns the relation of this development to the work of Dr. Theis. The boundary conditions are different. This one is arranged so that you can study the depletion of the stream piece-meal. I believe Dr. Theis took the entire amount.

DR. SCOTT: He took the entire amount to a given time and you are taking the variations. Is that right?

MR. GLOVER: Perhaps Dr. Theis can give me some help on this. In your case, and in the paper you prepared concerning the depletion of the stream, if I recall, you had an integral such as the one we have here, but you integrated from minus infinity to plus infinity and this one is arranged so that the depletion from the stream can be taken piece-meal. That is, I think, the only difference.

DR. SCOTT: Now, do we have any discussion, comments, etc., on any of the topics that were raised here?

MR. GLOVER: I would like to comment on a matter that has given me some concern. You have seen a number of cases where people have attempted to assess the validity of the Dupuit-Forchheimer approach by comparing the idealization with what they considered to be the reality and then making some decisions. I don't wish to say that that should should not be done, it should be done. However, I am pretty thoroughly convinced of one thing, and that is that the simple formulas that we derive by the use of the Dupuit-Forchheimer idealization are going to be the work horses we are going to have to use to solve most of our ground water problems. It seems to me that we could here return to some well accepted engineering principles. We have in our engineering work, from its beginning, had to deal with approximations. I believe if you review all of your engineering developments you will find most of them are of that nature. It has come to be conceded that the engineer must understand the nature of those formulas. He must be perfectly well aware of their limitations. He must use them with judgement and discretion and in cases of doubt, he must put the formula to actual test. It seems to me that this should be our philosophy here.

I have a slide, figure 3, which I think would be a case in point. This shows a test made to compare experimental data with the results of an analysis of the growth of a mound.(14) This analysis is worked out by using the Dupuit-Forchheimer principles. The test was made on a glass bead model by Professor Marmion of Texas Tech. You will notice the close approximation in the extreme case of a mound built up from the dry on an impermeable layer. I wonder if Professor Marmion would care to comment on the way this test was conducted?

DR. MARMION: The test was conducted with a model constructed of two parallel plates of plexiglass spaced about fifteen millimeters apart. The region between the plates was filled with 3 mm. glass beads to serve as the porous medium and a mineral oil, chosen to have the same index of refraction as the glass beads, was used as the fluid. With this combination of fluid and porous medium, the model becomes transparent in zones of complete saturation. A dark background was placed behind the model and a flood light illuminated the front of the model. Where the beads were saturated with oil, the light passed through and was absorbed by the black background while unsaturated zones refracted and reflected the light. Zones of complete saturation, including the growing mound, were well defined and were recorded photographically. I would be glad to answer any questions about the test set-up.

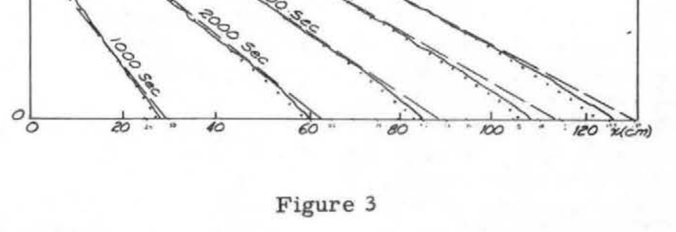


Figure 3

(14) Marmion, Keith R., 1962, Hydraulics of ground water mounds in artificial recharge. Paper presented at the winter meeting A.S.A.E. at Chicago.

MR. JACOB: I agree very heartily with what Dr. Theis has said about inhomogeneity. It is true we try to describe nature and we make some kind of a mathematical abstraction or over-simplification of what nature really looks like. But isn't homogenity, after all, a matter of scale? I think this leads to a favorite theme of mine, and that is, sometimes we get lost in details. In waste-disposal and dispersion problems, of course, you have to look at the details, but very often you get lost in the details where they are not involved. For example, in a whole irrigation project, it has to be appraised and the water balance obtained and its drainability determined, and sometimes we get lost in the details. There are techniques, not including pumping tests, for getting regional values of the transmissivity of an aquifer, for example.

I would like to recite two or three very pointed experiences that I have had in my career. We get an idea, as for example the exponential-integral solution of the heat-flow equation, which has become very useful in using one single equation that describes the hydraulics of wells best because wells are usually pumping at a steady rate, or nearly so. So we get carried away with the problems of determining permeabilities in the field by certain techniques that have been elaborated almost ad infinitum. We get these values of coefficients that we put in an equation and try to predict what the aquifers will behave like. Now, I have had examples where permeabilities so determined have been several times higher than the effective average permeability of the aquifer. We had a closed syncline, where we were attempting to get a water balance. When you look at the historic behavior of the water levels and the response to rainfall variations you get a certain parameter for the aquifer which is a small fraction of that which appears to be got by interference alone.

Just another example. This would be in the coastal plain of Israel where the successful wells hit a Pleistocene limestone called "Kurkar", which is very porous, and the well drillers know how to find it and finish a well in it. It has a very high transmissivity. You want to get the average transmissivity of the coastal plain as a whole, that is above the Saqiya Marl bed which forms the base of the aquifer. The wedge of sediments is several hundred feet thick at the coast. If you want to get the transmissivity of the whole series of beds, it comes to be a small fraction of what some of the well transmissivities are shown to be. And how do you get this transmissivity? Well, you get it again by analyzing the historical behavior of the aquifer.

So, I maintain that very often we get lost in the details. Just like we were trying to describe a table top. Somebody gets down and looks at it with a microscope, and really we just want to buy the top for its utility. It is a matter of scale. I think if we stay with the scale, sometimes inhomogeneities are not too troublesome.

DR. SCOTT: Mr. Bittinger, would you like to make a comment on the work that you and Mr. Glover did?

MR. BITTINGER: The small amount of theoretical work we have done here at Colorado State University has actually also been largely due to Mr. Glover. As many of you may not know, Mr. Glover has for several years taught some courses here. Dr. Scott is one of his students. Bill Nelson, I think, also took his course when he was here. I will just draw your attention to the two papers in Appendix B which are from CSU. One has to do with the drawdown around a well in a thin, unconfined aquifer. In the second, which does not include Mr. Glover's name as an author, but I assure you that he was in the background on it, is a development for the effect of a circular recharge basin over an unconfined aquifer. Like many others have done, this development was adapted from Carslaw and Jaeger's work in heat conduction.

DR. SCOTT: How about some questions from the audience?

MR. LAWRENCE BEER: I would like to direct a question to Mr. Jacob concerning bridging the gap between Geology and Civil Engineering in the technical hydrology field.

MR. JACOB: I do not know. I was trained as a Civil Engineer and therefore, I suppose, I am irretrievably lost to the scientist. But I have done graduate work in physics and in geology, and I became the head of a geophysics department, never having studied geophysics in any university, as many geophysicsts of this generation have had to do. We taught ground water in the University of Utah, at that time in the Geophysics Department, and we taught hydrology also. I think changes have been made since because of changes in personnel and so on. I know that quite a contest has been going on in the Earth Sciences between what some people call geohydrologists and what others call hydrogeologists.

The trouble with disciplines in the universities is that there is an overlap. I personally think that the study of ground water involves many disciplines and won't apologize if I write a paper and someone in soil mechanics takes me to task because I may have changed the notation, or someone in theoretical geology takes me to task because I may have made a few irresponsive remarks, or somebody in fluid mechanics, or in other fields. There are dozens interested in the subject of water.

Too often we shingle the roof from the crest down, and we have a lot of disciplines and a lot of jurisdictional disputes. We have people trying to decide whether we should call ourselves geohydrologists or hydrogeologists. And as long as we have these jurisdictional rivalries the roof does not get covered. Every part of a roof is covered with three shingles, and so if you cover the ground-water roof, you cover it that way. And so you have the field of Geological

Engineering. There is no such profession. There are a few schools like Princeton, University of Utah, and others which do have a curriculum, but there is no such profession. There is no journal, no society. Someday there may be. If there is, I hope they learn how to shingle the roof from the bottom up.

DR. SCOTT: This reminds me that among the list of societies I did miss one, The Association of Engineering Geologists. Mr. Robert Bean, do you wish to comment?

MR. BEAN: This is a little national pitch for AEG. We in California felt a while back there was a real need for an association of engineering geologists, and about five years ago we organized such an association. If you look in your list of organizations in Geo Times, you will find the geology teachers, the AAPG, the seismologists, the economic geologists, and other specialized groups, but you do not find engineering geologists. Let me say right now that as engineering geologists we include geologists who are working with civil engineers, in the overlap, Mr. Jacob, on civil engineering projects, dams, canals, highways, etc., and geologists who are working in ground water, with engineers, also in the overlap zone. So, I do want to let you know this: that after considerable internal discussion and some articles which have appeared in Geo Times, we have reincorporated under a new name, dropping California, and we are now the Association of Engineering Geologists.

This meeting here contains a lot of people who we would like very much to consider membership. I think you will recognize the desirability of a professional and technical association in the overlap zone between geology and civil engineering.

Our publications include the papers presented at our annual meeting, a brochure describing the organization, and a newsletter appearing every other month. We feel we have real group feeling, and also I think, we help each other both professionally and scientifically. You are eligible for associate membership if you are working in engineering geology, including ground water. For full membership, you need five years of experience in geology in some form, three of which have to be in engineering geology, including ground water. Interested engineers are encouraged to join as affiliates if they are not working in geology. Our membership requirements are definitely patterned after the ASCE membership requirements. We will send you on request a little brochure and also an application blank. For contacts after the meeting, our address is P. O. Box 21-4164, Sacramento, California.

DR. SCOTT: For all of that, we ought to get a free membership here. Are there any questions from those of you in the audience?

MR. DOMENICO: I have a question directed to Dr. Kashef. I would like to have him expand a little bit on some of the applications in soil mechanics that he is applying toward the coefficient of storage.

DR. KASHEF: You know the principle of effective pressures in soil mechanics and especially in the artesian aquifers. You know that the storage comes from the compressibility of the aquifer itself and this is the result of the change in the neutral pressure itself; so if you would study the effective pressure further, probably, I am not sure, you can get a very simple answer for the variation or the modes of variation of the coefficient of storage within the aquifer, especially in the artesian aquifer. I am not talking about the water table case. In the artesian aquifer the total pressure is constant at any time. The neutral pressure decreases, therefore the effective pressure increases, that is, the grain-to-grain pressure increases, and that is how we get the water out of the storage.

DR. THEIS: I would like to ask if you have considered the contribution of the confining beds. When you get compaction, you must be getting a whole lot of water from the confining beds.

DR. KASHEF: I will study that too, Dr. Theis, but after completing the idealized problem.

DR. VAN POOLLEN: I would like to elaborate a little more on this inhomogeneity. I agree that we should not always be disturbed about inhomogeneity in reservoirs if we are looking for the overall transmissibility or permeability. However, I would like to point out that in the petroleum industry we are particularly interested in this from the standpoint of secondary recovery. In many cases we are injecting water and pumping out oil, which is an old technique, but we are also injecting miscible fluids. In that instance we will get considerable differences in mixing zones and slug sizes required. We will have an oil reservoir and we will inject gas, but in between we will put a slug of material which is miscible with the oil and miscible with the gas. To determine the size of this slug, we have to know the mixing zone, and there inhomogeneity is very important. Also, we inject air and burn part of the oil, which is a relatively new technique, and, again we have to know much about inhomogeneity. To me the overall, the gross inhomogeneity of the reservoir is more important than are the microscopic forms. For example, one of the larger fields in the Sahara is called the Hassi Messaoud, or Happy Oil Field. It is about 30 miles in diameter, about 10, 000 feet deep and it has very high pressure oil and we find that there is no communication from one well to the other in many instances. Consideration is given to a secondary recovery project by injecting gas. Here we have gross inhomogeneity and I feel that transient behavior, transient methods, are very helpful in determining the gross inhomogeneities. I might also say that some of the oil companies have been going out in the Four Corners and they drill surface rock samples to study the overall picture of homogeneity or inhomogeneity, of the formations.

DR. SCOTT: That is a very interesting comment.

DR. YEVDJEVICH: As the input of water into a ground-water formation and the evaporation from soils are stochastic variables, and as the permeability and other characteristics of any naturally formed water-bearing formations are also stochastic variables (permeability of a formation is changing from place to place and from direction to direction in a place according to probability laws), the output of any ground-water formation must be also a stochastic variable. Therefore, the probability approach and the stochastic variable concept appears to be a feasible way of describing either a ground-water formation or its general response to water input.

Any study of sedimentation problems in deltas, reservoirs, and lakes shows that no homogeneity exists either in composition or in permeability of the deposited materials. The nonisotropic water-bearing formations are rather rule than exception.

There is a need for a suitable bridge between the classical (or deterministic) approach and the stochastic approach for the study of ground-water problems.

The usual classical way in approaching a problem of ground water is: (a) to approximate boundary conditions of ground-water formations by more or less simplified geometric shapes; (b) to assume that ground-water formations have either constant or simply varying characteristics; (c) to analyze boundary and initial conditions of a particular case; (d) to develop differential equations, ordinary or partial, or the mathematical models of ground-water response; and finally (e) to look for solutions of these equations.

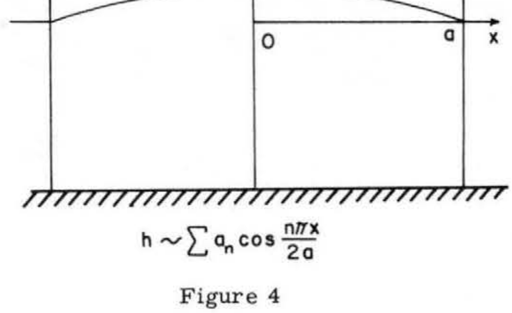
Due to the fact that solutions of differential equations are well systematized, it is usually a good approach to look through these systematic solutions and see how differential equations of a given type and of given boundary and initial conditions can be solved If there is no solution in closed form, the use of .inite difference methods and digital computers, or use of analog computers give the solution in an approximate form. This is the applied mathematics approach to ground-water problems. This approach, however, does not incorporate easily the stochastic characteristics of ground-water formations.

Looking from both a deterministic and a probablistic approach, the greatest problem of ground-water hydraulics and hydrology is in finding a bridge between stochastic characteristics of random input, random output, and properties of ground water formations, on one side, and the deterministic approach through the use of fluid mechanics laws as well as dynamic aspects in changes of the underground, on the other side. It seems that there is a research need to fill the bridge between the two lines of thought.

MR. JACOB: I think there are two things here. One is the randomness of the parameters - let us say, the permeability and the storage coefficient. A great deal of work has been done, as you may know, on the mathematics of elliptic equations and also parabolic equations that we use respectively in steady state problems and in nonsteady problems in ground-water. They are classified this way by mathematicians. You read a great deal of work in the mathematical literature on this, which arises again in diffusion problems.

Now, with regard to the stochastic nature of the input of the ground-water systems. This is an approach which needs to be worked on because very often our meteorological data does not go back far enough, does not go back much beyond our hydrological data. So to know what our input was, before the first epoch of our study, is extremely important. However, for historical records, we do take a deterministic approach on what the input to an aquifer is, that is, assuming that we can measure it. I think that there is a great need for work in this field - in the field of non-linear mechanics, which unconfined ground water actually is, and if I might, I would like to refer to the work of Boussinesq, which was referred to yesterday. What the problem is, is basically this. Boussinesq was basically a mathematical physicist. He was not trained in hydraulics, particularly, but he did some very interesting things in the theory of turbulence, and toward the close of his career, in ground water, more or less as a challenge to a mathematical physicist. He wrote two or three very interesting papers, and his work was put to good use by a French engineer by the name of Edmond Maillet, who was a contemporary of his and who made statistical studies of the flow of certain springs in the Paris Basin and elsewhere in France in limestone terrains, based upon these concepts. Now, fortunately, most of our ground water systems, even though in mountainous terrain, can be linearized very closely. But there are exceptions, and we see very many so-called ground-water hydrographs and the tail-end of the stream hydrographs plotted on a semi-log plot. Engineers are led to believe that somehow or other ground-water outflow of the basin has to be exponential, so you have all sorts of peculiar paradoxes arising when they try to apply these ideas to separate out surface and ground water run-off.

I am not going into detail, but merely point out that there is a lot of work to be done. Now, taking a very simple abstract case of an aquifer that is recharged continuously, we can draw a picture. (figure 4). We have what we call in German, the eigenwerte of a differential equation, or its characteristic numbers (n). Those of you who had elementary



$$h \sim \sum a_n \cos \frac{n\pi x}{2a}$$

Figure 4

ordinary differential equations learned that you could find general solutions, and that you could combine particular solutions to get general solutions of a differential equation. In partial differential equations this generally is not possible. The theory of linear partial differential equations is in a very rudimentary state even today. The theory of nonlinear partial differential equations is virtually nonexistent in our field. There are very few solutions available. One of them is the solution that Mr. Glover showed of filling at a point source when we initially have zero depth of flow. That has been solved analytically. One of them is this beautiful solution by Boussinesq, which in our office in our spare time we have studied in great detail on the computer. You know that you can have an equation like Laplace's equation or Poisson's equation. There are certain numbers you can get from the geometry for which that will give you solutions. Now, in a case of a system that is bounded by two parallel boundaries at which the head is maintained constant, this will be the cosine. We have a great number of harmonics; the n would be the successive harmonic numbers. This, of course, would be an aquifer of great depth. The "denivellation" would be small compared to the depth of flow.

Now consider an unconfined aquifer with zero head on the boundaries (figure 5). There would be a paradox on the outflow boundary if we look at it too close. We would get some kind of elliptic integral

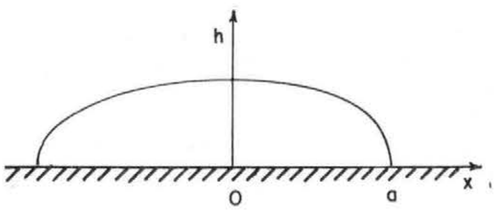


Figure 5

to get the shape of it. There is only one kind of characteristic solution to this state for this equation, and that, Boussinesq found by the separation of variables. Now, we say in this confined system (figure 4), that solutions are superposable, that is, there are all kinds of tricks that can be performed with the boundary conditions and with the solutions obtained, ad infinitum within the capacity of the system. Now there are certain limitations on the capacity of boundaries to deliver the water. But, whatever they may be, within those limitations there is linear superposability, linear solutions are added. Now, we hear many people say that there is no law of superposition in the unconfined, nonlinear case, but it is just that it is not linear and we have not discovered it, except for very simple states.

The decay of the profile in the thick aquifer is exponential in time. In handling these parabolic differential equations in one, two or three dimensions in bounded systems you come to a separation of variables. With the separation of variables, you always come out with $e^{-\alpha t}$ for the time variable, for the decay of any initial state that is put into the system. What is the decay law in the thin unconfined aquifer? Well, of course, it is hyperbolic in time. The law of decay is $1/(1 + \alpha t)$.

There are other solutions for this total differential equation $\frac{dh}{dt} \sim -\frac{1}{t^2}$ that one gets upon separation of variables. One is the hyperbolic cotangent of the time and the other is the hyperbolic tangent of the time. There are ways of putting a deterministic input, or stochastic input if you like, into a system of this kind. This, of course, is very idealized and does not look like any ground-water basin that I know of. But if I understand this idealized system, I will hope to understand the one in nature better than he who does not even understand this one. That is the value of long-haired approaches.

There is a law of superposition. Someday somebody in the field of mathematics is going to make a breakthrough. A whole new vista will be opened up.

Now, it would be extremely useful to engineers, if you could only solve the equation $\frac{K}{2} \nabla (h^2) = S\frac{dh}{dt}$.

You separate the variables and you get the two ordinary differential equations and the equation $\frac{dh}{dt} \sim -\frac{1}{t^2}$ is one of them, and the solution to the other one gives the profile. A whole field is awaiting to be moved into by our applied mathematicians who learn the key to the superposability of solutions. But, first of all, we have to start finding some particular solutions and then, having mastered the law of superposability, we will move forward. There is such a law because water knows it, and elastic beam knows it.

DR. YEVDJEVICH: A question can be legitimately raised: What are the chances that an engineer or a scientist working in the ground-water field would be able to discover solutions of some of the partial differential equations or nonlinear ordinary differential equations, which solutions do not exist in the field of differential equations and use full time to study these equations?

I still believe that the role of the physicist and other scientists and engineers in the ground-water research is more to apply the existing solutions or the procedures for approximate solutions of differential equations, than to try to compete with the specialized mathematicians in theoretical work for solution of differential equations.

MR. JACOB: I would like to reiterate what I said earlier. I really think that there needs to be cooperation, and there has not been enough cooperation. Universities such as this and others that are represented here can do a great deal, and are doing a great deal. I can name several institutions where they have very strong staffs in applied mathematics, and they are working on civil engineering and geological problems and vibration problems. Of course, most of the research is in missile and space technology and atomic energy. But there is a lot of work being done. There are a lot of books on calculation, just on the numerical solution of differential equations. You see, the engineers stepped out. Of course, the suggestion we had was made by the mathematician, John von Neuman. But, I mean, the engineers built these things to solve engineering problems, and physicists working in applied physics solved these problems and developed these techniques. The mathematician has come along a little bit belatedly and has taken up the chase and tried to verify and set the limits upon what has been done. In other words, he has tried to refine the techniques that have been used in numerical computation. The harmonic analyzer is not a new thing. Numerical integration, as has been mentioned here, was known to Newton, who had a "calculus of divided differences" before he discovered the infinitesimal calculus. So this is not a new thing. It is a matter of timing; it is a matter of what historical incidents have brought these things forth. And I think there is a tremendous gap, if you get into literature - and I am not qualified to read all of it - but if you get into the literature of mathematics you see a tremendous number of theorems that are established. But the mathematician's interest is gone as soon as he has done his mathematical work, because he is not interested in applications. There needs to be a bridging of this gap between the mind that is working upon the mathematical principle and the fellow that has to apply it. Now, I think that the engineers and the physicist and applied scientists can help the mathematicians a great deal in telling them what things are relevant, because this is what engineering is about. Engineering is about that that is relevant, timely, and economical. It used to be, but it is becoming less and less so. I have had mathematics professors, and you have had them, who could help you a great deal on techniques. But when it came to trying to make the decision, who could make the decision of what is relevant? How is he guided in making that decision? This is where field experience comes in, and this is where Dr. Theis mentions about what nature is really like. It needs to be studied in detail. But there is a time when you have to cut off the detail, and I think in ground water - and I will say it again: I spent too much time looking at the fabric rather than the whole cloth.

MR. GLOVER: I would like to comment very briefly on Dr. Yevdjevich's original question. One of the difficulties we have in dealing with ground water is that it runs below the surface and we cannot see it as we can see surface water. We cannot, for example, talk about points of diversion, we cannot measure flow with a current meter and do other things we are used to, and have always practiced with surface water. We know also, as Professor Yevdjevich pointed out, that these aquifers were laid down by erratic natural processes and there is very little chance that they will be uniform. We have nevertheless, for the sake of simplicity, idealized them as being uniform. The question comes then as to how to bring theory and practice together. It seems to me that, in the field, the best thing we can do is to assess the aquifer properties on the basis of tests made on the aquifer itself in some manner. Now, when we do that we get some constants which in a sense are not right. We put them in an equation, which is not right, but the situation is such that our answer is pretty good. There seem to be some cases also where we might say we are fortunate. We have, for example, layered aquifers where nature laid one bed down right on top of another. They are not of the same permeability and we worry about the effect of the components on the vertical permeability. In many cases it is unimportant for the reason that the areas available for vertical flow, and the vertical permeability can be much less than the horizontal permeability without causing the aquifer to behave as an actual many layered-aquifer. We get that relief from some of the difficulties that might otherwise afflict us. Another fortunate thing that we seem to find is that the solutions we get from the Dupuit-Forchheimer formulation seem to be strongly determined. They will give us closely the same answer even though we vary the quantities we put into them quite a little bit. Again, it is fortunate it just happens that way. But those things are in our favor and I think they will help us through some of our difficulties.

DR. SPIEGEL: I want to add to several points that Dr. Theis mentioned. The principal point is that there is a time-honored method of drawing water level contour maps. Water-level contour maps tell you, if you know a little bit about distribution of recharge, how the transmissivity is distributed through the aquifer. I think that we should perhaps do a little more work in the quantitative interpretation of water level contour maps to determine the degree of homogeneity of the aquifer. Another tool that we could use in the same general field is that of additional detailed analysis of base-flow recession curves. Because, if we know something about the general geometry of the aquifer which is producing base-flow recession curves, we can then compute the theoretical curve (which may or may not be an exponential one), and use the curve for the appropriate boundary condition to determine the transmissivity, storage coefficient, or the recharge rate from the aquifer. This can be done in a manner similar to that which we use for well pumping tests, either constant head or constant discharge.

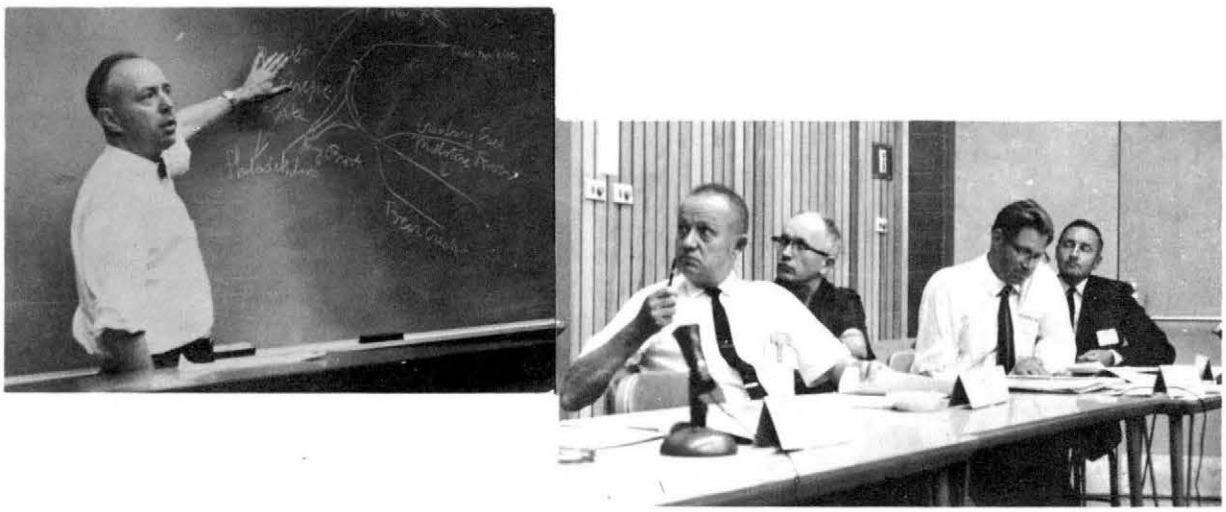
MR. SCOTT: Further comments?

MR. JACOB: This is a footnote. Following the suggestions of Boussinesq's work and the work by Maillet, in Israel in 1954 we analyzed all of the springs that issued from the limestone in Judeah. That country now relies upon the flow of those springs. When I first went there the question was, how long will these springs last if it ceases raining? And by statistical analysis of the outflow data, which was fairly good but not too good and not too long, knowing something about the geology and the structure and the origin of the water, and something about the solution phenomena in the limestone, we were able to arrive at a prediction of the volume of aquifer tributary to each spring above the threshold of the spring, and what the life of the spring would be. Most of them had an exponential decay. A very practical question, but the question had been solved way back in 1910 by the French. There are outflows on steep slopes,that can never become exponential,that will hyperbolic, and there is a bigger difference between hyperbolic and exponential outflow at infinity than there is in the beginning. You realize that they are two different orders of infinity. There is a great deal of work to be done that is not being done in the field of applied hydrology, and this requires correlation. Unfortunately, there are jurisdictional rivalries that prevent full correlation. But there needs to be a correlation so that hydrology is studied as hydrology and is not studied as ground water and surface water.

DR. YEVDJEVICH: If the volume of surface storage in a lake is expressed as the function of lake elevation and is approximated by a power function, and if the outflow rating curve (outflow discharge as function of the lake elevation) is also approximated by a power function, then free outflow of the lake under conditions of zero inflow gives die-away functions of different types. If the ratio of the exponent of two powers is less than a given value (unity), then the recession outflow curves are exponential functions; for other values of the ratio of exponents the functions are more complicated. Only in one case, when ratios of power functions is unity, the outflow recession curve is merely a simple exponential function. If inflow into the lake is constant, the functions are similar but more complex.

It can be also proved that the same is valid for ground-water formations. If the storage function of ground water in relation to ground-water elevation is approximated by a power function, and if outflow rating curve of this ground-water formation is also approximated by a power function, then the recession curve of this ground-water formation may be any function mentioned above, simple or more complex, depending upon the ratio of two powers.

MR. SCOTT: I would like to say that Dr. Abu-Zied and I worked on a decreasing discharge problem and this work is reported in a paper published in the last issue of the Journal of the Hydraulics Division ASCE. It includes values of a variable well function. These have been determined for a range of values and copies of the tables are available uopn written request.



Panel Session 3: Left to right: Dr. R. J. M. De Wiest, Mr. W. N. Tapp, Mr. C. E. Jacob, Dr. Zane Spiegel and Dr. N. A. Evans (moderator).

(Photographs by George **Palos**)

SESSION 3

MATHEMATICAL DEVELOPMENTS IN TRANSIENT GROUND WATER HYDRAULICS (LEAKY AQUIFER SYSTEMS)

DR. EVANS: Before I introduce the panel and while people are still coming in to take seats, I would like to make a few remarks. I heard a bit of information last night while attending a snow surveyors conference that is going on here on the Campus at the present time, and it goes something like this: What is the value of water? An analysis is that there are roughly 325,000 gallons in an acre-foot of water and an average farm of 160 acres will use 520 million quarts of water in a year. If that 520 million quarts were lined up they would make a row 45 feet wide stretching from San Francisco to New York. If those quarts were full of milk at 25 cents a quart, they would cost $130 million dollars. This is equivalent to what one farm would use in a year. If it were bourbon, it would be worth 2.6 billion dollars. Perhaps this points out that the cost of water is not in direct proportion to its worth.

We now have the panel which I want to introduce. On the far end Roger De Wiest, Princeton University; next to him Mr. Bill Tapp, Ground Water and Drainage Section, Chief Engineers Office, Bureau of Reclamation, Denver; C. E. Jacob, Consulting Engineer, Los Angeles; and Dr. Zane Spiegel, New Mexico State Engineer's Office.

The panel topic is given as: the Transient State Mathematical Developments for Leaky Aquifer Ground Water Systems. I will first call on the members of the panel individually in turn to give a quick ten minute review of their current work on recent developments, based on their backgrounds in this type of problem, then we will allow the panel members themselves to interchange questions, if there are any; and following that, we will ask the audience to participate with their questions.

Now, I would like to first introduce Dr. De Wiest.

DR. DE WIEST: Thank you, Dr. Evans.

Recently I worked as a consultant for the Elizabethtown Water Company which is responsible for the water supply of Princeton, New Jersey. In this capacity I examined a project of ground-water storage presented a few years ago by the late Homer Sanford, a consulting hydrologist. In his project, Sanford proposed to construct low earth dams on two rivers in the vicinity of Princeton, N. J. which drain ground water from a deep belt of alluvial deposits stretching between the Rariton Estuary and the Delaware in Trenton, N. J. By raising the water level in the rivers, the hydraulic gradient from the ground-water basin to the rivers would be decreased and hence the outflow of ground water would be retarded. Simultaneously the alluvial deposits should be decreased by intercepting trenches located parallel to the rivers and at some distance from the rivers. Sanford estimated that under certain conditions it would be possible to withdraw 5 million gallons per day from a given trench.

We analyzed the problem using the same techniques as described in the Geofisica paper (Vol. 54, 1963-I) to which I referred previously. The results of the analysis were published in the November bulletin of the Journal of the Hydraulics Division ASCE. They showed that Sanford's estimate could be over-optimistic because of lack of substantial data on the hydraulic conductivity of the aquifer.

MR. TAPP: Frankly, I was flattered when asked to take a place on this panel of distinguished experts. I am concerned with a different problem than most of these gentlemen; namely, engineering design. My function is to trap the problem and proceed with the design of engineering works. During the design studies, use is made of the equations discussed here.

My colleagues, Mr. Glover and Mr. Moody, have asked me to present two of our equations dealing with the leaky roof aquifer. You will find them in the Appendix C. The first is for the case of an aquifer overlain by a slowly permeable bed. For this case, the solution in dimensionless parameters is on the attached chart. We have the same solution in different form which we use primarily to recover the aquifer coefficients from field pumping tests. The solution shown here was redrawn to obtain a family of curves and a curve-fitting procedure worked out whereby we could plot the time-drawdown pattern from all of the observation wells on a graph and match all of them at the same time.

The other development is similar to that which Mr. Glover and Mr. Moody spoke of this morning. It applies to the drawdown at the center of a circular area. Again, the aquifer is assumed to be overlain by a slowly permeable bed. The equation yields a steady-state solution. It is used for estimating the drawdown at the center of the pumping area.

Now, I want to expand on my earlier remarks and point out to you that to sit and talk or read a manual about pumping tests is one thing; to go into the field and make a pumping test and recover the aquifer coefficients is something vastly different. I think of it this way: The equations are used in a backward solution starting from the field pumping test data to estimate the aquifer coefficients; namely, T and S and the vertical permeability k'/m'. With the coefficients and the same equations, we make a forward computation to predict what will happen to the ground-water regime in any particular area.

Before accepting the forward computations as a basis for design, we must make a judgment appraisal of the coefficients. In other words, how trustworthy are they. With this settled, we can proceed with the design. The forward computations are then used for theoretical operation studies to predict the ground-water behavior.

We always have to keep in mind that field tests come equipped with a price tag. There is a cost which we cannot exceed to collect field data. We must never collect field data just for the sake of the data. When we start to collect data, we must know how we are going to use them. The end product is an estimate to get an amount of money in an appropriation that will insure that we can build the facility when it is needed and stay within our cost ceiling.

Now, as an example of some of the things we have done with these backward and forward solutions, I recall eight water supply wells for a trial pump irrigation investigation. We drilled eight test wells at the sites with three or four observation wells and made pumping tests. We also reviewed all of the data for the area. We concluded the aquifer coefficients were of the right order of magnitude and represented actual conditions. Without any further ado, we started the forward computations for the wells. We designed the wells and made a theoretical operation study of what we could get out of them and of the drawdown. We let contracts for the wells, bought the pumps, and specified the pump settings. Perhaps we were lucky, every well was within the limits calculated. Of course, in such studies a great deal of judgment must be used in choosing the aquifer coefficients. I sometimes sum up this way: It is unfortunate that the equations used to attack pumping problems will always give an answer, regardless of the accuracy of the field data.

In our haste, we have overlooked two other Bureau developments. One has to do with pipe drains in which a fourth-degree parabola is used to represent the initial shape of the water table between the drains. It will be included in the Proceedings. The other development is by Mr. Glover for pipe drainage in a layered aquifer. It is a form of the leaky roof aquifer case under discussion this afternoon. It will also be included in the Proceedings.

DR. EVANS: Thanks, Mr. Tapp. Next we will hear from Mr. Jacob.

MR. JACOB: A little background, perhaps, on this leaky-aquifer theory might be in point. The people in the Netherlands, having to reclaim land from the sea - and having fought a winning battle, incidently - pioneered in this work, as you can imagine. And in the polders that had been drained and reclaimed there first arose the problem to describe this flow. We have an aquifer which is fairly transmissive and uniform in thickness overlain by one which is less transmissive, which in turn may be overlain by ponded water, or again overlain by a second transmissive aquifer. The first work that I know of was done by Steggewentz and Van Nes in the Netherlands - one steady-state problem and one nonsteady-state problem. And from there we began to work on radial flows to wells in cases where we had concentric boundaries. A paper was written on this with the idea of trying to describe what happens to a well or to a well field when there is some supply from above or from beneath coming from another aquifer. Now, this is just another model and, like all models, has its limitations. And since, of course, it has been greatly modified and elaborated upon, and will continue to be, but it is a useful concept.

Now, I might just say something about the velocity of flow. I think you realize that in an infinite two-dimensional space, finite potentials are not possible. It is fortunate that we live in a three-dimensional space. Thus the gravitational potential is bounded; electric and magnetic potentials and other potentials you can conceive of are all bounded. Logarithmic potentials in infinite two-dimensional space, are exceptional; they are unbounded. This leads of course, to the erroneous conclusion, which is a common conclusion, that you could have a steady flow in an isolated infinite aquifer. You get a certain degree of stability, but you never get absolute stability if you maintain a constant flux across the inner boundary. Now there are a lot of people in the art of engineering who do not believe this. All kinds of explanations are given why this is not true, but this is so. We always have aquifers with boundaries, and there are usually sources of inflow on those boundaries.

If the capacity of the boundaries to pass water into the aquifer is not exceeded, then conditions of steadiness of flow can be reached within that limit, and are reached in the neighborhood, for example, of an infiltrating stream. Moreover, if you have an aquifer that is overlain by ponded water and somehow you can maintain constant potential in the ponded water, then of course you can reach stability. The theoretical solution to this problem is a modified Bessel function of second kind of zero order which is called K_0 in most Bessel terminology, which, of course, goes asymptotically to zero at infinity. In other words, theoretically you do have an infinite area of influence, but you have a bounded influence. Of course, we never realize any of these conditions exactly in nature. In the first place we usually do not have wells that are controlled to pump at constant rates. They all pump at variable rates, which is due to the fact that turbine pumps have to adjust themselves to the declining water level. Nobody that I know of - somebody may have tried it, in fact we tried it once and it was not very successful - regulates the flow of the turbine pump, mechanically, so that it will be constant. This is not done. So you very rarely have these ideal situations on the internal boundary. Nor do you have ideal conditions on the outside boundaries or on the top and bottom surfaces. Now, some people may say, "What is it worth then, if you always have these limitations and then throw in on top of that anisotropy and inhomogeneity." Well, it is a useful model to be used as a guide, it can be used in many ways.

Many practical problems arise, as Mr. Tapp pointed out, in which the questions arise; can a project be drained or not, and what is the most feasible way of draining it? This came up in the Punjab, in that part of the former Punjab province in West Pakistan, and the question arose, how can waterlogged land, which was waterlogged over 60 years by leakage from earthen canals, be drained? The land slopes one foot to the mile downstream and slopes, on the average, much less cross-stream. The high ground between tributaries of the Indus River is in some places only 20 or 30 feet above the stream level. So it is very flat country. The problem as to how this could be done might be solved by a study of the geology and of the subsoil and classifications of the soil. But a study of the actual mechanics of the flow, to see how the water table is going to be lowered can be accomplished very simply by means of pumping tests. This has been done in several places. When a pump begins pumping, if initially the system is confined at all, it behaves as an elastic system, and as far as the water is concerned in the vicinity of the well, it can not tell if there is any leakage taking place or any contribution from outside that aquifer. So the aquifer will go through a period of response in which it behaves as a confined, elastic aquifer.

With the onset of the leakage - or if we do not want to think of leakage, think of the compaction of the confining beds, or it could be interbedded clay beds that are being compacted, or beds beneath - you can withdraw from storage within the clay much more water, volume-wise, than you can within the sand because the clay is obviously much more compactable. And so you will have, then, in the beginning an added contribution, which you might think of as a distributed inflow. For simplification this seems to be proportional to the head drop which occurs, and this, of course, is just a first approximation, because obviously if it comes from the compaction of clay it would not be a linear phenomenon at all. Still you can make some sort of approximation. If you have a thin diaphragm that is a confining layer or an aquiclude that is thin, across which you can impose a change of gradient very quickly, then the model is very good. If, on the contrary, you have thick confining beds, which themselves might have interbedded sands that are fairly continuous, then, of course, you have departures from this model. There is no such thing as a completely tight confining bed. Consider a flow regimen set up, as in the appended sketch, (figure 1) with the cases, (a) having ponded water maintained at constant potential, or as an alternative, (b) having another aquifer that is highly transmissive and unconfined, which can be replenished. Then you can set up this model of the flow. This was done by the Dutch merely as an approximation. The contrast in hydraulic conductivity (k/k') in this bed is high. The refraction is virtually complete, and we treat it mathematically as though water that passes through the aquiclude is added to the horizontal flow. The equation then becomes

$$\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t} - \frac{K'(h^0-h)}{b'T} .$$

Now this is a mathematical idealization as all mathematical models are.

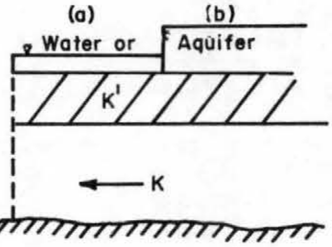


Figure 1

Another solution which has been worked out is heat conduction in a semi-infinite space with a slab under it and a line sink in the slab. Now, I began to tell you of the several phases of this regimen of flow. If you plot log t versus s , the drawdown (figure 2) you have the classical problem of a confined aquifer with a steady well.

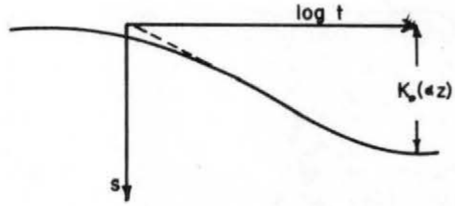


Figure 2

You have a solution that comes in asymptotically from zero time, which would make it infinity on the logarithmic scale. Then you have a straight line for awhile, and the slope of the straight line is inversely proportional to the transmissivity of the bed. From the coordinates of a point you can get the storage coefficient or "storativity." This, then, is the way the aquifer would behave during the initial phase, when water is being drawn from confined storage. Maybe a few minutes, or a few hours later, depending on the relative size of these parameters, there will be a departure. If head is maintained at constant level, by controlling the head, or recharging or whatever way, you can reach a steady state of flow in this bounded aquifer, if the capacity of the infiltration matches the capacity of the well. Now, the cross-section of this diagram, of course, is the K_o , which is a function of α times the radius. This has a asymptotic expression which involves the logarithm of the distance.

By pumping tests on single wells, you can determine the transmissivity, and if you know something about the characteristics of the well you can estimate the well loss from experience. With several wells of similar design in the same kind of formation, you can get yourself into the ball park and even someplace in the infield without having any observation wells, and enable yourself to estimate what we call the "leakance" by analogy to electrical circuits. Now, of course, this does not measure the average hydraulic conductivity of the shallow subsoil and so may be of little use to a drainage engineer, but on the other hand, in some cases it will effectively estimate the average hydraulic conductivity of the path of flow along which water will pass to a horizontal drainage system that may be built in that environment, and will be very useful. With a few observation wells this can be narrowed down very sharply in cases where the transmissive thickness of the aquifer is uniform and make possible, as Mr. Tapp has pointed out, the design of the wells and the prediction of their characteristics.

Thank you very much.

DR. EVANS: The last panel member is Dr. Zane Spiegel.

DR. SPIEGEL: I think before I discuss an extension of the theory of leaky aquifers, I would like to state that there are three of us here from the New Mexico State Engineer Office in Santa Fe, and we are part of a group of people who, like Mr. Tapp, use formulas and methods that have been devised by various people. I would like to make a short statement of the way in which we use these methods because I think we are probably the only state which does so. Several people here have asked me some questions, and perhaps others would like to hear some of it too. New Mexico's ground water law permits the State Engineer to declare, without any local knowledge or agreement by local groups beforehand, to declare certain areas of the state as being basins which are under his administrative control. He may continue to permit further drilling of wells and new appropriations of water within these basins, or he may not, in which case the areas are called closed basins. Now, in many areas there are already surface water appropriations for irrigation, municipal and industrial uses. In other areas there are existing ground water appropriations, but no surface water appropriations, because there are no streams from which surface water could be appropriated. In the former case the State Engineer has expressly recognized, and the statutes have implicitly recognized, that surface water and ground water are one and the same, and the pumping of the ground water will deplete the flow of the stream. In other basins where streams are a long distance from the pumping areas the areas are considered essentially as mining areas, and a different philosophy is used. In the areas in which there are no streams nearby, calculations have been made using Theis' equation, taking into account by image methods the boundaries that are present, to calculate the rate at which the mining will occur. The basic philosophy used in trying to determine at what level development should be stopped in the basin is that the water should last for 40 or 50 years, and when I say "should last" I mean the time that water can be withdrawn at what we now consider economical pumping lift for this period. Of course, it is recognized that economic conditions might change in the future, but decisions are made on present economic conditions. However, in the basins where streams are present which have surface water appropriations, then the effect of a well on the stream is computed and if a well owner wants to move a well, for instance, closer to a stream so that his well will affect the stream faster than the old one did, then it may be necessary for him to reduce the amount of this pumping to reduce the amount of effect on the stream. The main point is that in order for the State Engineer to administer the basins, he must be able to calculate what the effects will be to make decisions.

We have these two diagrams (figures 3 and 4) which I would like to use to show what happens in a long-range program of pumping from artesian aquifers.

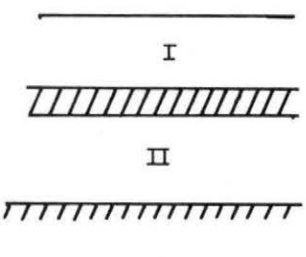


Figure 3

If we pump from a drain, or a line of wells, there is more or less unidirectional flow within the aquifer. Eventually we do affect the water level in the upper aquifer (referring to the upper aquifer by Roman numeral I and the lower aquifer by Roman numeral II). We find that we may have in the long run the simple result shown by Mr. Jacob's diagram. You may find that if you continue pumping long enough, the water level does not continue at a steady level because the water level in the upper aquifer has been lowered. Therefore the water level in the well must go down after a period of time. To take this into account, we have what we can call a mutually leaky aquifer system and this system can be drawn very generally as a prismatic block (figure 4); not an infinitesimally small block but just any block; and we consider that this block is divided into two parts, an upper aquifer I and a lower aquifer II by a leaky layer, and there is a water table in the upper one and a potentiometric surface in the lower one which we will call h_1 and h_2 respectively. We have the same kind of differential equation for each aquifer which we have to write for each of the aquifers separately. Each equation is going to have a form very much like the one written by Mr. Jacob

$$\nabla^2 h_2 = \frac{S}{T} \frac{\partial h_1}{\partial t} - \frac{K'}{Tb'} (h_1 - h_2)$$

Then we can also generalize this equation to the case where you have recharge to the upper aquifer by adding a term $-(W/T_1)\,h_1$ on the right. This gives us a general system of equations which will take into

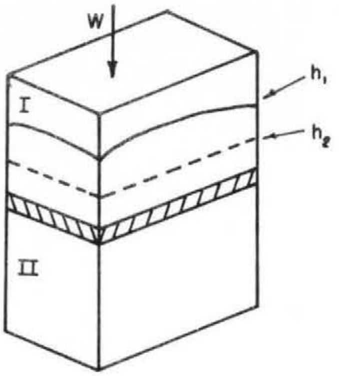


Figure 4

account mutual leakage between two aquifers. I might also add that, for this equation or for simpler ones, W does not necessarily have to be a constant; it can be a function of time or distance and many solutions are given in the area of heat conduction for different kinds of functions. Then in connection with the questions about physical applications in hydrology, we can make the recharge function some stochastic variable or statistical variable generated by any process that the statistician wants to generate. A way of doing this is to consider the recharge as a function of time, that is, a sequence of individual impulses, in rectangular form, (figure 5) could be rain on any number of days, or recharge; it does not have to be regular, it can be any time you want. In other words, you can approximate any function by a step function (jump function) using the Laplace transform. A solution of this equation can be obtained taking into account this arbitrary recharge. Hydrologists and statisticians thus have at least a starting point to relate their work in ground water hydrology.

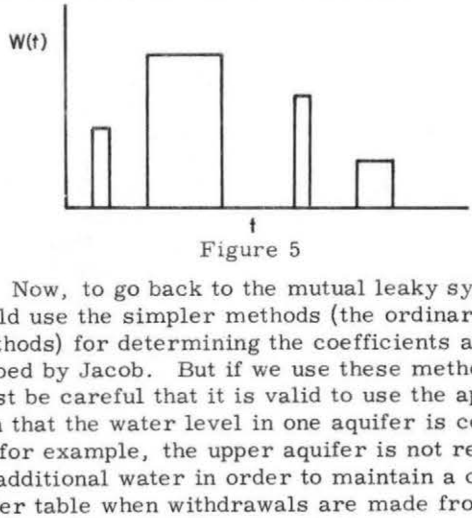


Figure 5

Now, to go back to the mutual leaky system, we could use the simpler methods (the ordinary leaky methods) for determining the coefficients as described by Jacob. But if we use these methods, we must be careful that it is valid to use the approximation that the water level in one aquifer is constant. If, for example, the upper aquifer is not recharged by additional water in order to maintain a constant water table when withdrawals are made from the lower aquifer, the potentiometric surface of the lower aquifer will not reach a stable level. Instead, it will decline at a rate dependent upon (a) the transmissivities of both the aquifers (b) the storage coefficient of the upper aquifer, and (c) the leakage coefficient. The drawdown curve may appear as in figure 6, where the solid portion represents the simpler case described by Jacob.

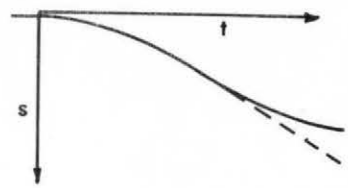


Figure 6

DR. EVANS: Now, I am going to give the members of the panel an opportunity to discuss with one another the questions that may have arisen in the course of these proceedings.

MR. JACOB: I would like to commend the contributions that are made here and apologize for not having any in written form. These elaborations that Dr. Spiegel has made are all practical. Probably in the end they may be very useful, so they are not just exercises. If you keep pumping an aquifer and if you have a well capacity that is large enough to permit this, you can start at a certain rate and maintain it, then you can dewater part of the formation. This is something that occurs in the petroleum business, so that you have mixed conditions of flow with a moving boundary between the confined flow and the unconfined flow. You get a fourth regimen of flow where the drawdown tends again towards a straight line, at about the same slope. In other words, you are still producing water with the same transmissivity. The horizontal capacity for flow is unchanged except in the immediate vicinity of the well. Out beyond,there is largely still the same transmissivity driving the flow, so you will get a parallel line that gives again the reciprocal of the transmissivity. One intercept will give you the storitivity of the bed when it was confined, the other will give you the storitivity when it began to be drained. In these cases where you have drainable land, or rather you want to know whether you can drain it or not, if you can run the pumping test long enough to begin to dewater the confining bed, then you can determine the storitivity of the underlying bed and you can make some calculations as to what is going to be feasible with any useful drainage well spacing. Now, sometimes this has to be a week, depending of course, whether your observations are made just in the pumping well itself. This is the technique. There is no other way of doing it; you have to wait until you can dewater.

DR. DEWIEST: Dr. Spiegel, may I ask you, what happens to the mathematic solution when you put wells in these aquifers? Do you still use the Laplace transform?

DR. SPIEGEL: I have not done any solutions for a well, however, I do not see why it can not be done.

DR. DEWIEST: I don't think the Laplace transform would work in the case where there are wells tapping the aquifer.

DR. SPIEGEL: Incidentally, I have said that many well fields can be approximated as a line sink and various linear one-dimensional solutions would apply in such cases.

MR. JACOB: You might be able to approximate the well field, that is the vicinity of the well field as a strip, with a more-or-less stationary state of flow within the whole field.

DR. EVANS: I think we are ready for audience participation at this time.

MR. NUZMAN: I would like to direct this question to Mr. Jacob. In your illustration you show the initial values as could be obtained by a short term pumping test. Could you give us a method we could use in the field to predict the length of time necessary during a pumping test to dissipate artesian pressures and begin dewatering under water table conditions?

MR. JACOB: The time that the dewatering will begin? Yes, I think you could. You would have to assume some value of the leakage coefficient, however. Then you would write an equation for the drawdown in the well itself, knowing that you started at a certain rate. One is going to get the drawdown equal to the distance of the static level from the top of the formation. You will have to have a preliminary estimate of the leakage factor and also knowledge of the internal resistance of the well itself, that is, what we call "well-loss coefficient." I think you realize that in most wells operating at what appears to be their optimum or design capacity, about half of the self-drawdown in the well is internal resistance, which approaches turbulent resistance. It does not establish full turbulence until it gets into the pipe, the casing itself, and flows upward, then of course, it becomes quite rough. It is still smooth flow through most casing perforations if the well is well constructed. Now, generally speaking, the wells I have seen will operate so that about half of the self-drawdown is well loss and the other half is formation loss, from a great distance to the well. This is just a rule of thumb.

MR. BEAN: I would like to ask Mr. Tapp if he will tell me if the Bureau has a Memorandum 657 and what the name is.

MR. TAPP: Technical Memorandum 657, "Studies of Ground Water Movement," is in print, as far as I know, and available from the Bureau's Publication Sales Office. It was issued at the Commissioner's Office at Denver, March 1960. At the moment, it is in the process of being reissued. There will be some additions and it will become a part of the Bureau's Engineering Monograph Series. I cannot tell you the Monograph number at this time.

MR. GLOVER: I wish to speak in regard to the relationships obtained from the solution described by Mr. Tapp: The elegant solution produced by Jacob and Hantush for the case of a well pumped at a constant rate, drawing the supply from an aquifer overlain by a semi-permeable bed is well known. They found a solution in the form of an integral. The chart described here (Appendix C) was not produced by the procedure used by Jacob and Hantush. The procedure employed here may be of interest because some of you may wish to use it at some time. The series solution for a finite outer boundary came from an earlier paper by Mr. Jacob, and we use it this way. We choose a ratio of the radii of the outer boundary and the inner boundary as for example 10 to 1. We then compute with the series until the disturbance reaches the outer boundary. When that happens we choose another ratio such as 100 to 1 and compute with it again until the disturbance reaches the new boundary. A new ratio, such as 1000 to 1 is then taken and the process continued. In this way the outer boundary can be made as remote as we desire. The advantage of this method is that only a few terms of the series is ever needed. In this way the solution for a finite boundary can be extended to cover the case of a very remote outer boundary. I have checked this solution against the one by Jacob and Hantush and I find them identical. I might say that I have also checked the solution of Jacob and Hantush independently.

DR. EVANS: Thank you very much, Mr. Glover. We welcome now further questions or comments from the floor.

MR. JENKINS: I would like to refer to the drawdown curve shown in figure 7. I have tried to analyze the "S" shape of the curve where A-A' has a slope in the general magnitude of B-B' which would result in a similar coefficient of transmissibility. However, a computed coefficient of storage for B-B' is considerably larger than for A-A'; but some tests are not run long enough to obtain the B-B' portion of the curve, which then causes complications in trying to obtain the correct value for the coefficient of storage. I feel that the A'-B portion of the curve is the result of slow drainage of an unconfined aquifer, leakage from an overlying aquifer, or a change in leakage in a semiconfined aquifer.

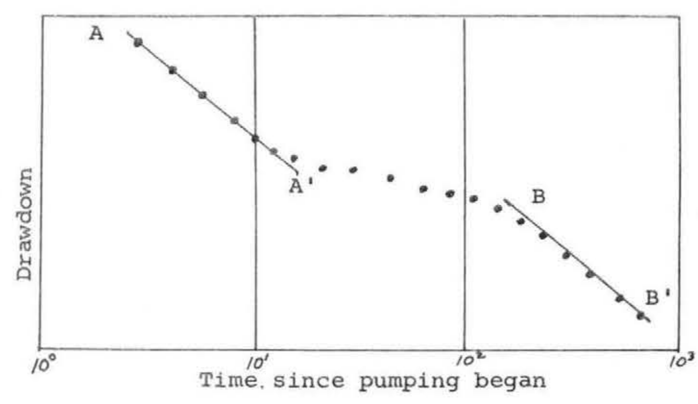


Figure 7

Ramsahoye and Lang (1961, Water - Supply Paper 1536-C) discuss a method of computing the coefficient of storage of a water-table aquifer by volume; I have also computed the approximate volume of the cone of depression for use in determining the coefficient of storage by dividing the volume of water pumped by the volume of the cone of depression.

I think this is a practical way of computing the coefficient of storage, and I wonder if any one on the panel has used this too.

DR. SPIEGEL: I think this method has been used in New Mexico by the U. S. Geological Survey Office for the Deming area in a closed drainage basin in southwestern New Mexico a few years ago.

MR. JENKINS: Do you think this has potential for getting a good storage value?

DR. SPIEGEL: The value obtained was close to what would have been guessed at anyway.

MR. TAPP: Mr. Jenkins, we used such calculations to check a storage value of some fine-grain materials overlying a coarse and highly transmissive aquifer. Mr. Glover did it as a check against some values for the storage that had been derived from shallow pumping tests and laboratory tests. The values in the laboratory tests and the other work ranged from about 10 to 18 percent. Mr. Glover computed on the basis of the dewatered cone of depression and obtained 15 percent; this, we believe, is a reasonable check.

MR. JACOB: This is in further answer to Mr. Jenkins' question. There is a Water Supply Paper (WSP 679a) by Wenzel, dated about 1934 or 35 in which he used this technique. And this technique is applied on tests that were run at that time. As you know, in those days, when the Thiem method was first introduced in this country, it was found necessary to use a great number of observation wells, and it was good that it was thought so, because great amounts of data were obtained by a number of persons; by Wenzel, by Lohman and others. All these tests were analyzed in detail in the U. S. Geological Survey, and cross checks were made between the different methods of looking at them. It involved estimates of actual dewatering in different annular areas concentric with the well. All of that has not been published, and I think it is some place in the U. S. Geological Survey.

MR. COOPER: Are we to believe that the coefficient one would obtain from the flat portion of the curve would be representative of the storage capacity of the aquifer? A coefficient obtained in this way would depend on time and on capture of water from sources other than storage. It could be much larger than the storage coefficient or specific yield of the aquifer.

MR. JACOB: That is true in a sense, Mr. Cooper, but that is an apparent coefficient. I can show you mixed cases of flow where you can get very high apparent specific yields. But, in turn, if you want to put it back into the equation that predicts what will happen, it is useful. Now, to change the conditions radically, it is no good. I think we should talk a little about specific yield. Dr. Meinzer had a useful, practical idea when he defined specific yield. Now, what do we mean by specific yield? Although this is not explicit in its definition, specific yield is for infinite time of drainage. In other words, we would get it if we had a homogeneous sand with an equilibrium moisture profile, and then slowly lowered it to a new position and allowed it to come to equilibrium again. Of course, he was talking about a field concept not a lab concept. You want to realize this in the laboratory. You maintain steady conditions before and after. You isolate the thing or you keep it in a thermostat. Then, of course, specific yield is a fixed quantity, if this is the concept. Now, we rarely can measure that. What we measure is a coefficient.

I use the term "storativity" merely because it allows you to use an adjective "storative." I coined this word. I went down to Venezuela and they had a hard time translating this into Spanish -- estaurativididad. Storativity -- this is a good word -- is Theis's storage coefficient. Only I do not know how to make an adjective out of storage coefficient. We have the word "restorative," in the English language which is an adjective, and we have restorativity, and so I thought I would just throw the "re" away and speak of storativity and storative. So, one aquifer may be more storative than another. So what is storativity? It depends upon the geometry. You can define this mathematically like so many coefficients in engineering and physics. It is defined by $Q = A \frac{\partial h}{\partial t} S\ (t, h)$. The rate of removal from storage is equal to some area times some rate of decline of head times some coefficient (S). It is a function of rate and of the history of the thing. It has variability, but it is no more variable than magnetic permeability, or many other coefficients in physics that are very useful, if you understand what you are doing with them. So, this is an instantaneous apparent value, it is the apparent rate at which water is coming out of storage per unit area per unit head decline with a certain given model and a certain geometry. Now, that should not trouble you because it is useful. It is true that you can sharpen it up, and you can relate it to simpler geometries so that you can turn around and, having got this variation in time, you can explain it by the actual phenomena taking place. But from the practical consideration of an engineer trying to design a gravity-well field, or testing gravity wells, he may want to drain some area or he may want to get a water supply in some situation. As a practical matter, he, of course, is interested in the ultimate value of the thing. This S ultimately approaches Meinzer's specific yield. The concept of specific yield is useful if you realize again how you are going to obtain it or how you are going to measure it. Now you do not measure it directly. You probably calculate instantaneous local values, but this S can temporarily exceed the specific yield. The S may have some initial value and go over and approach the specific yield from above. If you draw down most of your profile fast enough, you may have the moisture coming into the water table later on at a rate faster than it does in the beginning. The S for a particular situation is defined by the mathematical equation. And it is an apparent value if you are in a complicated geometry. And the more complicated the geometry the more this velocity $\partial h / \partial t$ determines really what it is.

MR. BEAN: We have a cooperative program with Ivan Johnson of the Geological Survey on specific yield that has been going on for about five years, so I want to make a comment or two with respect to what Mr. Jacob said in this regard. Our first concept was the thought that you just presented, that specific yield is the ultimate term only. Later, we had the thought of looking at Mr. Meinzer's original definition, and this concept does not appear there; in other words, Meinzer says the specific yield is the percent of the volume of a unit of material that is occupied by water which is drained by the force of gravity, and he does not say ultimate. So Mr. Johnson and the rest of us went along with your concept, Mr. Jacob, until within just the last year, and now we are beginning to think that specific yield is the same as your storativity, at any stage -- and the conditions at that stage not only should be, but have to be, defined. In other words, specific yield is dependent on a number of factors, one of which very definitely is time. Everyone knows that drainage is greater with time. Another is thickness of the bed. A third is the situation at the bottom of the bed, which is very important. For instance, is the bottom of the bed above clay or is it above sand?

Let me illustrate. (figure 8) During our cooperative investigation we took some samples in a field area. The water table there has dropped, and the materials have drained continuously without any vertical infiltration, as nearly as we can tell, for about 30-35 years. Suppose we have a silt bed with sand underlying it, as shown in the diagram. Moisture content by volume is shown on the abscissa, and the ordinate is depth. Now, after a long period of drainage the silt in the upper part of the bed drains to a moisture content of about 15-20 percent. Farther down, in a sample that is just above the sand bed, you find a much higher moisture content in that silt. Specific yield is shown as Sy , the difference between the porosity and moisture content. The specific yield of the silt in the upper part of the bed is maybe about 20 percent. The specific yield near the bottom of the bed is only about 10 percent. The reason for the difference, as I think you can see, is because of the much larger inter-granular openings in the sand. There is no capillary attraction to drain the lower part of the silt, while in the upper part of the silt the water will drain into the lower part of the stratum where it hangs up for years and years and years.

We have discovered a number of very interesting things of this type regarding specific yield. Most of this information is not yet published, although Mr. Johnson has put out a few preliminary or open file papers. But there is a lot to this specific yield, and Mr. Johnson and his crew are continuing to get more answers for us.

MR. JACOB: Although the writings of Meinzer will leave this unanswered, I think you can find correspondence in the files and memoranda of the USGS to show that this was the intention. Maybe C. V. Theis can comment on this. But before his death, speaking of Mr. Meinzer, he answered to my satisfaction that his intention was for it (specific yield) to be the ultimate value. Because, I can show you some early work on this thing you are working on out in California, in the very early stage when we talked about the drainage problem. Is this right, Dr. Theis?

DR. THEIS: I think you are right, Mr. Jacob, and I think Mr. Bean is wrong about the application of the concept. The definition was that it was the amount of water that could be drained from a porous medium by gravity, with an implication of long and indefinite time.

MR. BEAN: We went back to Meinzer's original published definition for our usage of the term

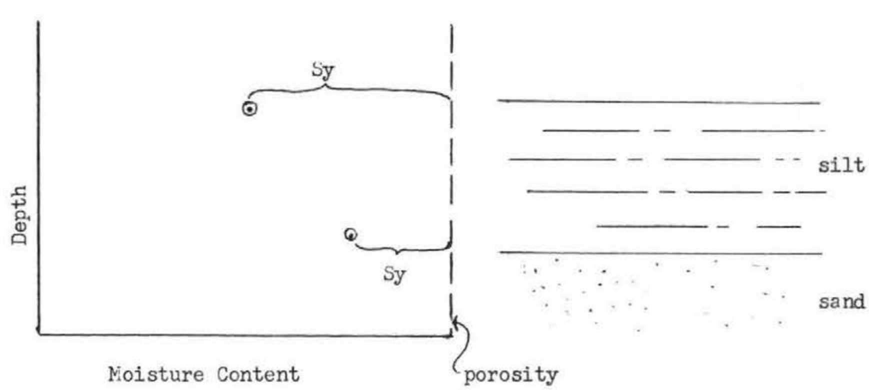


Figure 8

"specific yield." I have not read your secret correspondence on the subject!

MR. LOHMAN: I think the safest thing to do is to use the specific yield value qualified with the time in which it is determined, then you are on the safe side.

MR. JACOB: I think it takes more than time, Mr. Lohman. I think it takes a lot of boundary conditions, and there are some of them right here that you mentioned, in other words, you are looking at this as a practical profile. In the case of California, it is really not specific yield that they are interested in at all, it is the reverse of that. How much water are you going to be able to put in there on the first fill? The first cycle is going to be the fill-up, so there are papers in the literature that make a distinction between the "coefficient of drainage" and "coefficient of storage." There is a difference because there is hysteresis.

MR. NELSON: This phenomenon called "Specific Yield" is the result of a variety of boundary value problems of partially saturated flow. All of the factors which have been mentioned already enter the boundary value problem plus the interactions between soil characteristics i.e., different capillary conductivities, capillary pressures, and moisture contents. You are faced with the decision either to treat it as a partially-saturated flow system and analyze it that way, or you have to back off, as Mr. Jacob has so capably indicated and recognize that the coefficient is a composite of a wide variety of conditions, hence, incapable of precise definition.

MR. DOMENICO: I have had some experience along these lines in Canada, and know some people up there working on low permeability Upper Cretaceous sediments. In these formations, the ground water has definite characteristics of occurring under water table conditions, or unconfined, in that there is an unmistakable piezometric surface - surface topography confirmation, among other things. Recently, an article by Joe Toth and a reply by Mr. Stan Davis in the Journal of Geophysical Research described the flow system in these sediments. I have pumped a few wells finished in these formations and have always come up with an artesian, or confined, coefficient of storage -- point four zeros and something. Speaking again of Mr. Toth, he is of the opinion that the coefficient of storage in these sediments is controlled more or less by the position of the flow lines in the interbedded shales during pumping. Upon continued pumping, we could obtain a coefficient of storage representative of the water table, or unconfined system only when we bend these upper flow lines -- when we get vertical movement of water to our wells, and thereby violate one of the assumptions of the Theis equation. In this type of sediment, this seldom occurs. Further, in my limited experience in pumping limestone and basalt formations, as long as they are deeply buried, the result is always an artesian, or confined type of storage coefficient. To me it appears that we have a relationship between the vertical permeability of whatever is overlying the formation and the horizontal permeability of the material we are pumping. In other words, the value is more or less controlled by the geology, and may not reflect the confined or unconfined behavior of the system as a whole. The Canadian situation exemplifies this. The system has been recognized as unconfined, but the storage coefficient indicates confinement. Getting back to the basic question, how do we handle this coefficient? Can it be identified at all? Is it identified with our 24-hour pumping test?

MR. JACOB: Under certain idealized conditions, S is calculable from the mechanical properties of the rock, its porosity, etc. Now, was this upper-Cretaceous sandstone very deep?

MR. DOMENICO: The wells are up to 1000 ft deep. Some of the wells yield 10 gallons per minute, most yield less.

MR. JACOB: Well, you say you have a water table, but do you really have a water table?

MR. DOMENICO: I think so.

MR. JACOB: The water table is in Pleistocene deposits, is that not right?

MR. DOMENICO: It is. Both in the Pleistocene deposits and the Cretaceous sediments. If all points were plotted on one piezometric surface, they would be compatible.

DR. SPIEGEL: What are the depths of the sediments?

MR. DOMENICO: The depth of these sediments are deeper than the depth of the wells; well over 2000 ft. They are practically horizontal. As a matter of fact, the slight dip has no control on the piezometric surface or direction of movement, as indicated by the piezometric surface -- surface topography confirmation. Let me draw a diagram (figure 9).

The North Saskatchewan River is a line sink. The Pleistocene deposits average about 50 feet in thickness. The only outcrop areas for the Cretaceous sediments are in the river banks, and are discharge areas, not areas of recharge. The water levels in wells finished in Pleistocene deposits, or shallow and deep sandstone lenses, more or less conform with one another, and with the surface topography. The theoretical work by Toth described the flow system in these sediments, essentially an extension of the theoretical work of Hubbert, in 1941.

MR. JACOB: I would say you have a confined system if you get storativities that intimate confinement.

MR. DOMENICO: The feeling is that it is essentially an unconfined system.

MR. JACOB: How do you know?

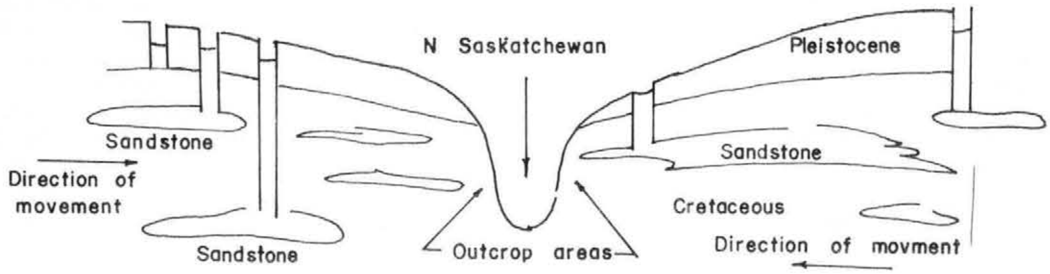


Figure 9

MR. DOMENICO: The diagram shows most of the factors. The most significant is the piezometric surface-surface topography confirmation. The outcrop areas functioning as discharge areas preclude the entrance of water there, and assures us that recharge occurs by downward movement of water from the Pleistocene deposits. The complete lack of control that the dip of the sediments plays on the direction of ground water movement is another factor. As this formation is barely pumped, the potential we measure at least approximates the pre-development potential.

MR. JACOB: It may be a semi-confined situation.

DR. SPIEGEL: I think the problem that is involved is this. The water table you talked about is the average potentiometric surface for each of the wells. There is a section penetrated by the wells which may be different for different wells, and therefore, this water table is not a water table, but a number of water levels which are related by the various sands that are penetrating it.

MR. JACOB: I think the explanation lies in this: That you have a very delicate balance between the supply of water and the evapotranspiration in the natural state and in the artificially cultivated state through consumptive use of water in the area. There is farming; it is wheat country. So you have a balance set up. Recharge occurs during the summertime from thunder showers and again when the frost melts in the spring. These are probably the two times when recharge occurs, I don't know whether Toth has made a water balance or not. We are involved in the potash mining operation in Saskatchewan, which is the same topography. It is not a leaky aquifer in the conventional sense, but is a semiconfined aquifer. There are gravel deposits in old channels, some in the buried Cretaceous landscape, some in Pleistocene channels. We have a gravel that is very extensive in the Boulder Lake area. We have begun to explore the lake beds also. These gravels are fairly continuous and very transmissive. Now, undoubtedly in the till ground to the west there are enough interbedded sands to provide quite a bit of continuity. There are Cretaceous sands under there, all of which have not been mapped, which have, I think, a slight dip to the west, very slight. I am not sure whether any flow has been contributed in this immediate area by the Cretaceous sands. We have not answered that yet, but, it suffices to say that we do have sands in the till, so that in the high ground, where you have till, when you get recharge the water table will then be conformable with the topography. And in the low lands, where you have some water-loving vegetation, there is waste from the area. You have some pastures also which use water. You have a very delicate balance, so you can get a water table that conforms to this glaciated landscape.

MR. DOMENICO: You are talking about the potential in the rocks?

MR. JACOB: Well, the potential in the Cretaceous rocks probably corresponds closely with the depth of burial, even in the shale. The whole thing has been standing there a long time, and I suppose the distribution of head obtained before cultivation reflected the weight of the overburden. I would imagine there would be in the Pleistocene beds some differences of head, however, between the areas immediately underlain by till (because of some of the high interbedded sands in the till where the water supply comes from) and the areas underlain by lake beds. You get high water levels when you get into these lenticular sands. In general you have a subdued replica of the landscape.

MR. DOMENICO: I suspect if we were to measure good control points, we would certainly find a difference between potentials. I was referring to a gross scale. Actually, the potential of the water in the rock depends upon the point of measurement in the flow system, high water levels in discharge areas, and low water levels in recharge areas, but always in the vicinity of the water levels in wells finished in Pleistocene deposits.

MR. JACOB: The thing breaks down to something like this: You have a balance between inflow and outflow plus storage. You try to get a water balance for a particular area, and the inflow depends upon the recharge rate (W) , which will vary with time and also vary in its distribution. The outflow will depend upon some hydraulic conductivity (K) or transmissivity. The storage change will depend on some average storativity (S). Now, if you want to take a look at this thing, think of the ratios. Which ones of these are important? Let's divide the equation* by the

* Inflow ($\sim$W) = Outflow ($\sim$K) + Storage ($\sim$S)

outflow. Then you will get the ratio of the recharge rate to the hydraulic conductivity (W/K) , and the ratio of the storativity to the hydraulic conductivity (S/K). What is the recharge rate compared to the vertical hydraulic conductivity on the average? You have a very large area and a relatively thin aquifer. Some attempts are being made using meteorological calculations of potential evaporation, and this is being studied in this case. Incidentally, a very fine symposium on groundwater hydrology was held in Calgary, sponsored by the National Research Council of Canada. This will be available in a few months and has a very interesting discussion of the hydrology of this kind of terrain, and a forerunner to the paper by Dr. Toth is in there.

I do not say we know the answer; it is a puzzling problem. You have a semiarid climate and you have most of your opportunity for recharge in the summer with the thunder showers, so W may be extremely low. It may be possible to set up a flow system -- and we tried to do it in a very highly idealized form -- in which you have basically an underlying transmissive layer with some kind of a till layer over it. Within the till are beds of sand, and you have an equilibrium with the atmosphere and with the plants that waste water in the lowlands which entails a different kind of surface boundary condition. We have set this up to treat it by numerical integration. The only thing is, we do not have enough data to plug in. There need, I think, to be more data on exactly what happens when it rains. How much recharge occurs? And what happens when the spring thaw occurs? In other words, what is the annual hydrological cycle?

DR. EVANS: Are there any other comments?

DR. SPIEGEL: Dr. Hantush asked me to mention a couple of points if they did not come up in other ways. The first point, related to the leaky aquifer problem, is the question whether the lateral boundary is leaky or not. This becomes important in relating the discharge of wells to the effect of the well on the stream. We start with a well in a simple aquifer with a stream nearby. This stream, let us say, has a constant water level. The stream has no impermeable lining in other words it has direct access to the aquifer. I will call this an unlined boundary condition. A well nearby can take water from the stream, as has been shown by Theis or Glover and Balmer in published papers. If however, the stream has a silt layer under the bed, or, for some reason there is a semi-permeable layer between the stream and the main aquifer, then there will be a difference in head or potential between stream level and the well after the well effects have reached out that far. Depending upon the size of the value of permeability of this leaky layer, we can generalize the stream bed as a vertical boundary for regions a short distance away. Then, the value of this boundary permeability, which we can call K_b becomes very important. I think that Hantush has made computations for a particular case in which it reduced the amount of water taken from the stream by a factor of 1/3. So, this is another application of leaky boundary problems. Now, many problems of this type have already been solved and if there is such a boundary condition in the field area in which you are working, it might be useful to you to refer to heat conduction literature for this problem (the radiation boundary problem). I think we might call it the leaky boundary condition. The next problem was equally presented by myself and Dr. Hantush involving the theoretical foundation for the concept of safe yield. I think this is a very important concept. It has not been mentioned so far and I do not want to expound on it but want anybody in the audience or panel to comment upon what safe yield really is.

DR. EVANS: Does anyone want to make a comment?

MR. JACOB: I think we ought to refer that to Mr. Raphael Kazmann, and he is not here.

MR. BEAN: I want to see if I can remember what was taught to me in ground water school, eight years ago. Safe yield is the largest amount of yield that you can get from a ground water basin, without something undesirable happening.

MR. JACOB: There was a paper in the Proceedings of the ASCE, (vol. 82, 1956) by Mr. Kazmann. You may have read it, but we had kicked it around about ten years ago, and I guess it still is being kicked around back in the USGS. There are so many criteria, such as encroachment of water, sea water, etc., in so many places, and also the allowability of mining water, which, in the days of the conventional safe-yield concept, were not quite contemplated, if you think of the economics of the 1920's. So we have different ideas now, but there is still the concept of safe yield. There are methods of determining it, and as you know, Raymond Hill devised a new method for what we call the safe yield of a basin, based on outflow.

DR. EVANS: I would like to request that discussion on safe yield be deferred, if you do not mind.

DR. DE WIEST: The first point mentioned by Dr. Spiegel is what the Russians refer to as the clogged river bed, and it has been treated analytically by Zhukovsky and is mentioned in the book by P. Ya. Polubarinova-Kochina in several places. (For example there is an approximate solution when the river bed is clogged on page 142.) At several different places in the book a reference to it and to the author of the original paper may be found because this book is like every other book, a compilation of research papers. The other point about the so-called radial boundary condition, is one that crops up naturally, as the one presented on figure 2 of the paper published in the November 1963 Journal of the Hydraulics Division, ASCE. (Replenishment of aquifers intersected by streams). If the continuity equation is applied here at the boundary where the discontinuity exists, then it is found that the head is indeed proportional to the derivative of the head, times a constant. A constant as a first approximation, actually a function of distance and other characteristics of the aquifer, and in the simplest case then, this may be called a radial boundary condition. So it crops up naturally.

DR. EVANS: Thank you, Dr. De Wiest. I know that Mr. Bittinger has some things to add from other sources.

MR. BITTINGER: I might add another reference on the subject of "safe yield," which I feel is a fairly good discussion, although quite general. This is the ASCE Handbook on Ground Water Management. Principally for the record, I wanted to mention a few other locations in which there is work going on in the transient ground water field. We have a good many of the people in the field assembled here, however, there are some who were not able to attend. I went through a recent issue of "Hydraulic Research in the U. S." which is published by the National Bureau of Standards, and I am just going to run through these briefly -- there are some you will be familiar with, and others that you may not be. At the University of Arkansas there is a project entitled Turbulent Flow in Porous Media, Professor John C. Ward in charge. At UCLA, Dynamics of Soil-Water Flow Towards and Into Subsurface Drainage Facilities, A. F. Pillsbury in charge. Some of these might or might not apply, since titles do not tell us everything. Pillsbury also has a project entitled Flow Through Anisotropic Porous Media. Professor Howe at the University of Iowa has a project entitled Mechanics of Bank Seepage in Natural Streams during Flood Flows. At the University of Michigan, Professor Streeter is in charge of a project called Unsteady Gravity Flow of Liquids Through Porous Media. Dr. Browzin at Ohio State University: Transient Flow Through Porous Incompressible Media with Various Boundary Conditions. Dale Swartzendruber at Purdue University, Department of Agronomy has projects entitled Analysis of the Dynamics of Moisture Flow in Soils, and Dynamics of Water Flow in Tile Drained Land. Dr. Toebes at Purdue is in charge of Hydromechanics of Fluid Collector Systems in Porous Media. Dr. Harr of Purdue, who had planned to be here, is working on Transient Development of the Free Surface in a Homogeneous Earth Dam. I think that catches most of those which appear applicable to the subject of this meeting. If there are any further comments on these, maybe you would like to discuss them.

MR. PAPADOPULOS: I would like to make a comment on the subject of collector wells. Some work on the subject has been also done at the New Mexico Institute of Mining and Technology. Dr. Hantush and I have reported our theoretical work on flow to collector wells in the September 1961 issue of the Journal of Hydraulics of the ASCE. Also, De Brine, a graduate student at N.M.I.M.T., is presently making some model studies of collector systems. However, there is still much to be done on the rather complicated mechanics of flow to collector systems and I would suggest that further work is encouraged.

DR. EVANS: Thank you for that suggestion. Any other comments? Mr. Jacob has been asked to review an item, and he said it would take him a very short time, let us turn this over to him.

MR. JACOB: Several have asked for the algorithm for calculating the law of superposition of states of flow in the example that I gave of a very thick unconfined aquifer with zero head on the boundaries. (Boussinesq's problem.) I thought it would be of interest. There is a law of superposition, and it is not linear. It turns out, if you have a certain average rate of recharge W^o, initially, and, if you have a certain variation of recharge W^i in time t^i. Then if you normalize W^o to 1 and call the successive steps ΔW^i, this becomes input. The problem is to get the output. What is the law of superposition? How does the water-table aquifer behave on the average? There is a simpler way of doing it; you can integrate by finite differences if you wish. This is a total differential equation. Let us say T is that part of the variation of the average depth of flow in time. You have an equation like this

$$\frac{1}{\alpha}\ \frac{dT(t)}{dt} = -T^2$$

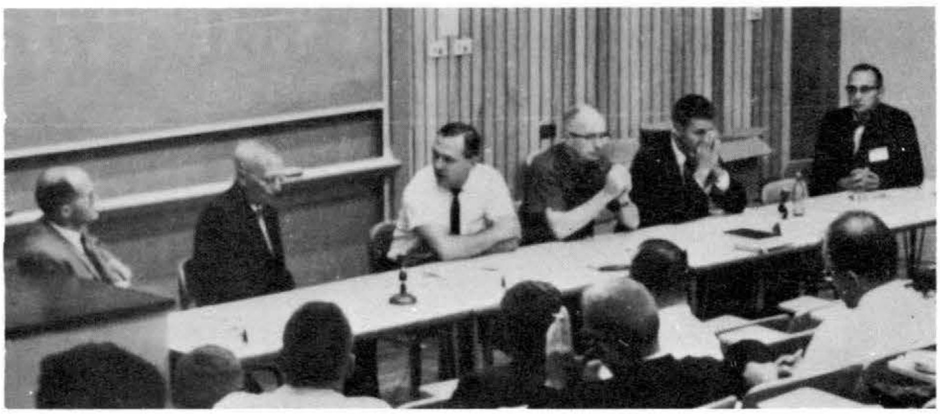
Now, in the linear case, of course, the T is to the first power, and that is why you get the exponential of $(-\alpha T)$. But in a nonlinear case this is T^2, giving hyperbolic variation. As I say, this can be integrated piecewise if you want to do it that way. But this does not throw any light on what the actual mathematics of the thing is. You merely get a numerical solution. Here is the actual analytical result. If I want to know what the shape of the curve is going to be during the first step,

$$T_1 = w_1^{1/2} \tanh \left[w_1^{1/2}(\tau_2 - \tau_1) + \tanh^{-1} w_1^{-1/2} \right]$$

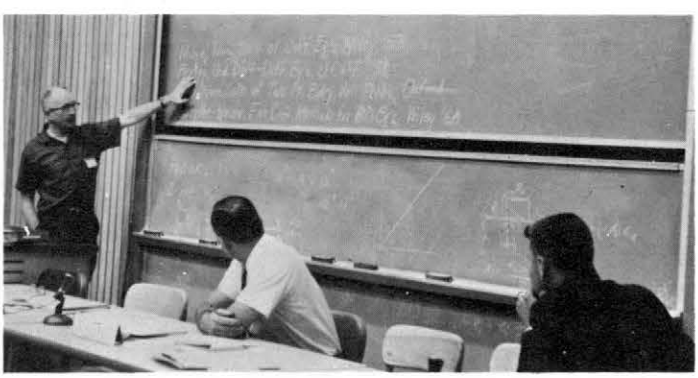
Before you have been through that once it looks like a lot of work. Actually, it needs to be set up in a table, and it goes very fast. Now, to go on to the next step all you do is to divide this through by $w_2^{1/2}$ or, if you want to look at it from this end, all you do is to multiply by $w_2^{-1/2}$ and start all over again. And you take the arc hyperbolic tangent, or if it is descending, you take the arc hyperbolic cotangent. This is the law of superposition of states.

DR. EVANS: Thank you, Mr. Jacob. This will wind up the session.

Congratulations to this panel for an excellent review of the present status of knowledge and practice relative to the mathematical treatment of leaky aquifer systems.



Panel Session 4: Left to right: Mr. D. A. Kraijenhoff van de Leur, Mr. R. E. Glover, Mr. P. A. Hurley, Mr. C. E. Jacob, Mr. R. W. Nelson and Mr. W. C. Walton (moderator).



(Photographs by George Palos)

SESSION 4

USE OF MODELS, ANALOGS, AND COMPUTERS

MR. WALTON: We have a very interesting and timely topic this morning called "Extension of Mathematical Developments in Transient Ground Water Hydraulics with Computers and Models." As you see, we have four speakers. Unfortunately Herb Skibitzke, who was to have discussed passive element electric analog computers, was unable to make it, and I think we are going to greatly miss his presence. Because of his absence I think during the discussion period I am going to call upon someone in the audience to perhaps help fill his boots. During the past several years there has been a great deal of progress made in applying electric analog computers and digital computers in solving ground water problems. Of course we all know sand tank models have aided us for a great number of years. I am going to call on the individual speakers for their comments first, and then at the end we will have a general discussion period. Our first speaker is Professor Kraijenhoff van de Leur who you have already met and know from the Netherlands whose subject will pertain to a sand tank model.

MR. KRAIJENHOFF: I have been asked to discuss some of the features of a sand tank, or granular model, that we are using for the study of non-steady groundwater flow to parallel drains. We apply certain time distributions of rainfall and measure the outflow rates. These outflow graphs can be compared with computed hydrographs such as I have already presented here. The idea was to test the range of applicability of the assumptions on which the computation method has been based. The main assumptions were:

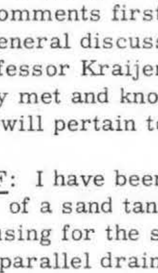
(a) Hooghoudt introduced the concept of the equivalent depth of horizontal flow in which the radial resistance near the drain has been accounted for. This concept was developed for steady flow and it has been assumed without further proof that it would also apply to the case of unsteady flow.

(b) The mean equivalent depth can be considered as a constant.

(c) The effect of the unsaturated zone has not been taken into account. In other words, as a first approximation, the rate of inflow into the saturated zone was assumed to be equal to the rate of rainfall on the soil surface.

Obviously only a granular model offers the opportunity to investigate the results of the assumption under (c). But in that case both the saturated and the unsaturated zone should be studied on the same scale and here we get involved with the effects of surface tension. Fortunately the soil scientists Miller and Miller have published a fundamental study on scale laws for unsaturated flow and these laws have been verified experimentally by Wilkinson and Klute. As the moisture content is no limiting factor these laws must also apply to saturated flow and they are consequently valid for the whole model.

Figure 1 shows a simplified explanation of the scale rules. The pore system in the granular medium is represented by a capillary tube with a diameter d and in this connection d is considered to be a representative dimension of the granular medium.



$$\rho g h_c \frac{\pi}{4} d^2 = \sigma \pi d$$

$$h_c = 4/g \ \sigma/d \ 1/\rho$$

$$\text{LENGTH SCALE: } m = \frac{h_{cm}}{h_c} = \frac{\sigma_m}{\sigma} \frac{d}{d_m} \frac{\rho}{\rho_m}$$

$$\text{POISEUILLE: } v = \frac{\rho g}{\eta} \frac{d^2}{32} s$$

$$s_m = s$$

$$\text{VELOCITY SCALE: } \frac{v_m}{v} = \frac{\rho_m}{\rho} \frac{\eta}{\eta_m} \left(\frac{d_m}{d}\right)^2$$

$$t = l/v \quad \text{TIME SCALE: } \frac{t_m}{t} = \frac{l_m}{l} \frac{v}{v_m} = \frac{\sigma_m}{\sigma} \frac{\eta_m}{\eta} \left(\frac{\rho}{\rho_m}\right)^2 \left(\frac{d}{d_m}\right)^3$$

FLOW THROUGH A POROUS MEDIUM:

$$\frac{v_m}{v} = \frac{k_m}{k} = \frac{p_m}{p}$$

Figure 1

The length scale is the ratio of the equilibrium capillary heights in the model and in the prototype. It follows that the length scale is determined by the surface tensions and densities of model-liquid and ground water and also by the ratio of the typical dimensions of the granular medium in the model and the granular soil in the prototype.

The velocity scale follows from Poiseuille's law and here we find that the viscosities play a role.

Finally the time scale can be directly derived from the length scale and the velocity scale.

A 50 percent ethanol-water mixture was used. This liquid has a low surface tension combined with a high viscosity. The high viscosity enables us to choose a coarse grained medium and this causes a reduction of the capillary rise. Since it is further reduced by a low surface tension we arrive at a small length scale.

Figure 2 shows the model. As usual it only represents half the drain spacing because the vertical midway between drains is a line of symmetry. The idea of using thermopane, a building material, was picked up at the Shell Laboratories at Amsterdam.

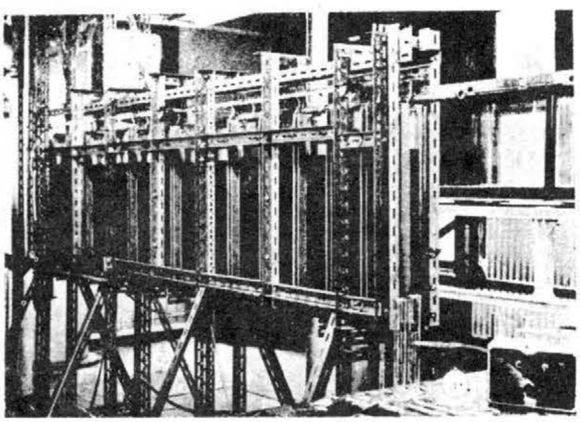


Figure 2

The price was less than 20 dollars. The half inch space between the glass sheets was partly filled with screened sand. On one side outlets were made to simulate drains or ditches and capillary tubes were inserted along the top to feed "rain" onto the surface. The rate of rainfall can be regulated by moving a reservoir, from which the capillary tubes are fed, up and down. As soon as the liquid has infiltrated it causes a flow of moisture to move downs from the surface to the saturated zone. This flow is like a wave that causes both changing storage and delay in the unsaturated zone.

Figure 3 shows the successive shapes of these waves, or moisture profiles, as computed for a medium of screened sand that was used by Childs. I am not going into great detail here, you can find all further particulars in the Journal of Geophysical Research, v. 67, no. 11, p. 4347-62, 1962.

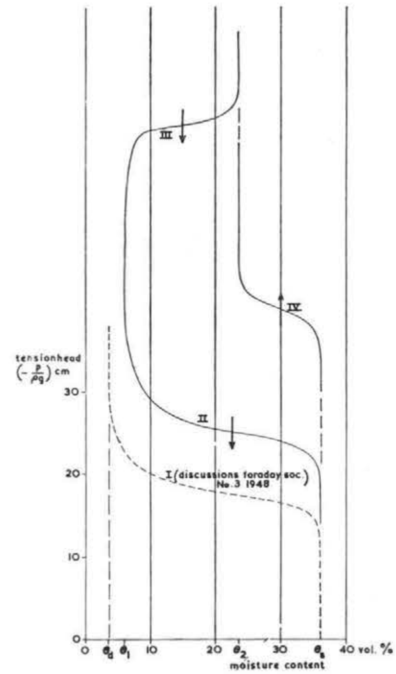


Figure 3

The line marked "I" represents the moisture profile when there is no flow, so when there is equilibrium. Profile No. II is moving down during a recession period. After rainfall has set in, profile No. III starts moving down the unsaturated zone until it overtakes No. II and merges with it. Only at this time the rain can begin to change the potential distribution in the saturated zone and cause outflow into the drains. After this moment profile No. IV is moving upwards against the downward moisture flow. When finally the rainfall stops, or its rate decreases, another moisture profile will move down the unsaturated zone and merge with profile No. IV.

This highly schematized picture is meant to show how changing storage and delay in the unsaturated zone will transform the time distribution of rainfall before it actually becomes inflow into the saturated zone. We must therefore expect that our third assumption of equality of rates of rainfall and inflow into the saturated zone may cause deviations between observed and computed hydrographs.

We have studied the relation between rainfall and groundwater runoff in our model (figure 4). Here the model simulates flow to a drain tube which is shown as a half circular dent on the right hand side of the model. Please note that there is a considerable convergence of flow that must be accounted for in the equivalent depth of horizontal flow. A "hyetograph" of three rainfall intensities was applied four times and the resulting outflow was both measured

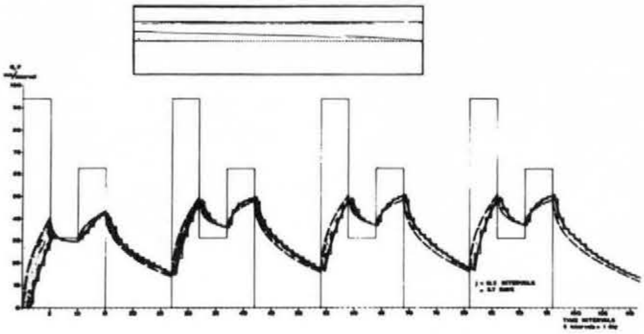


Figure 4

and computed. The experiment was started with a horizontal water table and an "empty" unsaturated zone. This explains the lag between the observed full line and the computed dashed line right at the beginning. For the three following waves the deviations are only slight but systematic. This good agreement also indicates that Hooghoudt's concept of an equivalent depth of horizontal flow, which he derived for steady flow, did not cause any appreciable deviation in this non-steady case.

suggest that we may stretch the presented computation method to a certain extent and expect not too inaccurate results even when the variation of the horizontal depth of flow is marked.

In the third experiment (figure 6) the position of the drain was almost at the bottom of the model and even then there was no excessive disagreement between computed and observed hydrographs in situation 4.

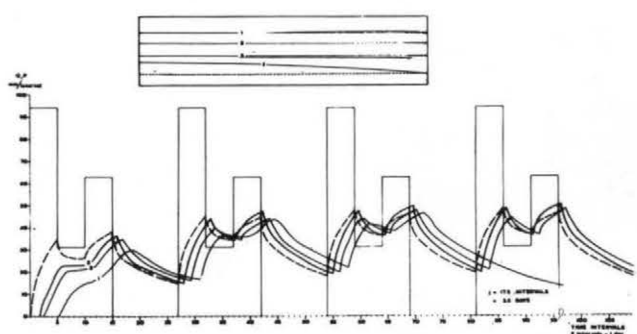


Figure 5

By the following experiments we tried to get an insight into the degree in which the unsaturated zone affects the hydrograph of outflow (figure 5).

In the schematic diagram of the model the curve marked 1 represents the highest position of the phreatic level and it indicates that the assumption of a constant depth of horizontal flow is not very close to reality. The computed and observed hydrographs for situation 3 are not too far apart. This seems to

These preliminary experiments clearly show that the unsaturated zone affects the rainfall-runoff relations and they illustrate the necessity of including the unsaturated zone in the study of transient groundwater flow.

Finally I want to make a few remarks about the computation of hydrographs. It is simple in its tabular form and as such practicable for the man on the job, nevertheless for long periods of many intervals it is tedious work. Moreover any local mistake will affect a subsequent series of computed values. We therefore mechanized the computation and used a very simple electrical device which is shown in figure 7.

I already mentioned that the outflow rate could be computed as the sum of 1, 2, or 3 terms and a rest term. Now the successive values of each of these terms were computed separately using circuits with a resistor and a capacitor. In the first circuit the RC-time stands for the reservoir coefficient and in the second and third circuits the RC-times are respectively 1/9 and 1/25 of the first. The inflow rates are simulated with the potentiometer and the successive values of the first, second or third term are read at the galvanometer on appropriate scales. For each interval these values for q_t^* , q_t^{**} and if necessary also q_t^{***} are entered in a table where finally the rest term is added.

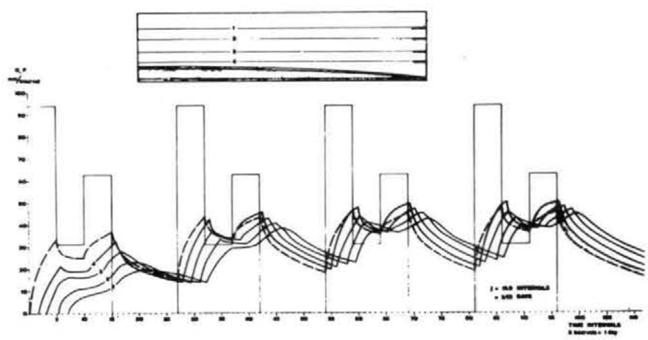


Figure 6

Figure 8 shows our latest effort. Here we have simulated half of the drain spacing by a network of resistors and capacitors. The resistors stand for the horizontal resistance against flow and the capacitors simulate the storage in the homogeneous medium. The ditch level is determined by the potentiometer. In order to simulate rainfall we have to insert currents at the network's nodal points at rates independent of the potentials at these points. We did not succeed however in realizing this in our "slow" model and we had to resort again to the concept of an instantaneous unit hydrograph. This appeared to help us out because, under the assumption of a constant depth of horizontal flow, an instantaneous change of the ditch level will affect flow in the same way as an instantaneous change of the phreatic level in the opposite direction. By use of the convolution integral the outflow can then be obtained by integration of the flow rates between the network and the potentiometer. This analog is under construction and we hope that it will soon enable us to convert lengthy series of inflow data into directly recorded hydrographs of outflow.

Coming to the end of my remarks, I would like to state, that I believe that the study of transient free surface groundwater flow is operating in the borderland of groundwater hydraulics and soil science. We should therefore cooperate with soil physicists who are interested and prepared to join us in our efforts.

In a general way I have tried to make clear how we are concerned with transforming the interesting and important results of mathematics into forms that can be used by the practical man in the field and shape them into tools for further research. Thank you.

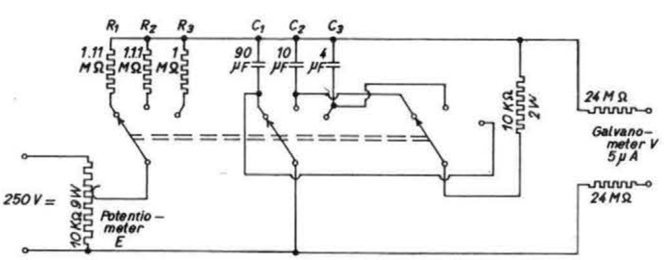


Figure 7

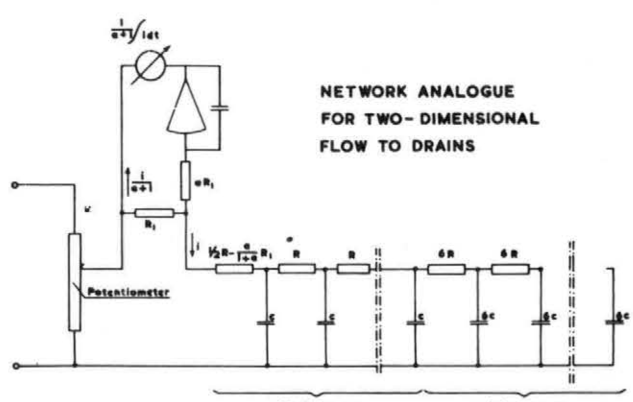


Figure 8

MR. WALTON: Our next speaker is Patrick Hurley, Engineer with the U.S. Bureau of Reclamation, Region 7, Denver Development Office. He is going to tell us what they have been doing with digital computers.

MR. HURLEY: The Denver Development Office is charged with making the investigation studies of large areas where ground water studies are a part of the over-all studies. We are trying to approach these studies on a statistical basis utilizing the digital computer to assist us in handling the data and makeing the numerous computations required. Previous to working in the Denver Development Office, I worked in the Office of the Chief Engineer where I had the opportunity to apply some of Mr. R. E. Glover's extensions of the Dupuit-Forchheimer Theory.

One of the current projects we have in the Denver Development Office is trying to set up a mathematical model of the entire South Platte River for all associated surface water and ground water developments. For those of you who are not familiar with this area, I will include a sketch (figure 9) of the geography of the basin.

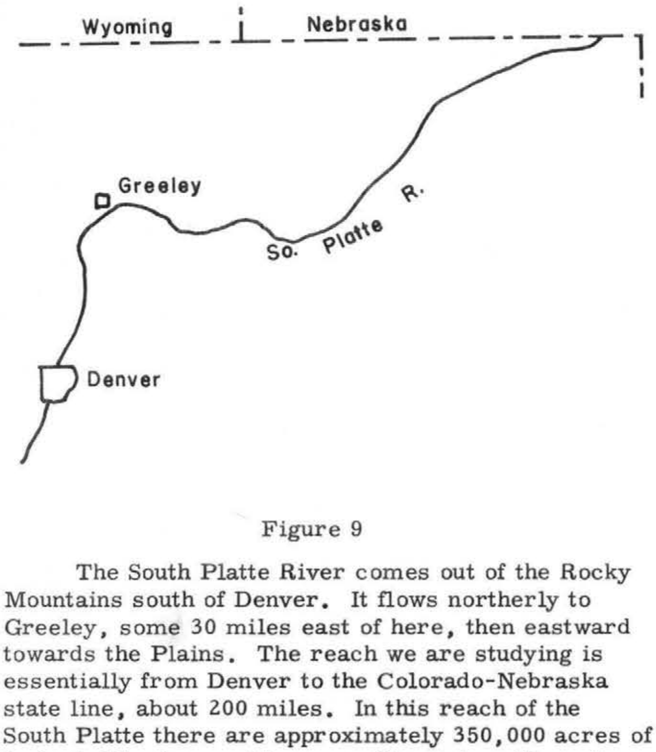


Figure 9

The South Platte River comes out of the Rocky Mountains south of Denver. It flows northerly to Greeley, some 30 miles east of here, then eastward towards the Plains. The reach we are studying is essentially from Denver to the Colorado-Nebraska state line, about 200 miles. In this reach of the South Platte there are approximately 350,000 acres of irrigated land served by some 60 canals. There is extensive well irrigation in some areas adjoining this reach of river.

We are going to study the historical records (31 years) to come up with a statistical analysis for the entire reach of river. This ambitious undertaking would be almost impossible if we did not have use of automatic data processing equipment to handle the data and make the computations. If we did not have the computer we would have to revert to shortcut approximations and reduced scope. The monthly historical irrigation operation of 350,000 acres for the 31 years will be studied, but that is only part of our anticipated study.

An interesting aspect of the problem is the increasing water supply to this area in the future due to the increasing imports of water through the Continental Divide by the large metropolitan area of Denver and surrounding communities to meet their expanding requirements. The future return flow from the City of Denver and surrounding communities will mean additional flow in the South Platte River. This raises interesting problems as to how the increased water supply can best be utilized. It is these problems we are attempting to solve in our investigations of the South Platte River.

I also want to make a point for studying historical records for such a long period of time by the Bureau of Reclamation. You might say that our clients are the farmers living in this valley. The questions which may be asked are generally based on past experiences. For example, some farmer might raise the point that back in 1934 he was drastically short of water and no crop was harvested. In the future, with the increasing water supply, if a similar year occurred again, what might he expect in terms of water supply? Our studies are geared to answer inquiries of this type.

One of the capabilities of the digital computer is data handling. Our studies utilize this capability. The historic diversions of the 60 canals throughout the 31 years amount to some 20,000 quantities, a mass of data that is far less formidable when punched into a stack of cards. We also have the effective rainfall and evapotranspiration calculated and on cards for the entire period. Mr. Earl Glenwright of our office has programmed the computation of consumptive use by the Lowry-Johnson method. The computation is not complicated but rather tedious, especially the determination of the beginning and ending dates of the growing season for 31 years at several points. A subroutine deducts the effective precipitation from the consumptive use to determine the irrigation requirement. Meteorological data for several stations were available from the Weather Bureau already punched on cards and were used in these programs.

The main part of our study of the South Platte River and of our computer application is essentially a ground water analysis. A canal seepage loss will be determined for each of the canals. As well as reducing the historic diversions to the amount effective in meeting the irrigation requirement for the shortage analysis, the canal seepage loss is one of the quantities recharging the ground water. The deep percolation losses from the irrigation applications are also a major source of ground water recharge. The return flows from these recharges to the ground water into the river will be approximated using the extensions presented by Mr. Glover earlier in this symposium. There are also a number of large reservoirs located off the river channel in this study area which contribute through seepage losses to the ground water recharge. The total recharge will also be determined through the computer program. The many pumps throughout this area represent a large withdrawal from the ground water. Our studies will include an analysis of the historical pumping throughout the 31 years period. Using these separate items, the main portion of the program will be to determine the gains to the river.

To determine the historical river gain, the precipitation upon the non-irrigated area and the resulting runoff, which is unmeasured, must be assumed. Our initial assumption is the rainfall on the non-irrigated area is consumptively used by the area. The rainfall is on the order of 14 inches per year while the evapotranspiration is some 26 inches per year so none can be assumed to enter the ground water reservoir and thereby contribute to the river gain. Of course large rainstorms result in runoff. This runoff contribution to the river gains will be found through trial and error as the "unexplained gain." This is the beauty of a computer. Innumerable solutions can be readily run by just changing a few cards for revised assumptions.

If through the statistical analysis, and a trial and error process, we can essentially duplicate the historic conditions mathematically, we can begin studies of how the future water may be best utilized. Our basic premise is that the higher in the basin the water can be used, the better use it is because more opportunity exists for reuse of the return flow or more simply "Highest use is the best use."

Once the studies of future conditions begin, the computer application will really start paying big dividends. With the basic program written and input data all ready, the many possible schemes for regulation and use of the future flows can be readily analyzed.

While I was in the Office of the Chief Engineer I applied some of Mr. Glover's work on return flow from irrigation. The graph included in the Appendix B entitled "Return Flow from Deep Percolations" is the basis for the application I will discuss. The digital computer was used to do the computations of return flow in the application.

The first step was to program the curve into a form the digital computer can rapidly handle. We approximated the curve by a 5th degree equation, the fit being made on the computer. Mr. Robert Main of the Office of the Chief Engineer did all the programming for this problem using an IBM 650. Although the curve on the referred graph is asymptotic in nature we zeroed the curve at $\frac{-\sqrt{4\alpha t}}{L} = 1.40$. This allowed an accounting for all the water entering the ground water table and simplified the results. The error introduced is only 0.6 of a percent.

We used the computer to analyze 13 years of historical operations of the Mesilla Valley in New Mexico. This valley lent itself ideally to a mathematical model. The records of diversions, land irrigated, and of the drain discharge are good. The drain discharge is a very good representation of the actual return flow that occurred historically. The deep percolation was determined from the precipitation and diversions. The flow of this quantity of water into the drains was then computed using the timing indicated on the curve. A comparison of the historic return flow as measured and the computed return flow showed a remarkably close correlation. This study of the Mesilla Valley and a write-up of the methods used are contained in Technical Memorandum Number 660 entitled "Predicting Return Flows from Irrigation" printed by the Office of the Assistant Commissioner and Chief Engineer, Bureau of Reclamation, Denver, Colorado, August, 1961. All the facts and figures are in it, including a brief description of the computer program used.

I hope to present in the near future a paper to the ASCE that will summarize the Technical Memorandum and explain in greater detail the application of this specific Dupuit-Forchheimer method of computing return flows.

MR. WALTON: Thank you Mr. Hurley. Our next speaker needs no further introduction - Mr. C. E. Jacob, Ground Water Consultant, Los Angeles, who will speak on active element electric analogs.

MR. JACOB: I am sorry that Mr. Herbert Skibitzke is not here, and I hope that the members of the USGS staff will forgive me if I overlap a little in what I shall say about analog computers. There are differences in terminology, but in the computer industry generally what is meant by an analog computer is an electronic device that has a number of very high-gain DC amplifiers and capacitors and resistances, fixed capacitors and variable resistances and also some fixed resistances, so wired together that it can simultaneously and continuously solve what we would call a system of difference-differential equations which approximates the differential equation of mathematical physics, whether it be for ground-water flow, heat conduction, diffusion, or elasticity.

Now, simulators of various kinds, have been used for many decades, and also resistance-capacitance networks, which in your outline here are distinguished as passive-element analogs. I think you realize that there is an analogy between the flow of electricity and the flow of other fluids such as water, heat, diffusion of molecules. This analogy holds for non-steady flow as well as steady flow because we have storage features in all these phenomena. That is, we can store electricity in a capacitor, we can store ground water in a compressible aquifer, we can store dissolved matter in a solvent, and we can store heat in a conductor. I think it would perhaps be well to briefly set up the basis and then to perhaps illustrate the use of an analog computer in the sense in which I defined an analog computer. I might say that historically, to my knowledge, the first resistance-capacitance network that was used in this country was used by Paschkis at Columbia University when I was a student there. You will find reference to this in the Transactions of ASME. (See Trans. ASME, 64, 105 (1942).)

Beginning in the early 1940's, culminating, I believe, in 1945, the first real serious attempt to simulate an underground flow system was made by Bruce of Carter Oil Co. in Tulsa, Oklahoma. (See AIME, Trans, 151, 112 (1943).) You will find a paper by him which goes into some detail, not into detail about the design, but into detail of the application of this device to petroleum reservoir problems, and this was done, I believe, in 1945. Many oil companies today have these simulators, in which there is a direct analogy between the fluid flow and flow of electricity. Many accessory circuits use control devices and electronic features.

Now, you are all familiar with what a Link trainer is. A Link trainer is a device which simulates the behavior of an airplane in response to manipulating the controls. It is used to train instrument pilots. This is a simulator which operates in what we call "real time." In other words, the time of the device is equal to the time in the control equipment. Computers are also used to control various processes, both in real time and in accelerated time (by computing ahead). Now, whether we use a resistance-capacitance network to solve unsteady flow problems or whether we use an analog computer, we first have to formulate or model our fluid-flow problem in electricity. Then, if we are going to use an analog computer we have to translate this, so to speak, into computer language, or rather into computer circuitry, and we have to draw a wiring diagram for the circuits that are going to be used. I would like to give you, as an illustration, an example that we have carried through on an analog computer which comes under the heading of semiconfined flow, which we talked about yesterday, and this is a radial case. However, two-dimensional or so-called unidirectional flow cases could be treated. In fact, we treated them first on a desk calculator before we ever did the radial problems on a computer. There are two very closely related and practical problems which, I see in the literature, others have tackled using different techniques, by using the Dupuit-Forchheimer approximation, etc. But here is a very interesting and practical problem in drainage engineering that has to do with this. Let us say we have an aquifer (figure 10) that is transmissive, for example a gravel bed (a) that is fairly deep. Overlying that we have, as we do in many irrigated valleys in the West, a series of sands and silts (b) in which very often the sand is the continuous matrix and the silts are usually lenticular. You might have a very high permeability in the bottom bed, relatively speaking, and a very low horizontal permeability in the top bed and an even lower vertical permeability. This is quite markedly anisotropic in the bulk. You can get an equivalent anisotropy. Let us say you want to see how the drains or wells behave in this set up.

This is not the leaky aquifer problem but is of different geometry. If we pump water out of the well (c), what is going to happen? Now let us say we have a "step well" and postulate that we have a level water table (d) and we have a potentiometric surface for the lower bed that coincides with it. In other words, neither upflow nor downflow in nature and with no irrigation yet. We begin pumping the well.

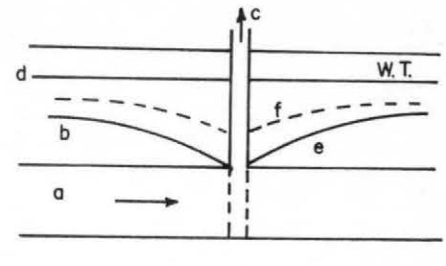


Figure 10

When you begin pumping, of course, you will begin to produce a cone of depression (e), and you will also produce a depression (f), in the water table in the top bed, and the water table will of course be horizontal at the well. That is one problem; now another problem that is related to it as far as computer circuitry is concerned.

This is a problem with the same structure, but now, instead of pumping, we have irrigating of the land or recharging it artifically, or by some means inducing infiltration at a high rate over a localized area. Let us say that we take a circular area, and we want to find out what happens in this circular area and beyond. We are going to recharge at a uniform rate, starting at a certain time. Actually, we can handle all kinds of inputs. But let us say we want to look at this simple problem, where suddenly somebody starts irrigating. What is going to happen now? These problems can be solved by other techniques, but we will put this on the computer and draw a set of curves that I think will be very useful. We produce a mound on the water table and we also produce a rising of the head in the lower bed. There would be a tendency for water just outside the circle to flow back up into the upper bed, whereas inside the circle it is descending into the more permeable formation. Now, this can be set up in circuitry, and I would be very glad, if we have the time and the permission of those who organized this symposium, to post some of these illustrations so that any of you who wish to see them can see them during the intermission. But, we have the computer circuitry here, and I have several curves that were actually recorder charts that were taken from the computer. I just reproduced the blue line prints. What will the circuitry be here? Well, here is where your engineering judgment comes in. What is important? You cannot model this thing literally. There is no point in going in and finding all the details of structure. You just take a general overall view of the thing as you are going to predict how it will behave in the bulk, not how it is going to behave in detail. So, knowing that the permeability of the upper bed is very low compared to that of the lower, everybody will agree that we can neglect vertical flows in the bottom bed except, perhaps, in the vicinity of a margin of the mound. We can also neglect horizontal flows in the top system. So we just describe these components of flow: horizontal flow in the bottom layer and vertical flow in the top layer. Now, the analog of the bottom bed is, of course, a rather small resistance. If we break this up into blocks- annular blocks - we think of the center of each block as a node in an electrical circuit, and we set up an analog of this type. You have already seen circuitry of this type shown by Mr. Kraijenhoff. Now, the effective resistance of the top bed is going to be variable. That is, the path of flow from the water table to the bottom of that bed is going to decrease as the head drops. The resistance will depend upon the potential difference or the head difference between those two points. We will have no horizontal resistance in the top layer. The coupling between the flows in the top and the bottom layers is through the variable resistors. We have storage only in the top layer. Now, just a remark about that.

The bottom layer is much less storative than the top one, certainly 1, 000-fold. All the storage is effectively in the top layer. We have a capacitor which simulates the storage in the top layer, and that is grounded. Somebody may have a question why that is so, but let us just pass it for the moment. Each one of our annular blocks will be a replication of this circuitry. I will write a little r for small resistance and a big R for large resistance. The time constant in an electrical circuit, I think you realize, in a resistance-capacitance network, is RC . Let us take a scale so you can control the time of operation. You can make something take place in a few seconds, say in a minute, that in nature would take decades to transpire over distances of a few miles. This is a tremendous advantage in electrical models over any kind of hydraulic models. You have greater flexibility in scaling.

It should be emphasized that this device requires appropriate circuits to control currents. For example, I have to put in a current at each upper node to maintain a steady current, even though the voltages are dropping, which represents a recharge. Or if it were a well I would have a single current going out the other way, which represents a draft upon the reservoir. It suffices to say that this electrical circuitry provides an analog, an electrical analog with electronic subsidiary circuits. This device does not solve the differential equations. Neither does it solve the difference equations. It solves the simultaneous equations we call difference-differential equations. I do not know if you are aware of it, but there is a rapidly developing field in mathematics in the theory of difference-differential equations. In other words, you see, we have modeled the flow system discretely in space, but continuously in time. If each capacitor gives up its stored electricity continuously when you lower the voltages, then this is a continuous process.

Let us look at the differential equation that we are dealing with: $\nabla^2 h = \frac{S}{T}\frac{\partial h}{\partial t}$. Talking about unidirectional flow, the Laplacian may be replaced by a second difference $\Delta^2 h/\overline{\Delta x}^2$, that is, the rate at which the slope changes.

$$\frac{\Delta^2 h}{\overline{\Delta x^2}} = \frac{S}{T}\frac{\partial h}{\partial t}$$

Leads are taken out and voltages read at the nodal points. I am solving a difference-differential equation. Now, a digital computer does not solve this.

A digital computer solves a second-order partial difference equation. The technique of putting very high-order difference equations into the digital computer will permit us, if we wish, to approach very closely the original differential equation. This is not ordinarily done in industry, because of the speed, the size, and the memory storage that are available. Most people prefer to use a great number of small steps, and therefore compute with the second-order difference equation. But there is a technique, using a desk calculator, where you can use a larger area of influence on a given node and go to higher and higher orders of difference equations.

The basis of an analog computer is the perfection of a very precise direct-current vacuum-tube amplifier that has a very high gain. But a very high-gain DC amplifier can do many things to an electrical circuit. Now there are ways of integrating just using an electrical circuit. With very high-gain amplifiers it can be done with extra precision. The gain, the ratio of the output voltage to the input voltage, runs up to 10^8 , or even higher. Successfully, you can run amplifiers to 10^4. Now, we have a computer that has 60 operational amplifiers which was built by a company that went out of business because the competition became so keen. Today there are very few companies manufacturing analog equipment. There were five times as many fifteen years ago, so equipment is available. You can get all the precision you need with the type of hydrological data available from the U. S. G. S. and the U. S. B. R.

Now, you need not only high gain, but also linearity. You need precision. The precision that is attainable is 10^{-4} to 10^{-5}. That means, that looking at the voltage, that you can reproduce it to four significant figures, or five, over the range of the voltage, which is usually 100 volts. So this is the range of precision. Now that is not like a digital computer, where you can get down to the 10^{-20} . We are not interested in this precision yet in ground water. I do not know when we will be.

I want to show you a number of circuits and show you what each does basically. The high-gain amplifier can integrate. It can differentiate. It can sum, and summing is the same thing as integrating in space. Differencing is the same thing as differentiating in space. Now, let us say we have an amplifier (figure 11) and we have an input resistor going into one of these very high-gain amplifiers where the gain is -A. That

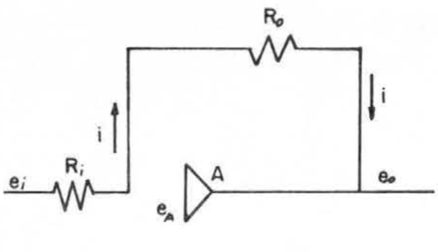


Figure 11

is, it is also an inverter. We have what we call the feedback resistance, R_o , e_o being the the voltage of the output, e_a being the voltage on the first grid, on the first tube, and e_i the input voltage. Now, let us call R_i the input resistance. I think this is enough for you to see why you have these high-gain amplifiers -- what this thing really does. This will be the device that will be a scale changer and inverter, it will invert the voltage and change the scale in proportion to these two resistances (R_o and R_i). Normally, a vacuum-tube grid draws negligible current, and most of the current will go through these two resistors and on around the amplifier. The current going through the resistances will merely be the inverse ratio of each resistance to its voltage drop.

$$\frac{e_i - e_a}{R_i} = \frac{e_a - e_o}{R_o} = i$$

This grid is very near to ground potential and e_o will be negative, and e_i will be positive. Because of the extremely high gain in the amplifier, the output voltage is equal to gain (-A), times the input voltage. This gain is so very high, even if only 10^4, that you can neglect e_a by comparison to e_o . So we get

merely: $e_o = -\frac{R_o}{R_i} e_i$

So what have we done? We have a multiplier, so we can change scale. But we have to multiply for other reasons than just changing scale. And we have an inverter.

If we want to integrate, we will merely have an input resistor and a feedback capacitor, and we have the amplifier. This has several vacuum tubes in its circuit. This is a package that you plug into the top of the device. By having a constant resistance R , we can integrate, and if we put a step input in here, it will give out a response e_o , which will look something like the curve in figure 12 -- strictly exponential. I think you realize that we are just charging up a condenser. From this step input you get an exponential output, but you work only on the straight-line part of this curve. This is where the gain is

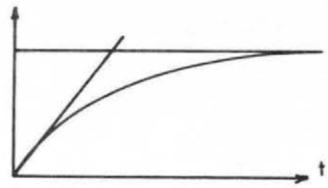


Figure 12

important, because you are working with a limited capacitance C . The effective capacitance (figure 13), here is not C , but A times C . The capacitance has been multiplied effectively by the gain. A unit step-input gives you a response that is:

$$A \left[1 - \exp(-\frac{t}{RCA})\right] \quad \text{or} \quad e_o \cong e_1 \frac{t}{RC}$$

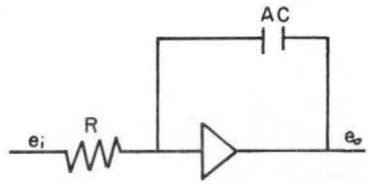


Figure 13

This is the same thing you have in ground water. If I put a step input into a linear groundwater system, I get an exponential output. Integrating in space with this device is done by feeding in several resistors, R_1, R_2, R_3, with different input voltages, which might represent feed-backs from different nodes in this system. This device merely sums these inputs and then they are integrated. So I can get the integral, for example, of $\frac{1}{C} \int (\frac{e_1}{r_1} + \ldots .) \, dt$

If I want to differentiate I just interchange the capacitor and the resistor.

These are the basic circuits of a computer. The results of the analysis enable us to demonstrate, for example, how a certain project can best be drained by means of deep drainage wells. We know the characteristics of the drainage wells in this situation, in the two-layer system with a declining water table in the top layer.

MR. WALTON: Thank you, Mr. Jacob. I think you have some references.

MR. JACOB: Several people asked me yesterday about difference equations. Numerical methods of computations were practiced long before the First World War and there is a lot of literature. It was done more or less by an empirical approach, without too much evaluation of what was being done.

Recently very good papers on ordinary difference differential equations have come out, but in what we call unidirectional flows, that simulate a circuit.

Here is a useful reference: Fox, Oxford Press, 1957, "The numerical solution of two-point boundary problems." There is a book by Milne, who was professor of mathematics at Oregon State University: "Numerical solutions of differential equations", Wiley, 1953. A more recent book by Pinney, who is at Stanford: "Ordinary difference-differential equations". This is published by the University of California at Berkeley, 1958. Another book is Forsythe and Wasow: "Finite difference methods for partial differential equations", Wiley, 1960. And there are many others. These are just a sample of the literature that is written for the physicist and engineer. There is a great deal of other literature that is written primarily for the mathematician.

MR. WALTON: Our next speaker is R. W. Nelson, Engineer at the General Electric Company, Richland, Washington. His subject is "Combination of digital computers and passive element electric analogs".

MR. NELSON: I will try to summarize some of the work we are doing and to give an overall picture of what our purpose is rather than spending a lot of time on detail studies that are underway. We might start from a comment made two days ago; namely, "it may be desirable to spend more effort on methods for determining permeability distributions in the field for use as inputs to any mathematical or analog model". Then, yesterday, Dr. Spiegel mentioned that more work is needed on techniques for obtaining field permeability distributions from water table contour maps. I would like to extend the latter idea somewhat and replace the water table map with the map of equal potentials, where the potential now contains two components, an elevation or body force component and a pressure component. It is in this area of obtaining permeability from field potential distributions that much of our research effort has been and is devoted.

Let us assume that the permeability is a continuous function of space, its first derivative is also differentiable, and let us see whether we can deduce a way of measuring the spatial distribution of permeability. As far as a theory along these lines, earlier work published in 1960 sets forth a general theory for steady flow. I might add that I think there is a tremendous area for extending this work to transient flow systems.

The method we are looking at for measuring the permeability in-place briefly stated is: Go to an existing field flow system and measure the present potential distribution. Study that potential distribution, i.e. the way in which that potential energy is dissipated, and deduce the spatial variation of permeability. The partial differential equations describing flow in heterogeneous soils lead the way and provide the requirements for a valid result. The existence and uniqueness conditions for the solution of the equations set some rather special conditions on the boundary condition in permeability. This boundary condition is very subtle and in one or two instances it has been overlooked. For example, in a paper in Petroleum Technology about a year ago the subtle nature of the boundary condition eluded consideration. The boundary condition requires a knowledge of the permeability distribution along one surface which intercepts every stream filament making up the flow system of interest. What are some of the problems one faces? The first problem is associated with the data from the field being tabular data rather than an analytic function. The potential data are taken from wells or piezometers at an irregular spacing, and from this we must deduce a permeability distribution. This turns out to be a rather challenging undertaking.

I will outline briefly some of these steps and indicate some of the problems. At the top of figure 14

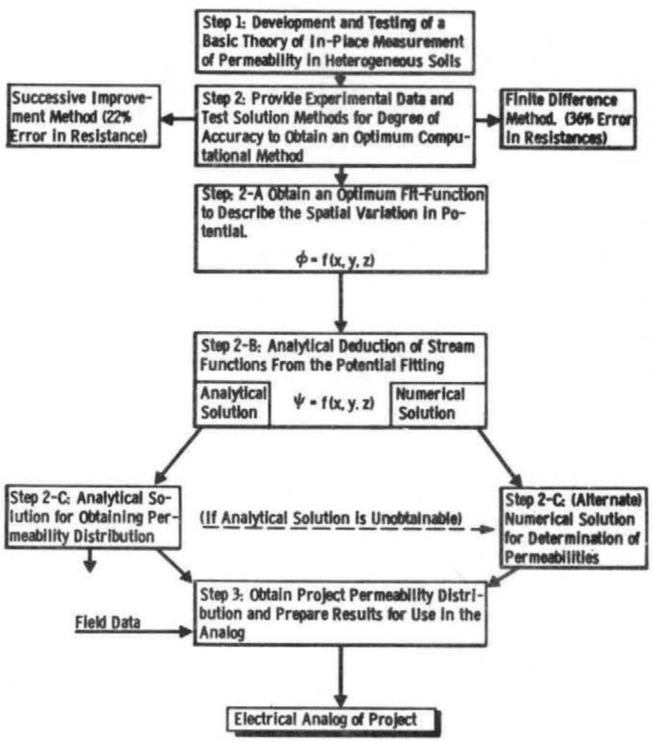


Figure 14

we have indicated the development of a basic theory for the in-place measurement of permeability, and as indicated, this has been completed with a very nice consistent theory at our disposal. The second step, moving downward, is to provide some experimental data and test solution methods to enable careful analysis for the degree of accuracy for any computation scheme. Initially a finite difference method was examined but an error of some 36 percent in resistance was found. The experimental data used were from inexpensive conductance paper analogs. Arbitrary patterns of punched holes were made in the conducting paper with no particular selections in pattern. The potential variation was observed on this surface. Later the analog was cut into small pieces and the resistance measured for each small square. From the previously measured potential the resistance distribution was computed and compared with the measured resistance data.

More recently we have left the resistance paper analogs and have been able to get analytic solutions for several boundary conditions for what we call the "Shoe Box Model." It is a shoe box in shape and contains soil with a permeability distribution of the form $e^{(ax + by + cz)}$, which can be solved in closed mathematical form. This enables us to use the potential function in any computational method to compute the permeability distribution for comparison with the known permeability. The newer analytical solution provides a very accurate means of testing any computational scheme.

To the left of the second step in figure 14 is shown the successive improvement method. Initially, this method seemed to hold promise in that it did reduce the error down to 22 percent, but it was not as successful as desired. Something might be said about the accuracy that is being sought. Some time was spent looking at this question. An empirical relationship based upon several test cases showed that if you went to a single location in a flow system and imposed an error in permeability on the order of 7 to 10 percent, then the error in observed potential would be on the order of one percent.

The variable usually needed in predicting new flow systems is the potential relationship. Therefore, if we are willing to stay within engineering accuracy in the potential (on the order of 10 percent), then as much as 70 or 80 percent error can be tolerated in the permeability factor. For our particular uses we are not satisfied with this large error since, even though the potential error is low, the permeability error comes back into travel time directly when integrating along the travel path. Our goal, then, is to find a method for deducing the permeability distribution to within a 10 percent error. When we get to 10 percent -- and we believe this can be done -- then we will likely not seek further refinement.

Referring again to figure 14, included in Step 2A is a procedure for moving around the computational problem. We return from the field with numerical data at irregular locations in three dimensions and have to deduce the permeability distribution. Step 2A involves finding an optimum-fitted function. Continuing downward in figure 14, the next step is to analytically deduce the stream functions from the potential fitting. There are two possibilities: An analytic solution was planned, if one could be found, or if not a second route numerically will be used. Incidentally, this figure was prepared for an outline of the work made about a year ago and this is why we have alternatives. I may mention that 6 months after the figure was made we arrived at the streamfunction step only to realize that the stream function for heterogeneous soils was not to be found in the literature. The stream function that had been talked about was the stream function from classical hydrodynamics, in which there is nothing similar to heterogeneity considered. Some time was required to come up with the stream function for heterogeneous media which I reported on in the first session.

One other thing in connection with the stream function, I was talking to Dr. van Schilfgaarde last night; two or three years ago he had raised the question of the orthogonality condition between equipotential and stream surfaces for partially-saturated flow. He mentioned that after he found the right approach he had shown orthogonality in about two steps. The streamlines and methods for partially-saturated flow are indicated in the paper. After looking at this brief outline, we examine what we are doing digitally in some cases and through analog networks in others.

In the upper righthand corner of figure 15 are shown the field data which include the field location, an x, y and z coordinate, time, t, and a potential observation, ϕ, for each piezometer. Let us take those field data and go into what we call the Computer Program, "Genoro." This is an adoption of some of the methods used by the Upper Atmospheric and Plasma Physics Unit of the Bureau of Standards at Boulder in connection with numerical mapping of the ionosphere. It has a very nice feature in that a set of ortho-normal fitting functions are generated, with the functions being orthogonal with respect to the irregular locations of the potential data. This special class of functions in effect uncouples the coefficient matrix of the linearly independent fitting functions. Accordingly, one can calculate the coefficients in a step-wise manner, rather than having to invert an extremely large matrix. It also has the advantage that you can add one more term without going back and recomputing the previous coefficient.

This outlines the fitting method; let us see what the program does. In figure 15, "Genoro," enables reduction of data on an irregular tabulated interval to functional form. The inputs needed are the tabular field data and the desirable fitting function. I would not want to minimize the importance of desirable function forms.

We are spending quite a bit of time getting functional forms that perform satisfactorily for us. From classical theory the potential function can have saddle points, in the geometrical sense; however, it cannot have maxima and minima. This helps in setting up certain classes of functions to accomplish fitting.

The computer program is written in such a manner that essentially as many dimensions or independent variables as desired can be used with the upper limit only being on the combined number of fitting functions and independent variables.

DATA AND COMPUTER OPERATIONS REQUIRED FOR DETERMINING THE IN-PLACE PERMEABILITY DISTRIBUTION FOR HETEROGENEOUS SOILS

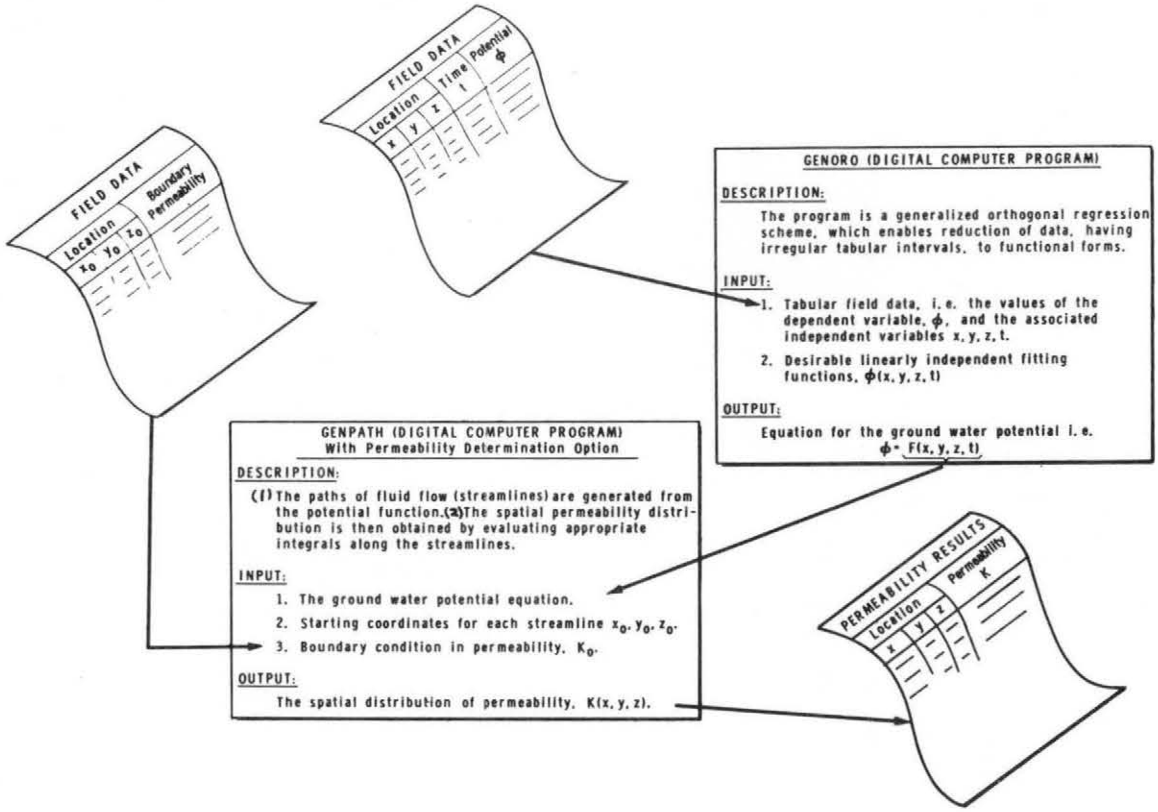


Figure 15

The output of the Genoro Computer Program is an equation for the potential. This may be a horrendous-looking equation; however, it does boil down to a very nice list of coefficients which are efficiently handled by the computer. Usually these coefficients are used only internally, or are stored on magnetic tape and do not have to be used on the outside of the machine.

The output from "Genoro" goes to a second computer program entitled "Genpath." The formulation of this program is completed at present and programming is underway. Some preliminary work done in checking the numerical methods showed excellent accuracy in the method for determining the flow paths.

The computer program "Genpath" uses the potential function to deduce the paths of fluid flow or the streamlines. The input is illustrated in figure 15. The output from the program gives the spatial distribution of permeability in tabular form.

Figure 16 shows the preparatory steps for simulation and use of electrical analog techniques. The program "Reperm" utilizes the permeability distribution and boundary conditions to give the size of resistors needed for an electrical analog. The resistance values required for analog simulation are calculated from two inputs -- the permeability distribution and the geometrical configuration of the flow system. In figure 16 is indicated an input chart of field data. It is descriptive of the Hanford Project where we are concerned with the configurations of the Columbia River and Yakima River, the basalt boundaries beneath the project, elevations of the water table, and other geometrical boundary effects that are of concern. The digital computer program "Reperm" is nearing completion. The outputs of the program are the resistor purchase list, an analog wiring diagram, and a decoding list to enable conversion from the analog read-out scanner numbers. The components for an analog model can be ordered and assembled using the output of the program.

The analog is described in figure 16. Preliminary work on equipment is underway at the present time. It will be a small analog of some 1800 elements and will be used to test and develop equipment and techniques.

Now, let us take a look ahead at what we will do with the output potential results from the final analog; in this case some of the same programs will be utilized that were already described.

In figure 17, the results from the analog are shown in the upper left-hand corner; these are the three coordinates of location and the new predicted potential at that point. The predicted potential in tabular form can be inputted to "Genoro," which is the same program used earlier, only this time we are going to get a fitted expression for the potential as the answer to the problem solved by the electrical model. The output then goes to a more comprehensive version

PREPARATORY STEPS FOR SIMULATION AND USE OF ELECTRICAL ANALOG

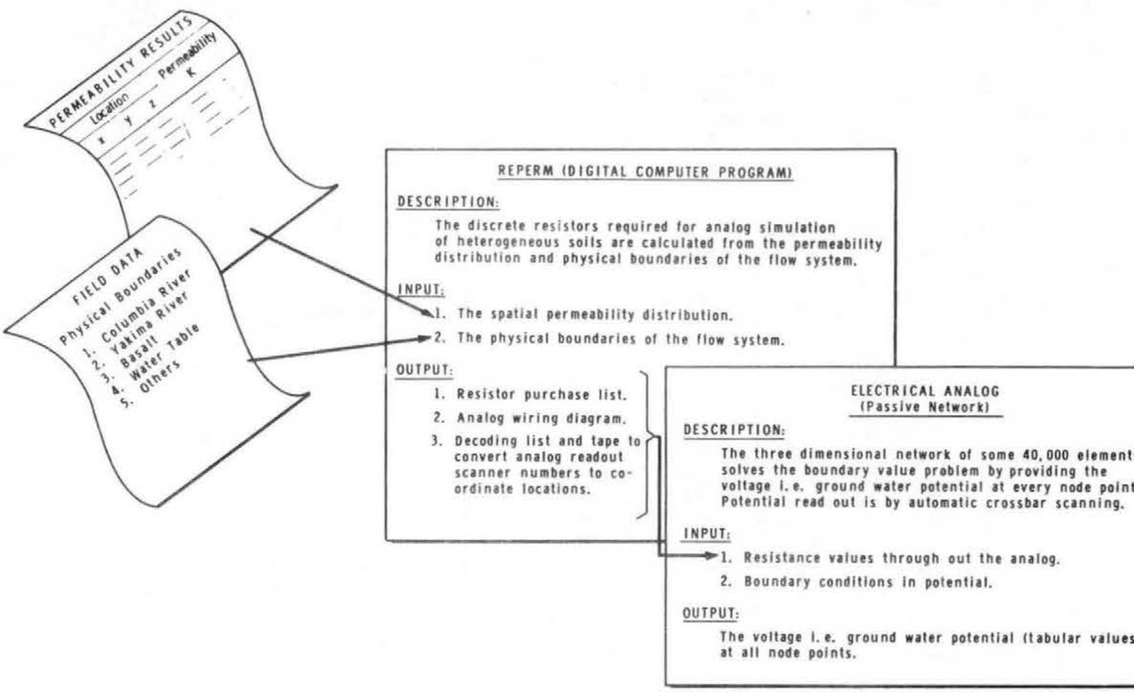


Figure 16

OPERATIONS ON ANALOG RESULTS TO ENABLE WASTE DISPOSAL ANALYSIS

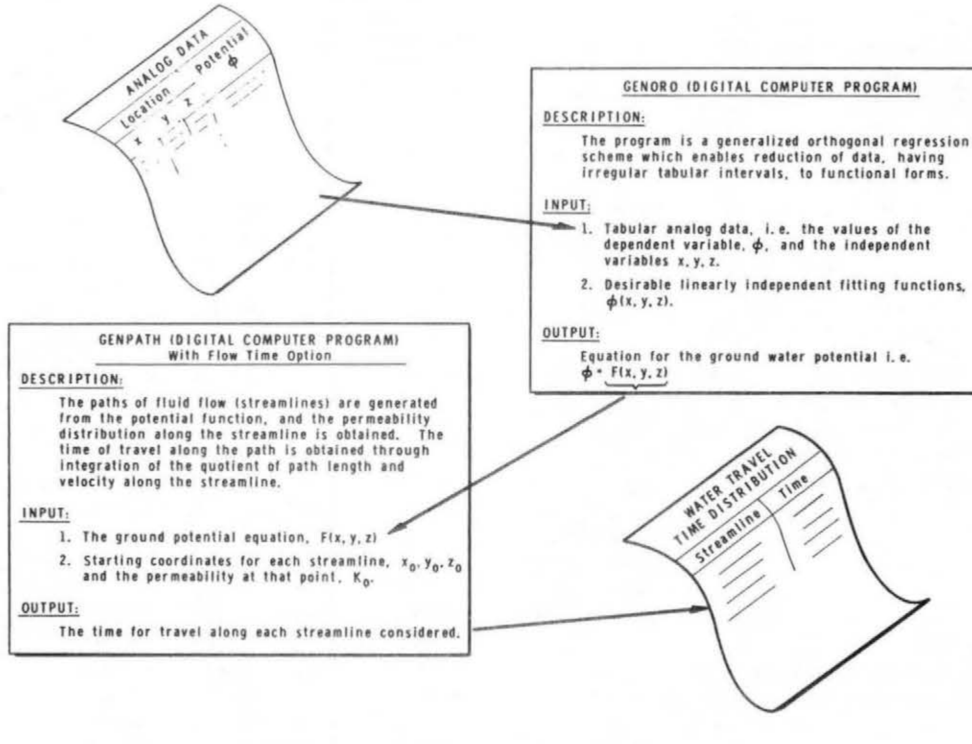


Figure 17

of the program "Genpath" which now will tell what the paths of fluid flow are and the time of travel along each path. These are the results which are vital to the analysis of the complete ground water flow system.

MR. WALTON: Before we get into a discussion period, R. E. Glover, Engineer with the U. S. Bureau of Reclamation in Denver has consented to give us about 10 minutes on the passive element electric analog computer.

MR. GLOVER: First, let us see what we are talking about -- first slide, please. This is a passive element analog. It was built by the Bureau of Reclamation for studying the effect of pumping for drainage in a valley. This valley is about 40 miles long. The width across the valley itself is about 4 miles and this, where you see the different colored elements is a mesa area which is also irrigated. The red dots represent the existing wells. Here is the conduit that comes down through the valley and the wells are connected to it. The yellow elements represent the mesa area which has a different permeability as it is of different geologic age. These elements are actually plugs in which a connection can be made with a jack. This is the read-out equipment, in this case a Brush oscillograph, and here are the input units -- these with the red pilot lights are input units. These will put a given current into a node point regardless of the voltage change in the analog network. As to the nature of this analog, figure 18 is a drawing showing the detail -- with node points, conductances, capacitances and electrical condensers. The relationships between the hydraulic and the electrical quantities are: (1) The flow of water is represented by a flow of current, (2) the water table elevation is represented by a voltage, (3) the transmissivity is represented by a conductance and (4) storage of water due to a rise of the ground water table is represented by a capacitance. Time is represented by time in the analog, but to a different scale. In this particular analog, the ratio of times is such as that one second of analog time represents 100 days of prototype time.

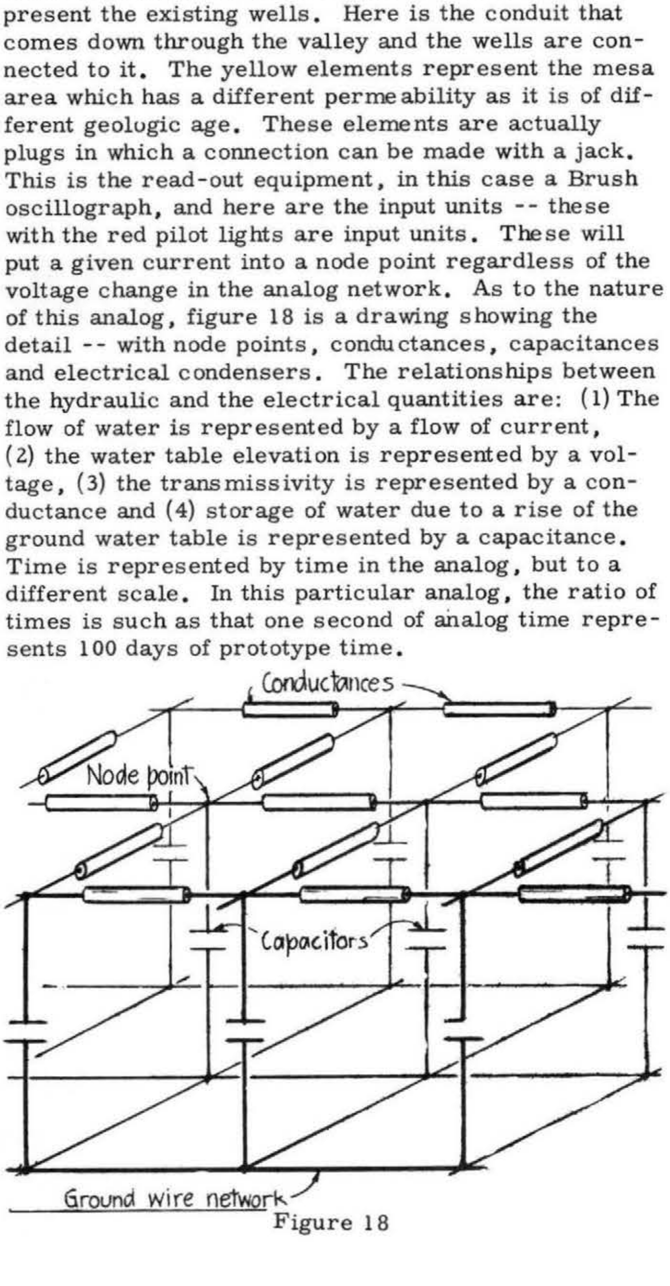


Figure 18

MR. WALTON: I also understand that you have 2 or 3 slides showing some other models that you've been working on.

MR. GLOVER: Yes - those are sand tanks - (shown schematically on figure 19).

MR. WALTON: Will you identify the area you have modeled?

MR. GLOVER: It is on the border - along the Colorado River. In size this analog panel is 8 ft by 4 ft.

This is a sand-model type of experiment and it is of somewhat different nature from the other things described here. This is an aquifer sweetening investigation. As we more nearly completely encumber our water supplies we find that the salt load of the rivers is becoming increased. For example, if you find an area where salt has been allowed to accumulate, say, because of bad drainage practices, and then the time comes when something must be done to drain it they then must put in drains and the water has to be disposed of generally into a stream. It may already be carrying a heavy salt load so that this is a matter which is becoming of importance and I think will become more important as time goes on. The slide shows a terminal state. The saline water has been dyed blue so that you can see what happens to it. This experiment represented a two-part aquifer, with an upper sand and a lower gravel where the ratio of permeability was about 50 to 1. The two parts were of equal depth. The salt water occupied the lower bed at the beginning of the test. The fresh water came in from the top as by irrigation. Notice the line of separation; it separates the flow in the upper bed from the flow which comes through the lower bed. The behavior is that the water comes down, gets into the lower bed, flows to the vicinity of the drain and then breaks directly up through to it. The water which came through the lower bed, comes through inside this line. The fresh water applied at the surface goes to the drain by two routes. A small part, applied near the drain goes directly to the drain, but if it comes in a little farther back, it goes under the line of separation and comes up. The first model represented an aquifer 80 ft deep with a distance between drains of 1320 ft or thereabouts, and this amount of salt water remains in permanent storage. The situation is similar to what happens near the sea coast. There is an interface between the stagnant sea water in the aquifer and the moving fresh water.

A later experiment was made to test the behavior under the condition of a drain spacing of 640 ft. In the first case there was no drain in one end of the tank. The bulkhead at the undrained end of the tank then represented a line of symmetry across which there was no flow. Again, we see the residual salt remaining -- this is the terminal state. The stored salt volume extends clear to the center. There is a small amount down below the frame of the tank which cannot be seen but in this case, the salt storage does come across the full span.

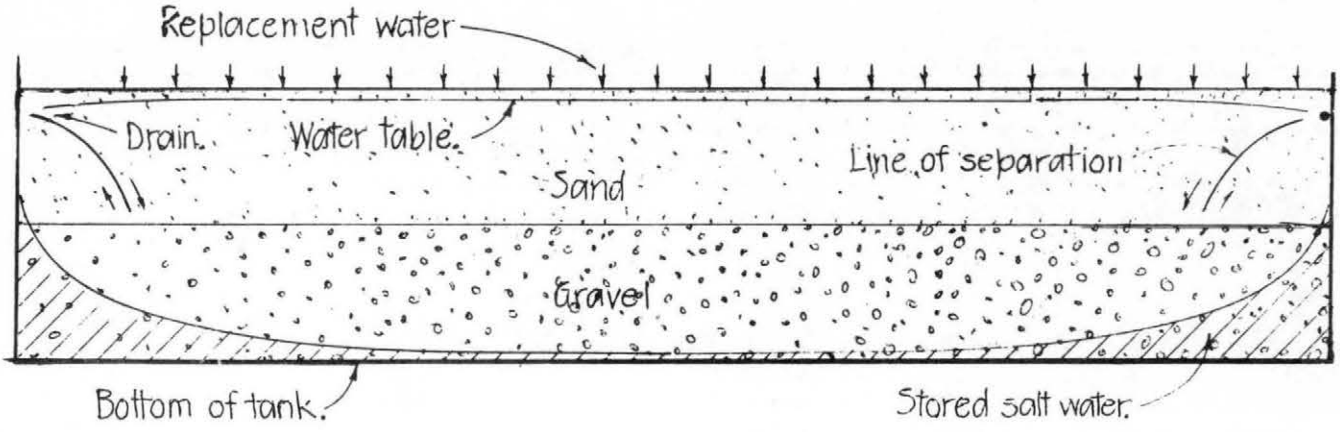


Figure 19

MR. NELSON: Is there a difference in density in the two materials?

MR. GLOVER: The saline water has about 6,000 parts per million of salt. The fresh water is Denver city water.

In another case an additional drain has been placed at the middle of the tank so that we now have a drain spacing representing about 320 ft. Here again, an almost steady state is reached and this amount of salt water remains in almost permanent storage. Thank you.

MR. WALTON: Thank you, Mr. Glover. If you will just join our panel here, I'm sure there will be some questions.

I understand Dr. DeWiest has about 3 or 4 slides describing a model he has been working with at Princeton.

DR. DE WIEST: These are close-up photographs of the model (Hele-Shaw apparatus or parallel plate model) that was built to test the results of the analytical study described in ASCE Transactions, Paper No. 3362, (vol 127, 1962, I). They are also reproduced in that paper.

Two 1/2-inch thick plexiglass plates resting on a sump are kept at a constant distance of 0.484 cm by means of aluminum strips at top and bottom and by spacers whose interference in the flow picture is negligible. The aqueous glycerol solution is circulated in a closed cycle from the sump by a gear pump. It is pumped to a storage tank suspended at the ceiling of the air conditioned room, and flows by gravity into the entrance reservoir attached to the plates. This flow is controlled by a fine regulating valve.

A movable overflow in the entrance reservoir is raised or lowered by a variable-speed motor capable of continuous speed control in a 2.3:1 range. With the fine regulating valve, the flow into the entrance reservoir can be easily adjusted so that the level in this reservoir rises at the same speed as the funnel of the overflow. The discharge through the plates is drained into a tank. This tank is supplied with a variable overflow which can be adjusted accordingly to the discharge through the plates, so that the boundary conditions of the underdrain can be exactly simulated at all times. From the drainage tank, the liquid flows back to the pump.

The photographs show free surfaces developed for high and low waterlevels. Computed free surfaces for different conditions of water level rise are drawn in pencil on the plates.

MR. WALTON: Now, we have come to the discussion period.

MR. HURLEY: There was one point I forgot to mention in my talk which relates to a point that was mentioned yesterday. What is the effect of the sensitivity of the aquifer characteristic that you put into the formulas on the Mesilla Valley problem that we ran. The general return flow to the river had a pattern as indicated in figure 20, increasing during the irrigation season and reaching a maximum in August or so.

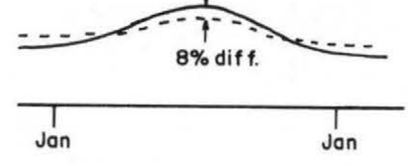


Figure 20

If we would have maybe a 25 percent error in the determination of the average permeability it would have made about an 8 percent difference at the high point as indicated in figure 20. A little lower than at the peak and a little higher for larger return flow before the irrigation season began again. So 25 percent reduction gave about an 8 percent reduction in the peak amount at the height of the irrigation season.

MR. KRAIJENHOFF: May I ask Dr. De Wiest a question in relation to his Hele-Shaw model? I think he is familiar with the work of Luthin and Day on the lateral flow above a sloping watertable in dune sand. They found that the transport above the phreatic level may contribute significantly, especially when the depth of horizontal flow below the water table is small. I remember in his last slide, Dr. DeWiest had one line very low down. I think in this case a Hele-Shaw model does not give you the possibility to judge accurately the total flow. There is another thing that I should like to add: Especially in a nonsteady situation the moisture distribution in the unsaturated zone will influence the movement of the watertable. The Hele-Shaw model has no unsaturated zone and this may also influence the results.

DR. DE WIEST: What you say, Mr. Kraijenhoff, is quite correct. If you recall the figure, we only studied the unsteady state flow with a very steep advancing front of the free surface, in other words, when we had a very high head behind the dam. The low head that I have shown was just the steady state picture. We computed the actual discharge there and compared it with the discharge from the mathematical formula for the same height to length ratio of the dam and found close agreement between the two discharges. But I know that for very low flow rates, if you make a small error, the relative error becomes very important and this could be indeed because of the fact that the unsaturated flow had been neglected. We did not study the contribution of flow in the unsaturated zone. A model can be representative of different prototypes at different scales and it is clear that problems in agricultural drainage apply to smaller prototypes than those in civil engineering, as for example seepage through dams. Flow in the unsaturated zone is more important for the former than for the latter prototypes.

MR. KRAIJENHOFF: May I use this opportunity to stress the necessity for cooperating with the soil physicists.

DR. MARMION: I have two questions -- one for Mr. Kraijenhoff and one for Mr. Hurley. The first on this ethanol-water mixture that you used in your model, can you tell me something about the height of the capillary fringe, in that particular model? My other question to Mr. Hurley is: What kind of computer was used and what was its storage capacity?

MR. KRAIJENHOFF: May I answer the first question. I must confess that my presentation was a little bit casual at that point but the diagram giving the conductivity as a function of soil moisture content was derived from a publication by Childs, and does not relate to the actual sand I used. The sand I used for the model had a mean grain diameter between 1 mm and 1-1/2 mm. The actual height of the capillary fringe in equilibrium was about 2-1/2 centimeters. That is one inch but since it is a fringe this is only an approximate measure. I may add to this answer that in cases where we try to use this model for comparing it with structured soil, we use this 2-1/2 centimeters, the equilibrium capillary rise if you like, to compare it with equilibrium capillary rise in a structured soil. That determines the length scale. And then we have a velocity scale, which is the ratio of the permeability in the field and the one found in the model by using Kirkham's ponded water case. Finally, the time scale is determined by a combination of the length scale and the velocity scale.

MR. HURLEY: The computer used for the return flow studies in the Mesilla Valley, following Mr. Glover's work, was the IBM 650 which has a maximum storage I think, of 2000. Earl Glenwright could tell us exactly what machine we were using for the South Platte studies.

MR. GLENWRIGHT: The IBM 7090.

MR. WALTON: Yes.

MR. AKIN: I would like to ask Mr. Hurley why the operation study on the South Platte River was done on the computer, rather than on an analog model, and what would be the advantage of this?

MR. HURLEY: The very mass of the data we thought would require a digital computer. The diversion records, the stream flow records, the pumping records, the farm requirements, the farm canal losses, just to handle all these data would require a computer even though the calculations are simply adding and subtracting. To handle this tremendous mass and to go through it again and again would require the digital computer. However, after we get a mathematical model and more-or-less convince ourselves what the average permeability within a whole reach of river may be, then we might think about making an analog of this entire area. We have to know more about the actual facts first, it is a very complex area.

MR. GLOVER: I have an additional answer to add to that. It would be possible to put that thing on an analog, but it would require input devices which are not available at the present time.

MR. AKIN: The U. S. Geological Survey is working at the present time to make a model of the Roswell artesian basin. We hope to be able to us all this model constantly, day by day and our hope is to get an analog model as the final result. However, I am not sure now but what we might start with the computer, using mathematical models and eventually work into a final analog model.

MR. HURLEY: Maybe Mr. Glover could enlighten me on this, but after we had a pretty good idea from using the digital computer we could use the analog computer to give us such answers as the rise and fall in the water table.

MR. GLOVER: I think you might find an advantage in doing it the other way around, use the analog first and apply the digital computer afterwards. The reason is that from the analog you can get an influence function, that you can put into a digital machine and add up. The alternative requires an input device, which is not available at the present time.

DR. BRUTSAERT: Having authorities on electrical analogs in the audience, such as Dr. Bouwer, maybe I should not make these comments; but in connection with the brief discussion between Mr. Kraijenhoff and Dr. De Wiest, I would like to add that at the University of California with Dr. Luthin and Dr. Taylor, we developed an electrical analog, to solve nonsteady state flow problems, such as a falling water table. Now, the method was essentially to reduce the nonsteady problem to a succession of steady states. The analog was only used to determine then the potential distribution for the solution of Laplace's equation of each steady state. The unsteady state aspect of the problem came into the picture through the boundary conditions. That is, in the calculation of the next step. The unsaturated aspect of the flow came also in through the boundary conditions. The agreement between the data of Luthin and Worstell which Mr. Kraijenhoff mentioned and our results of the analog study was perfect, but only when we considered the unsaturated flow. In other words, we tried both methods, and by introducing the unsaturated flow only could we get the agreement.

DR. DE WIEST: How did you take care of the free surface boundary condition with the electric analog? You can do it with the Hele-Shaw model because a free surface boundary is developed.

DR. BRUTSAERT: We had to make some approximations, of course. The main thing we did, though, was to consider that drainage water was contributed also by the unsaturated zone, i.e., the zone above the capillary fringe.

DR. DE WIEST: Did you assume Dupuit-Forchheimer simplifications?

DR. BRUTSAERT: No, this is why we went to the electrical resistance network, in order to determine a potential distribution.

DR. DE WIEST: In the case of the electric analog model, you solve a difference equation. In the Hele-Shaw model, we try to test solutions of a differential equation. There are advantages and shortcomings to both techniques.

MR. JACOB: I am not familiar with the work, but I think this problem would be to model with a fine mesh in a vertical plane, then handle your water table by successive approximation. You probably have a continuous succession of steady states, and then your storage is taken up by the difference between the steady states.

DR. DE WIEST: But that is not quite accurate enough.

MR. JACOB: The gentleman states that it fitted the data if they took some allowance for the flow in the unsaturated zone above it, which is mostly parallel to the water table, and which can be done by approximation. I have seen some work on this, but I am not up-to-date on it.

MR. NELSON: What is the boundary condition you are imposing, are you imposing the condition that the free surface is a surface at atmospheric pressure?

DR. DE WIEST: Yes.

MR. NELSON: I do not see the problem, if there are two conditions to meet at that surface, what is the problem? Assume a geometrical surface for the network; obtain the voltage along the surface and see whether that agrees with the second condition, which is that the potential is equal to the elevation.

This is rigorous to within the limits of the finite difference.

DR. DE WIEST: However, we want a continuous solution here, not the finite difference solution.

MR. NELSON: I have the impression that the discussion hinges on the fact of whether you do or do not want to go to a finite mesh.

DR. VAN SCHILFGAARDE: The question here seems to be whether or not you like a finite mesh, on the other side you would like a continuous distribution, but you are satisfied to accept the poor differential equation to get a continuous distribution. I think one would be better off to accept the discontinuity that comes from a good theoretical model that comes from a finite mesh, than we are to accept a poor continuous approximation.

DR. DE WIEST: I do not agree here. The differential equation is not a poor one, since you transform it into a difference equation. We constructed this model without distorting the scale because we did not want to make the Dupuit-Forchheimer simplification. We were interested in the unsteady displacement of the free surface front in the vicinity of the drain. I admit that we had to make an approximation in regard to the final free boundary condition when we were trying to find the perturbation potential, but now the error introduced was of a smaller order.

DR. VAN SCHILFGAARDE: The additional approximation you have made is that the free surface is a rigorous boundary between the flow region and the region where there is no flows which is a very serious approximation from the physical point of view.

DR. DE WIEST: This could be a serious approximation only if the contribution of flow rate due to the capillary fringe were a significant fraction of the flow rate in the saturated zone.

DR. VAN SCHILFGAARDE: I think it goes far beyond that. You are trying to use the model as a model of a prototype which is made of geologic material, soil material, whichever is your background. And in this case we do not have a simple capillary fringe, as such, we have a continuous distribution of pressure and moisture content and potential.

DR. DE WIEST: Unless we accept uniform conditions of capillarity along the free surface, the problem would not be treatable in the method that we used. We did not feel that capillarity would be important in the study of dams.

DR. BRUTSAERT: Perhaps with dams that are 30 or 40 feet high, this is all right, but in agricultural drainage where you have a drain at a depth of 3 feet, the unsaturated zone is often the main region of flow.

DR. DE WIEST: This is not a problem of drainage, this is a problem of stability of a dam. Neglecting the unsaturated zone is an error here, yes, I agree with that. But this seems to be general practice in civil engineering design of dams and I would like to hear the opinion of the delegates of the U.S.B.R.

MR. GLOVER: In our case we are seeking for engineering answers. In the case of an earth dam we would be concerned, I think primarily with stability. And when we come to the water table, there we have the zone of complete saturation and below it a zone of increased pressure. I think we would solve it simply without consideration of the capillary fringe.

DR. DE WIEST: That is what I did.

MR. KRAIJENHOFF: I do not think you can draw a line between drainage problems and stability problems. You are probably familiar with the work being done with the Deltadienst in the Netherlands on the pressure behind asphalt facings of sand-cored dikes caused by tidal movement on the outside. This is definitely a stability problem, but here, if you do not include the capillary zone or the unsaturated zone you get the wrong answer, or at best you get an approximate answer. Comparison of computed answers and measurements on the site have caused them to include the unsaturated zone in their studies.

MR. BLATCHLEY: I would like to ask Mr. Hurley when he simulated the precipitation versus the consumptive use do you take into account excess precipitation - that is, runoff by the surface, if so, how?

MR. HURLEY: The precipitation we take into account is what we call the effective precipitation, which is worked out on the idea that if you get one inch of rainfall, it is nearly all effective. If you get two inches of rainfall it is proportionately less effective. If you get up to say six inches of rainfall, you figure only 3-1/2 inches of the six are effective for meeting the evaporation. The effective precipitation is that which can go towards meeting the evaporation. We have not considered in our model this other precipitation we call the non-effective precipitation. Our first assumption is that this non-effective precipitation is going to be used by plants other than those on the irrigated area.

MR. BLATCHLEY: You account for some running off over land?

MR. HURLEY: Yes, this is what we call the non-effective precipitation runoff and we make the assumption that this will be taken up through the consumptive use by let us say cottonwood trees down along the river bottom and other areas that are not irrigated.

DR. VAN SCHILFGAARDE: My question is directed toward Mr. Nelson. In the program you described, how did you obtain the input information on conductivity you needed to arrive at your answer?

MR. NELSON: We do not have all the information at present, but we have a field program underway. We are using a combination of pumping tests, a packer device which is packed in wells and then measured. The latter method is essentially the same as Kirkham's piezometer method but with appropriate geometry factors. This problem of the boundary condition in permeability is probably the greatest area of difficulty in the proposed method. This is the reason I suggested that time matching may be an extremely important feature in the future. I am sorry today I cannot tell you whether we can by time matching analytically determine the relative conductance for the boundary condition; I think there are chances. The transient system needs to have more study. I think there is a possibility that time matching can enable us to reduce and essentially eliminate this problem. I apologize for the vagueness. That vagueness is based on lack of knowledge of the system. It is an area, I think, where more study needs to be done, because it is a tremendously promising area.

MR. JACOB: You say you can handle no extreme points, only saddle points. Does this have to do with the parameters or does this have to do with the variables in the problem?

MR. NELSON: Neither. The velocity that is the gradient of a potential function can have stagnation points or points of zero velocity, hence, the potential function can have only saddle points.

MR. JACOB: You cannot handle distributed sources or sinks, in other words, unless you make a new boundary out of it. Right?

MR. NELSON: Yes, you have to have a continuous region, and we do just what is done analytically. We use a branch out and then 3-dimensional analogs to remove singularities from the interior of a flow problem.

MR. JACOB: Is this planned to be done literally in three dimensions or in two dimensions?

MR. NELSON: It is literally going to be three dimensional.

MR. JACOB: How many nodal points can you handle?

MR. NELSON: Right now our read-out equipment is shooting at 13,600 nodes.

MR. JACOB: This is a little more than 23 cubed.

MR. NELSON: Right. There is a problem here and it goes back to the distortion of vertical and horizontal dimensions in practical flow systems. In practice you cannot take the cube root because none of our flow systems are that deep, they are not cubical.

MR. JACOB: How about handling discontinuities of permeability on the boundary? How are these going to be treated?

MR. NELSON: I maintain that permeability is a continuous functional of location.

MR. JACOB: I take it you are going to draw a plane through space, and this is going to be the so-called output boundary of the system? In other words, you are not going to have a bank of a stream where you have a seepage face or something like that, but you are going to draw an arbitrary boundary within the medium.

MR. NELSON: No, we are going to actually try to model a bank.

MR. JACOB: You are going to actually take samples out of the bank?

MR. NELSON: This is an interesting possibility and maybe a practical one. One problem occurs to me since often when flow comes out of the bank you would need to measure 10 miles; if you pick an optimum path back to your sources of flow, you may be able to measure along a smaller distance. That is a good point.

MR. WALTON: I wish Mr. Skibitzke were here to discuss a passive element electric analog computer. I have had a little bit of experience with it and on a practical basis, it is really something. The few models that we have in Illinois, we have checked with field conditions, and we are able now to go ahead in a very complex aquifer situation and try to tell people what the effects of future development will be. To me, this passive element electric analog computer, is the first thing I have ever had in my possession to insert geology on a practical basis. I am not an electrical engineer, but I can understand this gimmick that Herb Skibitzke has introduced. It is a wonderful educational tool because you are manipulating electrical components to simulate geology, you feel it, you see it. We have made checks on its accuracy with cheap components, and rather inexpensive excitation response apparatus, and you really get your money's worth. It is not as difficult to understand as I think many people are led to believe. I spent about a week with Herb Skibitzke and came back, later he sent up Robbie to construct a analog model and we went through the mathematics that Bermes has in an unpublished report, and believe me, it is understandable, you can get your teeth into it. I do want to suggest that you look into the literature and that you consider passive element electric analog models.

MR. AKIN: I was wondering if you have any particular trouble with these analog models.

MR. WALTON: We have no trouble whatsoever with our excitation response apparatus. We use a DuMont Type 304A oscilloscope -- it only has about a dozen knobs and I can play around with about three of them and actually understand the oscilloscope trace. We have Tektronix wave form and pulse generators -- they have never konked-out except on the first probe when all went off, so we were quite wise, when one thing stopped, we took all of our apparatus over to our Electrical Engineering Department, and they replaced every tube and I think every component. I might qualify these statements by saying that we got all of our apparatus on surplus, the State of Illinois has had to date but a few hundred dollars to invest in the computer. Talking about an investment, and we are in pretty deeply, in several large models involving as many as 10,000 resistors and capacitors, we have invested less than $1500 total. As I stated earlier, with this equipment, all the lights went from green to red, then blue and faded off. We replaced all the tubes and have had no trouble whatsoever since then, and we run the equipment day in and day out because we only have a one-step function generator, we do not have an arbitrary function generator.

MR. JACOB: I might mention that it is generally known that function generators can be built mechanically, that is, electromechanical generators with sufficient accuracy for most hydrological work. I am talking about field hydrological studies. I am not talking about trying to model something you are trying to explain analytically, but modeling something with the precision and the accuracy of data we have from hydrology. There are very simple ways of generating functions, so this should not limit anybody working with passive-network analysis, that they have no function generators. There are all kinds available on surplus, all kinds of potentiometers that can be driven by cams or by templates, etc. This is very easy to do, with sufficient accuracy.

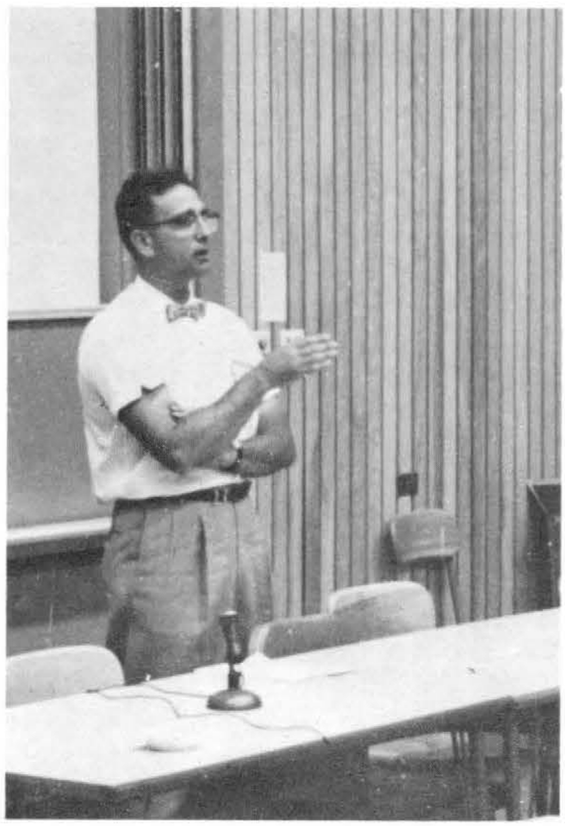
MR. GLOVER: Do you think that you could take 31 years by months with the canal demands, the river flows and the ground water pumping and put it all on one of those things?

MR. JACOB: Yes, you could do it on a long template. It could be done, but you would lose precision. It can be done accurately on a potentiometer. This is a simple way of doing it. The cost of good high-quality function generators in the analog business is extremely high. This is one of the biggest drawbacks in going to analog computers -- the cost of generating the function. It is much easier done digitally.

MR. WALTON: I might mention one of my experiences. I went to Penn State as a guest lecturer and met a physicist - electrical engineer there. I did not have all of my equipment with me, I was missing a particular pulse generator and I asked if he could perhaps look around the University and come up with the required component and put a computer together for me. He said he would rather perhaps make his own equipment, so within an hour and a half I went down to his laboratory and he had a little metal black box with batteries and transistors which functioned as a pulse generator. He was apologetic for the fact that he had spent $25.00 on a combination power source, wave form generator and pulse generator. This experience leads me to believe that perhaps we could in the future develop equipment specifically for our ground water field. Surely there must be overlap in all of the components that we buy from Tektronix and other companies, and if he could put together excitation response equipment in a small black box by using transistors and batteries, we should be able to come up with some very compact equipment in the future.

I think our panel has done a marvelous job on a very technical subject, and they deserve a round of applause.

Thank you for being so attentive.



Dr. H. K. van Poollen

Photograph by George Palos)

TRANSIENT TESTING IN THE OIL INDUSTRY

By H. K. Van Poollen

For a number of years scientists in both the water industry and the petroleum industry have studied fluid flow. In many instances these studies are made independent of one another and frequently the two disciplines are unaware of mutual developments. Although the problems are similar, certain differences exist. For one, the petroleum industry deals with depths ranging from as little as 500 feet to over 20,000 feet. Very high pressures are encountered. Sometimes as many as seven production strings occur in one hole. These multiple completions call for difficult instrumentation. The purpose of my talk is essentially to point out some correlation between the two disciplines and to point out to you some of the difficulties and developments we have had over the last few years.

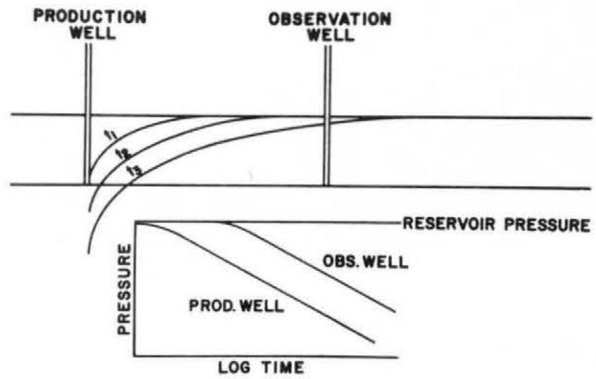


Figure 1

In general, all flow problems in the petroleum industry are of artesian nature. We do not have problems you have been discussing, where the water table is falling and consequently the transmissibility reduces with time. We do have the same problem of saturated or unsaturated media where differences in relative permeability effects are important.

When studying single phase oil or water reservoirs we can use simple heat flow equations. In gas reservoirs we have a different problem. The compressibility of the system is not independent of pressure. As a matter of fact the compressibility of the gases is, roughly speaking, inversely proportional to the pressure. A somewhat modified continuity equation is used which has its limitations. We have to work with semi-empirical relationships rather than strictly mathematical ones when gas reservoirs are involved.

There are reservoirs which contain both oil and free gas, which may show saturation differences with time and pressure. For example, if a reservoir contains undersaturated oil at reservoir pressures, gas may come out of solution near the well bore due to the pressure reduction while producing the well; consequently, close to the well bore we may have different relative permeabilities than further in the reservoir and this ring of free gas may grow with time or may grow with increased rate of production. Nevertheless, the usual assumption is that oil and gas saturations are constant throughout the reservoir and this assumption seemingly renders reasonable answers. Even in this situation simple heat conductivity equations are used.

Figure 1 shows a production well and an observation well in a reservoir. The production well is being pumped at a constant rate and pressure profiles occur surrounding the production well as indicated for times t_1, t_2, and t_3. The pressure behavior at the producing well and the observation well are given in this figure. We are assuming presently that we are dealing with an infinite reservoir. In the observation well the relationship would be that of the exponential integral which ground water hydrologists refer to as the Theis equation. For longer times in the observation well and at the producing well the relationship becomes essentially logarithmic. This logarithmic behavior will occur within a matter of a few minutes at the producing well. Already I mentioned to you that we frequently deal with very deep wells therefore we will not have observation wells to deal with. Consequently, we have to work entirely with transient presure observations at the production well.

With the information available at the producing wells only, we can readily determine permeability which is unrelated to well bore size. From information available at the producing well only we cannot determine the ϕc product in which ϕ is the porosity and c the compressibility of the reservoir, because we do not know the exact value for the well bore radius. We can measure it with a caliper but we do not know the effective well bore radius so we are rather limited in what we can learn under most circumstances as compared to what you might be able to learn from your observation wells. However, if one is dealing with a finite reservoir there is a way around this. Then, the method advanced by the late Park Jones is used. He was a follower of your work and has put it to practice in the petroleum industry.

In addition to the lack of observation wells we have the problem of discontinuities within the reservoir. Undoubtedly you also have many discontinuities in the reservoir itself. In the petroleum industry we have additional discontinuities of oil-water tables and gas-oil contact. One type of discontinuity is of particular interest. Figure 2 shows normal drawdown behavior and also the drawdown behavior observed on a few wells. This problem is definitely not common

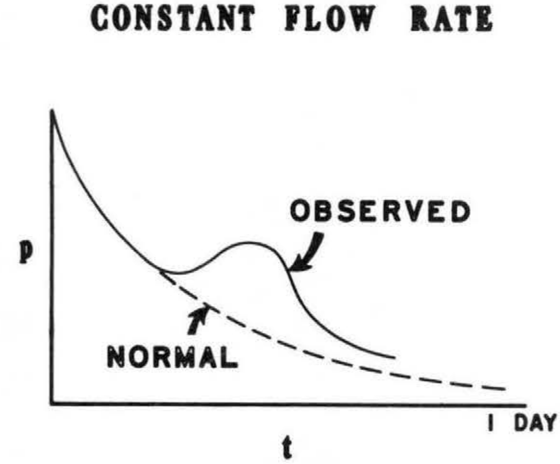


Figure 2

but very interesting. We are sure of a constant flow rate, nevertheless, a sudden pressure rise may be observed. This behavior occurs mainly on gas wells and was observed in the Gulf Coast area. This curve should mean that sudden injection of fluids or injection of energy occurs in the reservoir.

To explain the anomalous drawdown behavior as shown in figure 2 we have an explanation which includes a capillary pressure phenomenon. Figure 3 shows a well connected to a main reservoir surrounded by a water bearing strata which contains lenses with hydrocarbon. When producing from this reservoir we will get a pressure reduction in the major reservoir. It is assumed that the reservoir will contain

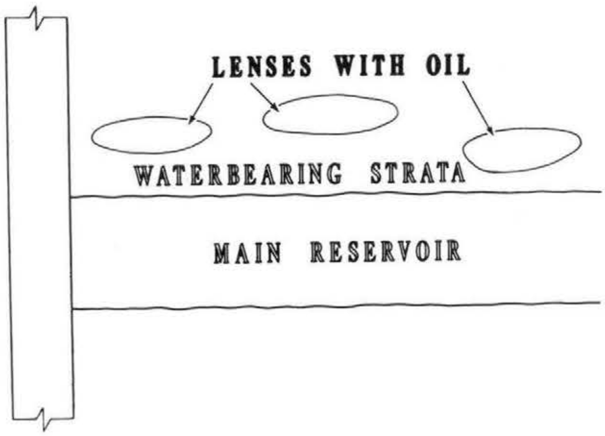


Figure 3

water as the wetting phase. The surrounding material is of lower permeability and is also wetted by water. Following the pressure drop in the main reservoir, a pressure gradient is established between the lenses and the main reservoir. As soon as the pressure gradient between the lenses and the water bearing strata becomes such that the entry pressure is overcome, the gas from the lenses may enter the main reservoir. Now communication exists between the lenses and the main reservoir and we suddenly have additional energy due to expansion available or we might consider this as a sudden influx of fluid into the reservoir, which will result in the increase of pressure on the drawdown curve. We have carried this explanation further than merely a postulation. Various laboratory tests have been performed and a publication on this matter is forthcoming.

Next I would like to show you some reasons why petroleum engineers perform transient tests. First of all, there is the determination of well bore efficiency. This type of test is rather simple and of short duration. We merely like to know how efficient the well bore could have been under ideal situations. We may have an area of reduced permeabilities surrounding the well bore due to drilling mud or scales or sometimes the perforations are not sufficient or at erroneous intervals. Another possibility is that the well has not fully penetrated the reservoir. For these calculations we merely calculate the permeability from drawdown or buildup curves and next calculate the permeability from deliverability observations assuming a certain radius of drainage. The ratio of the two permeabilities calculated is indicative of the well bore damage or the well bore efficiency.

Figure 4 shows a different application of transient testing. Various States set different rules for well spacing. These rules frequently consider reservoir continuity considerations. For example, the initial

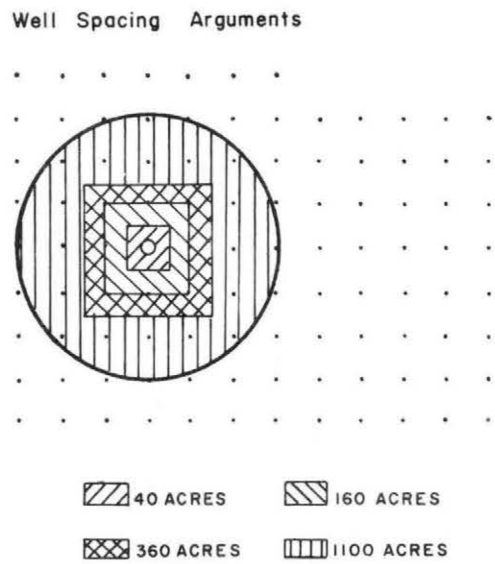


Figure 4

ruling calls for 40 acre spacing as indicated by the dots in figure 4. However, if one can prove that good reservoir continuity exists up to the large circle, one may have a better possibility to go to a larger spacing which means fewer wells and consequently may mean the difference between a profitable or a marginal operation.

Transient testing is also used to calculate distances to possible barriers which might be faults or pinchouts or they might be oil-water contacts, gas-oil-contacts or the like. We are fully aware that we will only get an approximate number but this is something exploration people are used to.

One instance may be cited where a company drilled a number of gas wells in an area assuming that original discoveries were connected. A large drilling program was started, only to find that the formation was barren between the wells. A similar instance arose, transient testing was performed on wells and these tests indicated to a certain accuracy that at least a square mile of oil was connected to each well. Consequently, one can say that instead of getting point information, we are getting "square mile" information and one is somewhat more certain that no discontinuities existed between these wells. Consequently, transient testing was of help in such a probability game.

Figure 5 shows a problem of mutual interest to the oil industry and the water industry. The upper left sketch shows a formation containing vugs which are coated with a low permeability liner. The upper

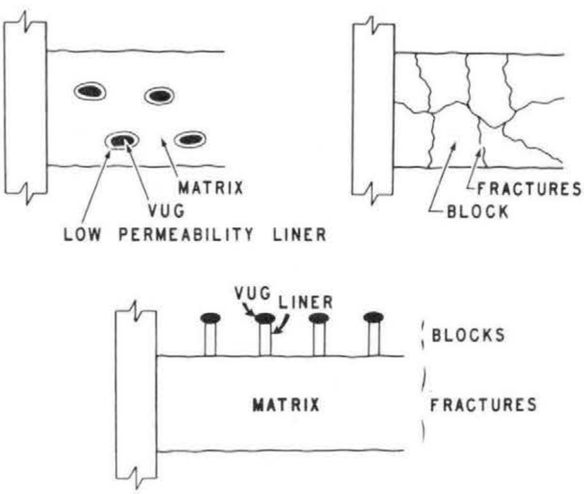


Figure 5

right shows a formation of relatively low permeability intersected by fractures of higher permeability. Mathematically, one can show that these two problems are essentially identical. They may be represented by the lowermost sketch where vugs are connected to a matrix. No flow will occur through the vugs, however, only flow will be out of the vugs during production of the well. The liner is represented by small connections from the vug to the matrix. To make this analogous to a fracture system, one would look at the vug and liner combination as being the matrix blocks and the large matrix as being the fracture system. To evaluate this type of a problem, one can use the usual continuity equation for the main flow channels and add additional source functions to be representative of either the matrix blocks in a fractured formation or the vugs in a regular formation. Figure 6 shows the result of this type of an analysis.

CONSTANT FLOW RATE

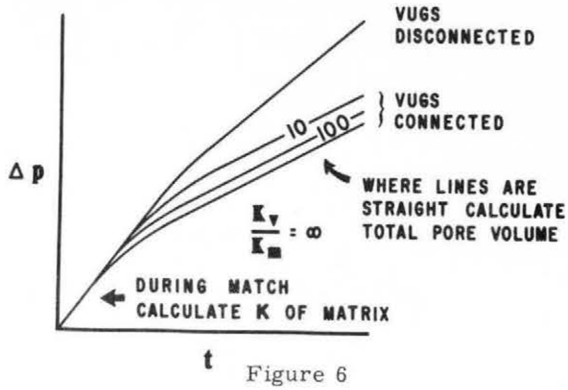


Figure 6

Figure 7 shows the equations which are applicable in an infinite reservoir (I) and in a finite reservoir (III). In between a transition (II) is shown. Park Jones differentiated these equations which in the infinite reservoir (during the ln approximation) would give:

$$Y = \frac{dp}{dt}\Big/ qB = \frac{70.6\,\mu}{t\,k\,h}$$

Rearrangement of this equation renders:

$$\frac{kh}{\mu} = \frac{70.6}{t\,Y}$$

in which

Y is in psi/bbl

$\frac{dp}{dt}$ is in psi/day

q is in bbl/day

k is in md

h is in ft

μ is in cp

t is in days

Examination of the above equations indicates that a plot of log Y versus log t should result in a straight line with a slope of 45^0. Such a plot has been prepared in Figure 8.

To obtain data to make plots as indicated in figure 8, one first plots observed pressure versus time on rectangular coordinate paper. Measuring slopes of lines tangent to this plot renders values for $\frac{dp}{dt}$. These data are divided by the rate in reservoir barrels and plotted as log Y versus log t .

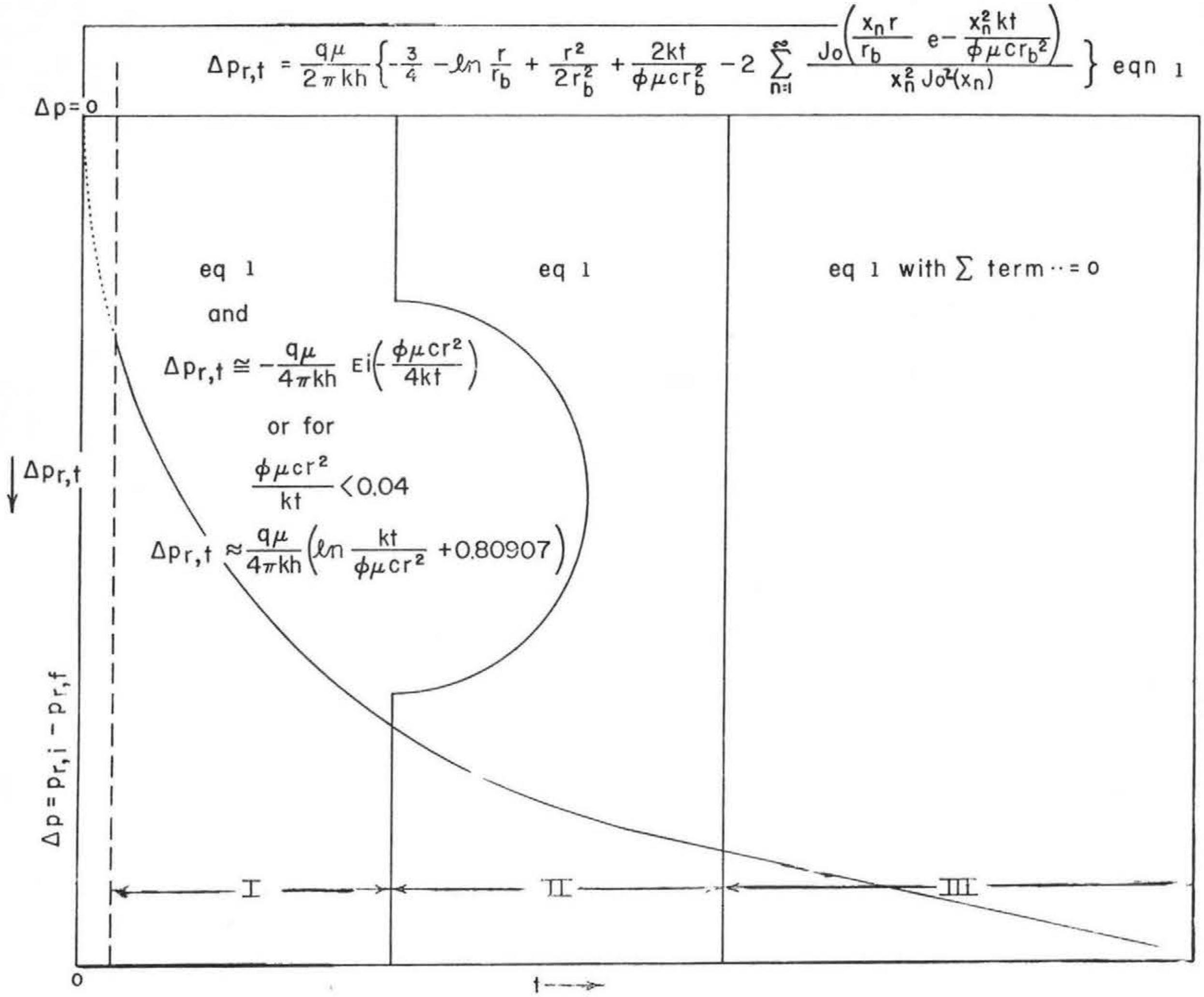


Figure 7

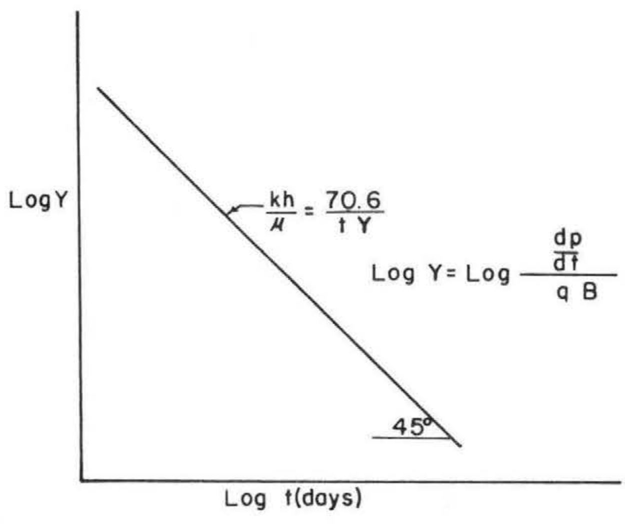


Figure 8

Differentiation of the finite equation in Figure 7 (with the series sum equal to zero) and using practical field units, renders

$$Y = \frac{dp}{dt}\bigg/ qB = \frac{1}{cN}$$

in which

$\frac{dp}{dt}$ is in psi/day

q is in bbl/day

B is formation volume factor

c is in psi^{-1}, compressibility

N is the connected pore volume, bbls

Consequently, one should be able to calculate connected reserves by the use of this method. Had the pressure been maintained at the outer radius, the change in pressure would also have been constant, but in addition, it would have been zero. When no change in pressure occurs, the differential of pressure with respect to time would go to zero as indicated in figure 9.

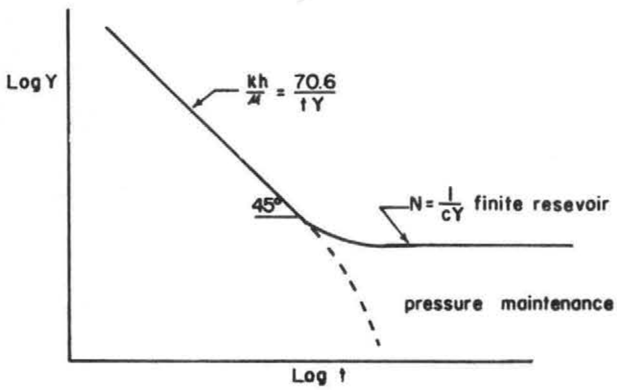


Figure 9

The drawdown will be reduced if larger volumes of vugs become connected. From this study it has been learned that during the early transient period one may be able to calculate the transmissibility of the main flow system. The total pore volume may be determined from the late production history. This work was originally started by Irving Fatt of the California Company and his students when he later went to teach at Berkeley. Additional work was shown by Pan American Oil Company.

Next I would like to discuss with you the Jones method of reservoir testing. Already it was mentioned that an infinite reservoir would exhibit an essentially logarithmic function when pressures are observed at the well bore. Jones took the time derivative of this function.

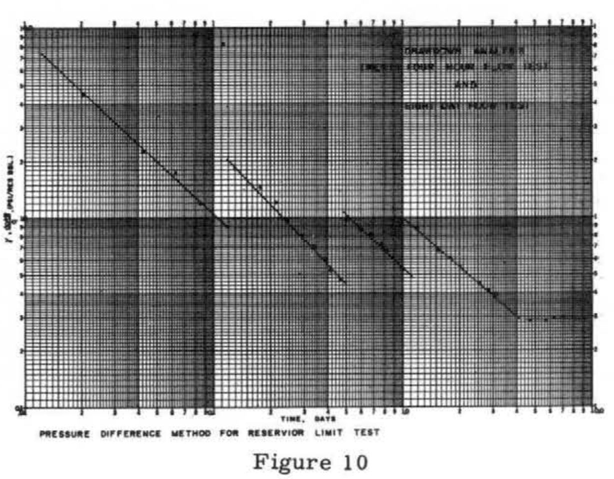


Figure 10

Figure 10 shows an example of a well located near intersecting faults. This well had been drilled for approximately $490,000.00 at a depth of 15,000 feet and two more wells had been planned. During the testing period it was found that even this well could not pay for itself. Consequently, additional drilling was stopped and approximately $1,000,000 was saved due to the performance of this test. This test cost less than $5,000 which included all expenses, such as evaluation, service companies, etc.

Figure 11 shows one interpretation of this particular test. The distance to the first fault was estimated to be 142 feet, to the second 282 feet, and a third at approximately 424 feet. By planimetering this area or by calculating the reserve due to the constant Y-value, or by using a material balance calculation following depletion of this well similar answers were obtained.

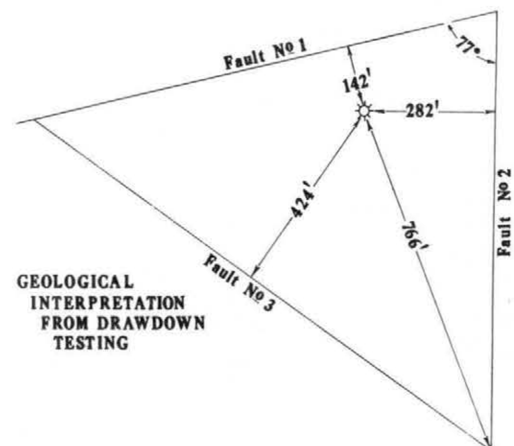


Figure 11

In many instances wells are placed on production to determine their productivity index. This index is the ratio of the amount of production divided by the drawdown. If wells were allowed to stabilize or if these measurements were performed at constant time increments since initiation of production one should always obtain a straight line relationship when plotting pressure drawdown versus oil production. However, if rates are randomly changed, and one consequently randomly obtains productivity index values one may obtain erratic numbers such as given in figure 12. The conclusion from this erratic test indicates that at zero oil production one would have a pressure drawdown. Or, the test would indicate that the higher the rate of production the better the productivity of the well. Consequently, it is recommended that prior to testing a well at a different rate the reservoir should be allowed to return to equilibrium.

Whereas, the deliverability of an oil well can be expressed by the ratio of rate divided by drawdown, the deliverability of a gas well may be expressed by the equation $Q = C(p_1^2 - p_2^2)^n$. Q is the flow rate, C is the deliverability constant, p_1 is the reservoir pressure, p_2 the well bore pressure and n an exponent. To get this type of information, one produces a well at a known constant rate of flow for a predetermined length of time, and observes the pressure. Then the well is again shut in and produced at a different rate with pressure observation, etc. In this way one may obtain the data necessary to construct figure 13. Figure 13 will then give us the

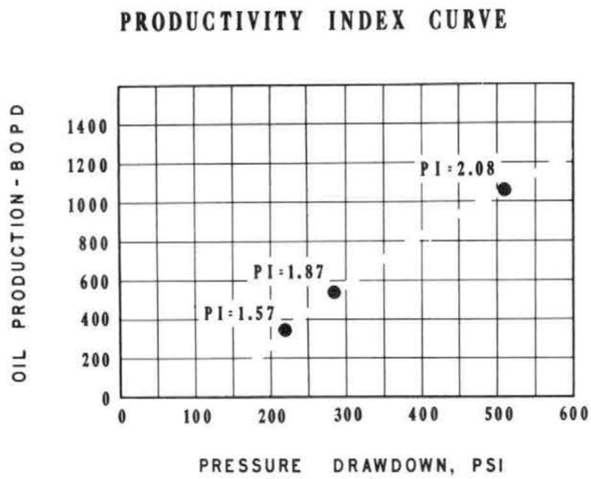


Figure 12

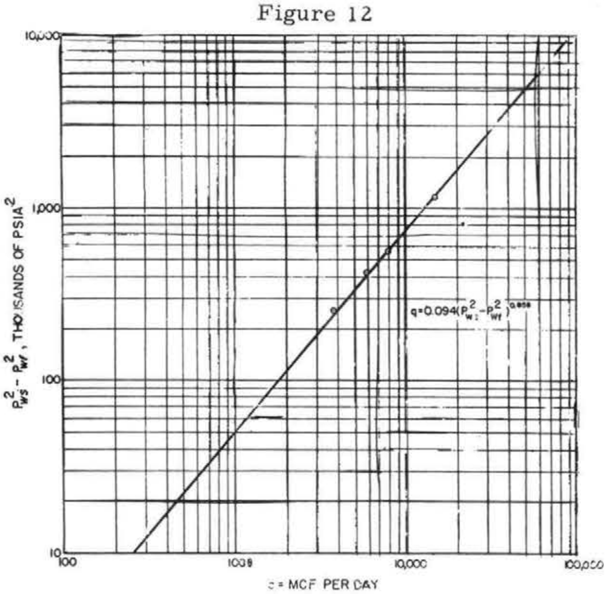


Figure 13

deliverability of a well at any pressure which is controlled by pipeline requirement. The exponent n is the result of one type of nondarcian flow. Explanations have met with much controversy. Apparently, a mixture of flow types occurs. There is turbulent and laminar flow within each little segment of the reservoir. If that is ture, n ought to be a function of rate and n has not necessarily been proven to be a function of rate. The relationship shown is empirical at best.

The following equations show how the performance coefficient C is related to time

$$\frac{C_1}{C_2} = \left[\frac{\ln \frac{\alpha}{a}\, t_2^{1/2}}{\ln \frac{\alpha}{a}\, t_1^{1/2}}\right]^n \quad \text{and} \quad \frac{\alpha}{a} = \frac{t_2^{\left[\frac{C_2^{1/n}}{2(C_1^{1/n} - C_2^{1/n})}\right]}}{t_1^{\left[\frac{C_1^{1/n}}{2(C_1^{1/n} - C_2^{1/n})}\right]}}$$

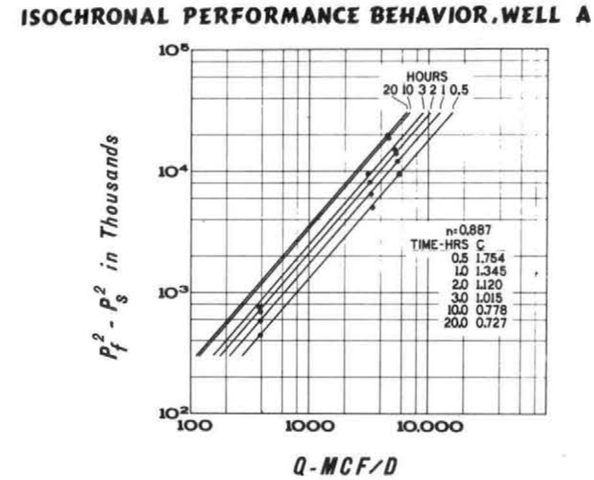


Figure 14

This relationship is a curve fitting technique which is again empirical.

If the performance coefficient is a function of time, different back pressure curves should be necessary at different times. This is shown in Figure 14.

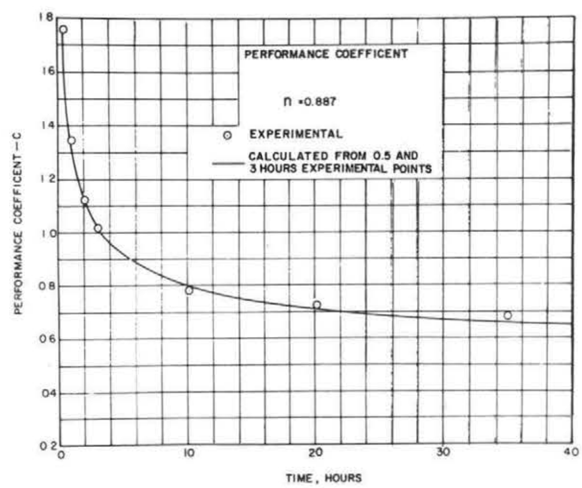


Figure 15

Figure 14 shows back pressure curves for times of 1/2, 1, 2, 3, 10, and 20 hours. To obtain these data, one should produce a well at a constant rate q_1 and pressures should be observed. Then the well should be shut in for a considerable time such that the pressure will return to the original reservoir pressure. Next a new rate shold be established and the pressures observed. Thereafter, the pressure measurements and rate measurements at 1 hour following initiation of the test should be measured and these data are used to calculate the 1 hour back pressure isochronal curve. This is repeated for different times. Thereafter, the relationship for c given above should be used to extrapolate the performance coefficients for stabilization time. The stabilization time is that moment where pressure change with respect to time becomes constant. Figure 15 shows the performance coefficient with respect to time. Figure 16 shows the pressure drawdown and buildup behavior from which the data were obtained.

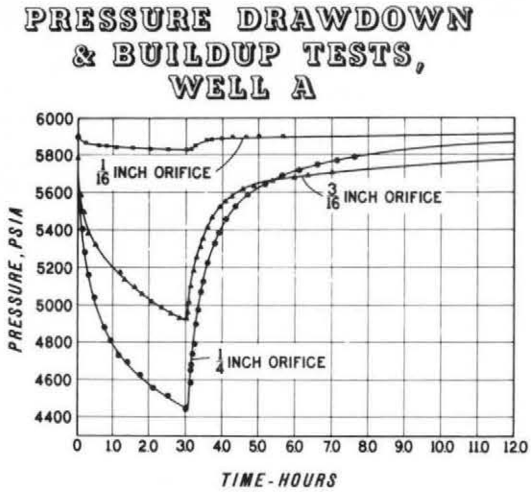


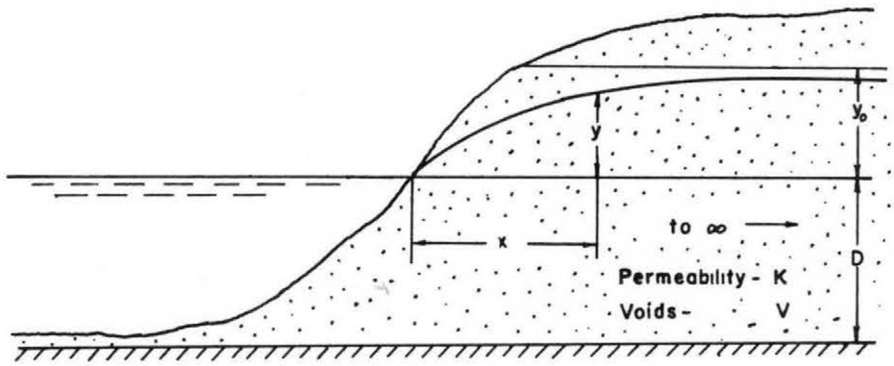
Figure 16

APPENDIX A

SUMMARIES OF SOLVED CASES IN RECTANGULAR COORDINATES

TABLE OF CONTENTS

A - 1. BANK STORAGE



REFERENCE:

Glover, R. E., 1953, Methods of computation of quantity and monthly distribution of return-flow - Kanopolis Unit - Missouri River Basin Project, Memorandum to Regional Director, in U. S. Bureau of Reclamation Technical Memorandum 657, 1960, Section N, p. 136-46.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$h^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$$

BOUNDARY CONDITIONS:

$y = y_o$ for $x > 0$ when $t = 0$.

$y = 0$ for $x = 0$ when $t > 0$.

SOLUTION:

$$\frac{y}{y_o} = \frac{2}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4 h^2 t}}} e^{-\beta^2} \, d\beta \qquad (y \ll D)$$

SYMBOLS: (Consistent units).

$$h^2 = \frac{KD}{V}$$

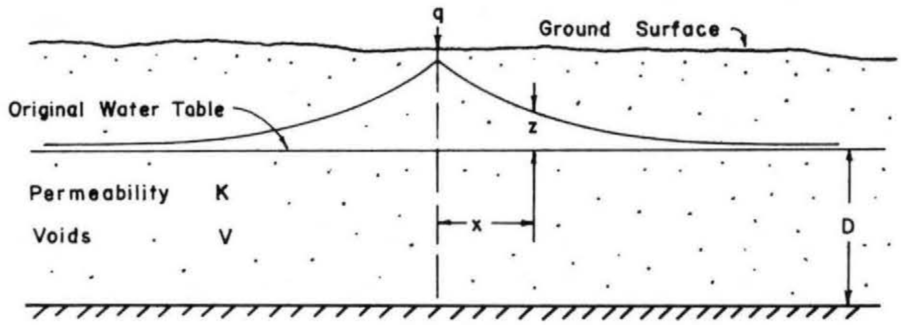
t = time

COMMENTS:

Typical use: Bank storage.

Tabulations: Widely distributed as "Probability Integral".

A - 2. LINE SOURCE



REFERENCES:

Glover, R. E., 1952, Methods of estimating the depletion of flood flows in Ladder Creek resulting from well pumping, in U. S. Bureau of Reclamation Technical Memorandum 657, 1960, Section G, pp 73-80; and Moody, W. T., 1952, Drawdown in a one-dimensionally infinite aquifer, in U. S. Bureau of Reclamation Technical Memorandum 657, 1960, Section P, p. 147-152.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$h^2 \frac{\partial^2 Z}{\partial x^2} = \frac{\partial Z}{\partial t}$$

BOUNDARY CONDITIONS:

$Z = 0$ when $t = 0$ for $x > 0$.

$-KD \frac{\partial Z}{\partial x} = \frac{q_1}{2}$ when $x = 0$ for $t > 0$.

SOLUTION:

$$Z = \frac{q_1 x}{2\, KD} \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4h^2 t}}}^{\infty} \frac{e^{-u^2}}{u^2}\, du$$

SYMBOLS: (Consistent units).

$h^2 = \frac{KD}{V}$

t = time.

COMMENTS:

Typical use: canal seepage.

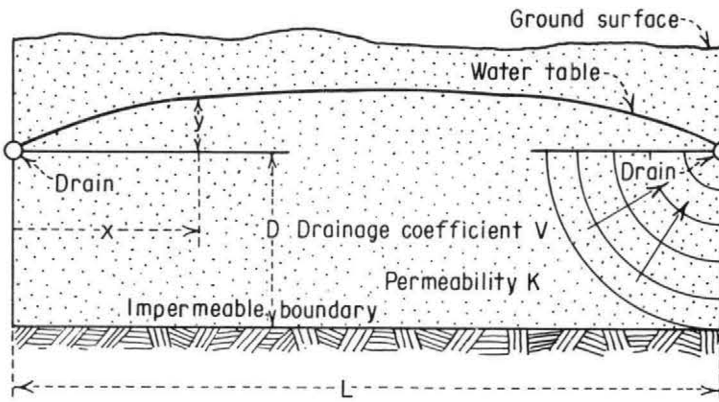
Integral has been tabulated by M. W. Bittinger at Colorado State University, Fort Collins.

Line Source Integral: $$I_x = \sqrt{\pi} \int_{\frac{x}{\sqrt{4\,h^2\,t}}}^{\infty} \frac{e^{-u^2}}{u^2}\,du$$

$x/\sqrt{4\,h^2\,t}$	I_x	$x/\sqrt{4\,h^2\,t}$	I_x	$x/\sqrt{4\,h^2\,t}$	I_x
0.0000	∞	0.34	2.6628	0.78	0.38848
0.0005	3541.8	0.35	2.5306	0.79	0.37294
0.001	1769.3	0.36	2.4065	0.80	0.35804
0.002	883.07	0.37	2.2901		
0.003	587.68	0.38	2.1805	0.81	0.34373
0.004	439.98	0.39	2.0774	0.82	0.33000
0.005	351.36	0.40	1.9802	0.83	0.31681
0.006	292.28			0.84	0.30415
0.007	250.08	0.41	1.8885	0.85	0.29199
0.008	218.43	0.42	1.8018	0.86	0.28032
0.009	193.81	0.43	1.7199	0.87	0.26911
0.01	174.12	0.44	1.6424	0.88	0.25834
		0.45	1.5689	0.89	0.24800
0.02	85.516	0.46	1.4993	0.90	0.23807
0.03	55.993	0.47	1.4333		
0.04	41.241	0.48	1.3706	0.91	0.22853
0.05	32.396	0.49	1.3110	0.92	0.21936
0.06	26.506	0.50	1.2544	0.93	0.21056
0.07	22.303			0.94	0.20210
0.08	19.156	0.51	1.2005	0.95	0.19397
0.09	16.712	0.52	1.1493	0.96	0.18616
0.10	14.760	0.53	1.1004	0.97	0.17866
		0.54	1.0539	0.98	0.17146
0.11	13.166	0.55	1.0096	0.99	0.16453
0.12	11.841	0.56	0.96728	1.00	0.15788
0.13	10.722	0.57	0.92692		
0.14	9.7661	0.58	0.88840	1.1	0.10414
0.15	8.9397	0.59	0.85162	1.2	0.06820
0.16	8.2186	0.60	0.81647	1.3	0.04426
0.17	7.5845			1.4	0.02843
0.18	7.0227	0.61	0.78289	1.5	0.01806
0.19	6.5219	0.62	0.75078	1.6	0.01133
0.20	6.0728	0.63	0.72008	1.7	0.00702
		0.64	0.69070	1.8	0.00429
0.21	5.6682	0.65	0.66260	1.9	0.00259
0.22	5.3018	0.66	0.63570	2.0	0.00154
0.23	4.9688	0.67	0.60994		
0.24	4.6650	0.68	0.58527	2.1	0.00090
0.25	4.3868	0.69	0.56164	2.2	0.00052
0.26	4.1313	0.70	0.53900	2.3	0.00029
0.27	3.8959			2.4	0.00016
0.28	3.6785	0.71	0.51730	2.5	0.00009
0.29	3.4772	0.72	0.49651	2.6	0.00005
0.30	3.2905	0.73	0.47657	2.7	0.00003
		0.74	0.45745	2.8	0.00001
0.31	3.1168	0.75	0.43912	2.9	0.00001
0.32	2.9550	0.76	0.42153	3.0	0.00000
0.33	2.8040	0.77	0.40466		

Computed from National Bureau of Standards, Tables of Probability Functions, Vol.I, MT8, U.S. Government Printing Office, 1941.

A - 3. PARALLEL DRAINS ABOVE IMPERMEABLE BOUNDARY



REFERENCE:

Glover, R. E., 1953, Formulas for movement of ground water, Oahe Unit Missouri River Basin Project, in Bureau of Reclamation Memorandum No. 657, Section D, p. 35-46.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$h^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}$$

BOUNDARY CONDITIONS:

$y = y_o$ when $t = 0$ for $0 < x < L$

$y = 0$ when $x = 0$ and $x = L$ for $t > 0$

SOLUTION:

$$y = y_o \frac{4}{\pi} \sum_{n=1,3,5,\text{ etc.}}^{n=\infty} \frac{e^{-\frac{h^2 n^2 \pi^2 t}{L^2}}}{n} \cdot \sin \frac{n \pi x}{L} \qquad (y_o << D)$$

SYMBOLS: (Consistent units).

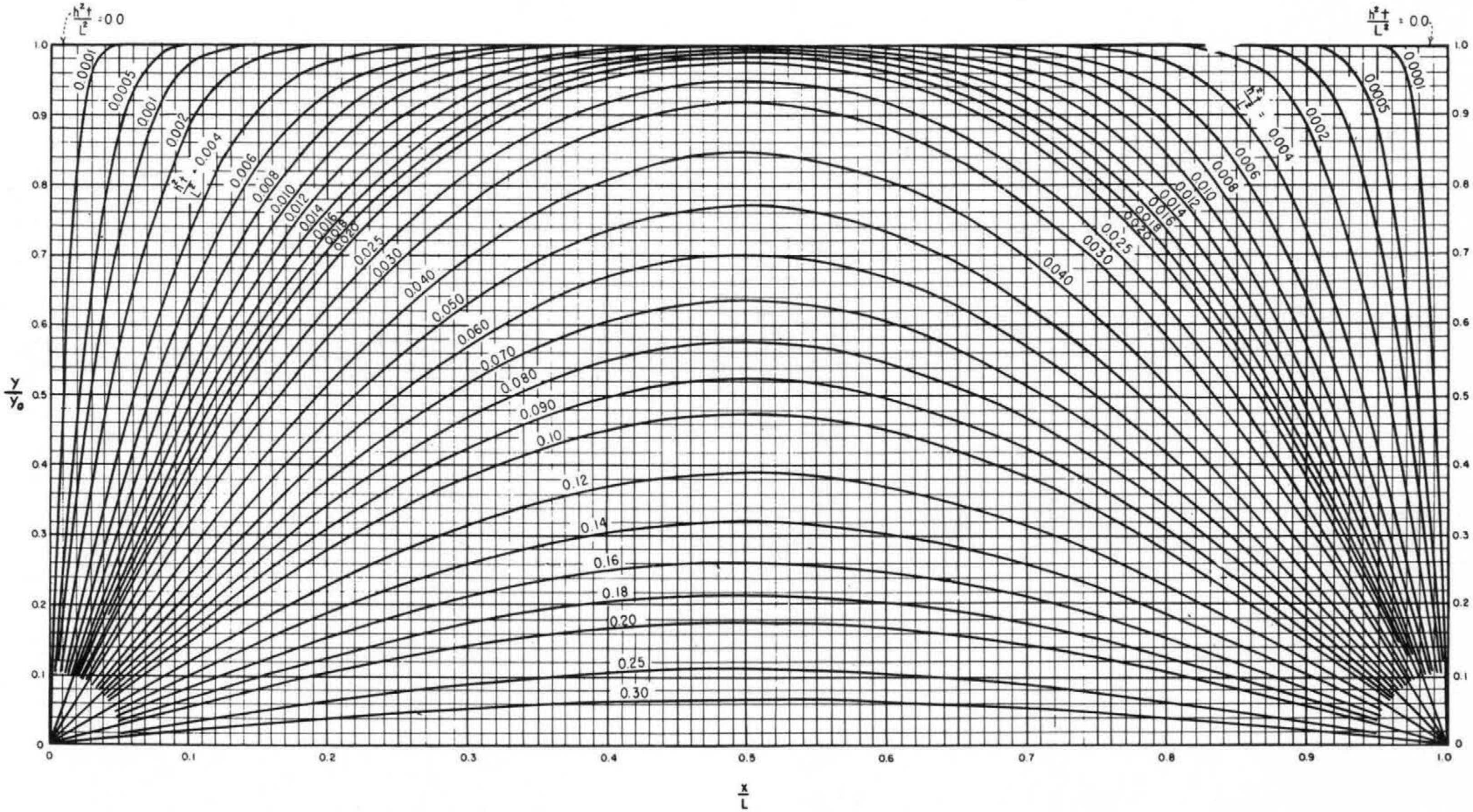
$h^2 = \frac{KD}{V}$

t = time.

COMMENTS:

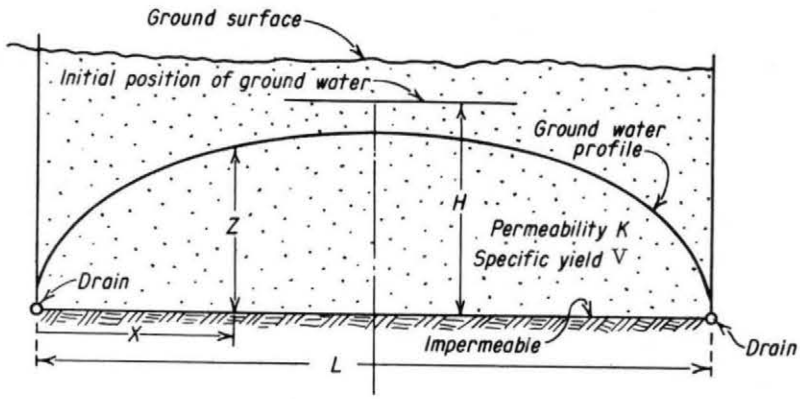
Typical use: Tile drainage.

Graph - see next sheet.



Note: This figure was originally published as Figure 4 of "Cooling of Concrete Dams," Bulletin 3, Part VII, Boulder Canyon Project Final Reports.

A - 4. DRAINS ON THE IMPERMEABLE BOUNDARY



REFERENCE:

Glover, R. E., 1954, Well pumping and drainage formulas, in Bureau of Reclamation Technical Memorandum No. 657, Section A, p. 2-27, Case 8 (Nonlinear) p 23.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation.

DIFFERENTIAL EQUATION:

$$\frac{\partial}{\partial x}\left(KZ\,\frac{\partial Z}{\partial x}\right) = V\,\frac{\partial Z}{\partial t} \qquad \text{or} \qquad \frac{\partial}{\partial \xi}\left(u\,\frac{\partial u}{\partial \xi}\right) = \frac{\partial u}{\partial \eta}$$

BOUNDARY CONDITIONS:

$Z = 0$ for $x = 0$ and $x = L$, for $t > 0$.

$Z = H$ at $x = \frac{L}{2}$ when $t = 0$.

SOLUTION: (approximate. See Boussinesq, 1904).

$$U = WY \quad \text{where} \quad \int_0^W \frac{W\,dW}{\sqrt{1 - W^3}} = \sqrt{3}\,\xi \quad \text{and} \quad Y = \frac{1}{\frac{9}{2}\left(\frac{\alpha t}{L^2}\right) + 1}$$

SYMBOLS: (Consistent units).

$\alpha = \frac{KH}{V}$; $U = \frac{Z}{H}$, $\xi = \frac{x}{L}$, $\eta = \frac{KH}{VL^2}\,t$

L = drain spacing

x = distance measured from one drain towards the other

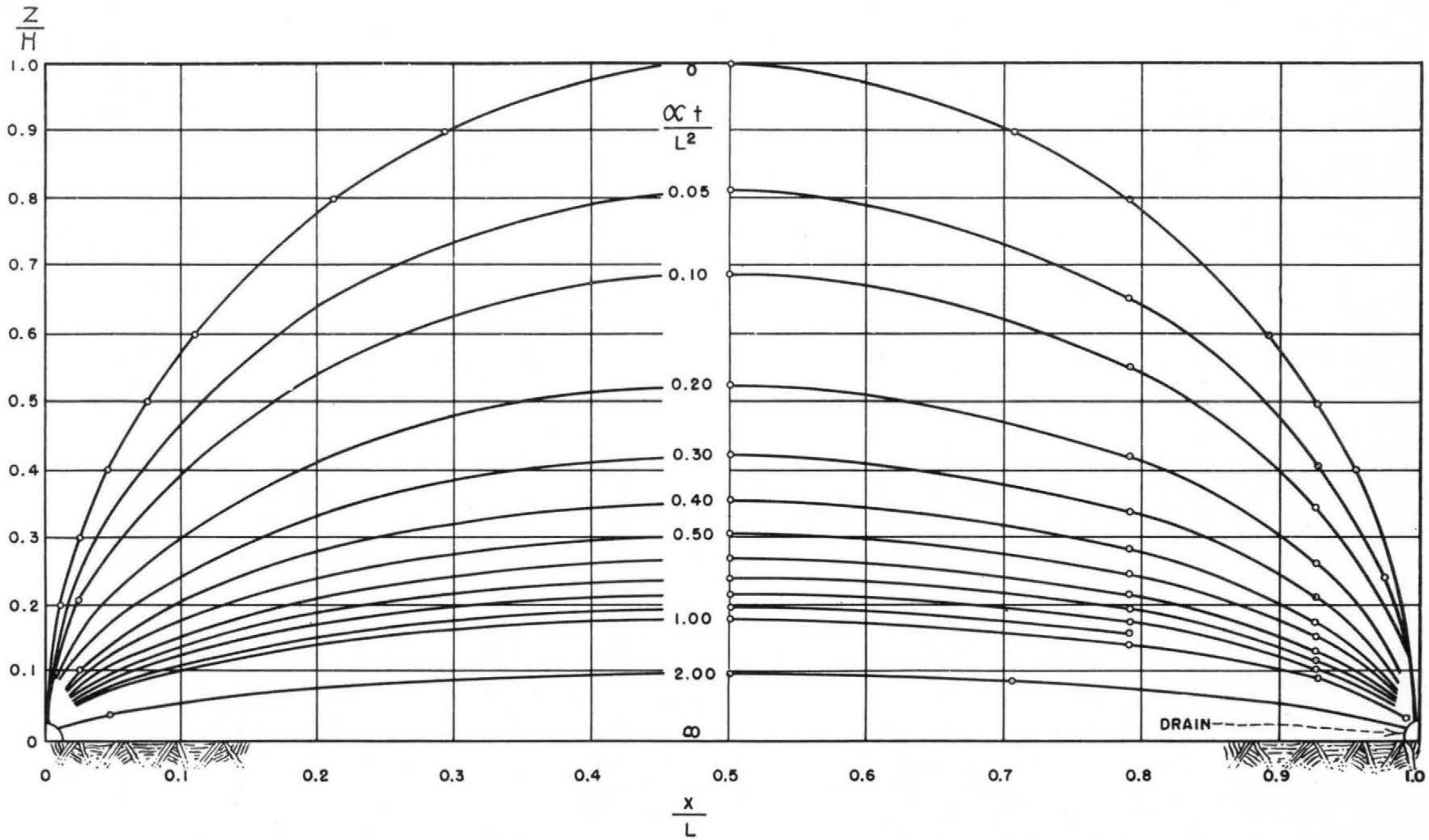
H = drainable depth midway between drains when $t = 0$.

t = time

COMMENTS:

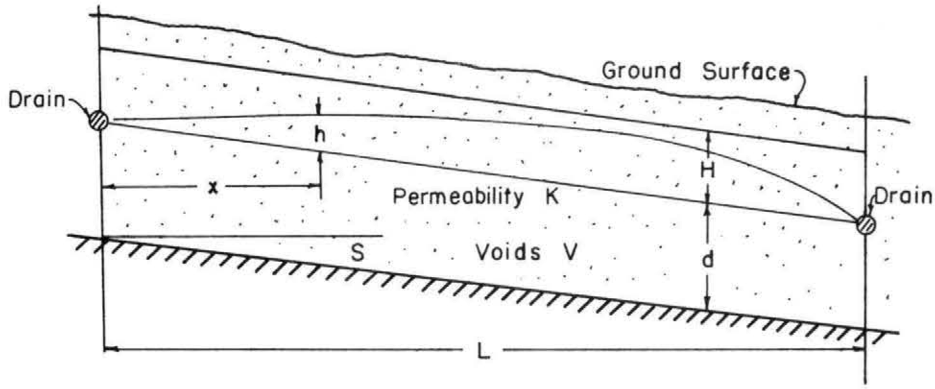
For drains on the barrier, the drain spacing is given by $L = \sqrt{\frac{[illegible]}{2\left(\frac{H}{Z_c} - 1\right)}}$

Here Z_c represents the drainable depth at $x = \frac{L}{2}$ at the time t.



Z/H
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
αt/L²
0
0.05
0.10
0.20
0.30
0.40
0.50
1.00
2.00
∞
DRAIN
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x/L

A - 5. PARALLEL DRAINS IN SLOPING AQUIFER



REFERENCE:

Glover, R. E., 1959-60, Effect of slope on drainage rates, Bureau of Reclamation Informal Memorandums to Chief, Office of Drainage and Groundwater Engineering.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$K(d + h)\frac{\partial^2 h}{\partial x^2} + K\frac{\partial h}{\partial x}\left(\frac{\partial h}{\partial x} - S\right) = V\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

$h = H$ when $t = 0$, for $0 < x < L$

$h = 0$ when $x = 0$ and $x = L$, for $t > 0$

SOLUTION:

Three approximate solutions obtained:

(1) From a simplified form of the differential equation.

(2) By the method of P. W. Werner.

(3) By the method of E. Picard.

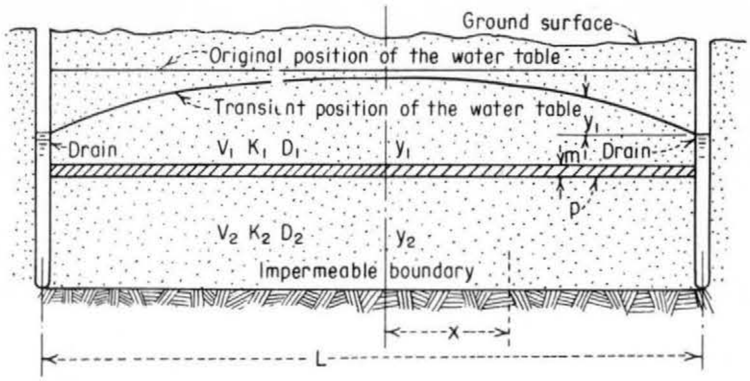
SYMBOLS:

See figure.

COMMENTS:

Computed values compared for $\frac{x}{L} = 0.6$, $S = 0.05$ in the memo of Aug. 12, 1960.

A - 6. DRAINAGE OF A STRATIFIED AQUIFER



REFERENCE:

Glover, R. E., 1953, Limitations of drainage formulas, in Bureau of Reclamation Technical Memorandum No. 657, Section E, p. 47-67.

SUMMARIZED BY :

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

Parametric form.

BOUNDARY CONDITIONS:

Uniform drainable depth at time zero.

SOLUTION:

Exponential-trigonometric .

SYMBOLS: (Consistent units).

See figure.

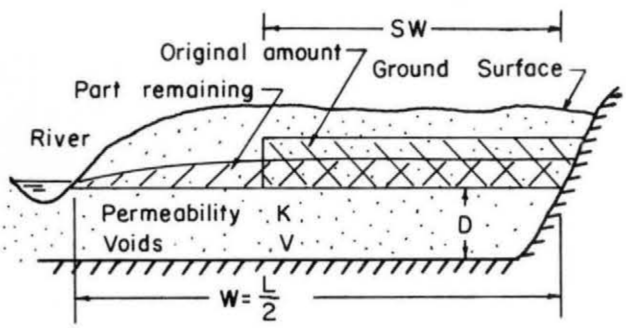
COMMENTS:

Applies to drainage of a three part aquifer. Approximate criteria :

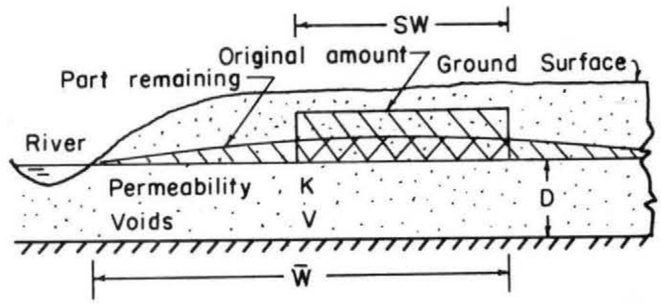
If $\frac{pL}{2m} >> \frac{4K_2D_2}{L}$, both beds act together. If $\frac{pL}{2m} << \frac{4K_2D_2}{L}$, upper bed acts alone.

If quantities are nearly equal special treatment is needed.

A - 7. RETURN FLOWS FROM AN IRRIGATED STRIP



Case I



Case II

REFERENCE:

Glover, R. E., 1962, Return flow from an irrigated strip, from Bureau of Reclamation Informal Memorandum to Head, Water Resources and Utilization Section.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

See figures

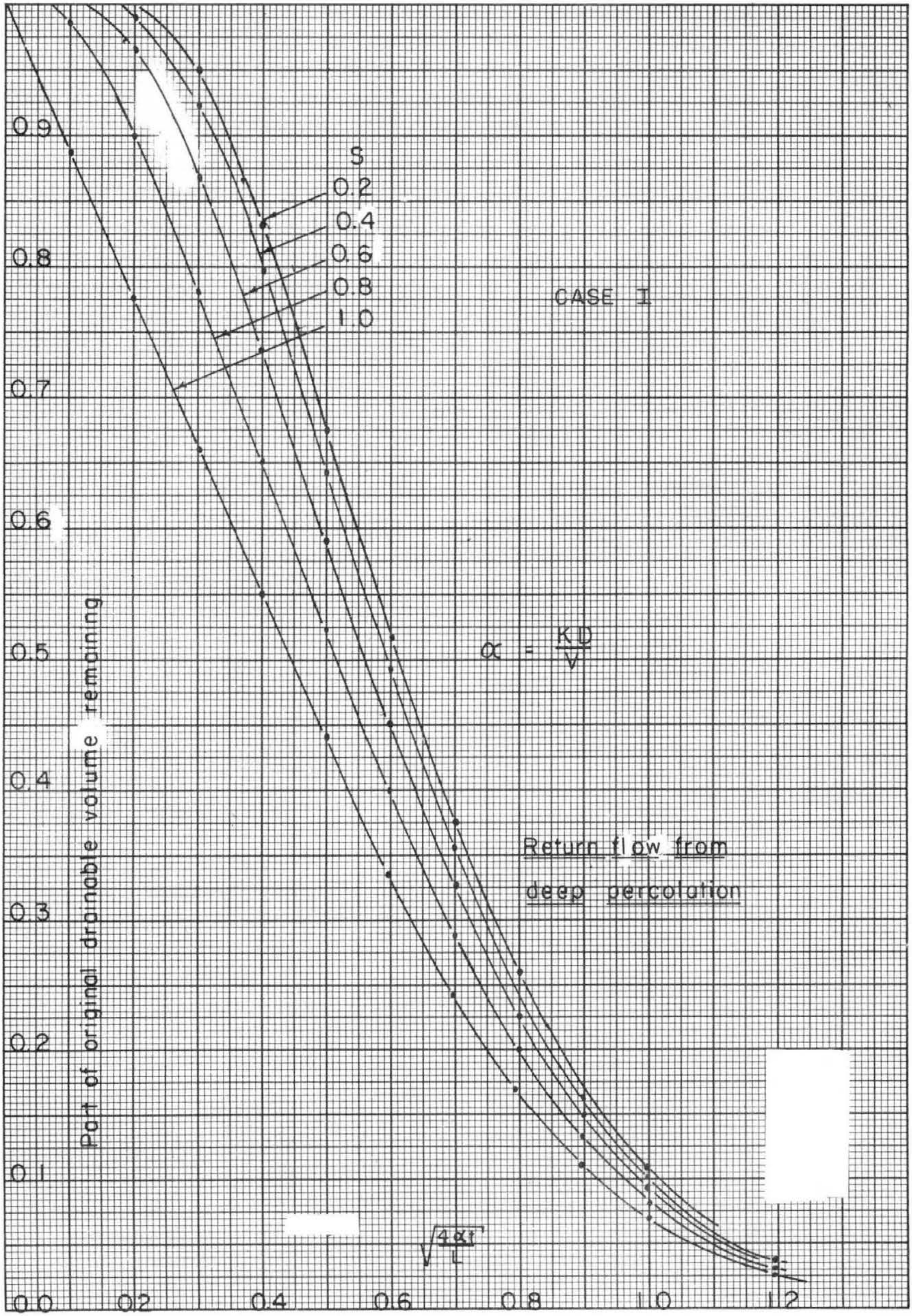
SOLUTION:

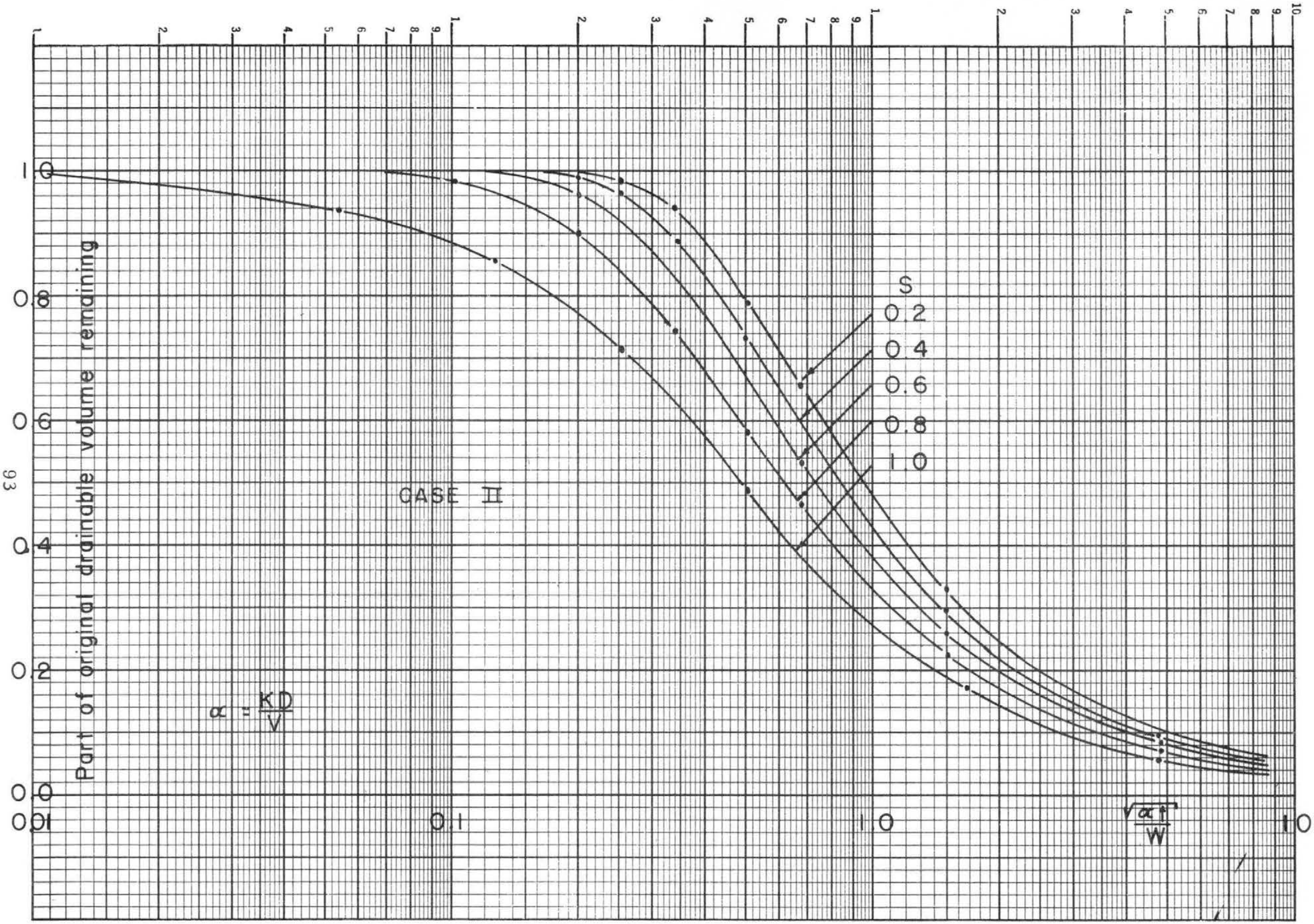
Exponential-trigonometric and probability integral types.

SYMBOLS: (Consistent units).

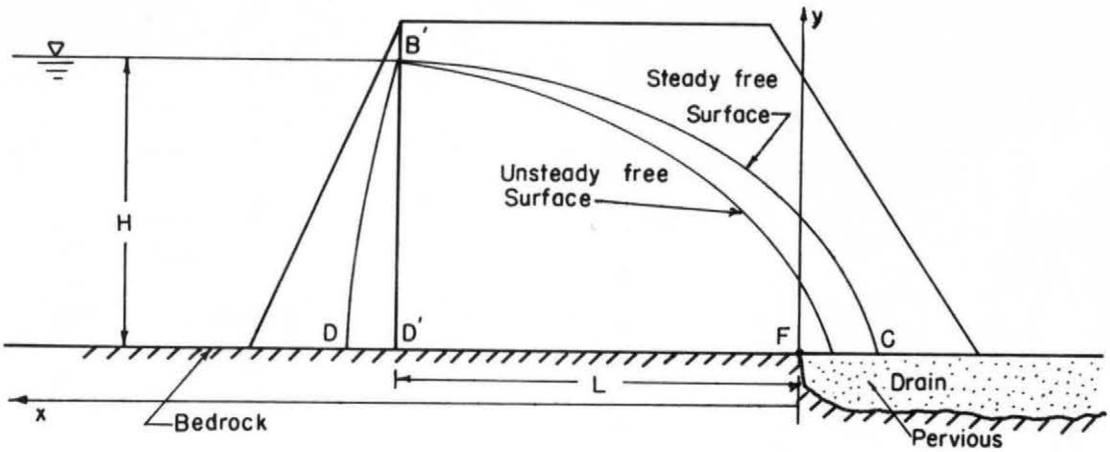
COMMENTS:

Typical use: Estimates of return flows.


S
0.2
0.4
0.6
0.8
1.0
CASE I
α = KD/V
Return flow from deep percolation
Part of original drainable volume remaining
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.2
0.4
0.6
0.8
1.0
1.2
√(4αt/L)


Part of original drainable volume remaining
1.0
0.8
0.6
0.4
0.2
0.0
0.01
0.1
1.0
10
CASE II
S
0.2
0.4
0.6
0.8
1.0
α = KD/V
√(αt)/W

A - 8. UNSTEADY FLOW THROUGH EARTH DAM



REFERENCE:

De Wiest, R. J. M., 1960, Unsteady flow through an underdrained earth dam: Jour. Fluid Mech., v. 8, pt. I, p. 1-9.

SUMMARIZED BY:

Roger J. M. DeWiest, Princeton University

DESCRIPTION OF CASE TREATED:

The damping of the unsteady flow through a dam or levee of uniform hydraulic conductivity and with horizontal underdrain is examined. A slow rise from low level to final full level is considered. The essential idea used in the analysis is consideration of the unsteady flow as a time-dependent perturbation of the final steady flow. The unsteady potential $\phi(x, y, t)$ is expanded in a power series of $e^{-\lambda t}$, of the form

$$\phi(x, y, t) = \phi_0(x, y) + \phi_1(x, y)\, e^{-\lambda t} + 0(e^{-2\lambda t})$$

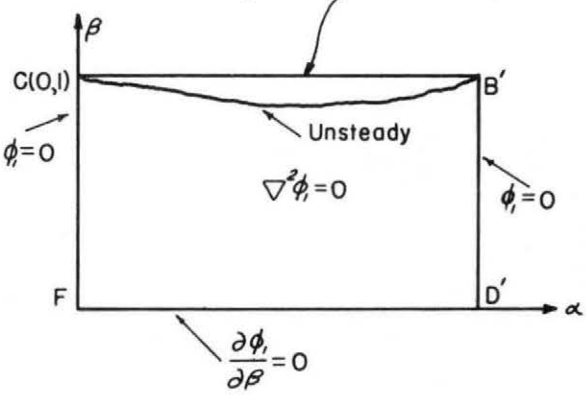
in which $\phi_0(x, y)$ is the known steady state potential and $\phi_1(x, y)$ is the unknown perturbation potential. λ is an eigen value which is found by the solution of a determinantal equation.

The differential equation to solve in the hodograph plane is

$$\nabla^2 \phi_1 = 0$$

with the boundary conditions as indicated on the sketch.
A numerical example is given.

$$\frac{\partial \phi_1}{\partial \beta} = (a\lambda)^2 (1 + \alpha^2)\phi_1 + (a\lambda)\frac{\partial \phi_1}{\partial \alpha}$$



SOLUTIONS:

See original paper.

SYMBOLS:		Dimensions
$\phi(x, y, t)$	= potential	L
ϕ_o	= steady state potential	L
ϕ_1	= perturbation potential	L
λ	= eigen value	$\frac{1}{T}$
a	$= \frac{\epsilon Q_{H_o}}{K^2}$	T
ϵ	= porosity	dimensionless
Q_{H_o}	= steady flow rate for maximum pool level	$\frac{L^2}{T}$
K	= hydraulic conductivity	$\frac{L}{T}$
α	$= \frac{Ku_o^2}{q_o^2}$; $\beta = \frac{Kv_o}{q_o^2}$ Hydrographs coordinates	dimensionless
u_o, v_o, q_o	= steady state velocities	$\frac{L}{T}$

A - 9. UNSTEADY FLOW THROUGH EARTH DAM

REFERENCE:

De Wiest, R. J. M., 1961, Free surface flow in homogeneous porous medium: Am. Soc. Civil Engineers Trans., v. 127, pt. I, p. 1045-89.

SUMMARIZED BY:

Roger J. M. De Wiest, Princeton University

DESCRIPTION OF CASE TREATED:

This paper consists of an analytical and experimental extension of previous paper by the author. Rapid rises from low level to fixed full level behind an earth embankment are treated. The analytical results are tested in a Hele-Shaw viscous flow model. Numerical examples are included. The analytical and experimental results compare reasonably well.

DIFFERENTIAL EQUATION IN HODOGRAPH PLANE:

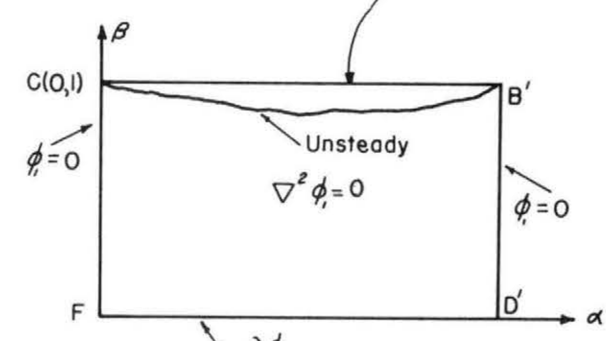
$$\nabla^2 \phi_1 = 0$$

where ϕ_1 is a perturbation potential.

BOUNDARY CONDITIONS:

See sketch.

$$\frac{\partial \phi_1}{\partial \beta} = \left[(a\lambda)\,\alpha\,(1+\alpha^2)^{-1} + (a\lambda)^2\,(1+\alpha^2)\right]\phi_1 + \left[2\,(a\lambda) - \alpha\,(1+\alpha^2)^{-2}\right]\frac{\partial \phi_1}{\partial \alpha} + (1+\alpha^2)^{-1}\,\frac{\partial^2 \phi_1}{\partial \alpha^2}$$



Hodograph with Boundary conditions

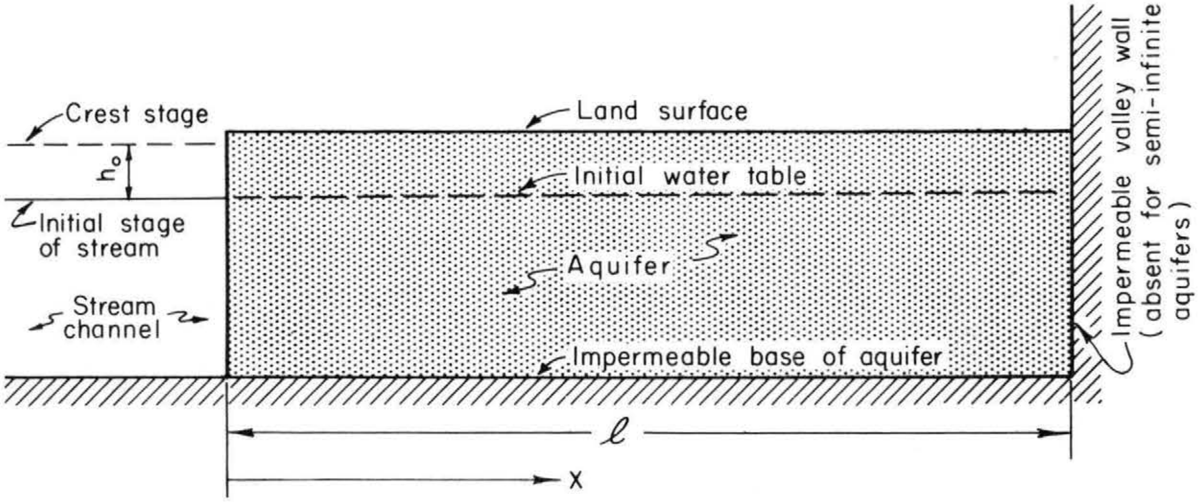
SOLUTIONS:

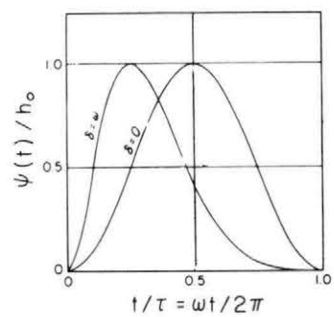
See original paper.

SYMBOLS:

See A-8.

A - 10. BANK STORAGE DUE TO FLOOD WAVES





$$\psi(t) = \begin{cases} Nh_0 e^{-\delta t}(1-\cos\omega t), & 0 \le t \le \tau \\ 0; & t \ge \tau \end{cases}$$

Function $\psi(t)$ approximating flood-wave hydrograph

REFERENCE:

Cooper, H. H. , Jr., and Rorabaugh, M. I., 1963, Ground-water movements and bank storage due to flood stages in surface streams: U. S. Geol. Survey Water-Supply Paper 1536, in press.

SUMMARIZED BY:

H. H. Cooper, Jr.

DESCRIPTION OF CASE TREATED:

Solutions are derived for the changes in ground-water head, ground water flow, and bank storage caused by flood waves in surface streams, assuming that the stage hydrographs of the streams can be approximated by damped sinusoidal curves.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial x^2} - \frac{S}{T}\frac{\partial h}{\partial t} = 0$$

BOUNDARY CONDITIONS:

$$h(x, 0) = 0; \quad 0 \le x \le \ell$$

$$\frac{\partial h(\ell, t)}{\partial x} = 0; \quad t \ge 0$$

$$h(0, t) = \psi(t)$$

SOLUTION:

For change in ground-water head:

$$h_{t \leq \tau} = Nh_o \left\{ e^{-\eta\omega t} \left[\frac{\cos[(\ell - x)\pi\sqrt{\eta}/2\ell]}{\cos[\pi\sqrt{\eta}/2]} - A\cos(\omega t + \theta) \right] + \frac{4}{\pi} \sum_{n=1}^{\infty} \sin[(2n-1)\pi x/2\ell] \frac{(2n-1)\, e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2} \right\}$$

$$h_{t \geq \tau} = \frac{4Nh_o}{\pi} \sum_{n=1}^{\infty} \sin[(2n-1)\pi x/2\ell] \frac{(2n-1)\left[1 - e^{-[\eta - (2n-1)^2]\, 2\pi\beta}\right] e^{-(2n-1)\beta\omega t}}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2}$$

$A = A(x, \ell, \delta, \omega, T, S)$ $\qquad$ $\beta = \dfrac{\pi T \tau}{8 S \ell^2}$

$\theta = \theta(x, \ell, \delta, \omega, T, S)$ $\qquad$ $\eta = \dfrac{4\delta S \ell^2}{\pi^2 T}$

For ground-water flow into stream:

$Q = T \dfrac{\partial h(o,t)}{\partial x}$ = flow into stream per unit length

$$Q_{t \leq \tau} = Nh_o \sqrt{\omega TS} \left\{ e^{-\eta\beta\omega t} \left[\eta\beta \tan\frac{\pi\sqrt{\eta}}{2} + B\cos(\omega t + \phi) \right] + \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{(2n-1)^2\, e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2} \right\}$$

$$Q_{t \geq \tau} = Nh_o \sqrt{\omega TS} \cdot \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{(2n-1)^2 \left[1 - e^{-[\eta - (2n-1)^2]\, 2\pi\beta}\right] e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2] + [\eta - (2n-1)^2]^3 \beta^2}$$

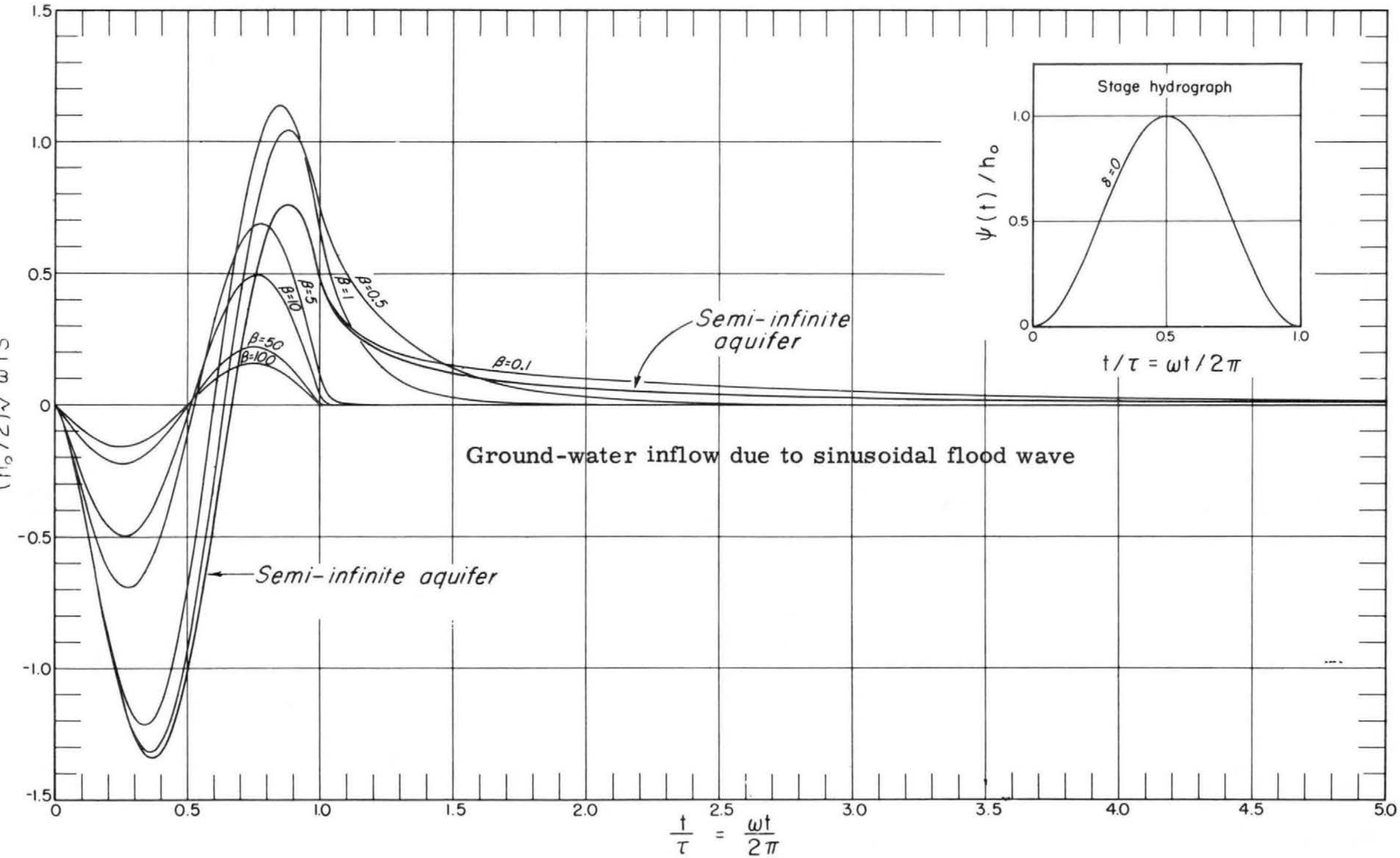
$B = B(\ell, \delta, \omega, T, S)$ $\qquad$ $\phi = \phi(\ell, \delta, \omega, T, S)$

For bank storage:

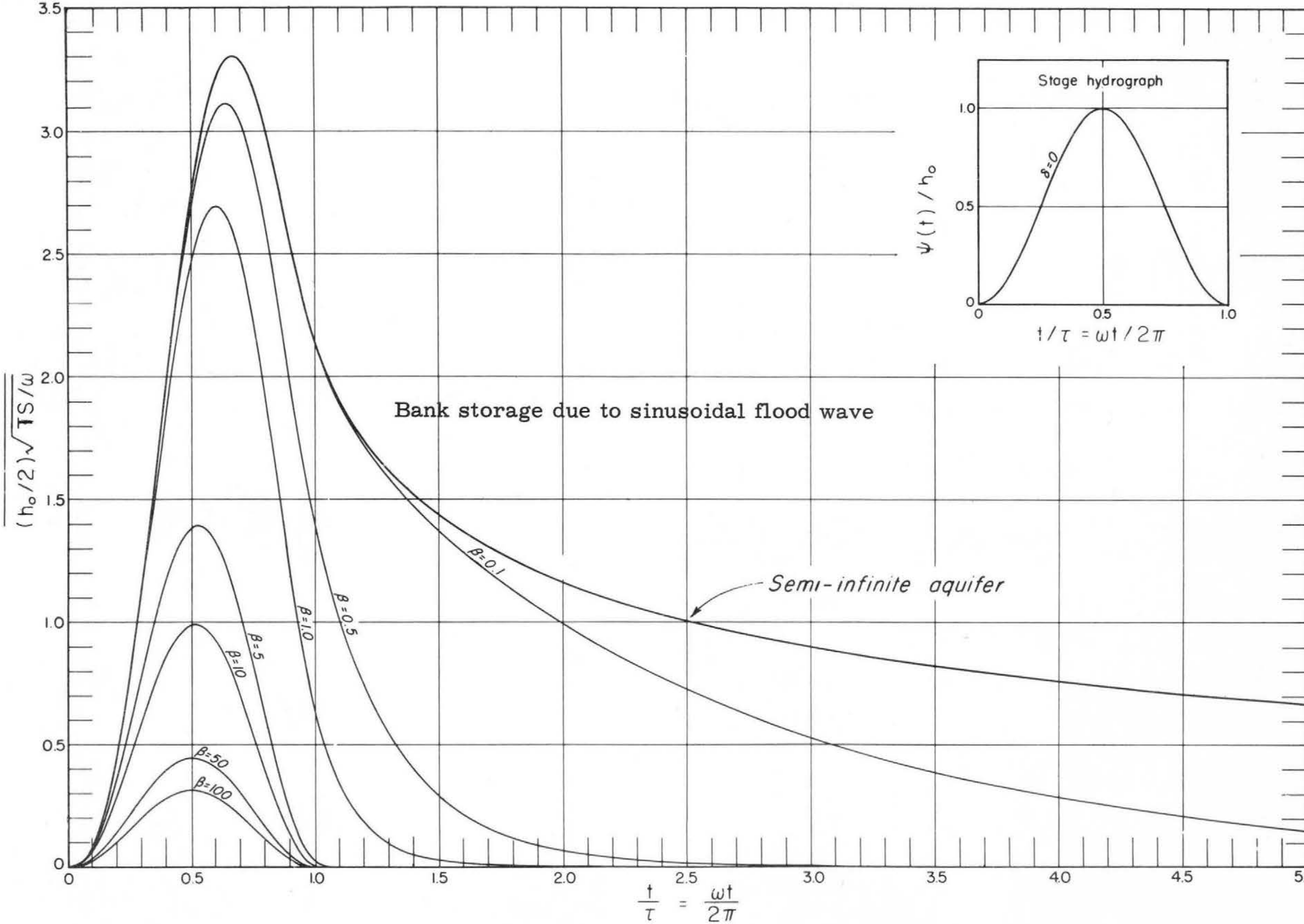
$V = -\displaystyle\int_0^t Q\,dt$ = Bank storage per unit length.

$$V_{t \leq \tau} = Nh_o \sqrt{\frac{TS}{\omega}} \left\{ e^{-\eta\beta\omega t} \left[\frac{1}{\sqrt{\eta\beta}} \tan\frac{\pi\sqrt{\eta}}{2} - \frac{B}{\sqrt{\eta^2\beta^2 + 1}} \sin(\omega t + \phi - \arctan \eta\beta) \right] + \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2]\beta + [\eta - (2n-1)^2]^3 \beta^3} \right\}$$

$$V_{t \geq \tau} = Nh_o \sqrt{\frac{TS}{\omega}} \cdot \frac{4\sqrt{\beta}}{\pi} \sum_{n=1}^{\infty} \frac{\left[e^{-[\eta - (2n-1)^2]\, 2\pi\beta} - 1\right] e^{-(2n-1)^2 \beta\omega t}}{[\eta - (2n-1)^2]\beta + [\eta - (2n-1)^2]^3 \beta^3}$$



Ground-water inflow due to sinusoidal flood wave

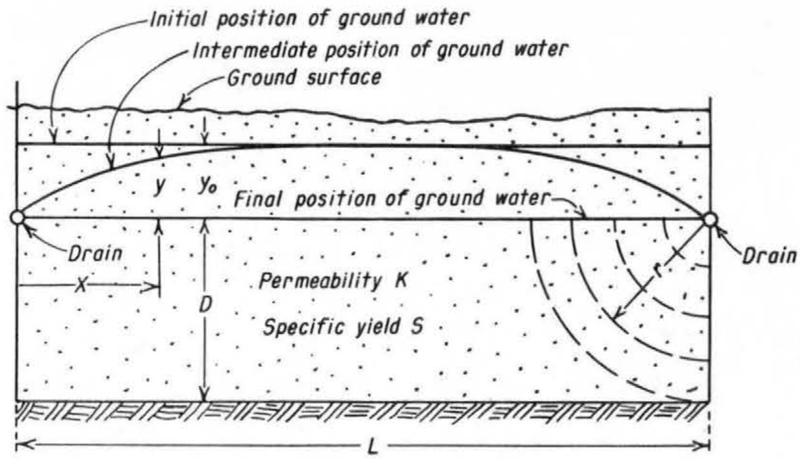

Bank storage due to sinusoidal flood wave
$\frac{V}{(h_o/2)\sqrt{TS/\omega}}$
$\frac{t}{\tau} = \frac{\omega t}{2\pi}$
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0
β=0.1
β=0.5
β=1.0
β=5
β=10
β=50
β=100
Semi-infinite aquifer
Stage hydrograph
β=0
ψ(t) / h_o
t / τ = ωt / 2π

LIMITATIONS: Assumes (1) stream channel fully penetrates the aquifer and has vertical banks, (2) changes in ground-water head are small in relation to thickness of the saturated zone, so that transmissibility remains practically constant in space and time, and (3) stream does not overflow its banks.

SYMBOLS WITH UNITS:

h		Change in ground-water head at time t and distance x, (L)
h_o		Maximum rise of stream stage, (L)
ℓ		Distance from bank of stream to valley wall (L)
N	=	$1/\left[e^{-\delta t_c}(1-\cos\omega t_c)\right]$, (Dimensionless)
Q		Ground-water flow into stream per unit length at time t, (L^2T^{-1})
S		Coefficient of storage of aquifer, (Dimensionless)
T		Transmissibility of aquifer, (L^2T^{-1})
t		Time since beginning of stage oscillation, (T)
t_c		Time of flood crest, (T)
V		Bank storage per unit length of stream at time t, (L^2)
x		Distance from bank of stream, (L)
α	=	$\pi^2\sigma/4\ell^2$, (T^{-1})
β	=	$\alpha/\omega = \pi\tau T/8\ell^2 S$, (Dimensionless)
δ	=	$\omega\cot(\omega t_c/2)$ Constant determining degree of asymmetry of curves, ψ, (T^{-1})
η	=	$\delta/\beta\omega$ (Dimensionless)
θ	=	Function defined in reference, (Radians).
ξ	=	$(\ell - x)/\ell$ (Dimensionless)
σ	=	T/S Hydraulic diffusivity of aquifer, (L^2T^{-1})
τ		Period or duration of stage oscillation, (T)
ϕ		Constant defined in reference, (Radians)
$\psi(t)$		Function representing stage hydrograph of stream, (L)
ω	=	$2\pi/\tau$ Frequency of stage oscillation, (T^{-1})

A - 11. EFFECT OF LOCAL DRAIN RESISTANCE



REFERENCE:

Glover, R. E., 1953, Formulas for movement of groundwater - Oahe Unit - Missouri River Basin Project, in Bureau of Reclamation Technical Memorandum no. 657, Section D, p 41., Tuthill, L. H., Glover, R. E., Spencer, G. H., and Bierce, W. B., 1951, Insulation for protection of new concrete in winter, Journal of the American Concrete Institute, vol 23, no. 3, p. 262-264.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\frac{dp}{dr} = \frac{-2q}{\pi K r} \quad \text{(R. E. Glover)}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0 \quad \text{(W. T. Moody)}$$

BOUNDARY CONDITIONS:

$p = 0$ when $r = D$.

$p = 0$ when $x = \infty$ $\qquad S_a(x) \begin{cases} = 0, & 0 \leq x < a \\ = 1, & a \leq x \end{cases}$

$$\left(\frac{\partial p}{\partial x}\right)_{x=0,\ y\neq 0} = 0 \qquad \left(\frac{\partial p}{\partial y}\right)_{y=0,\ x\neq 0} = 0 \qquad \left(\frac{\partial p}{\partial y}\right)_{y=-D} = 0$$

SOLUTION:

$$p = \frac{2q}{\pi K} \log_e\left(\frac{D}{r}\right)$$

$$p_a = \frac{2q}{\pi K} \log_e\left(\frac{D}{a}\right)$$

$$p = \frac{q}{K}\left[\frac{xS_{a(x)}}{D} - \frac{1}{\pi}\left\{\log_e\left(\operatorname{Cosh}\frac{\pi x}{D} - \operatorname{Cos}\frac{\pi y}{D}\right) + \log_e 2\right\}\right]$$

$$p_a = \frac{q}{K}\left[\frac{a}{D} + \frac{2}{\pi}\log_e\left(\frac{D}{\pi a}\right)\right]$$

SYMBOLS: (Consistent units).

$h^2 = \left(\frac{KD}{V}\right)$, $\quad r$ = radius, $\quad t$ = time, $\quad a$ = drain radius, $\quad u = \frac{y}{y_o}$

q = flow to unit length of drain from one side

p = pressure decrease causing flow toward the drain

p_a = value of p at $r = a$

L = distance between drains

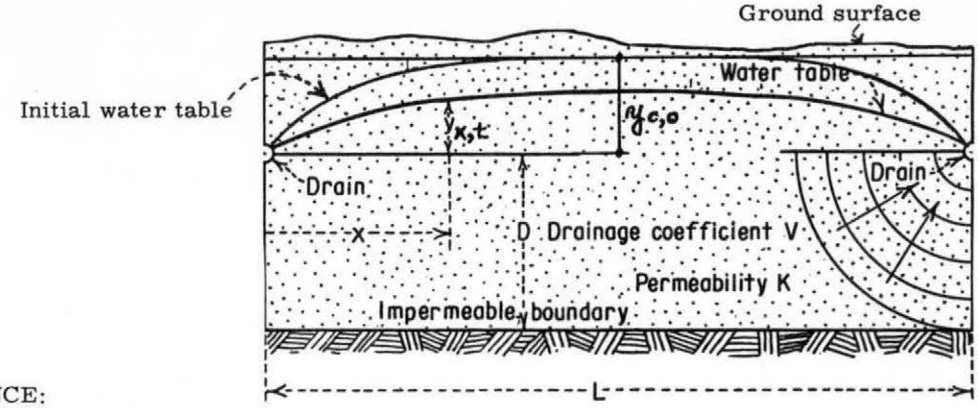
$$E = \frac{\pi K}{2D \log_e(\frac{D}{a})} \quad \text{(Glover)} \qquad E = \frac{\pi K}{\pi a + 2D \log_e(\frac{D}{\pi a})} \quad \text{(Moody)}$$

COMMENTS:

The notation given here adapts the A.C.I. charts for computation of drainable depths y when the local resistance to flow to the drain is to be accounted for. $E = \frac{q}{p_a D}$.

The flow to the drain is $q = E D Y_{x=o}$.

A - 12. WATER TABLE BETWEEN PARALLEL DRAINS



REFERENCE:

Moody, W. T., and Tapp, W. N., Unpublished development presented by L. D. Dumm at 1960 Winter Meeting, American Society of Agricultural Engineers, Memphis, Tenn.

SUMMARIZED BY:

W. T. Moody, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\frac{\partial y}{\partial t} = \alpha \frac{\partial^2 y}{\partial x^2}$$

BOUNDARY CONDITIONS:

$$t = 0, \quad 0 \le x \le L; \quad y_{x,o} = \frac{8y_{c,o}}{L^4}\left(L^3x - 3L^2x^2 + 4Lx^3 - 2x^4\right)$$

$$t \ge 0, \quad x = 0 \text{ or } L; \quad y_{o,t} = 0 \text{ and } y_{L,t} = 0$$

SOLUTION:

$$y_{x,t} = \frac{192y_{c,o}}{\pi^5} \sum_{m=0}^{\infty} \frac{(2m+1)^2\pi^2 - 8}{(2m+1)^5} \exp\left(-\frac{(2m+1)^2 \pi\alpha t}{L^2}\right) \sin\frac{(2m+1)\pi x}{L}$$

SYMBOLS:

$$\alpha = \frac{KD}{V}$$

See sketch for other symbols.

COMMENTS:

Initial condition using a 4th degree parabola has been found to check field and experimental results closely.

A - 13. PATH FUNCTIONS FOR TRANSIENT FLOW THROUGH HETEROGENEOUS POROUS MEDIA

REFERENCE:

Nelson, R. W., 1963, Stream functions for three-dimensional flow in heterogeneous porous media: Internat. Union Geodesy and Geophys., 13th Gen. Cong. Proc.

SUMMARIZED BY:

R. William Nelson, General Electric Company

CASE TREATED:

Path functions are derived for transient flow in three-dimensions of nondiffusive fluids through heterogeneous porous media.

DIFFERENTIAL EQUATION:

Notation: Whenever a lower case subscript appears twice in the same monomial, this monomial stands for the sum of four terms obtained by assigning the values of 1, 2, 3 and 4. The three Cartesian space coordinates and time are designated as x_1, x_2, x_3, and x_4, respectively, or x_i, i = 1, 2, 3, 4. Where less than 4 integers values of the subscript are needed, the range will be explicitly given. Partial differentiation with respect to x_i will be indicated by the operator ∂_i .

Basic equation

$$\frac{D\nu}{Dx_4} = v_i \partial_i \nu + \partial_4 \nu = 0 \qquad i = 1,2,3 \tag{1}$$

where:

$\frac{D}{Dx_4}$	is the convective derivative operator,	(T^{-1})
ν	designates the path function, and	(L^3)
v_i	is velocity in the x_i coordinate direction	(LT^{-1})

BOUNDARY CONDITION:

Consider some surface which is time dependent and intercepts every path filament in the flow system of interest. (An equipotential surface which is itself moving through space in time is a convenient surface). Let such a surface be:

$$\Gamma(x_i) = C \tag{2}$$

The appropriate boundary condition specifies the flux distribution across the above surface, i.e. :

$$Q = F(\Gamma_2) \tag{3}$$

where:

Q is the flux or flow rate,	(L^3T^{-1})
F is the function describing Q along the surface $\Gamma(x_i)$	(L^3T^{-1})

RESULTS:

I. The general equation of a path line is:

$$f(x_i) = a_1 \tag{4}$$

$$g(x_i) = a_2 \tag{5}$$

$$\eta(x_i) = a_3 \tag{6}$$

where f, g, and η are respectively set equal to constants a_1, a_2, a_3, and are independent integrals of the systems of differential equations :

$$\frac{dx_1}{\partial_1 \psi} = \frac{dx_2}{\partial_2 \psi} = \frac{dx_3}{\partial_3 \psi} = K \; dx_4 \tag{7}$$

where:

$\psi = p/\gamma + x_3$ is the hydraulic potential head or potential function. (L)

p/γ is the pressure head. (L)

x_3 is the coordinate direction oriented parallel with the gravitational vector.

$K = K(x_1, x_2, x_3)$ is the spatial distribution of permeability for the heterogeneous soil. (LT^{-1})

II. The velocity expressed in terms of the path functions - f, g, and η is:

$$v_i = \frac{\partial^3 [F(\Gamma)]}{\partial f \, \partial g \, \partial \eta} \; (\epsilon_{ijkm} \, \partial_j f \; \partial_k g \; \partial_m \eta), \quad i = 1, 2, 3 \tag{8}$$

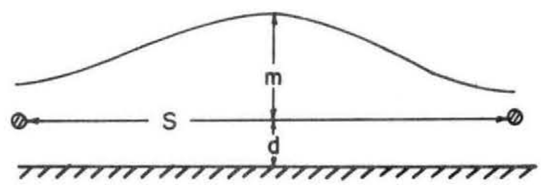
where: ϵ_{ijkm} is the epsilon permutation symbol.

III. The material distribution ν (analogus to flux in steady flow) is:

$$\nu = \int_{f_o}^{f} \int_{g_o}^{g} \int_{\eta_o}^{\eta} \frac{\partial^3 [F(\Gamma)]}{\partial f \; \partial g \; \partial \eta} \; df \; dg \; d\eta^{1/} \tag{9}$$

1/ Alternate simpler forms of equations 8 and 9 can be obtained through using the works of Schouten and the translation from Russian of Goluber's lectures on integrability and post multiplier, as shown in complete manuscript.

A - 14. FALL OF WATER TABLE BETWEEN PARALLEL DRAINS



REFERENCE:

Bouwer, Herman, and Van Schilfgaarde, Jan, 1962, Simplified prediction method for the fall of the water table in drained land: Presented at Am. Soc. Agricultural Engineers, Dec. 1962, (no. 62-728).

SUMMARIZED BY:

Jan van Schilfgaarde, North Carolina State College

DIFFERENTIAL EQUATION: $P = -f\,C\,\frac{dm}{dt}$

Method: Combine with any P - m relation for steady state conditions, be it in the form of equations, tables, graphs or nomographs.

Examples: Hooghoudt's ellipse equation reads

$$P = 4\,K\,m\,(2d_e + m)/\,S^2,$$

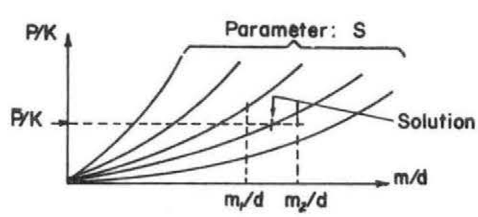
and integration yields

$$Kt/f = (CS^2/8d_e)\ \ln\,[m_o(m_t + 2d_e)\,/m_t(m_o + 2d_e)].$$

Similarly, from Toksoz and Kirkham,

$$Kt/f = CS\,[\ln(m_o/m_t)\,]\quad F(r/S,\ d/S).$$

Graphically, from Ernst's relaxation solution nomographs:



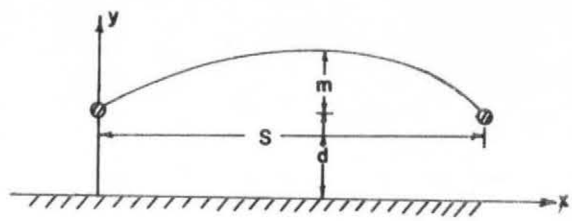
Choose m_o, m_t and Δt. Then $\overline{P} = C\,f\,\Delta m/\Delta t$. Mark lines for $\overline{P}$, m_o/d and m_t/d ; read S.

Limitations: The major advantages of this technique are its simplicity and general applicability. Through the introduction of C, it loses sophistication and some accuracy. The transient results can never be better than the steady-state starting point. The graphic procedure lends itself to field operations.

SYMBOLS:

d	=	depth of impervious layer below drain (L)
d_e	=	"equivalent depth", per Hooghoudt's convergence correction (L)
f	=	drainable pore space, fraction (Dimensionless)
K	=	hydraulic conductivity (L/T)
C	=	correction for change in shape of water table; generally $0.8 < C < 1.0$ (Dimensionless)
m_o	=	initial midpoint height of water table above drains (L)
m_t	=	water table height at time t (L)
P	=	instantaneous drainage rate, corresponding to steady state rainfall (L/T)
S	=	drain spacing (L)
t	=	time (T)

A - 15. DESIGN OF TILE DRAINAGE FOR FALLING WATER TABLES



REFERENCE:

Van Schilfgaarde, Jan, 1963, Design of tile drainage for falling water tables: Am. Soc. Civil Engineers Proc., v. 89, no. IR2.

SUMMARIZED BY:

Jan van Schilfgaarde, North Carolina State College

DIFFERENTIAL EQUATION:

$$f\ \partial y/\partial t = \frac{\partial}{\partial x}(K\ y\ \partial y/\partial x)$$

BOUNDARY CONDITIONS:

$y = y_o = d + m_o$ at $x = S/2$ for $t = 0$

$y = d$ at $x = 0, S$ for $t \geq 0$

$dy/dx = 0$ at $x = S/2$ for $t \geq 0$

SOLUTION:

$$S = 3\ A \left[\frac{K(d+m)(d+m_o)\,t}{2f(m_o - m)} \right]^{1/2}$$

where A is related to the incomplete beta function. $A = A\,[d/(d+m_o)]$, given graphically.

Within 3 percent error,

$$A = [1 - (d/y_o)^2]^{1/2}$$

Limitations:

Based on DF assumptions: hence requires convergence correction for greater d. Although simple in form, the Hooghoudt type of "equivalent depth" correction requires trial and error. Caution is needed in interpreting initial condition: It is not equivalent to Brooks' flooded (or at least horizontal water table) initial condition.

SYMBOLS:

A = a function involving the incomplete beta (Dimensionless)

d = depth of impervious layer below drains (L)

f = drainable pore space, fraction (Dimensionless)

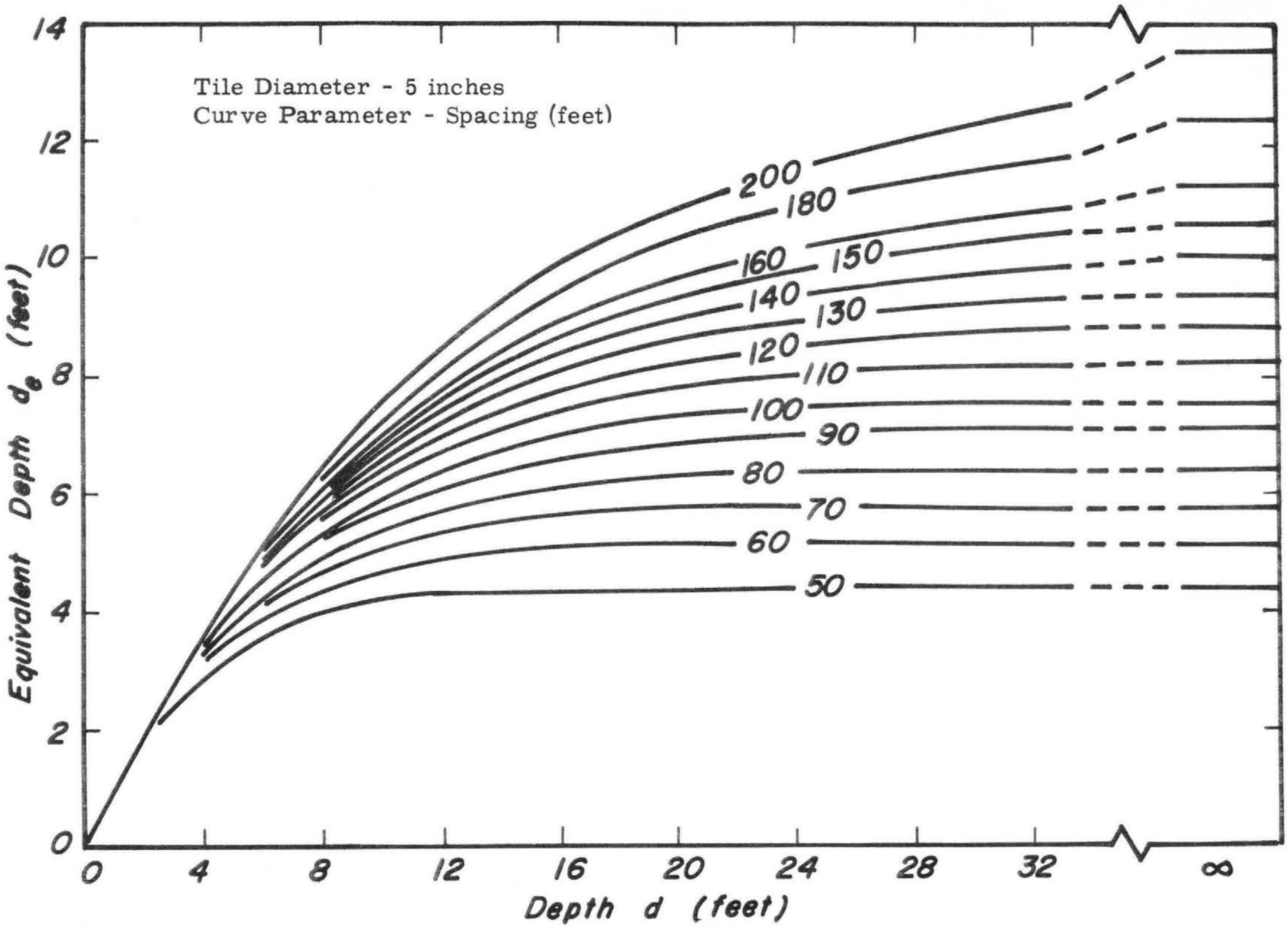
K = hydraulic conductivity (L/T)

m = midpoint water table height above drains (L)

S = drain spacing (L)

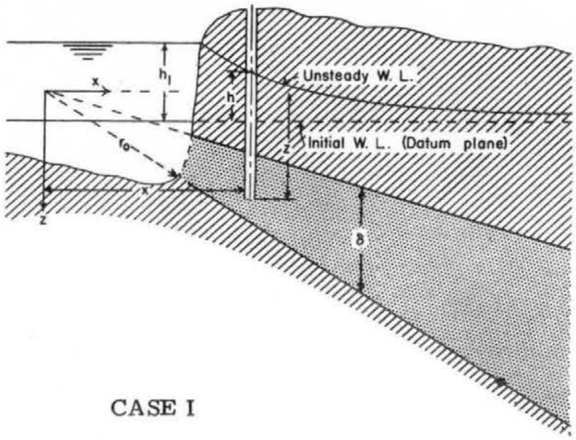
t = time (T)

x, y coordinates with origin at impervious layer immediately below drain center.

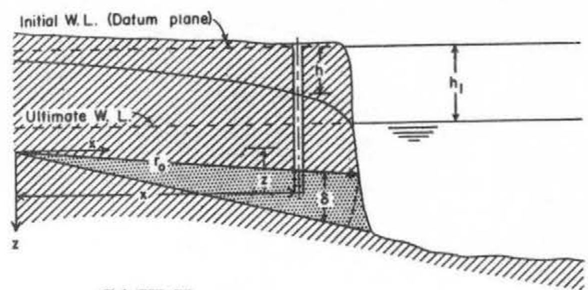


Hooghoudt's equivalent depth as a function of d for different drain spacings (on the curves).

A - 16. FLOW OF GROUND WATER IN SANDS OF NONUNIFORM THICKNESS (Part 1)



CASE I



CASE II

REFERENCE:

Hantush, M. S., 1962, Flow of ground water in sands of nonuniform thickness, 1, Flow in a wedge-shaped aquifer: Jour. Geophys. Research, v. 67, no. 2, p. 703-709.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

CASE I. FLOW IN A SEMI-INFINITE WEDGE-SHAPED AQUIFER

BOUNDARY CONDITIONS AND SOLUTIONS:

(a) Sudden change in channel water level,

Boundary Conditions

$$h(x, z, 0) = 0 \quad \text{and} \quad h(r_o, t) = h_1, \ 0 < t$$

Solutions:

$$h(x, z, t) = h_1 \ A(\rho, \tau) \qquad q = K \ \delta \ h_1 G(\tau)$$

For $\tau < 0.01$:

$$h \approx \frac{h_1}{\sqrt{\rho}} \left[\operatorname{erfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) + \frac{\rho-1}{4\rho} \sqrt{\tau} \ \operatorname{ierfc}\left(\frac{\rho-1}{2\sqrt{\tau}}\right) \right] \qquad q \approx K\delta h_1 \left(0.5 + \frac{1}{\sqrt{\pi\tau}}\right)$$

For $\tau > 500$:

$$h \approx 0.434 \, h_1 \, W(\rho^2/4\tau) \, / \, \log_{10}(2.25\tau) \qquad q \approx (0.87 \, K \, \delta \, h_1) / \log_{10}(2.25\tau)$$

(b) Intermittent sudden changes in channel water level

Boundary Conditions		Solutions		
$h(r_o, t) = h_1$	for $0 < t < t_1$	$h(x,z,t) = f_1$	$q \ K\delta = F_1$	$0 < t < t_1$
$= h_2$	for $t_1 < t < t_2$	$= f_1 + f_2$	$= F_1 + F_2$	$t_1 < t < t_2$
$= h_n$	for $t_{n-1} < t < \infty$	$= f_1 + f_2 + \ldots + f_n$	$= F_1 + F_2 + \ldots + F_n$	$t_{n-1} < t < \infty$

(c) Sinusoidal fluctuation in channel water level

Boundary Conditions:

$$h(r_o, t) = h_o \sin(2\pi t/t_o) \text{ for } 0 < t$$

Solution:

$$h(x,z,t) = h_o \frac{N_o(\alpha\rho)}{N_o(\alpha)} \sin \frac{2\pi t}{t_o} + \phi_o(\alpha\rho) - \phi_o(\alpha) - h_o A_1(\rho,\tau,\tau_o)$$

where

$$A_1(\rho,\tau,\tau_o) = \frac{4}{\tau_o} \int_0^\infty \frac{J_o(u\rho)\, Y_o(u) - Y_o(u\rho)\, J_o(u)}{\left[J_o^2(u) + Y_o^2(u)\right]\left[(2\pi/\tau_o)^2 + u^4\right]} e^{-\tau u^2} u\, du$$

The function $A_1(\rho,\tau,\tau_o)$ should be tabulated before the above equation for $h(x,z,t)$ is used.

Of interest is the steady periodic head distribution which is given by the above equation with $A_1(\rho,\tau,\tau_o) = 0$.

CASE II . FLOW IN A FINITE WEDGE-SHAPED AQUIFER

SOLUTIONS:

(a) Sudden change in channel water level

$$h = h_1 A'(\rho,\tau) \qquad q = K \delta h_1 G'(\tau)$$

(b) Intermittent sudden changes in channel water level

Solution same as in the case of semi-infinite wedge-shaped aquifer with f_n and F_n replaced by f'_n and F'_n .

(c) Channel water level varying linearly with time , i.e. $h(r_o,t) = ct \qquad t > 0$

$$h = \frac{c r_o^2 S_s}{K} \left[\tau - \frac{1-\rho^2}{4} + 2 \sum_{m=1}^{\infty} \frac{J_o(\beta_m \rho)}{\beta_m^3 J_1(\beta m)} e^{-\tau \beta_m^2} \right]$$

$$q = 2(\delta r_o)(c r_o S_s) \left[\frac{\rho}{4} - \sum_{m=1}^{\infty} \frac{e^{-\tau\beta_m^2}}{\beta_m^2} \right]$$

(d) Sinusoidal fluctuation in channel water , i.e. $h(r_o,t) = h_o \sin(2\pi t/t_o)$

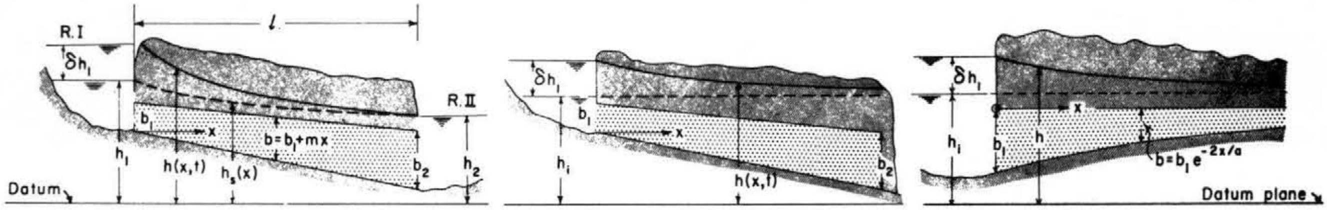
$$h = h_o \left\{ \left[\frac{M_o(\alpha\rho)}{M_o(\alpha)}\right] \sin\left[\left(\frac{2\pi t}{t_o}\right) + \theta_o(\alpha\rho) - \theta_o(\alpha)\right] + \frac{4\pi}{\tau_o} \sum_{m=1}^{\infty} \frac{\beta_m J_o(\beta_m \rho)\, e^{-\tau\beta^2_m}}{\left[\left(\frac{2\pi}{\tau_o}\right)^2 + \beta_m^4\right] J_1(\beta_m)} \right\}$$

The steady periodic solution can be obtained by neglecting the series.

SYMBOLS:

α	$= \sqrt{2\pi/\tau_o}$
β_m	= The m^{th} zero of J_o
δ	= The central angle of the wedge-shaped **aquifer**.

$\theta_o(y)$	=	The argument of the complex number $J_o(y\, i^{3/2})$.
ρ	=	$(1/r_o)(x^2 + z^2)^{1/2}$.
τ	=	$Kt/ r_o^2 S_s$
τ_o	=	$Kt_o/r_o^2 S_s$
$\phi(y)$	=	The argument of the complex number $K_o(y\sqrt{i})$
$A(\rho,\tau)$	=	The flowing well function for nonleaky aquifers. (See Jaeger, J. C. - "Numerical values for the temperature in radial heat flow" J. Math. Phys., 34, 1956)
$A'(\rho,\tau)$	=	$\left[1 - 2\sum_{m=1}^{\infty} \frac{J_o(\beta_m \rho)}{\beta_m J_1(\beta_m)} e^{-\tau\beta_m^2}\right]$
$A_1(\rho,\tau,\tau_o)$	=	The function defined in Case I(c) .
c	=	A constant of proportionality.
erfc(x)	=	The complementary error function.
ierfc(x)	=	The first repeated integral of the error function. (See Carslaw and Jaeger - "Heat Conduction in Solids", Oxf. Univ. Press, 1947).
f_n	=	$(h_n - h_{n-1})\ A(\rho,\tau-\tau_{n-1})$
F_n	=	$(h_n - h_{n-1})\ G(\tau - \tau_{n-1})$
f'_n	=	$(h_n - h_{n-1})\ A'(\rho,\tau-\tau_{n-1})$
F'_n	=	$(h_n - h_{n-1})\ G'(\tau - \tau_{n-1})$
$G(\tau)$	=	The flowing well discharge function for nonleaky aquifers (see Jacob and Lohman - "Non-steady flow to a well of constant drawdown in extensive aquifers", Trans. AGU, 33, 1952).
$G'(\tau)$	=	$2\sum_{m=1}^{\infty} e^{-\tau\beta^2_m}$
h(x, z, t)	=	The hydraulic head at any point (x, z) and any time t .
h_n	=	The channel water level during the period following the nth sudden change. $(n \neq 0)$.
h_o	=	One-half amplitude of the sinusoidal fluctuation of channel water level.
J_o, J_1, Y_o	=	Bessel functions of the first kind,(zero and first order) and second kind (zero order).
K	=	The hydraulic conductivity of the aquifer.
K_o	=	The zero-order modified Bessel function of the second kind.
$M_o(y)$	=	The modulus of the complex number $J_o(y\, i^{3/2})$.(see McLachlan - "Bessel Functions for Engineers", Oxf. Univ. Press, 1955) .
$N_o(y)$	=	The modulus of the complex number $K_o(y\sqrt{i})$. (see McLachlan, 1955).
q	=	Rate of seepage per unit length of channel.
r_o	=	Radius of cylindrical face of aquifer in contact with channel bed.
S_s	=	Specific storage of aquifer.
t	=	Time since flow is set in motion.
t_{n-1}	=	Time at which the nth sudden change in channel water level takes place.
W(x)	=	The well function for nonleaky aquifers available in tabular form.



CASE I CASE II CASE III

REFERENCE:

Hantush, M.S., 1962, Flow of ground water in sands of nonuniform thickness, 2, Approximate theory: Jour. Geophys. Research, v. 67, no. 2, p. 711-720.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

CASE I. FLOW IN WEDGE-SHAPED AQUIFERS BETWEEN TWO PARALLEL RESERVOIRS

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial x^2} + \frac{m}{b_1 + mx}\frac{\partial h}{\partial x} = \frac{1}{\nu}\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

IA. Sudden change of water level in Reservoir I

$$h(x,0) = h_s$$
$$h(0,t) = h_s(0) + \delta h_1 = h_1 \quad \delta h_1$$
$$h(1,t) = h_s(1) = h_2$$

IB. Water level in Reservoir I varying linearly with time

$$h(x,0) = h_s$$
$$h(0,t) = h_1 + C_1 t$$
$$h(1,t) = h_2$$

SOLUTIONS (IA):

$$h = h_s + \delta h_1\left[\frac{\ln ky}{\ln k} - \pi\sum_{n=1}^{\infty} C(\beta_n k,\tau)\, V_o(\beta_n k y)\right]$$

$$q_1 = q_s - K_m \delta h_1\left[\frac{1}{\ln k} + 2\sum_{n=1}^{\infty} C(\beta_n k,\tau)\frac{J_o(\beta_n)}{J_o(\beta_n k)}\right]$$

$$q_2 = q_s - K_m \delta h_1\left[\frac{1}{\ln k} + 2\sum_{n=1}^{\infty} C(\beta_n k_2,\tau)\right]$$

where

$$h_s = h_1 - (h_1 - h_2)\frac{\ln[(b_1 + mx)/b_1]}{\ln[(b_1 + ml)/b_1]} \qquad q_s = Km(h_1 - h_2)/\ln[(b_1 + ml)/b_1]$$

SOLUTIONS (IB):

$$h = h_s + \frac{c_1 t}{\tau k^2}\left\{\frac{\tau k^2 \ln ky}{\ln k} + \frac{k^2(y^2-1)\ln(ky-1)\ln k + (k^2-1)\ln y}{(2\ln k)^2} + \pi\sum_{n=1}^{\infty}\frac{1}{\beta_n^2} C(\beta_n k,\tau)\, V_o(\beta_n k y)\right\}$$

$$q_1 = q_s - Km\frac{c_1 t}{\tau k^2}\left\{\frac{\tau k^2}{\ln k} + \frac{(k^2-1) + 2k^2(\ln k - 1)\ln k}{(2\ln k)^2} - 2\sum_{n=1}^{\infty}\frac{1}{\beta_n^2} C(\beta_n k,\tau)\frac{J_o(\beta_n)}{J_o(\beta_n k)}\right\}$$

$$q_2 = q_s - Km\frac{c_1 t}{\tau k^2}\left\{\frac{\tau k^2}{\ln k} \mp \frac{(k^2-1)-(k^2+1)\ln k}{(2\ln k)^2} - 2\sum_{n=1}^{\infty}\frac{1}{\beta_n^2}\ C(\beta_n k, \tau)\right\}$$

CASE II. FLOW IN A WEDGE-SHAPED CLOSED AQUIFER

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial x^2} + \frac{m}{b_1 + mx}\frac{\partial h}{\partial x} = \frac{1}{\nu}\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

II A. Sudden change in channel water level

$$h(x,0) = h_i$$

$$h(0,t) = h_i + \delta h_1$$

$$\frac{\partial h}{\partial x}(1,t) = 0$$

II B. Water level in channel varying linearly with time

$$h(x,0) = h_i$$

$$h(0,t) = h_i + ct$$

$$\frac{\partial h}{\partial x}(1,t) = 0$$

SOLUTIONS (II A):

$$h = h_i + \delta h_1 + \pi\delta h_1 \sum^{\infty} E(\epsilon_n R, \tau)\ V_o(\epsilon_n y)$$

$$q_1 = 2Km\,\delta h_1 \sum_{n=1}^{\infty} E(\epsilon_n R, \tau)$$

SOLUTIONS (II B):

$$h = h_i + \frac{ct}{\tau}\left\{\tau + \left[\frac{(y^2-1) - 2R^2\ln y}{4}\right] - \pi\sum_{n=1}^{\infty}\frac{1}{\epsilon_n^2}\ E(\epsilon_n R,\tau)\ V_o(\epsilon_n y)\right\}$$

$$q_1 = \frac{K\,mct}{2\tau}\left[(R^2-1) - 4\sum_{n=1}^{\infty}\frac{1}{\epsilon_n^2}\ E(\epsilon_n R,\tau)\right]$$

CASE III. FLOW IN SEMI-INFINITE SAND BECOMING EXPONENTIALLY THIN

DIFFERENTIAL EQUATION: (For $b = b_1 \exp(-2x/a)$)

$$\frac{\partial^2 h}{\partial x^2} - \frac{2}{a}\frac{\partial h}{\partial x} = \frac{1}{\nu}\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

III-A. Sudden change in channel water level

$$h(x,0) = h_i$$

$$h(0,t) = h_i + \delta h_1$$

$$h(\infty,t) = h_i$$

III-B. Channel water level varying linearly with time.

$$h(x,0) = h_i$$

$$h(0,t) = h_i + ct$$

$$h(\infty,t) = h_i$$

SOLUTIONS (III-A):

$$h = h_i + \frac{1}{2}\delta h_1\left[\operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}} - \frac{\sqrt{\nu t}}{a}\right) + e^{2x/a}\operatorname{erfc}\left(\frac{x}{\sqrt{4\nu t}} + \frac{\sqrt{\nu t}}{a}\right)\right]$$

$$q_1 = \frac{Kb_1\delta h_1}{a}\left[\frac{\exp(-\nu t/a^2)}{\sqrt{\pi\nu t}/a} - \operatorname{erfc}\left(\frac{\sqrt{\nu t}}{a}\right)\right]$$

SOLUTIONS(III-B):

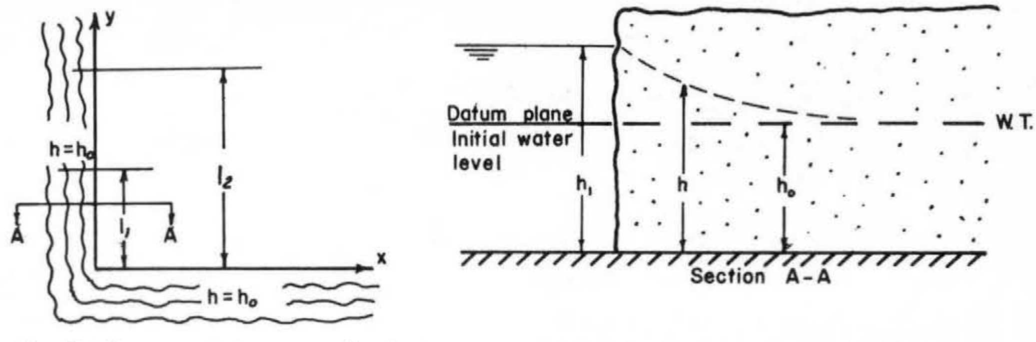
$$h = h_i + \frac{ct}{2(\sqrt{\nu t}/a)}\left[\left(\frac{x}{\sqrt{4\nu t}} + \frac{\sqrt{\nu t}}{a}\right) . e^{2x/a}\,\mathrm{erfc}\left(\frac{x}{-\sqrt{4\nu t}} + \frac{\sqrt{\nu t}}{a}\right) - \left(\frac{x}{\sqrt{4\nu t}} - \frac{\sqrt{\nu t}}{a}\right)\mathrm{erfc}\left(\frac{x}{\sqrt{4\nu t}} - \frac{\sqrt{\nu t}}{a}\right)\right]$$

$$q_1 = \frac{Kb_1 ct}{a}\left[\frac{\exp(-\nu t/a^2)}{-\sqrt{\pi \nu t}\;/a} + \frac{\mathrm{erfc}\,(\sqrt{\nu t}/a)}{2\,\nu t/\,a^2} - \mathrm{erfc}\,(\sqrt{\nu t}/a)\right]$$

SYMBOLS:

$A(x, y)$	= The flowing well function available in tabular form.
a	= A geometric parameter defining the exponential variation of aquifer thickness.
b, b_1, b_2	= The aquifer thickness at x, at x = 0, and at x = 1, respectively.
c, c_1	= Constants defining linear variation of water levels with time at some boundary.
$C(\beta_n k, \tau)$	$= \left\{J_o(\beta_n)\, J_o(\beta_n k)/[\,J_o^2(\beta_n) - J_o^2(\beta_n k)]\right\} . \exp(-\tau k^2 \beta_n^2)$.
$E(\epsilon_n R, \tau)$	$= \left\{J_1^2(\epsilon_n R)/[J_o^2(\epsilon_n) - J_1^2(\epsilon_n R)]\right\} \exp(-\tau\,\epsilon_n^2)$.
erfc	= The complementary error function.
$h(x, y, t)$	= The average piezometric head in a vertical column of the aquifer at any point(x, y) and any time t.
h_1, h_2	= Constant water levels in Reservoir I and Reservoir II, respectively.
h_i	= Initial head in a flow system.
h_s	= The average head during the initial steady state flow.
J_o, J_1	= Bessel functions of the first kind, zero and first order, respecitively.
k	$= b_1/(b_1 + m1)$.
K	= The hydraulic conductivity of the aquifer.
1	= Horizontal distance between Reservoir I and II.
m	$= m_2 - m_1$
m_1, m_2	= The slopes of the upper and lower confining beds of a wedge-shaped aquifer.
q_1, q_2	= Rate of seepage from or into Reservoir I and II respectively.
q_s	= The rate of seepage during initial stead-state flow.
R	$= 1/k$
S_s	= Specific storage.
t	= Time since an initial flow condition.
$V_o(\beta_n k y)$	$= Y_o(\beta_n)\, J_o(\beta_n k y) - J_o(\beta_n)\, Y_o(\beta_n k y)$
$V_o(\epsilon_n y)$	$= Y_o(\epsilon_n)\, J_o(\epsilon_n y) - J_o(\epsilon_n)\, Y_o(\epsilon_n y)$
y	$= (b_1 + m x)/b_1$
Y_o, Y_1	= Bessel functions of the second kind, zero and first order, respectively.
β_n	= The nth zero of $J_o(\beta_n)\, Y_o(\beta_n k) - Y_o(\beta_n)\, J_o(\beta_n k) = 0$
δh_1	= Amount of sudden change in water level of Reservoir I.
ϵ_n	= The nth zerc of $J_o(\epsilon_n)\, Y_1(\epsilon_n R) - Y_o(\epsilon_n)\, J_1(\epsilon_n R) = 0$
ν	$= K/S_s$
τ	$= \nu t m^2/b_1^2$.

A - 18. BANK STORAGE IN AN AQUIFER BETWEEN TWO PERPENDICULAR STREAMS



REFERENCE:

Hantush, M. S., Bank storage in an aquifer between two perpendicular streams, Unpublished lecture notes.

SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

STATEMENT OF THE PROBLEM:

To determine water level fluctuation and the rate and total volume of seepage into or out of the aquifer in response to a sudden change in water levels in the streams.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h^2}{\partial x^2} + \frac{\partial^2 h^2}{\partial y^2} = \frac{1}{\nu}\frac{\partial h^2}{\partial t}$$

BOUNDARY CONDITIONS:

$$h = (0, y, t) = h(x, 0, t) = h_1$$

$$h(x, y, t) = 0$$

$$\frac{\partial h}{\partial x}(\infty, y, t) = \frac{\partial h}{\partial y}(x, \infty, t) = 0$$

SOLUTION:

(a) Equation for water table

$$h^2 - h_o^2 = (h_1^2 - h_o^2)\left[1 - \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right) \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)\right]$$

(b) Equation for rate of bank storage along the y-axis

$$q = \frac{K(h_1^2 - h_o^2)}{\sqrt{4\pi\nu t}} \operatorname{erf}\left(\frac{y}{\sqrt{4\nu t}}\right)$$

Note: For that along the x-axis, replace y with x .

(c) Equation for total volume of bank storage in a stretch of stream along the y-axis between l_1 and l_2 during a period t_o from the start.

$$Q = \frac{2Kt_o}{\sqrt{\pi}}\left\{\left(\frac{l_2 - l_1}{\sqrt{4\nu t_o}}\right) + \operatorname{ierfc}\left(\frac{l_2}{\sqrt{4\nu t_o}}\right) - \operatorname{ierfc}\left(\frac{l_1}{\sqrt{4\nu t_o}}\right)\right.$$

$$+ \frac{1}{2\sqrt{\pi}}\left[\frac{l_2^2}{4\nu t_o} W\left(\frac{l_2^2}{4\nu t_o}\right) - \frac{l_1^2}{4\nu t_o} W\left(\frac{l_1^2}{4\nu t_o}\right)\right.$$

$$\left.\left. + \exp\left(-\frac{l_1^2}{4\nu t_o}\right) - \exp\left(-\frac{l_2^2}{4\nu t_o}\right)\right]\right\}$$

Note: The same applies if the stretch is along the x-axis.

SYMBOLS:

$\bar{b}$ $\approx \dfrac{h_o + h}{2} \approx \dfrac{h_o + h_1}{2}$

erf(x) $= \dfrac{2}{\sqrt{\pi}} \int_0^x \exp(-\beta^2)\, d\beta$ = the error function of x, tabular values of which are available (Dwight, 1961; Carslaw and Jaeger, 1959).

erfc(x) = $1 - \mathrm{erf}(x)$ = the complement of the error function, tabular values of which are are available (Carslaw and Jaeger, 1959).

h = height of the water table above the base of the aquifer.

h_o = initial height of the water table above the base of the aquifer.

h_1 = elevation of water level in streams for $t > 0$.

$i^n \mathrm{erfc}(x)$ $= \int_x^\infty i^{n-1} \mathrm{erfc}(\beta)\, d\beta$, with $n = 1, 2, \ldots\ldots$, and $i^0 \mathrm{erfc}(x) = \mathrm{erfc}(x)$ = the nth repeated integral of the error function, which is available in tabular form (Kaye, 1955; Carslaw and Jaeger, 1959).

K = hydraulic conductivity of the aquifer

l_1, l_2 = see figure

Q = total volume of bank storage in a stretch of stream between l_1 and l_2 during a period t_o from the start.

q = rate of bank storage at any distance along the stream and at any time t since the start.

t = time since change of water level in the stream.

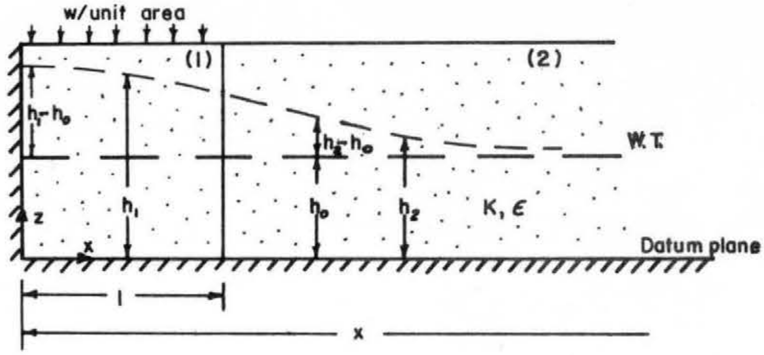
t_o = length of period of continuous flow since the start.

$W(u)$ $= \int_u^\infty \dfrac{e^{-y}}{y}\, dy$ = well function for non-leaky aquifers; tabular values are avaliable.

ϵ = specific yield of the aquifer.

ν $= \dfrac{K\bar{b}}{\epsilon}$

A - 19. GROWTH OF GROUND WATER RIDGE IN RESPONSE TO DEEP PERCOLATION



REFERENCE:

Hantush, M. S., Growth of ground water ridge in response to deep percolation, Unpublished lecture notes.

SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

STATEMENT OF THE PROBLEM:

The problem is to determine the growth of a ground-water ridge in response to a uniform rate of deep percolation over a limited area. The aquifer extends indefinitely in the x- direction.

DIFFERENTIAL EQUATIONS:

$$\frac{\partial^2 h_1^2}{\partial x^2} + \frac{2W}{K} = \frac{1}{\nu_1}\frac{\partial h_1^2}{\partial t} \qquad \frac{\partial^2 h_2^2}{\partial x^2} = \frac{1}{\nu_2}\frac{\partial h_2^2}{\partial t}$$

BOUNDARY CONDITIONS:

$$h_1(x,0) = h_o \qquad h_2(x,0) = h_2(\infty,t) = h_o$$

$$\frac{\partial h_1}{\partial x}(0,t) = 0 \qquad h_1(l,t) = h_2(l,t)$$

$$K\,h_1\frac{\partial h_1}{\partial x}(l,t) = K\,h_2\frac{\partial h_2}{\partial x}(l,t)$$

SOLUTION:

$$h_2^2(x,t) = h_o^2 + \frac{W\nu t}{K}\left[4\,i^2\operatorname{erfc}\left(\frac{x-l}{\sqrt{4\nu t}}\right) - 4\,i^2\operatorname{erfc}\left(\frac{x+l}{\sqrt{4\nu t}}\right)\right]$$

$$h_1^2(x,t) = h_o^2 + \frac{2W\nu t}{K}\left\{1-\frac{1}{2}\left[4\,i^2\operatorname{erfc}\left(\frac{l-x}{\sqrt{4\nu t}}\right) + 4\,i^2\operatorname{erfc}\left(\frac{l+x}{\sqrt{4\nu t}}\right)\right]\right\}$$

SYMBOLS:

$\operatorname{erfc}(x)$ = $1-\operatorname{erf}(x)$ = the complement of the error function, tabular values of which are available (Carslaw and Jaeger, 1959).

h_o = original height of the water table above the base of the water-table aquifer.

h_1, h_2 = height of the water table above the base of the water-table aquifer after percolation has started in sections 1 and 2 respectively.

$i^n\operatorname{erfc}(x)$ = $\int_x^\infty i^{n-1}\operatorname{erfc}(\beta)\,d\beta$, with $n=1,2,\ldots$, and $i^0\operatorname{erfc}(x) = \operatorname{erfc}(x)$ the nth repeated integral of the error function, which is available in tabular form (Kaye, 1955; Carslaw and Jaeger, 1959).

K_1, K_2 = hydraulic conductivity of the aquifer in sections 1 and 2 respectively.

l = horizontal length from the origin (x, z) to $x = l$.

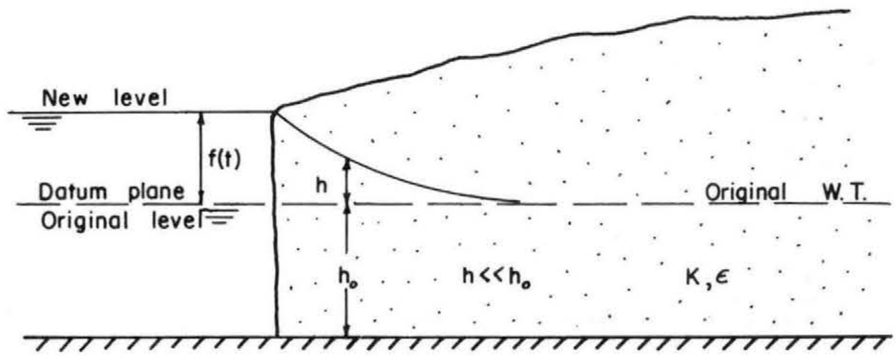
t = time since the initial condition of flow.

W = uniform rate of percolation per unit area.

ϵ = specific yield of the aquifer.

ν = $\frac{Kh_o}{\epsilon}$

AN EQUATION FOR ESTIMATING TRANSMISSIBILITY AND COEFFICIENT OF STORAGE FROM RIVER-LEVEL FLUCTUATIONS



REFERENCE:

Hantush, M. S., 1961, Discussion of - An equation for estimating transmissibility and coefficient of storage from river-level fluctuations, by P. P. Rowe: Jour. Geophys. Research, v. 66, no. 4, 1310-11.

SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

STATEMENT OF THE PROBLEM:

The problem is to find solutions for the fluctuation of ground water levels in response to water level fluctuations in streams that cut through the water-bearing materials. The stream is assumed to flow in fairly straight and effectively long course.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial x^2} = \frac{1}{\nu}\frac{\partial h}{\partial t} \; ; \qquad \nu = T/\epsilon$$

BOUNDARY CONDITIONS:

$$h(x, 0) = 0$$
$$h(\infty, t) = 0$$
$$h(0, t) = f(t)$$

SOLUTION:

For $h(0,t) = f(t) = ct$,

$$h(x,t) = ct\,[\,4i^2 \operatorname{erfc}(U)\,]$$

$$= ct\left\{(1 + 2U^2)\operatorname{erfc}(U) - (2/\sqrt{\pi})\,U \exp(-U^2)\right\}$$

For $h(o,t) = f(t) = c_1\sqrt{t}$,

$$h(x,t) = (c_1\sqrt{\pi t}/2)\,[\,2i \operatorname{erfc}(U)\,]$$

$$= c_1\sqrt{t}\left\{\exp(-U^2) - \sqrt{\pi}\;U \operatorname{erfc}(U)\right\}$$

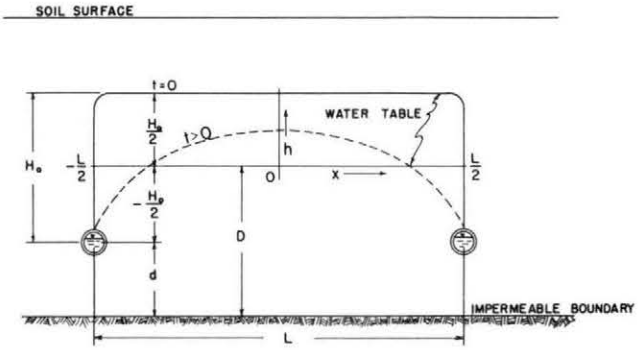
For $h(0,t) = f(t) = c_2 t(t_1 - t)$,

$$h(x,t) = c_2 t\left\{t_1\,[\,4i^2 \operatorname{erfc}(U)\,] - 4t\,[\,8i^4 \operatorname{erfc}(U)\,]\right\}$$

SYMBOLS:

c	=	Constant.
erfc	=	Complementary error function; tabular values are available (Dwight, 1961; Carslaw and Jaeger, 1959).
h	=	Water level at any time t and any distance x from the stream.
i^nerfc	=	The nth repeated integral of the error function; tabular values are available (Carslaw and Jaeger, 1959; Kaye, 1955).
K	=	Hydraulic conductivity.
T Kh_o	=	Transmissivity of the aquifer.
t	=	Time since pumping started.
U	=	$x / \sqrt{4 \nu t}$
x	=	Distance of the point of observation from the intersection of the aquifer with the stream.
ϵ	=	Specific yield.
ν	=	T/ϵ

A - 21 PARALLEL DRAINS ABOVE IMPERMEABLE BOUNDARY



REFERENCE:

Brooks, R. H., 1961, Unsteady flow of ground-water into drain tile: Am. Soc. Civil Engineers Proc., v. 87, no. IR 2, p. 27-37.

SUMMARIZED BY:

R. H. Brooks, Agricultural Research Service

DIFFERENTIAL EQUATION:

$$\alpha \frac{\partial^2 h}{\partial x^2} - \frac{\partial h}{\partial t} = -\frac{\alpha}{D}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{\alpha}{D} h \frac{\partial^2 h}{\partial x^2}$$

BOUNDARY CONDITIONS:

$$h\left(\pm \frac{L}{2}, t\right) = -\frac{H_o}{2}, \quad (t \geq 0)$$

$$h(x,0) = \frac{H_o}{2} \quad \left(-\frac{L}{2} \leq x \leq \frac{L}{2}\right)$$

SOLUTIONS:

$$\text{(1)} \quad h(x,t) = \left(1 + \frac{H_o}{2D}\right)\left(h_o + \frac{H_o}{2}\right) - \frac{\alpha}{2D} H_o t \frac{\partial^2 h_o}{\partial x^2} + \frac{1}{2D} \frac{\partial h_o}{\partial x} \int \left(h_o + \frac{H_o}{2}\right) dx$$

$$- \frac{1}{2D}\left(h_o + \frac{H_o}{2}\right)^2 - \int_0^t \frac{\partial F}{\partial T}(H_o - G)\,dT - \frac{H_o}{2} \qquad (1)$$

where

$$h_o = \frac{4H_o}{\pi} \sum_{n=1,3,5..}^{\infty} (-1)^{(n-1)/2} \frac{1}{n} \exp\left(-\frac{\alpha n^2 \pi^2 t}{L^2}\right) \cos \frac{n\pi}{L} x - \frac{H_o}{2}$$

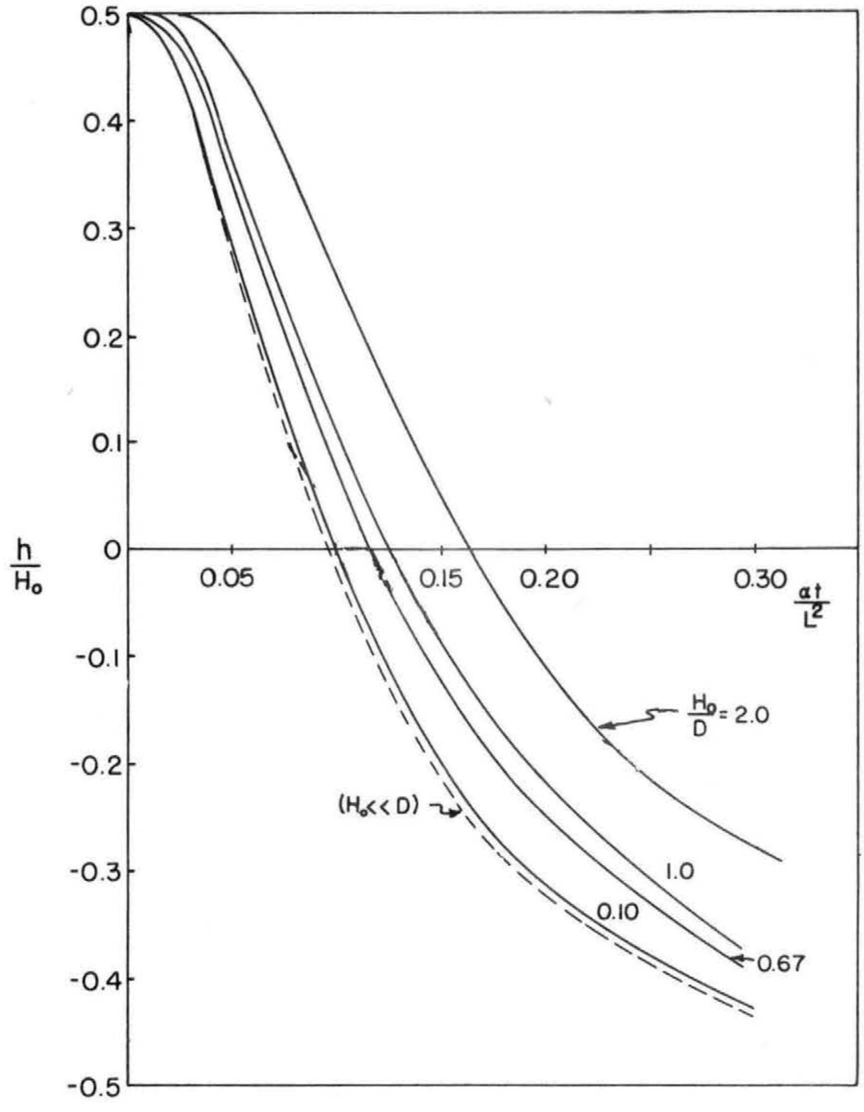
$$F = \frac{8 H_o^2}{\pi^2 D} \sum_{n=1,3,5..}^{\infty} \sum_{m=1,3,5..}^{\infty} \frac{1}{2} \exp\left[-\frac{(m^2+n^2)\,\alpha\,\pi^2 T}{L^2}\right]$$

The term, G, under the time integral is equal to $\left(h_o + \frac{H_o}{2}\right)$ that in turn is a function of $[(t-T), x]$ in which T is a time variable that takes on the range of values $t \geq T \geq 0$. The particular integral

$$\frac{1}{2D} \quad \frac{\partial h_o}{\partial x} \int (h_o + \frac{H_o}{2}) \, dx$$

did not satisfy the boundary condition, hence, the time integral in Eq. 1, that was evaluated numerically, was introduced to permit restoration of the boundary condition.

$$h_1 = -D + \sqrt{D^2 + 2 D h_o + (\frac{H_o}{2})^2} \qquad (2)$$



—THEORETICAL CURVES OF RELATIVE WATER TABLE HEIGHT, h/H_0, AS A FUNCTION OF THE TIME PARAMETER $\alpha t/L^2$ FOR TILE AT VARIOUS RELATIVE DISTANCES H_0/D ABOVE THE IMPERMEABLE BOUNDARY AND FOR $x = 0$

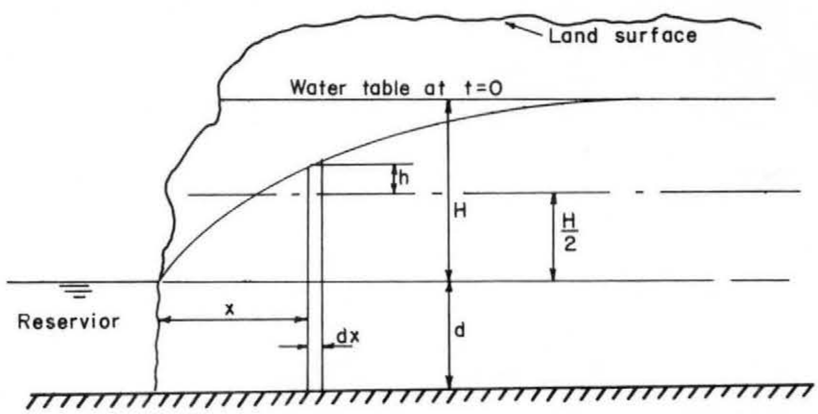
A - 22 FLOW OF GROUND WATER INTO A SURFACE RESERVOIR

REFERENCE:

Haushild, W. L. and Kruse, E. G., 1960, Transient flows through an infinite saturated aquifer of zero slope: Am. Soc. Civil Engineers Proc., v. 86, no. HY 7, p. 13-20.

SUMMARIZED BY:

D.E.L. Maasland, Colorado State University



SOLUTION I:

for $d \gg H$

Differential Equation:

$$\alpha \frac{\partial^2 h}{\partial x^2} = \frac{\partial h}{\partial t}$$

Boundary Conditions:

$$h(0, t) = -H/2$$
$$h(\infty, t) = H/2$$
$$h(x, 0) = H/2$$

Solution: (See Carslaw and Jaeger)

$$h_1 = -\frac{H}{2} + \frac{2H}{\sqrt{\pi}} \int_0^{x/\sqrt{4\alpha t}} e^{-u^2}\, du$$

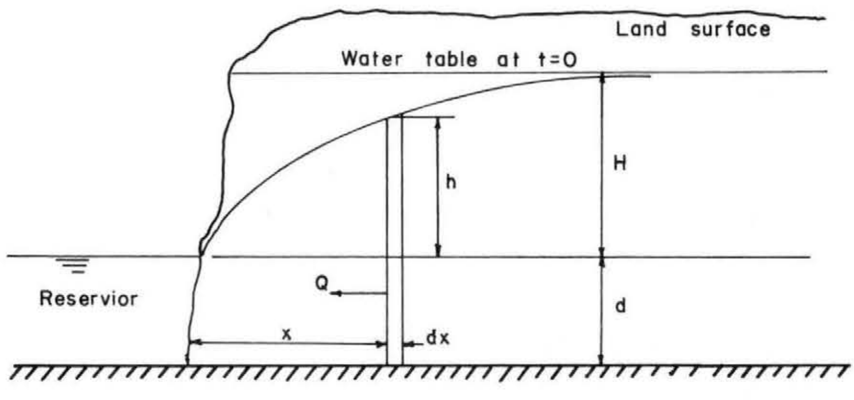
where $\phi = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^{x/\sqrt{4\alpha t}} e^{-u^2}\, du$ is the error function

SOLUTION II:

By applying the method of Picard.

$$h_2 = -\frac{1}{2D}\left(-\frac{H}{2} + H\phi\right)^2 - \frac{H}{2}\left(-\frac{H}{2D}\,\frac{2}{\sqrt{\pi}}\,\frac{x}{\sqrt{4\alpha t}}\, e^{-\frac{x^2}{4\alpha t}} + \frac{H}{D}\,\frac{2}{\pi}\, e^{-\frac{2x^2}{4\alpha t}}\right.$$

$$\left. + \frac{H}{D}\,\frac{2}{\sqrt{\pi}}\,\frac{x}{\sqrt{4\alpha t}}\, e^{-\frac{x^2}{4\alpha t}}\,\phi\right) - \frac{H}{2}\left[1 - \left(\frac{H}{4D} + \frac{2}{\pi}\,\frac{H}{D}\right)\right][1 - \phi] + \frac{H}{2}\left[1 + \frac{H}{4D}\right]\phi$$



SOLUTION III:

Identical to Solution I, but reference level changed (see sketch)

$$h_3 = H\phi$$

SOLUTION IV:

By integration of $K(d+h)\frac{\partial h}{\partial x} = KD\frac{dh_3}{\partial x}$

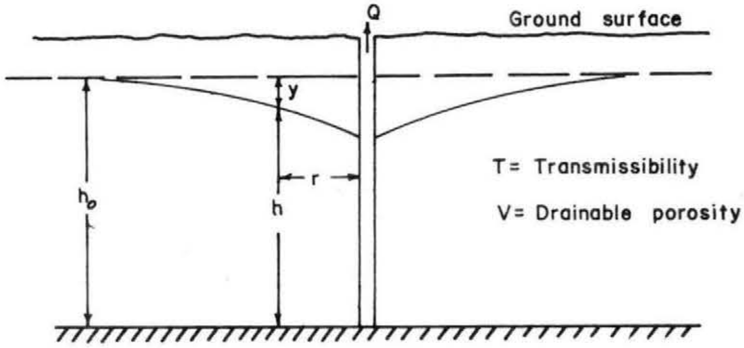
$$h_4 = -\sqrt{2D\,h_3 + d^2} - d$$

APPENDIX B

SUMMARIES OF SOLVED CASES IN CYLINDRICAL COORDINATES

TABLE OF CONTENTS

B - 1. WELL PUMPED AT CONSTANT RATE Q (THEIS EQUATION)



REFERENCE:

Theis, C. V., 1935, The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground water storage: Am. Geophys. Union Trans., v. 16, p. 519-524.

SUMMARIZED BY:

D.E.L. Maasland, Colorado State University

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{V}{T}\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

$$h(r,0) = h_o$$

$$h(\infty,t) = h_o$$

SOLUTION:

$$y = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-u}}{u}\,du \qquad \text{where} \quad u = \frac{r^2 V}{4Tt}$$

$$\int_u^\infty \frac{e^{-u}}{u}\,du = W(u) = 0.577216 - \ln u + u - \frac{u^2}{2\cdot 2!} + \frac{u^3}{3\cdot 3!} - \frac{u^4}{4\cdot 4!} + \ldots\ldots$$

COMMENTS:

Theoretically, the equation applies rigidly only to water bodies

(1) which are contained in entirely homogeneous sediments

(2) which have infinite areal extent

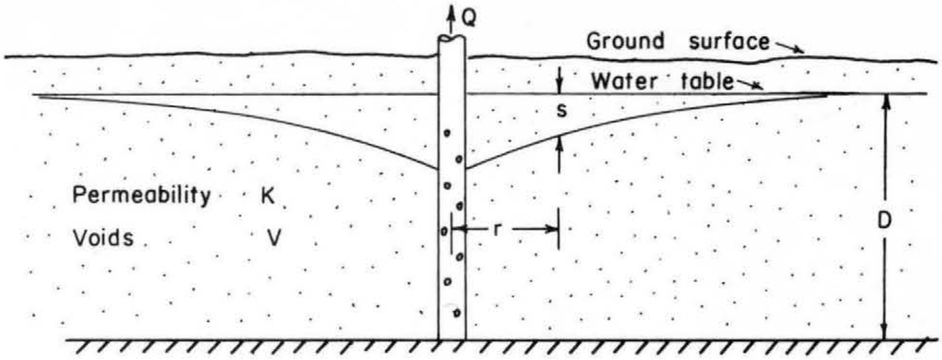
(3) in which the well penetrates the entire thickness of the water body

(4) in which the transmissibility is constant at all times and in all places

(5) in which the well has an infinitesimal diameter

(6) (applicable only to unconfined water-bodies) - in which the water in the volume of sediments through which the water table has fallen is discharged instantaneously with the fall of the water table.

B - 2. WELL DRAWING WATER FROM AN AQUIFER



REFERENCE:

Glover, R. E., Well pumping and drainage formulas, in Bureau of Reclamation Technical Memorandum, No. 657, Section A, p 2 - Case 4 page 7.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation.

DIFFERENTIAL EQUATION:

$$\frac{\partial s}{\partial t} = \alpha \left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right)$$

BOUNDARY CONDITIONS:

$s = 0$ when $t = 0$ for $r > 0$

$s \to 0$ when $r \to \infty$

SOLUTION:

$$s = \frac{Q}{2 \pi K D} \int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u} du \qquad (s \ll D)$$

SYMBOLS (Consistent units):

$\alpha = \frac{KD}{V}$

Q = the flow of the well.

COMMENTS:

Integral is tabulated in T. M. 657., pages 9 and 10, which are reproduced on the following pages.

Typical use: Pumped well.

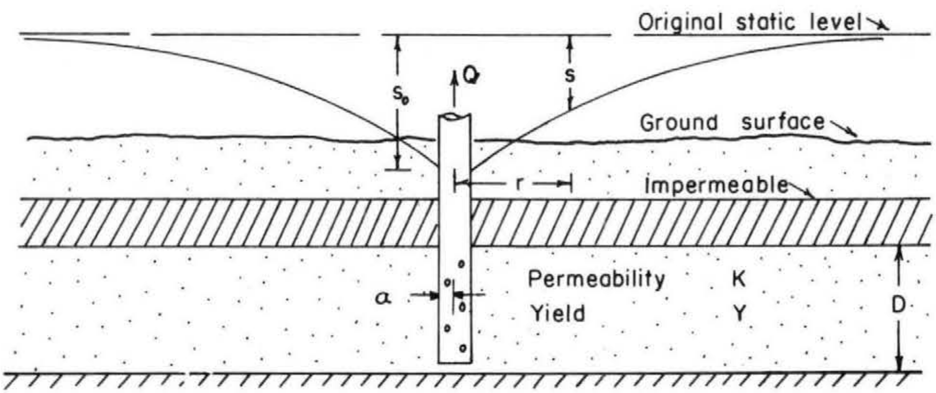
Tabulation of pumped well integral,
from p. 9 and 10 of reference .

$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$
0	∞	0.050	2.7084	0.091	2.1124	0.42	0.6634
0.010	4.3166	0.051	2.6886	0.092	2.1016	0.43	0.6437
0.011	4.2213	0.052	2.6692	0.093	2.0909	0.44	0.6247
0.012	4.1343	0.053	2.6502	0.094	2.0803	0.45	0.6062
0.013	4.0543	0.054	2.6316	0.095	2.0698	0.46	0.5884
0.014	3.9802	0.055	2.6133	0.096	2.0594	0.47	0.5710
0.015	3.9112	0.056	2.5953	0.097	2.0491	0.48	0.5542
0.016	3.8467	0.057	2.5777	0.098	2.0390	0.49	0.5380
0.017	3.7861	0.058	2.5604	0.099	2.0289	0.50	0.5221
0.018	3.7289	0.059	2.5434	0.100	2.0190	0.51	0.5068
0.019	3.6749	0.060	2.5266	0.11	1.9247	0.52	0.4919
0.020	3.6236	0.061	2.5101	0.12	1.8388	0.53	0.4774
0.021	3.5748	0.062	2.4939	0.13	1.7600	0.54	0.4634
0.022	3.5284	0.063	2.4780	0.14	1.6873	0.55	0.4498
0.023	3.4839	0.064	2.4623	0.15	1.6197	0.56	0.4365
0.024	3.4414	0.065	2.4469	0.16	1.5567	0.57	0.4237
0.025	3.4006	0.066	2.4317	0.17	1.4977	0.58	0.4112
0.026	3.3614	0.067	2.4167	0.18	1.4423	0.59	0.3990
0.027	3.3237	0.068	2.4020	0.19	1.3900	0.60	0.3872
0.028	3.2873	0.069	2.3874	0.20	1.3406	0.61	0.3758
0.029	3.2523	0.070	2.3731	0.21	1.2938	0.62	0.3646
0.030	3.2184	0.071	2.3590	0.22	1.2494	0.63	0.3538
0.031	3.1856	0.072	2.3451	0.23	1.2072	0.64	0.3433
0.032	3.1539	0.073	2.3313	0.24	1.1669	0.65	0.3330
0.033	3.1232	0.074	2.3178	0.25	1.1285	0.66	0.3231
0.034	3.0934	0.075	2.3045	0.26	1.0917	0.67	0.3134
0.035	3.0644	0.076	2.2913	0.27	1.0565	0.68	0.3040
0.036	3.0363	0.077	2.2783	0.28	1.0228	0.69	0.2949
0.037	3.0089	0.078	2.2655	0.29	0.9904	0.70	0.2860
0.038	2.9823	0.079	2.2528	0.30	0.9594	0.71	0.2774
0.039	2.9563	0.080	2.2403	0.31	0.9295	0.72	0.2690
0.040	2.9311	0.081	2.2280	0.32	0.9007	0.73	0.2609
0.041	2.9064	0.082	2.2158	0.33	0.8731	0.74	0.2529
0.042	2.8824	0.083	2.2037	0.34	0.8464	0.75	0.2452
0.043	2.8589	0.084	2.1919	0.35	0.8206	0.76	0.2377
0.044	2.8359	0.085	2.1801	0.36	0.7958	0.77	0.2305
0.045	2.8135	0.086	2.1685	0.37	0.7718	0.78	0.2234
0.046	2.7916	0.087	2.1570	0.38	0.7486	0.79	0.2165
0.047	2.7701	0.088	2.1457	0.39	0.7262	0.80	0.2098
0.048	2.7491	0.089	2.1345	0.40	0.7046	0.81	0.2033
0.049	2.7285	0.090	2.1234	0.41	0.6836	0.82	0.1970

Table --Continued

$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$	$\frac{r}{\sqrt{4\alpha t}}$	$\int_{\frac{r}{\sqrt{4\alpha t}}}^{\infty} \frac{e^{-u^2}}{u}\,du$
0.83	0.1909	1.24	0.04730	1.65	0.00932	2.6	0.00008
0.84	0.1849	1.25	0.04559	1.66	0.00892	2.7	0.00004
0.85	0.1791	1.26	0.04394	1.67	0.00855	2.8	0.00002
0.86	0.1735	1.27	0.04235	1.68	0.00819	2.9	0.00001
0.87	0.1680	1.28	0.04080	1.69	0.00784	3.0	0.00001
0.88	0.1627	1.29	0.03931	1.70	0.00751		
0.89	0.1575	1.30	0.03787	1.71	0.00719		
0.90	0.1525	1.31	0.03647	1.72	0.00688		
0.91	0.1476	1.32	0.03512	1.73	0.00658		
0.92	0.1429	1.33	0.03382	1.74	0.00630		
0.93	0.1383	1.34	0.03256	1.75	0.00603		
0.94	0.1339	1.35	0.03134	1.76	0.00576		
0.95	0.1295	1.36	0.03016	1.77	0.00551		
0.96	0.1253	1.37	0.02903	1.78	0.00527		
0.97	0.1212	1.38	0.02793	1.79	0.00504		
0.98	0.1173	1.39	0.02687	1.80	0.00482		
0.99	0.1134	1.40	0.02584	1.81	0.00460		
1.00	0.10969	1.41	0.02486	1.82	0.00440		
1.01	0.10607	1.42	0.02390	1.83	0.00420		
1.02	0.10255	1.43	0.02298	1.84	0.00402		
1.03	0.09914	1.44	0.02209	1.85	0.00384		
1.04	0.09583	1.45	0.02123	1.86	0.00366		
1.05	0.09262	1.46	0.02041	1.87	0.00350		
1.06	0.08950	1.47	0.01961	1.88	0.00334		
1.07	0.08648	1.48	0.01884	1.89	0.00319		
1.08	0.08355	1.49	0.01810	1.90	0.00304		
1.09	0.08071	1.50	0.01738	1.91	0.00290		
1.10	0.07796	1.51	0.01669	1.92	0.00277		
1.11	0.07529	1.52	0.01603	1.93	0.00264		
1.12	0.07270	1.53	0.01538	1.94	0.00252		
1.13	0.07020	1.54	0.01477	1.95	0.00240		
1.14	0.06777	1.55	0.01417	1.96	0.00229		
1.15	0.06541	1.56	0.01360	1.97	0.00218		
1.16	0.06313	1.57	0.01305	1.98	0.00208		
1.17	0.06092	1.58	0.01252	1.99	0.00198		
1.18	0.05878	1.59	0.01200	2.00	0.00189		
1.19	0.05671	1.60	0.01151	2.1	0.00115		
1.20	0.05470	1.61	0.01104	2.2	0.00069		
1.21	0.05276	1.62	0.01058	2.3	0.00041		
1.22	0.05088	1.63	0.01014	2.4	0.00024		
1.23	0.04906	1.64	0.00972	2.5	0.00014		

B - 3. FLOWING ARTESIAN WELL



REFERENCE:

Glover, R. E., Well pumping and drainage formulas, in Bureau of Reclamation Technical Memorandum No. 657, Section A, p. 13 - case 5.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation.

DIFFERENTIAL EQUATION:

$$\frac{\partial s}{\partial t} = \alpha \left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r} \frac{\partial s}{\partial r} \right)$$

BOUNDARY CONDITIONS:

$s = 0$ when $t = 0$ for $r > a$

$\frac{\partial s}{\partial r} = 0$ when $r = b$

SOLUTION:

$$s = s_o \sum_{n=1}^{n=\infty} A_n U_o (\beta_n r) e^{-\alpha \beta_n^2 t} \qquad U_o(\beta_n r) = J_o(\beta_n r) \, Y_o(\beta_n a) - J_o(\beta_n a) \, Y_o(\beta_n r)$$

$$A_n = \frac{\frac{a}{\beta_n} U'_o (\beta_n a)}{\frac{1}{2}\left[\left(b U_o (\beta_n b) \right)^2 - \left(a U'_o (\beta_n a) \right)^2 \right]} \qquad U'_o (\beta_n b) = o$$

SYMBOLS (Consistent units):

$\alpha = \frac{KD}{Y}$

Y = the yield of the aquifer per unit horizontal area per unit of drawdown (Dimensionless).

$Q = 2\pi KDs_o \, G \left(\frac{\sqrt{4\alpha t}}{a} \right)$

COMMENTS:

Typical use: Flowing artesian well.

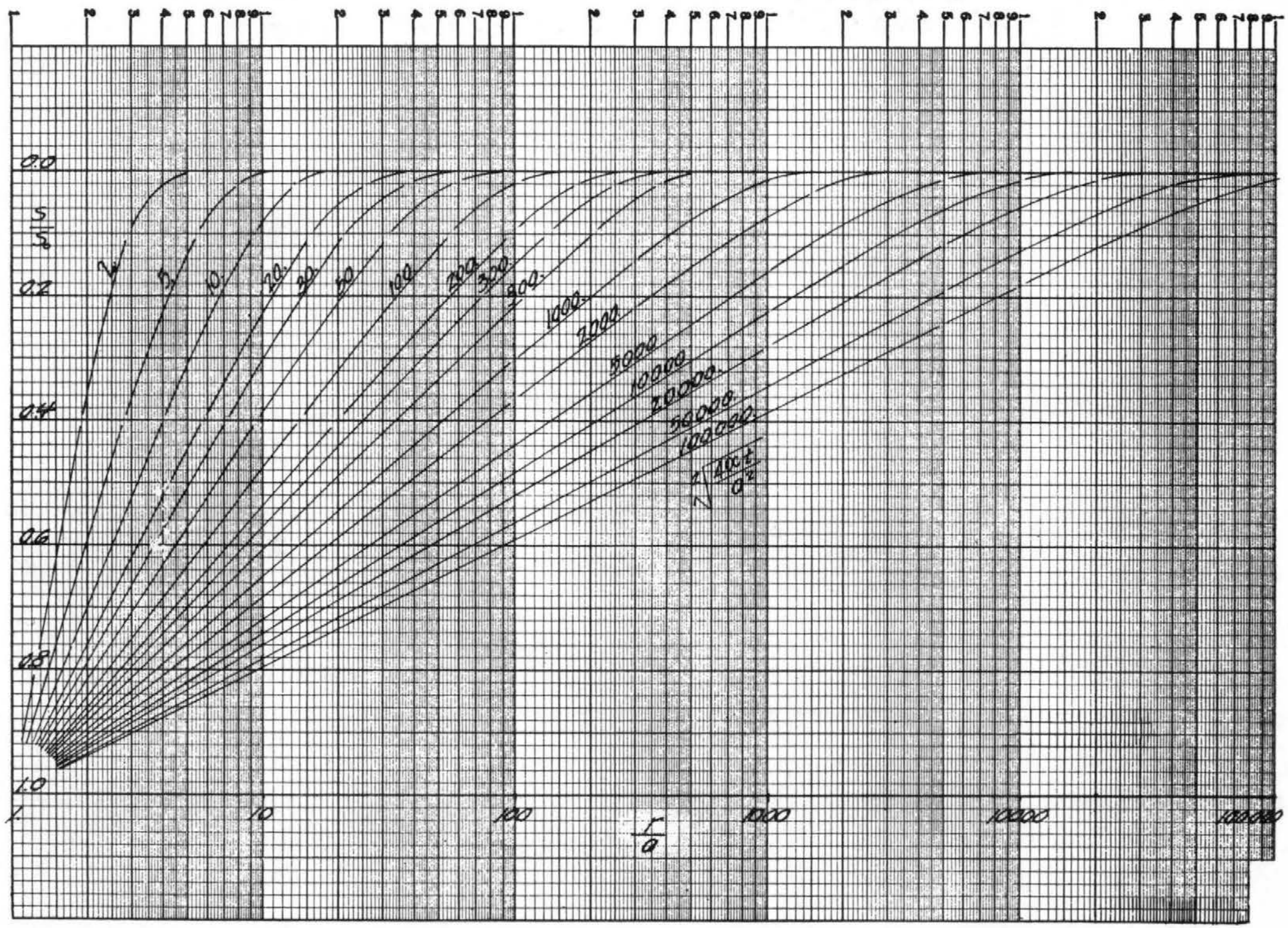
Tabulations of G $\frac{-\sqrt{4\alpha t}}{a}$, from p. 15 and 16 of reference.

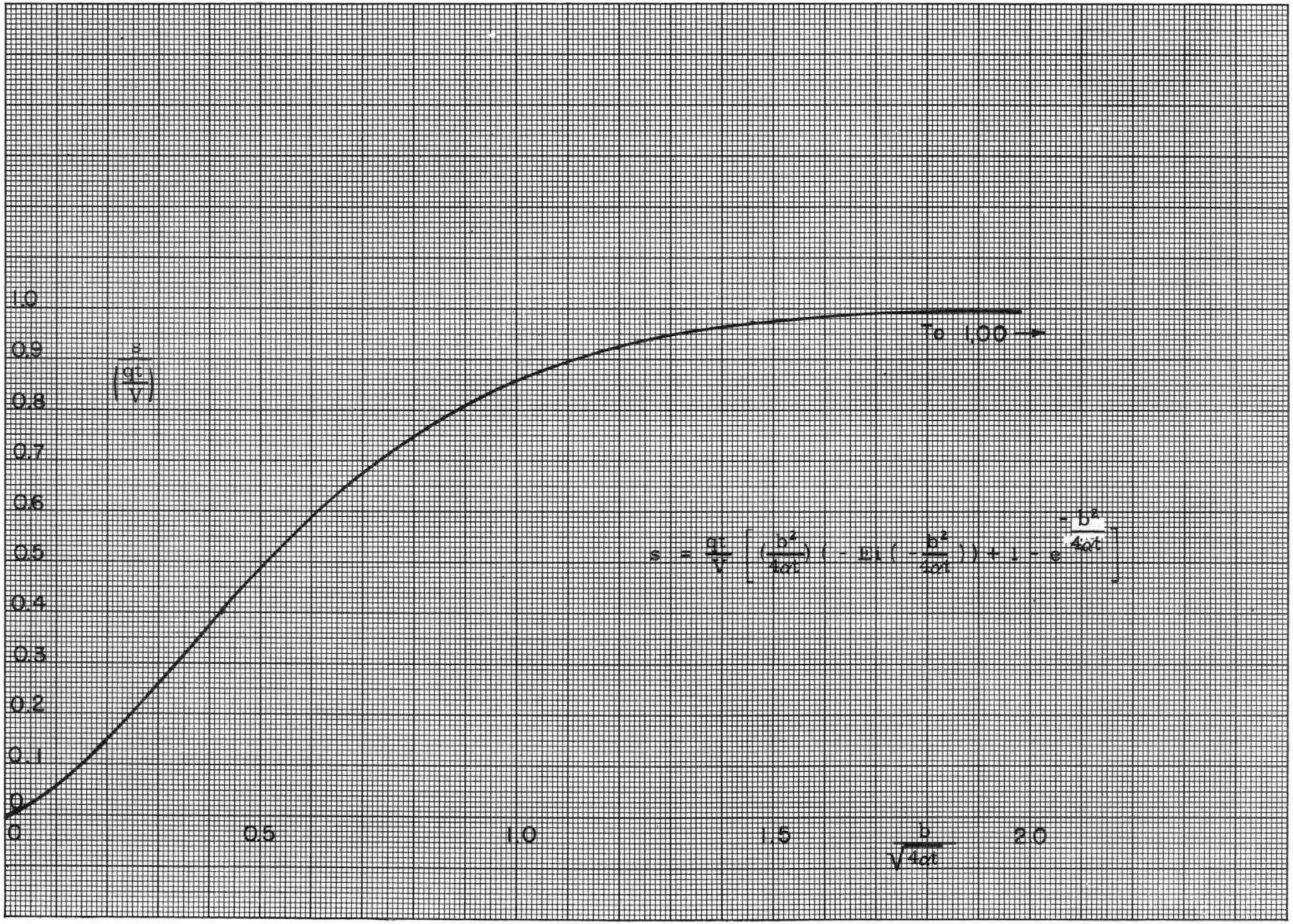
$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$
0.001	1,128.88	0.042	27.37	0.083	14.09	0.34	3.798	0.75	1.964
0.002	564.69	0.043	26.74	0.084	13.93	0.35	3.703	0.76	1.944
0.003	376.63	0.044	26.14	0.085	13.77	0.36	3.613	0.77	1.924
0.004	282.60	0.045	25.58	0.086	13.61	0.37	3.528	0.78	1.905
0.005	226.18	0.046	25.03	0.087	13.46	0.38	3.447	0.79	1.886
0.006	188.56	0.047	24.51	0.088	13.31	0.39	3.370	0.80	1.868
0.007	161.70	0.048	24.01	0.089	13.17	0.40	3.298	0.81	1.850
0.008	141.55	0.049	23.53	0.090	13.03	0.41	3.227	0.82	1.833
0.009	125.88	0.050	23.07	0.091	12.89	0.42	3.161	0.83	1.816
0.010	113.34	0.051	22.63	0.092	12.76	0.43	3.098	0.84	1.799
0.011	103.08	0.052	22.20	0.093	12.62	0.44	3.038	0.85	1.783
0.012	94.53	0.053	21.79	0.094	12.49	0.45	2.980	0.86	1.767
0.013	87.30	0.054	21.40	0.095	12.37	0.46	2.926	0.87	1.752
0.014	81.10	0.055	21.02	0.096	12.24	0.47	2.873	0.88	1.736
0.015	75.73	0.056	20.65	0.097	12.12	0.48	2.822	0.89	1.722
0.016	71.02	0.057	20.30	0.098	12.00	0.49	2.774	0.90	1.707
0.017	66.88	0.058	19.95	0.099	11.89	0.50	2.728	0.91	1.693
0.018	63.19	0.059	19.62	0.100	11.78	0.51	2.683	0.92	1.679
0.019	59.88	0.060	19.30	0.11	10.751	0.52	2.640	0.93	1.665
0.020	56.92	0.061	18.99	0.12	9.895	0.53	2.599	0.94	1.652
0.021	54.23	0.062	18.70	0.13	9.171	0.54	2.559	0.95	1.639
0.022	51.79	0.063	18.41	0.14	8.551	0.55	2.520	0.96	1.626
0.023	49.56	0.064	18.13	0.15	8.013	0.56	2.483	0.97	1.614
0.024	47.52	0.065	17.86	0.16	7.542	0.57	2.447	0.98	1.602
0.025	45.64	0.066	17.59	0.17	7.126	0.58	2.412	0.99	1.590
0.026	43.90	0.067	17.34	0.18	6.757	0.59	2.379	1.00	1.578
0.027	42.29	0.068	17.09	0.19	6.427	0.60	2.346	1.01	1.566
0.028	40.80	0.069	16.85	0.20	6.129	0.61	2.316	1.02	1.555
0.029	39.41	0.070	16.61	0.21	5.860	0.62	2.285	1.03	1.544
0.030	38.11	0.071	16.39	0.22	5.615	0.63	2.256	1.04	1.533
0.031	36.90	0.072	16.17	0.23	5.391	0.64	2.227	1.05	1.523
0.032	35.76	0.073	15.95	0.24	5.186	0.65	2.200	1.06	1.512
0.033	34.69	0.074	15.74	0.25	4.998	0.66	2.173	1.07	1.502
0.034	33.69	0.075	15.54	0.26	4.824	0.67	2.147	1.08	1.492
0.035	32.74	0.076	15.34	0.27	4.662	0.68	2.122	1.09	1.482
0.036	31.84	0.077	15.14	0.28	4.513	0.69	2.098	1.10	1.472
0.037	31.00	0.078	14.96	0.29	4.373	0.70	2.073	1.20	1.383
0.038	30.19	0.079	14.78	0.30	4.243	0.71	2.050	1.30	1.307
0.039	29.43	0.080	14.60	0.31	4.121	0.72	2.028	1.40	1.242
0.040	28.71	0.081	14.42	0.32	4.007	0.73	2.006	1.50	1.185
0.041	28.02	0.082	14.25	0.33	3.899	0.74	1.984	1.60	1.136

Table --Continued

Value of $G\left(\frac{\sqrt{4\alpha t}}{a}\right)$

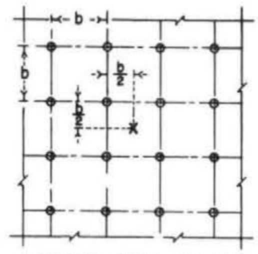
$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$	$\frac{\sqrt{4\alpha t}}{a}$	$G\left(\frac{\sqrt{4\alpha t}}{a}\right)$
1.70	1.091	5.90	0.551	11	0.428	80	0.239	500	0.167
1.80	1.052	6.00	0.547	12	0.416	90	0.232	600	0.162
1.90	1.016	6.1	0.543	13	0.404	100	0.226	700	0.158
2.00	0.984	6.2	0.539	14	0.394	110	0.221	800	0.155
2.10	0.954	6.3	0.535	15	0.384	120	0.217	900	0.152
2.20	0.928	6.4	0.531	16	0.375	130	0.213	1,000	0.150
2.30	0.903	6.5	0.528	17	0.367	140	0.210	2,000	0.136
2.40	0.880	6.6	0.524	18	0.359	150	0.207	3,000	0.129
2.50	0.860	6.7	0.521	19	0.352	160	0.204	4,000	0.124
2.60	0.840	6.8	0.517	20	0.346	170	0.202	5,000	0.121
2.70	0.822	6.9	0.514	21	0.340	180	0.200	6,000	0.118
2.80	0.805	7.0	0.511	22	0.335	190	0.198	7,000	0.116
2.90	0.800	7.1	0.508	23	0.330	200	0.196	8,000	0.114
3.00	0.775	7.2	0.505	24	0.326	210	0.194	9,000	0.113
3.10	0.762	7.3	0.502	25	0.322	220	0.192	10,000	0.112
3.20	0.748	7.4	0.499	26	0.318	230	0.191	20,000	0.104
3.30	0.735	7.5	0.496	27	0.314	240	0.189	30,000	0.099
3.40	0.723	7.6	0.493	28	0.311	250	0.188	40,000	0.097
3.50	0.712	7.7	0.491	29	0.308	260	0.187	50,000	0.095
3.60	0.701	7.8	0.488	30	0.306	270	0.185	60,000	0.093
3.70	0.691	7.9	0.486	31	0.303	280	0.184	70,000	0.092
3.80	0.681	8.0	0.483	32	0.300	290	0.183	80,000	0.091
3.90	0.672	8.1	0.480	33	0.297	300	0.182	90,000	0.090
4.00	0.664	8.2	0.478	34	0.295	310	0.181	100,000	0.089
4.10	0.656	8.3	0.476	35	0.292	320	0.180	200,000	0.084
4.20	0.648	8.4	0.473	36	0.290	330	0.179	300,000	0.081
4.30	0.640	8.5	0.471	37	0.288	340	0.178	400,000	0.080
4.40	0.633	8.6	0.469	38	0.286	350	0.177	500,000	0.078
4.50	0.626	8.7	0.466	39	0.284	360	0.176	600,000	0.077
4.60	0.619	8.8	0.464	40	0.282	370	0.175	700,000	0.076
4.70	0.613	8.9	0.462	41	0.280	380	0.174	800,000	0.075
4.80	0.607	9.0	0.460	42	0.278	390	0.174	900,000	0.074
4.90	0.602	9.1	0.458	43	0.277	400	0.173	10^6	0.074
5.00	0.596	9.2	0.456	44	0.275	410	0.172		
5.10	0.590	9.3	0.454	45	0.274	420	0.172		
5.20	0.585	9.4	0.452	46	0.272	430	0.171		
5.30	0.580	9.5	0.450	47	0.270	440	0.171		
5.40	0.574	9.6	0.448	48	0.269	450	0.170		
5.50	0.570	9.7	0.446	49	0.267	460	0.169		
5.60	0.565	9.8	0.444	50	0.266	470	0.169		
5.70	0.560	9.9	0.443	60	0.256	480	0.168		
5.80	0.556	10.0	0.441	70	0.247	490	0.168		


0.0
0.2
0.4
0.6
0.8
1.0
s/s0
2
5
10
20
30
50
100
200
300
500
1000
2000
5000
10000
20000
50000
100000
$\sqrt{\frac{4\alpha t}{a^2}}$
1
10
100
1000
10000
100000
r/a



$\dfrac{s}{\left(\frac{qt}{V}\right)}$
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1.0
1.5
2.0
$\dfrac{b}{\sqrt{4\alpha t}}$
To 1.00
$s = \dfrac{qt}{V}\left[\left(\dfrac{b^2}{4\alpha t}\right)\left(-Ei\left(-\dfrac{b^2}{4\alpha t}\right)\right) + 1 - e^{-\frac{b^2}{4\alpha t}}\right]$

B - 5. DRAWDOWN IN A SQUARE ARRAY OF WELLS



TYPICAL WELL ARRAY

LEGEND

O Well

X Point for which drawdowns are given.

REFERENCE:

Moody, W. T., 1955, Determination of minimum drawdown in a square array of wells, in Bureau of Reclamation Technical Memorandum No. 657, Section R, p 159.

SUMMARIZED BY:

W. T. Moody, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\frac{\partial s}{\partial t} = \frac{T}{S}\left(\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r}\right)$$

BOUNDARY CONDITIONS:

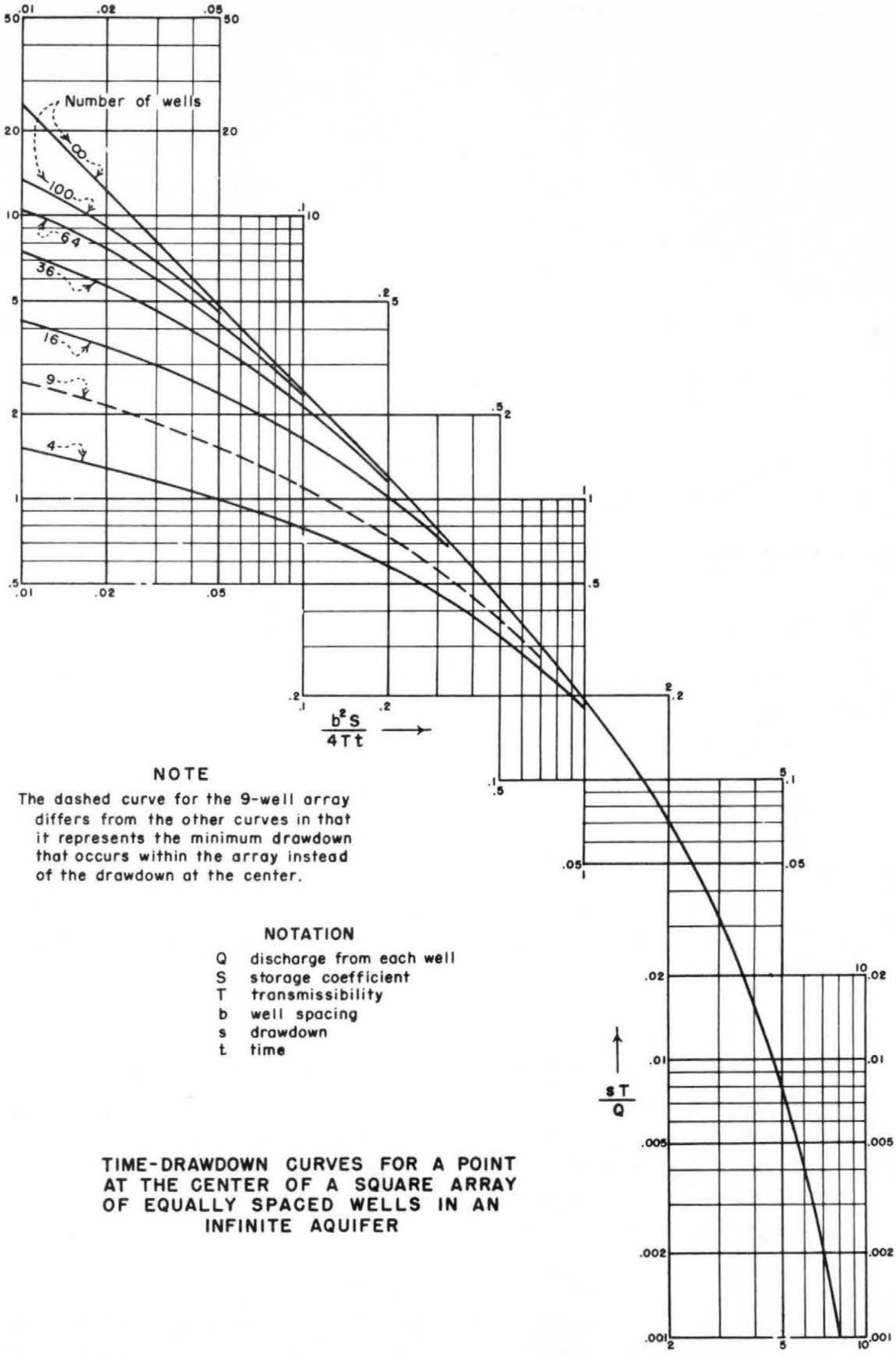
$s = 0$ when $t = 0$ for $r > 0$.

SOLUTION:

Based on the Theis equation $$s = \frac{Q}{4\pi T}\int_{u^2}^{\infty}\frac{e^{-x}}{x}dx$$ See attached graph.

SYMBOLS (Consistent units):

u^2	=	$\frac{r^2 S}{4Tt}$
r	=	Radius measured from the well to a point at which the drawdown s is to be computed.
t	=	Time.
T	=	Transmissibility.
Q	=	Discharge from each well.
b	=	Well spacing.
S	=	Storage coefficient.



NOTE

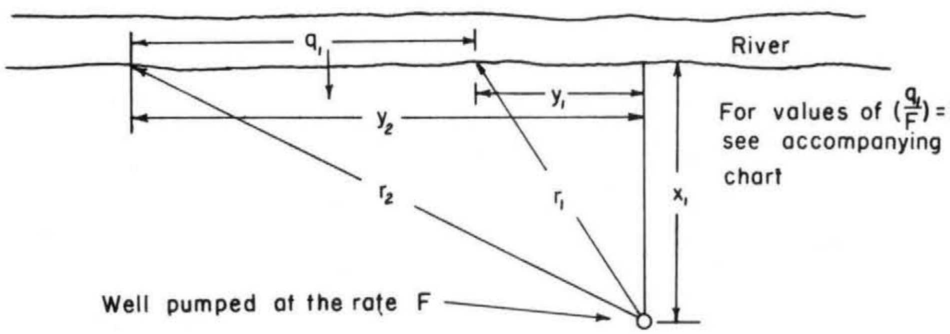
The dashed curve for the 9-well array differs from the other curves in that it represents the minimum drawdown that occurs within the array instead of the drawdown at the center.

NOTATION

Q discharge from each well
S storage coefficient
T transmissibility
b well spacing
s drawdown
t time

TIME-DRAWDOWN CURVES FOR A POINT AT THE CENTER OF A SQUARE ARRAY OF EQUALLY SPACED WELLS IN AN INFINITE AQUIFER

B - 6. RIVER DEPLETION DUE TO PUMPING



REFERENCE:

Glover, R. E., 1952, Methods of estimating possible depletion of flows in the Smoky Hill and North Solomon Rivers in Kansas resulting from well pumping, in Bureau of Reclamation Technical Memorandum No. 657, Section I, p. 88 - 97.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$q_1 = -\int_{y_1}^{y_2} KD \frac{\partial P}{\partial x} dy \quad \text{where} \quad P = \frac{F}{4\pi KD} \int_{\frac{r^2}{4h^2t}}^{\infty} \frac{e^{-u}}{u} du$$

BOUNDARY CONDITIONS:

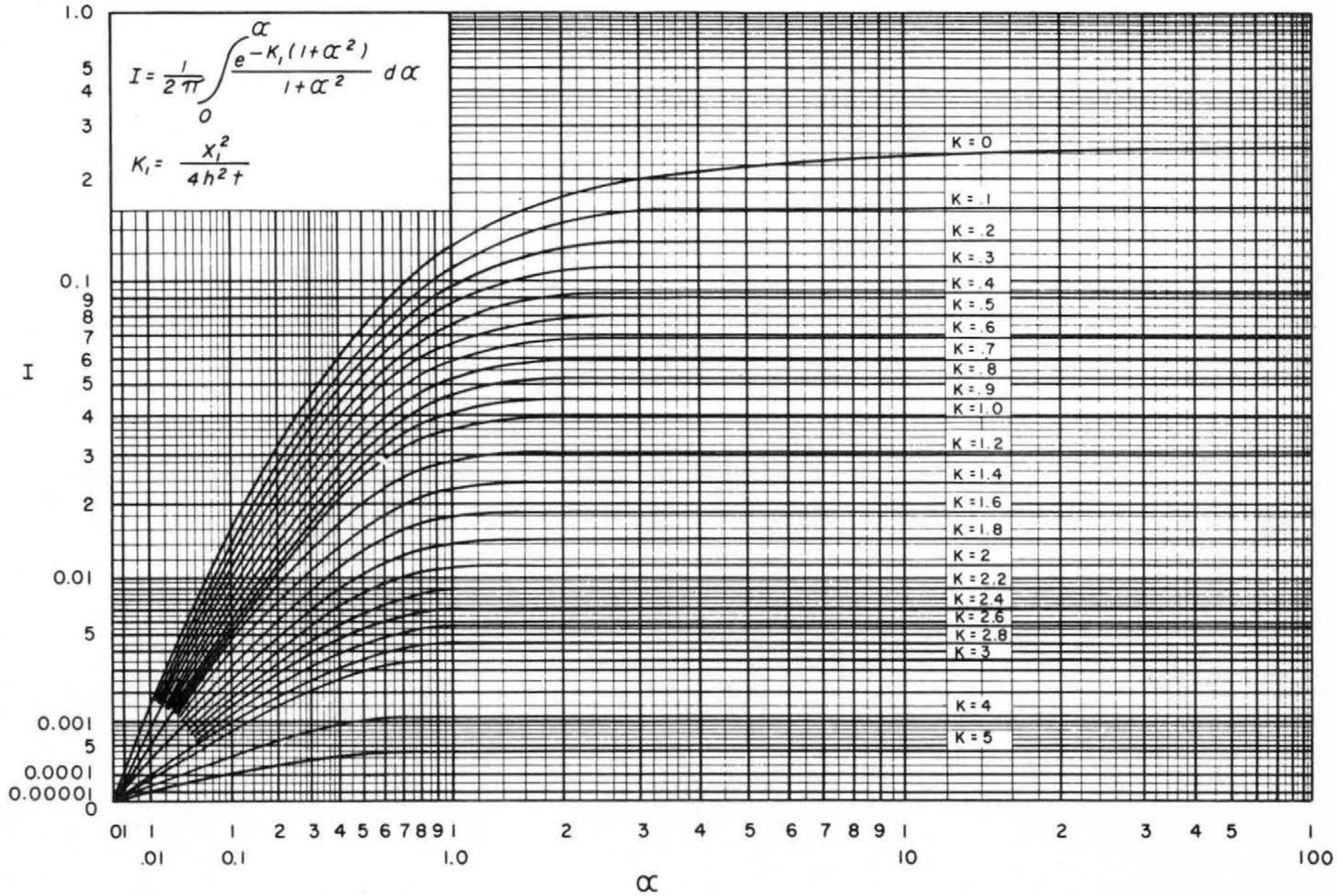
Pumping begins at $t = 0$.

SOLUTION:

$$\frac{q_1}{F} = \frac{1}{2\pi} \int_{\alpha_1}^{\alpha_2} \frac{e^{-\frac{x_1^2}{4h^2t}(1+\alpha^2)}}{(1+\alpha^2)} d\alpha$$

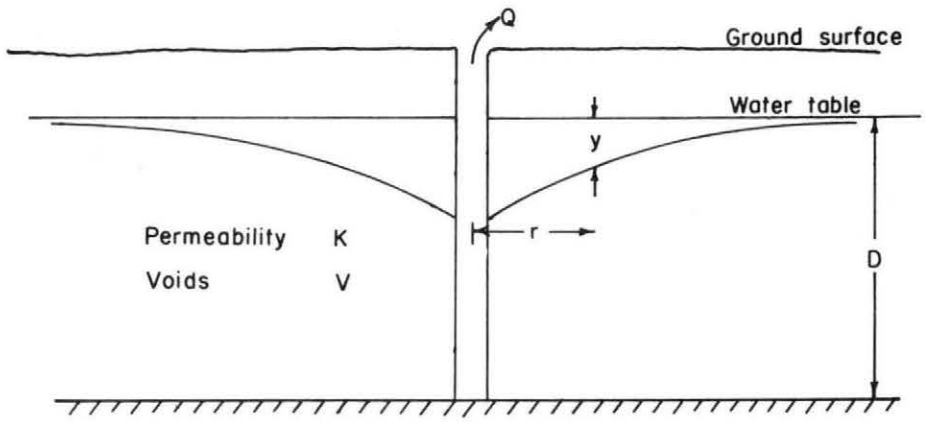
SYMBOLS (Consistent units):

α	$= \frac{y}{x_1}$	h^2.	$= \frac{KD}{V}$
α_1	$= \frac{y_1}{x_1}$	P	= Drawdown.
α_2	$= \frac{y_2}{x_1}$	r	= Radius.
		r	$= \sqrt{x^2 + y^2}$
t	= Time	q_1	= Flow to the well between y_1 and y_2



$I = \frac{1}{2\pi}\int_0^{\alpha} \frac{e^{-K_1(1+\alpha^2)}}{1+\alpha^2}\,d\alpha$
$K_1 = \frac{X_1^2}{4h^2t}$
K = 0
K = .1
K = .2
K = .3
K = .4
K = .5
K = .6
K = .7
K = .8
K = .9
K = 1.0
K = 1.2
K = 1.4
K = 1.6
K = 1.8
K = 2
K = 2.2
K = 2.4
K = 2.6
K = 2.8
K = 3
K = 4
K = 5
I
α

B - 7. DRAWDOWN DUE TO PUMPING FROM AN UNCONFINED AQUIFER



REFERENCE:

Glover, R. E., and Bittinger, M. W., 1960, Drawdown due to pumping from an unconfined aquifer: Am. Soc. Civil Engineers Proc., v. 86, no. IR 3, p. 63-70.

SUMMARIZED BY:

D. E. L. Maasland, Colorado State University.

DIFFERENTIAL EQUATIONS:

$$F = -2\pi r\,(D-y)\;K\,\frac{\partial y}{\partial r} \qquad (1) \qquad\qquad F = \int_r^{\infty} 2\pi r\; dr\; V\,\frac{\partial y}{\partial t} \qquad (2)$$

BOUNDARY CONDITIONS:

$y \to 0$ when $u \to \infty$

SOLUTION:

An iteration procedure provides an effective means for obtaining the second and successive improved approximations. An excellent starting point for the procedure can be obtained by substituting a value of $\frac{\partial y}{\partial r}$ obtained from

$$y = \frac{Q}{2\pi KD}\int_u^{\infty}\frac{e^{-\beta^2}}{\beta}\,d\beta \qquad (3)$$

into equation (1) with the quantity (D-y) replaced by D

This yields $\frac{F}{Q} = e^{-u^2}$. Substituting this in

$$\psi = \frac{1}{\sigma}\left(1 - \sqrt{1 - 2\sigma\int_{u_1}^{\infty}\frac{F}{Q}\,\frac{du}{u}}\right) \qquad (4)$$

gives a second approximation for ψ of the form:

$$\psi = \frac{1}{\sigma}\left(1 - \sqrt{1 - 2\sigma\int_{u_1}^{\infty}\frac{e^{-u^2}}{u}\,du}\right) \qquad (5)$$

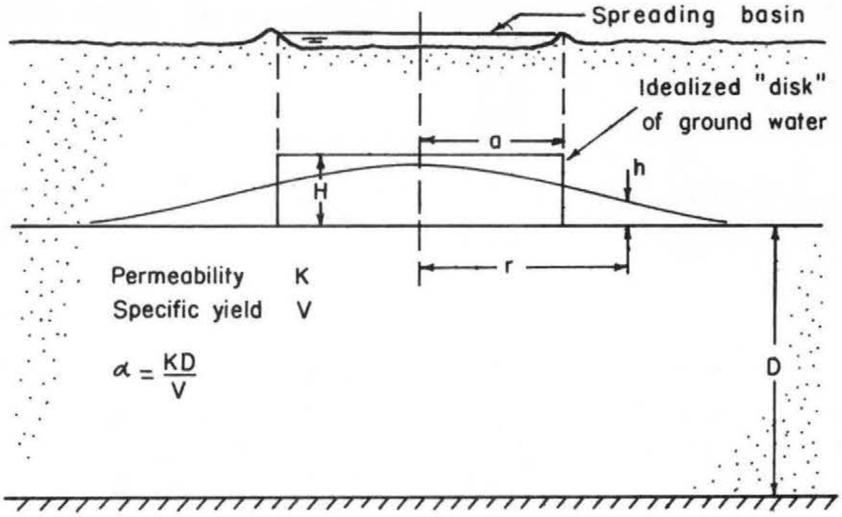
A third approximation may be obtained by graphical integration based on equation (4) and

$$\frac{F}{Q} = -2\int_{u_1}^{\infty} u^2\,\frac{\partial\psi}{\partial u}\,du \qquad (6)$$

SYMBOLS (Consistent units):

$$u = \frac{r}{\sqrt{4\alpha t}} \qquad \sigma = \frac{Q}{2\pi KD^2} \qquad \psi = \frac{2\pi yKD}{Q}$$

B - 8. CIRCULAR RECHARGE BASIN OVER AN UNCONFINED AQUIFER



REFERENCE:

Bittinger, M. W., and Trelease, F. J., 1960, The development and dissipation of a ground water mound beneath a spreading basin: Presented at Am. Soc. Agricultural Engineers, Dec. 1960.

SUMMARIZED BY:

M. W. Bittinger, Colorado State University

DIFFERENTIAL EQUATION:

$$\frac{\partial h}{\partial t} = \alpha \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r} \frac{\partial h}{\partial r} \right)$$

BOUNDARY CONDITIONS:

$h = H$, when $t = 0$ and $0 \leq r \leq a$,

$h = 0$, when $t = 0$ and $r > a$.

$h \rightarrow 0$ when $t \rightarrow \infty$ and $r \rightarrow \infty$

SOLUTION:

$$\frac{h}{H} = \frac{1}{2\alpha t} \cdot e^{\frac{-r^2}{4\alpha t}} \int_0^a \frac{-r'^2}{4\alpha t} \, I_o \left(\frac{rr'}{2\alpha t} \right) r' dr'$$

at $r = 0$, $\frac{h}{H} = 1 - e^{\frac{-a^2}{4\alpha t}}$

Tables of Solutions:

Germond, H. H., The Circular Coverage Function, The Rand Corporation, RM330, January 26, 1950.
Masters, J. I., Some Applications in Physics of the P-Function, The Journal of Chemical Physics, V23, N10, October 1955, 1865-1874.

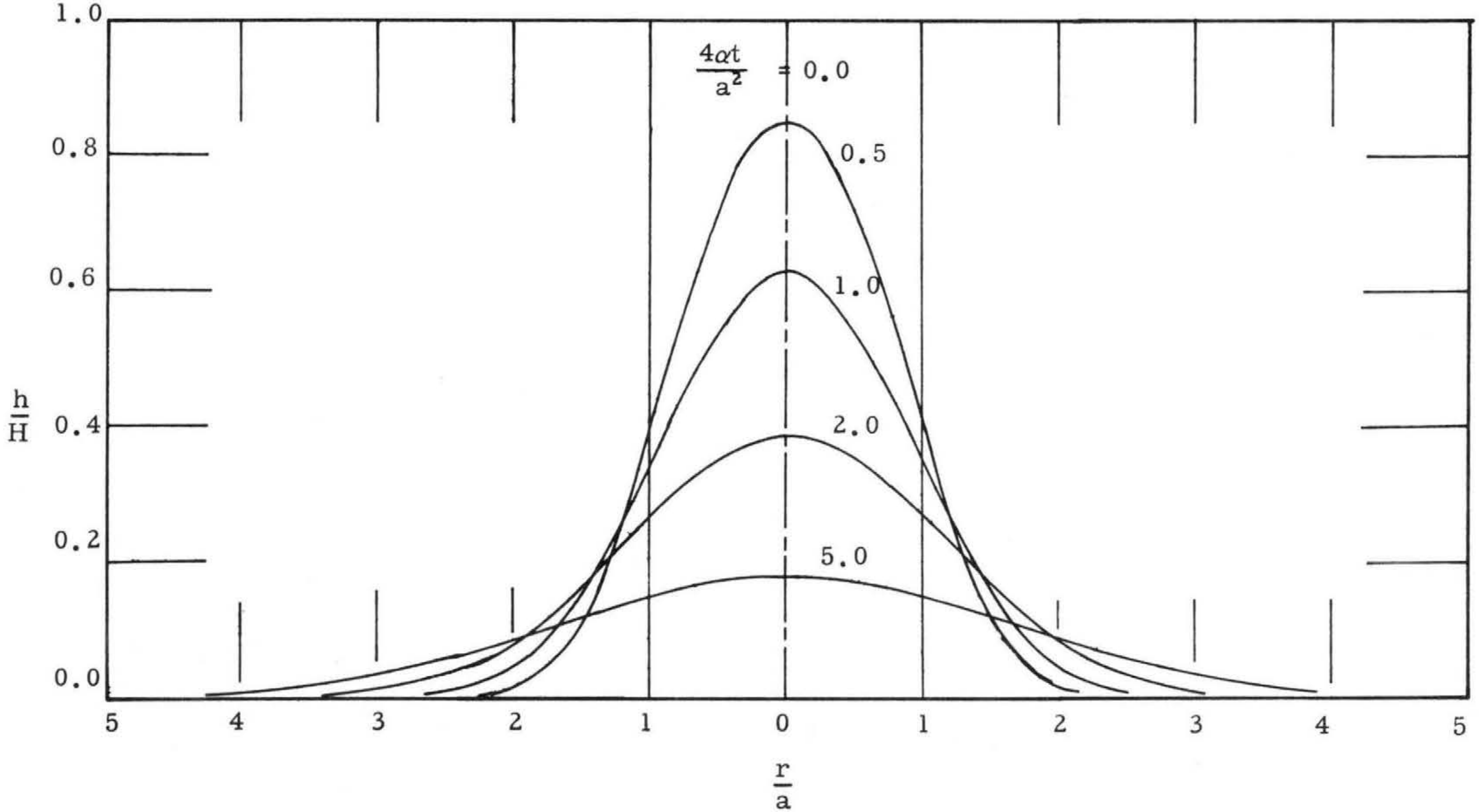
LIMITATIONS:

(1) $h \ll D$

(2) Top of ground water mound not in contact with bottom of spreading basin.

COMMENTS:

Solution as presented gives change of h in time and space after an instantaneous release of water at time zero. Summation of periodic instantaneous releases may be used to simulate continuous recharge.


$\frac{h}{H}$
1.0
0.8
0.6
0.4
0.2
0.0
$\frac{4\alpha t}{a^2} = 0.0$
0.5
1.0
2.0
5.0
5 4 3 2 1 0 1 2 3 4 5
$\frac{r}{a}$

B - 9. NONSTEADY FLOW FOR WELLS WITH DECREASING DISCHARGE

REFERENCE:

Abu-Zied, Mahmoud A., and Scott, Verne H., 1963, Nonsteady flow for wells with decreasing discharge: Am. Soc. Civil Engineers Proc., v. 89, no. HY3, p. 119-132.

SUMMARIZED BY:

Mahmoud A. Abu-Zied, University of California, Davis, California

STATEMENT OF THE PROBLEM:

In general, the analytical solutions for nonsteady confined flow to a well available for determination of drawdown or aquifer characteristics assume that the discharge of the well is essentially constant over the period of pumping. In many cases, however, some change takes place. This is particularly true during short-term pumping tests.

The decrease in the discharge of a completely penetrating well pumping from a homogeneous, isotropic, elastic, artesian aquifer was considered in developing an analytical solution for the drop in the piezometric head caused by the well.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}$$

BOUNDARY CONDITIONS:

1. $h = 0$ for $t \leq 0$
2. $\lim_{r \to 0} h = 0$ for $t > 0$
3. $\lim_{r \to 0} \left(r\frac{\partial h}{\partial r} \right) = -\frac{Q(t)}{2\pi T}$ for $t > 0$
4. $Q(t) = Q_o(\alpha + \beta e^{-at})$

SOLUTIONS:

$$h(t) = \frac{Q_o}{4\pi T}\left[\alpha W(B) + \beta e^{-A} f(A,B) \right]$$

Values of $f(A,B)$ were computed and tabulated for the following ranges of A and B:

$$0.0001 \leq A \leq 900$$

$$10^{-8} \leq B \leq 90$$

This equation may also be expressed as follows:

$$h(t) = -\frac{Q_o}{4\pi T} \mathrm{Ei}(-B)\left[\alpha + \beta \exp(-A) J_o(2\sqrt{c}) \right]$$

$$+ \frac{BQ_o}{4\pi T} \exp(-A-B) \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{(-1)^m (n-m-1)!\, A^n}{(n!)^2 \quad B^{-m}}$$

Graphical Solution for the Transmissibility and Storage Coefficients

A type curve solution was suggested for the determination of S and T . For any value of A as A_o the drawdown data around the well may be plotted against $\frac{r^2}{t_o}$ on log-log graph paper (Field data curve).

A type curve for A_o can be constructed also using the same scale of the field data curve for $\phi(B)$ vs B, where

$$\phi(B) = \alpha W(B) + \beta e^{-A_o} f(A_o, B)$$

By a method of superposition a matching position may be found from which values can be determined to calculate the aquifer characteristics.

SYMBOLS:

h	=	Change in head.
r	=	Radial distance.
S	=	Storage coefficient.
T	=	Transmissibility.
t	=	Time.
$Q(t)$	=	Pumping rate at any time.
Q_o	=	Maximum discharge of well.
α, β, a	=	Discharge parameters.
A	=	at
B	=	$\frac{r^2 S}{4Tt}$
$W(B)$	=	$\int_B^\infty \frac{e^{-x}}{x}\, dx$
$f(A, B)$	=	$\int_B^\infty \frac{e^{-x+\frac{c}{x}}}{x}\, dx = \int_0^1 \frac{e^{Ay-\frac{B}{y}}}{y}\, dy$
c	=	AB (independent of time)

B - 10. MODIFIED NONSTEADY SOLUTIONS FOR DECREASING DISCHARGE WELLS

REFERENCE:

Abu-Zied, Mahmoud A., and Scott, Verne H., 1963, Modified non-steady solutions for decreasing discharge wells: Submitted for publication in Am. Soc. Civil Engineers Proc. (no. HY).

SUMMARIZED BY:

Mahmoud A. Abu-Zied, University of California, Davis, California

STATEMENT OF PROBLEM:

Modified solutions are proposed for the problem of nonsteady flow to a decreasing discharge well completely penetrating an extensive artesian aquifer.

Field data, obtained from an experimental well with a decreasing discharge, were used to test the validity of the general and modified solutions. A controlled constant discharge test was conducted on the same well for comparison purposes.

Modifications were developed for specific ranges of the parameters A and B (see previous summary).

SOLUTIONS:

Case 1. $B \leq 0.01$ and $A \leq 0.1$

For this range of A and B it was found that the general drawdown equation can be approximated by

$$h = \frac{Q_o}{4\pi T}\left[(\alpha + \beta e^{-A}) \ln \frac{e^{-0.57}}{B}\right] \tag{1}$$

For a time, drawdown data (r constant), Eq. 1 plots as a straight line for $\frac{h}{\alpha + \beta e^{-A}}$ vs log t. The transmissibility of the aquifer and the storage coefficient can be calculated from the slope of the straight line and its intercept with the time axis respectively.

In case of a distance drawdown data (t constant), Eq. 1 plots as a straight line for h vs log r . T and S can be calculated as before.

Case 2. $B > 1$ and $c < 1$

In this range, the general solution may be approximated by

$$h = \frac{Q_o}{4\pi T}\left[W(B)\left(\alpha + \beta \exp(-A)\, J_o(2\sqrt{c})\right) + \beta A \exp(-A-B)\right]$$

For a distance-drawdown data a type curve solution is possible for the determination of T and S.

Type curve: $\psi(A_o, B)$ vs B

Field data curve: h vs $\frac{r^2}{t_o}$

where $\psi(A_o, B) = \left[W(B)\left(\alpha + \beta \exp(-A_o)\, J_o(2\sqrt{c})\right) + \beta A_o \exp(-A_o - B)\right]$

Case 3. β is very small compared to α

For this case it is shown that the general solution reduces to the Theis Non-equilibrium formula for h .

Case 4. After a long period of pumping

The general drawdown expression reduces to the equilibrium solution for a steady discharge after a long period of pumping.

Accordingly the choice of the method to be used in calculating the drawdown and the aquifer characteristics is dictated by criteria given in the following table.

Range of Band A	Solution to be used
1. $B \leq 0.01$, $A < 0.1$	Modified solution Case 1
2. $B > 1$, $c < 1$	Modified solution Case 2
3. $A \leq$, $B \geq 8$	Theis nonequilibrium solution
4. All values not included in 1, 2 or 3	General solution

Applications on field data showed the following:

1. The decreasing discharge time data confirmed the validity of the exponential relationship that was used in developing the general solution
2. The general and modified solutions gave satisfactory values for the drawdown and aquifer characteristics.
3. The modified solutions proposed simplify the computations and are appropriate within the ranges indicated.

B - 11. NONSTEADY FLOW TO A WELL OF CONSTANT DRAWDOWN IN AN EXTENSIVE AQUIFER

REFERENCE:

Jacob, C. E. and Lohman, S. W., 1952, Non-steady flow to a well of constant drawdown in an extensive aquifer: Am. Geophys. Union Trans., v. 33, no. 4, p. 559-69.

SUMMARIZED BY:

S. W. Lohman, U. S. Geological Survey

DESCRIPTION OF CASE TREATED:

This is a method of determining T and S from discharge test of a single flowing artesian well, during which drawdown is constant but discharge (flow) diminishes with time. Based upon analogous heat flow equation developed by L. P. Smith in 1937.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} = \frac{S}{T}\frac{\partial h}{\partial t}$$

INITIAL AND BOUNDARY CONDITIONS:

$h = h_o$ for $t = 0$ and for all values of r.

$h \longrightarrow h_o$ as $r \longrightarrow \infty$, for $t > 0$

$h = h_o - s_w$ for $r = r_w$ and for $t > 0$

SOLUTION:

$$Q = 2\pi T(h_o - h)\,G(\alpha) = 2\pi T s_w G(\alpha) \qquad (1)$$

where

$$\alpha = \frac{Tt}{Sr_w^2}, \text{ and } G(\alpha) = \frac{4\alpha}{\pi}\int_0^\alpha Xe^{-\alpha X^2}\left[\frac{\pi}{2} + \tan^{-1}\frac{Y_o(X)}{J_o(X)}\right]dX \qquad (2)$$

where $J_o(X)$ and $Y_o(X)$ are Bessel functions of zero order of the first and second kinds, respectively. This integral is not tractable by integration, so was solved for the values given below by a series of summations.

Table of solutions:

Table -- Values of $G(\alpha)$ for values of α between 10^{-4} and 10^{12}

	10^{-4}	10^{-3}	10^{-2}	10^{-1}	1	10	10^2	10^3
1	56.9	18.34	6.13	2.249	0.985	0.534	0.346	0.251
2	40.4	13.11	4.47	1.716	.803	.461	.311	.232
3	33.1	10.79	3.74	1.477	.719	.427	.294	.222
4	28.7	9.41	3.30	1.333	.667	.405	.283	.215
5	25.7	8.47	3.00	1.234	.630	.389	.274	.210
6	23.5	7.77	2.78	1.160	.602	.377	.268	.206
7	21.8	7.23	2.60	1.103	.580	.367	.263	.203
8	20.4	6.79	2.46	1.057	.562	.359	.258	.200
9	19.3	6.43	2.35	1.018	.547	.352	.254	.198
10	18.3	6.13	2.25	.985	.534	.346	.251	.196

	10^4	10^5	10^6	10^7	10^8	10^9	10^{10}	10^{11}
1	0.1964	0.1608	0.1360	0.1177	0.1037	0.0927	0.0838	0.0764
2	.1841	.1524	.1299	.1131	.1002	.0899	.0814	.0744
3	.1777	.1479	.1266	.1106	.0982	.0883	.0801	.0733
4	.1733	.1449	.1244	.1089	.0968	.0872	.0792	.0726
5	.1701	.1426	.1227	.1076	.0958	.0864	.0785	.0720
6	.1675	.1408	.1213	.1066	.0950	.0857	.0779	.0716
7	.1654	.1393	.1202	.1057	.0943	.0851	.0774	.0712
8	.1636	.1380	.1192	.1049	.0937	.0846	.0770	.0709
9	.1621	.1369	.1184	.1043	.0932	.0842	.0767	.0706
10	.1608	.1360	.1177	.1037	.0927	.0838	.0764	.0704

The type curve may conveniently be prepared by plotting values of $G(\alpha)$ versus α on 3 by 5 cycle logarithmic paper (convenient size 11 x 14-1/2 inches). Values of Q and t from field tests are then plotted on translucent logarithmic paper to the same scale. Superposition of the curves allows a matching point to be found from which T and S may be determined from equations (1) and (2).

Asymptotic solutions:

When t is large relative to $\frac{Sr_w^2}{T}$, which occurs early for an artesian well, T may be determined from the slope of the straight line that results from plotting on semilog paper

$$T = \frac{2.30}{4\pi\Delta(s_w/Q)/\Delta\log_{10} t/r_w^2} \tag{3}$$

S may be determined from the intercepts on the ordinate $(s_w/Q) = 0$, by $S = 2.25\ Tt/r_w^2$ (4)

or S may be determined from the data region of the plot by

$$S = \frac{2.25\ Tt/r_w^2}{\log_{10}^{-1}\left[\frac{s_w/Q}{\Delta(s_w/Q)}\right]} \tag{5}$$

If r_w is not known, T, but not S, may be determined from

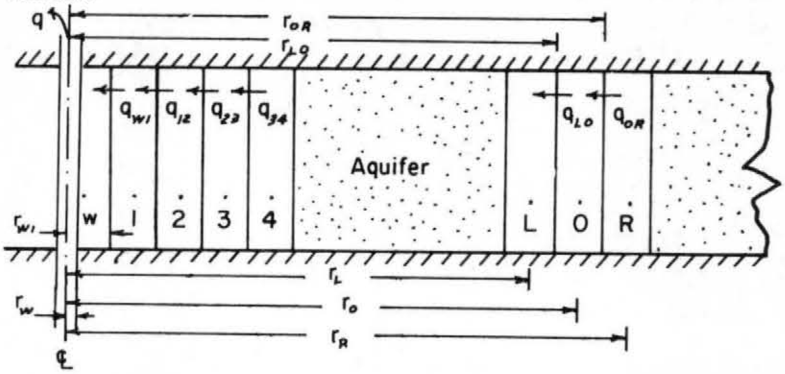
$$T = \frac{2.30}{4\pi s_w \Delta(1/Q)/\Delta\log_{10} t} \tag{6}$$

These straight line solutions are recommended over the curve-matching method.

SYMBOLS (Consistent units):

T	= Coefficient of transmissibility.	(L^2/T)
S	= Coefficient of storage.	dimensionless
h	= Head.	(L)
t	= Time.	(T)
r_w	= Well radius.	(L)
s_w	= Drawdown in discharging well.	(L)
Q	= Discharge.	(L^3/T)

B - 12. NUMERICAL SOLUTIONS OF STEADY-STATE AND TRANSIENT FLOW PROBLEMS - ARTESIAN AND WATER - TABLE WELLS



REFERENCE:

Kashef, A. I., Toulouklan, Y. S. and Fadum, R. E., 1952, Numerical solutions of steady-state and transient flow problems - artesian and water-table wells: Purdue Univ. Eng. Expt. Sta. Bull. 117, Lafayette, Ind. 116 p.

SUMMARIZED BY:

A. I. Kashef, North Carolina State College

DESCRIPTION OF CASES TREATED:

Artesian Wells

A finite difference equation is determined either from the physical aspects of the nature of flow, or from the transformation of the fundamental mathematical equation (Theis equation) to its finite difference form.

Let any three successive points L, 0 and R of radii r_L, r_0 and r_R from the well center be chosen through the aquifer within the well influence. It is assumed that the water head at each of these points represents the head of the concentric shell containing the point. The intermediate shell represented by the point 0, has an internal radius of r_{L0} and an external radius of r_{0R}. These two radii are the averages of r_L and r_0, and r_0 and r_R respectively.

Considering the Point 0 - or strictly speaking the circle 0 - then, the problem is to determine the piezometric head h_0' at the end of a given time interval Δt from a knowledge of the piezometric heads h_L, h_0 and h_R at L, 0 and R respectively at the beginning of this interval. The initial gradient at shell 0 is assumed to remain constant throughout the assumed time interval Δt which should in practice be small. The smaller the values of both the shell thicknesses and Δt, the more accurate are the results and the more labour will be involved in finding a solution.

Considering these finite three consecutive shells L, 0 and R, then $h'_0 = F_L h_L + F_0 h_0 + F_R h_R$ (1)

where: $F_L = r_{L0}/M(r_0 - r_L)$, $F_0 = 1 - (F_L + F_R)$, $F_R = r_{0R}/M(r_R - r_0)$

$M = SA_0/2\pi T \cdot \Delta t$, A_0 = area of the base of shell 0.

Applying the drawdowns $s = h_e - h$ rather than the head values h, Equation (1) reduces to:

$$s'_0 = F_L s_L + F_0 s_0 + F_R s_R \qquad (2)$$

The sum of the factors F is equal to unity and M is arbitrarily chosen depending upon the required degree of precision within a certain limiting value:

$$M \geq \left[r_{0R}/(r_R - r_0) + r_{L0}/(r_0 - r_L) \right] \qquad (3)$$

Dividing the aquifer into successive concentric shells w, 1, 2, 3 . . . . . L, 0, R . . . etc. starting from the well surface, then the drawdowns within any shell after a certain time interval can be obtained by applying equation (2). In order to start a solution for this equation, the drawdown at the well surface - represented by shell w - at the end of any time interval should be calculated. Since the quantity of water that leaves any shell is larger than that entering into it during a given period by an amount equal to the quantity of water that is released from storage in the same shell due to the decrease of head, then the condition at the well surface can be determined as follows:

The ratio of the discharge q_{w1} (see Figure) entering the first shell w defined by r_w and r_{w1} and the discharge q pumped out of the well is given by:

$$q_{w1}/q = e^{-u} \tag{4}$$

where: $u = r^2_{w1} \; S/4Tt = r^2_{w1} \; S/4Tn\Delta t, \; n = 1, 2, 3, \ldots \text{ etc.}$

but:

$$q = q_{w1} + \int_0^r S(\partial h/\partial t)\, 2\pi r \, dr , \tag{5}$$

or, in a finite difference form:

$$q(1 - e^{-u}) = S\pi(r^2_{w1} - r^2_w)(h_w - h_w')/\Delta t$$

or,

$$s'_w = s_w + q\Delta t(-e^{-u})/S\pi(r^2_{w1} - r^2_w) , \tag{6}$$

where h_w and h'_w are the heads at the well surface at the beginning and end of a given time interval Δt. s'_w is the drawdown at the end of Δt. Equation (6) can be used to determine the head at the well surface after any time t since the start of pumping. The treatment of the conditions at the well surface as presented in this summary varies from that given in the original paper. This modification was given by the same authors in the 8th International Congress of Theoretical and Applied Mechanics, Istanbul 1952.

The physical meaning of equation (5) could be applied for any two successive shells within the aquifer. Equations (1) or (2) may be derived on that basis. The application of equation (2) should be repeated together with equation (6) during the successive time intervals up till the steady-state condition is reached or the heads at the desired elapsed time since the start of pumping have been found.

Water -Table Wells

In this case the numerical method is applied by dividing the water-bearing formation into consecutive shells concentric with the center line of the pumped well as in the artesian case. The solution is simplified by introducing Dupuit's assumption. In most of the approximate mathematical solutions given for this problem, the variation of the depth of the saturated zone with time is neglected. In the numerical method however, this variation is accounted for.

Following the same notations as in the artesian case, and introducing the specific yield Y rather than the coefficient of storage S , the finite difference equation is given by:

$$h_0' = F_R h_R^2 + F_L h_L^2 + h_0 - F_0 h_0^2 \tag{7}$$

where

$$F_R = r_{0R}/M(r_R - r_0), \quad F_L = r_{0L}/M(r_0 - r_L),$$

$$F_0 = \Sigma F = F_R + F_L , \quad M = Y(r_{0R}^2 - r_{0L}^2)/k\Delta t = YA_0/\pi k(\Delta t)$$

Equation (7) is based on the fact that the difference between the water leaving a shell and the water entering it, through a certain time interval, is equal to the water drained from the saturated zone within that particular shell. Equation (7) may be derived from transforming the following equation (8) to its finite-difference form:

$$\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\,\frac{\partial(h^2)}{\partial r}\right)\right] = \frac{2Y}{k}\left(\frac{\partial h}{\partial t}\right) \tag{8}$$

In order to start the solution h'_w at the well surface (first shell) is determined from the following equation:

$$h'_w = \sqrt{(h'_1)^2 \, q(r_1 - r_w)/\pi r_{w1} k}$$

At the end of the first time interval, it is assumed that $h'_1 = h_1$.

B - 13. ARTESIAN WELL, SEMI-GRAPHICAL SOLUTIONS

REFERENCE:

Kashef, A. I., 1961, A semi-graphical solution of artesian well problems under the transient condition: 5th Internat. Conf. Soil Mech. and Foundation Engineering Proc., v. 2, div. 3B-7, p. 637-640.

SUMMARIZED BY:

A. I. Kashef, North Carolina State College

DESCRIPTION OF CASE TREATED:

This paper includes a proposed semi-graphical procedure for determining the piezometric level of an artesian well at any time measured from the start of pumping. The method is based on both the Theis solution and the numerical method (see the summary on "Numerical Solutions of the Transients Flow Problems"). The procedure is thought to be a trial to eliminate some of the idealized assumptions made in mathematical solutions, the laboriousness of numerical solutions and a step forward in establishing a simple method of solving practical and complicated problems such as that presented by a group of wells.

From the Theis equation, the following equation can easily be derived:

$$ds/dr = \frac{q}{2\pi Tr} e^{-u} \tag{1}$$

where: $u = \frac{r^2 S}{4Tt}$

Transforming equation (1) to its finite-difference form:

$$\Delta s = \frac{Se^{-r^2}}{4T(n\Delta t)} \frac{q\Delta r}{2\pi Tr} \quad \text{and} \quad s = \sum_{r=r}^{\infty} \Delta s \tag{2}$$

where n expresses the number of time intervals since the start of pumping. Draw the curve e^{-u} vs. r for, say, the first time interval (n = 1) and make the various chosen shells. "Δs" within each shell lower than the next shell to its right can thus be calculated from equation (2).

After the nth time interval, the same Δa values hold true but for new radii = n $\sqrt{r}$, because:

$$\Delta s = \frac{q}{2\pi T(r\sqrt{n})} \cdot (\Delta r\sqrt{n}) \cdot e^{-(r\sqrt{n})^2 S/4T\cdot n\Delta t} = \frac{q}{2\pi Tr} \cdot \Delta r \cdot e^{-r^2 S/4T\Delta t}$$

Thus the curve (or polygon) may be drawn once and for all the first time interval and the radii shortened to $r/\sqrt{2}$, $r/\sqrt{3}$, $4/\sqrt{4}$, . . . etc. to measure the drawdowns corresponding to the 2nd, 3rd, 4th, ... etc. time intervals respectively, rather than shifting the curve itself. Near the well surface, an adjustment should be made for Δs , between $r = r_w$ and $r = r_w\sqrt{n}$ at any time interval n and calculated from:

$$\Delta s \approx \frac{q}{\pi T} \frac{(\sqrt{n} - 1)}{(\sqrt{n} + 1)} \tag{3}$$

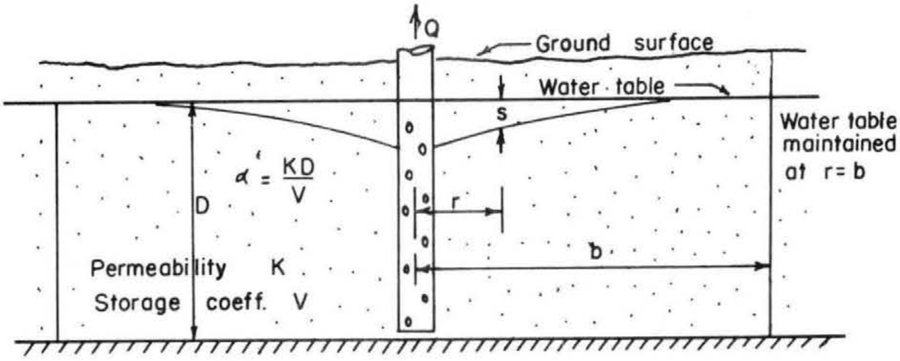
because $e^{-u} \approx 1.0$ in the vicinity of the well. Δs has to be calculated separately from equation (3) and added to that of shell w.

This method allowed the selection of time intervals much greater than those in the numerical method. Besides, the drawdown curve at any time may be obtained directly without drawing the successive drawdowns of the successive time intervals as in the case of the numerical procedures.

Any other drawdown curve may be drawn and applied for the entire solution and not necessarily that corresponding to the first time interval. The radii are thus increased or decreased accordingly.

In the original paper an example is solved and compared with the results obtained from the numerical solution.

B -14. DRAWDOWN AROUND A WELL WITH CONSTANT WATER LEVEL MAINTAINED AT RADIUS b



REFERENCE:

Bureau of Reclamation. Unpublished result derived by R. E. Glover.

SUMMARIZED BY:

W. T. Moody, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{1}{\alpha}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

$s = 0$ when $t = 0,\ r > 0$

$s = 0$ when $r = b,\ t > 0$

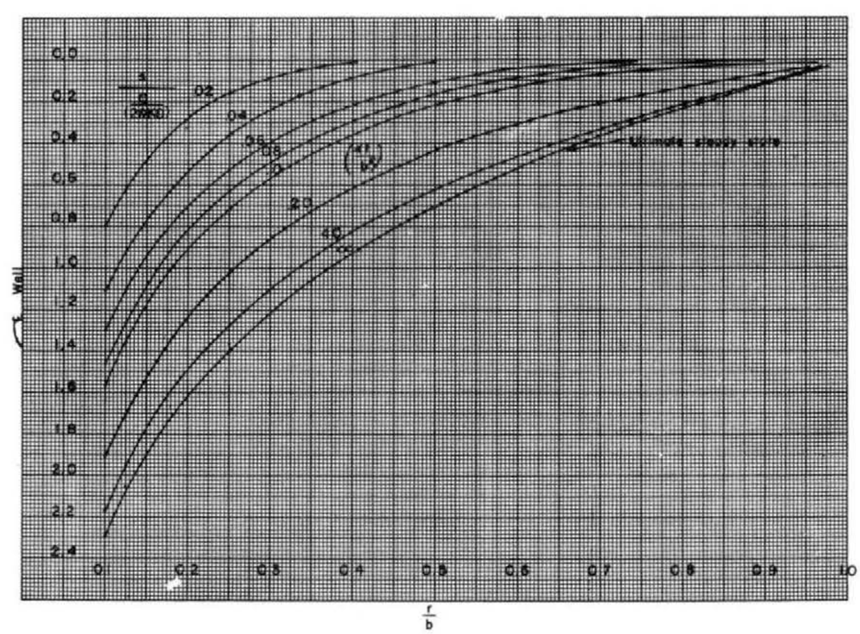
$-2\pi KDr \longrightarrow Q$ as $r \longrightarrow 0,\ t > 0$

SOLUTION:

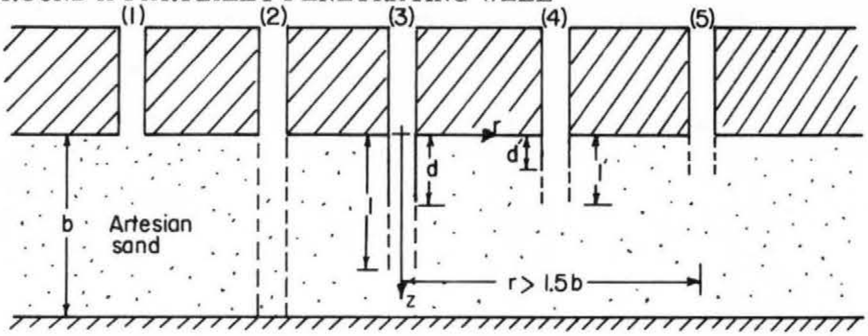
$$s = \frac{Q}{2\pi KD}\left[\ln\frac{b}{r} - 2\sum_{n=1}^{\infty}\frac{\exp\left(-\alpha X_n^2\ t/b^2\right) J_0(X_n\, r/b)}{X_n^2\ J_1^2\ (X_n)}\right] \qquad J_o\,(X_n) =$$

SYMBOLS:

See sketch above.



B - 15. DRAWDOWN AROUND A PARTIALLY PENETRATING WELL



REFERENCE:

Hantush, M. S., 1961, Drawdown around a partially penetrating well: Am Soc. Civil Engineers Proc., v. 87, no. HY 4, p. 83-98.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

$$s(r, z, 0) = 0$$

$$s(\infty, z, t) = 0$$

$$\frac{\partial s}{\partial z}(r, 0, t) = \frac{\partial s}{\partial z}(r, b, t) = 0$$

$$\lim_{r \to 0} \left\{ 2\pi K r \int_0^b \frac{\partial s}{\partial r}\, dz \right\} = -Q$$

SOLUTIONS:

I. Drawdown in Piezometers

$$s = \frac{Q}{4\pi K b}\left[W(u) + f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right)\right] \qquad \text{or}$$

$$s = \frac{Q}{8\pi k(l - d)}\left[M\left(u, \frac{l + z}{r}\right) + M\left(u, \frac{l - z}{r}\right) + f'\left(u, \frac{b}{r}, \frac{l}{r}, \frac{z}{r}\right)\right.$$

$$\left. - M\left(u, \frac{d + z}{r}\right) - M\left(u, \frac{d - z}{r}\right) - f'\left(u, \frac{b}{r}, \frac{d}{r}, \frac{z}{r}\right)\right]$$

in which

$$f\left(u, \frac{r}{b}, \frac{l}{b}, \frac{d}{b}, \frac{z}{b}\right) = \frac{2b}{\pi(l - d)} \sum_{n=1}^{\infty} \frac{1}{n}\left(\sin\frac{n\pi l}{b} - \frac{n\pi d}{b}\right) \cdot \cos\frac{n\pi z}{b}\, W\left(u, \frac{n\pi r}{b}\right) \qquad \text{and,}$$

$$f'\left(u, \frac{b}{r}, \frac{x}{r}, \frac{z}{r}\right) = \sum_{n=1}^{\infty}\left[M\left(u, \frac{2nb + x + z}{r}\right) - M\left(u, \frac{2nb - x - z}{r}\right) + M\left(u, \frac{2nb + x - z}{r}\right) - M\left(u, \frac{2nb - x + z}{r}\right)\right]$$

and where

$$u = \frac{r^2 S_s}{4 K t}$$

For values of the function $M(u, \beta)$ see Table 1 of reference.

Asymptotic Solutions:

(a) Small values of time $t < \frac{(2b-1-z)^2 S_s}{20K}$

$$s = \frac{Q}{8\pi K(1-d)}\left[M(u, \frac{1+z}{r}) + M(u, \frac{1-z}{r}) - M(u, \frac{d+z}{r}) - M(u, \frac{d-z}{r})\right]$$

The above solution applies for any t in the case of infinitely deep aquifers, i.e. for $b \longrightarrow \infty$

(b) For large values of time, $t > \frac{b^2 S_s}{2K}$

$$s = \frac{Q}{4\pi Kb}\left[W(u) + f_s(\frac{r}{b}, \frac{1}{b}, \frac{d}{b}, \frac{z}{b})\right]$$

in which

$$f_s = \frac{4b}{\pi(1-d)} \sum_{n=1}^{\infty} \frac{1}{n} K_o\left(\frac{n\pi r}{b}\right)\left(\sin\frac{n\pi 1}{b} - \sin\frac{n\pi d}{b}\right) \cos\frac{n\pi z}{b}$$

II. Average Drawdown in Observation Wells

The average drawdown $\bar{s}$ in an observation well screened between the depths $1'$ and d' ($1' > d'$) can be obtained by integrating the equation of drawdown in piezometers with respect to z between the limits d' and $1'$ and then dividing the result by ($1' - d'$).

Therefore

$$\bar{s} = \frac{Q}{4\pi Kb}\left[W(u) > \bar{f}\left(u, \frac{r}{b}, \frac{1}{b}, \frac{d}{b}, \frac{1'}{b}, \frac{d'}{b}\right)\right]$$

in which

$$\bar{f} = \frac{2b^2}{\pi^2(1-d)(1'-d')} \sum_{n=1}^{\infty} \frac{1}{n^2}\left(\sin\frac{n\pi 1}{b} - \sin\frac{n\pi d}{b}\right) \cdot \left(\sin\frac{n\pi 1'}{b} - \sin\frac{n\pi d'}{b}\right) W(u, \frac{n\pi r}{b})$$

Asymptotic Solutions:

(a) Small values of time $t < \frac{2(b-1-1')^2 S_s}{20K}$

$$\bar{s} = \frac{Q}{8\pi K(1-d)(1'-d')}\left[F(u, \frac{1+1'}{r}, \frac{1-1'}{r}) - F(u, \frac{d+1'}{r}, \frac{d-1'}{r})\right.$$

$$\left. + F(u, \frac{d+d'}{r}, \frac{d-d'}{r}) - F(u, \frac{1+d'}{r}, \frac{1-d'}{r})\right]$$

in which

$$F(u, \beta, \alpha) = r\left\{\beta M(u,\beta) - \alpha M(u,\alpha) + 2\left[-\sqrt{y}\ \mathrm{erfc}(-\sqrt{yu}) - x\ \mathrm{erfc}\ -\sqrt{xu} + \frac{e^{-xu} - e^{-yu}}{-\sqrt{\pi u}}\right]\right\}$$

in which

$$x = 1 + \beta^2 \ ; \quad y = 1 + \alpha^2$$

(b) Large values of time, $t > \frac{b^2 S_s}{2K}$

$$\bar{s} = \frac{Q}{4\pi Kb}\left[W(u) + \bar{f}_s\left(\frac{r}{b}, \frac{1}{b}, \frac{d}{b}, \frac{1'}{b}, \frac{d'}{b}\right)\right]$$

in which

$$\bar{f}_s = \frac{4b^2}{\pi^2(1-d)(1'-d')} \sum_{n=1}^{\infty} \frac{1}{n^2} K_o\left(\frac{n\pi r}{b}\right) \cdot \left(\sin\frac{n\pi l}{b} - \sin\frac{n\pi d}{b}\right)\left(\sin\frac{n\pi l'}{b} - \sin\frac{n\pi d'}{b}\right)$$

III. Drawdown in Piezometers or Wells for $r/b > 1.5$

For $r/b > 1.5$

$$s = \bar{s} = \frac{Q}{4\pi Kb} W(u)$$

IV. Recovery Equations.

If t and t' are the time, reckoned respectively from the commencement and end of pumping, the residual drawdown s' in a piezometer during recovery can be shown to be

$$s' = s(t) - s(t')$$

Similarly, the average residual drawdown in an observation well is

$$\bar{s}' = \bar{s}(t) - \bar{s}(t')$$

in which

$t = t_o + t'$ and t_o is the time at which the pumping ceased.

Note: Equivalent solutions for leaky, for anistropic aquifers, or both, can be found in "Advances in Hydro-Science", Ch. on Hydraulics of Wells by M. S. Hantush, edited by V. T. Chow, Acad. Press (in preparation).

SYMBOLS

ν = K/S_s, dimension L^2T^{-1}

b = Thickness of aquifer, L.

d = Depth from top of aquifer of the unscreened portion of pumped well, L.

d' = Depth from top of aquifer of the unscreened portion of observation well, L.

$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy$ = the error function.

$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ = the complement of the error function.

K = Hydraulic conductivity of the aquifer, LT^{-1}.

$K_o(x)$ = The zero-order modified Bessel function of the second kind.

l = Depth of penetration of pumped well, L.

l' = Depth of penetration of an observation hole, L.

$M(u,x) = \int_u^{\infty} \frac{e^{-y}}{y} \mathrm{erf}(x\sqrt{y})\, dy$, tabular values of which are given in Table 1 - reference.

Q = Discharge of pumped well, L^3T^{-1}.

r = Radial distance from pumped well, L.

s = Drawdown in piezometers, L.

$\bar{s}$ = Drawdown in observation holes, L.

s' = Residual drawdown in piezometers, L.

$\bar{s}'$ = Residual drawdown in observation holes, L.

S = Coefficient of storage.

S_s = S/b = Specific storage, L^{-1}.

t = Time since pumping started, T.

t_o = Period of pumping, T.

t' = Time since pumping stopped, T.

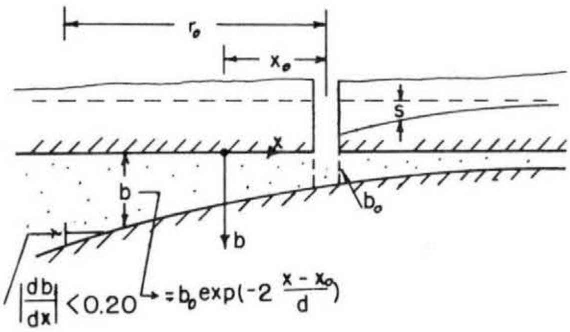
u = $(r^2S_s/4Kt)$.

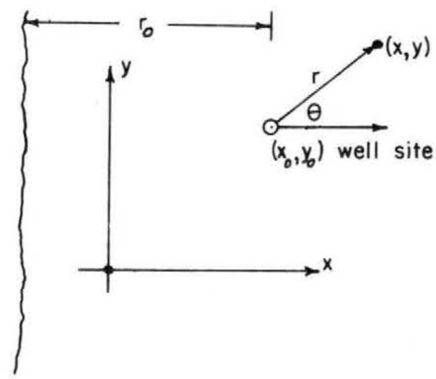
$W(u) = \int_u^{\infty} \frac{e^{-y}}{y} dy$ = The well function, tables of which are available.

$W(u,x)$ = The well function for leaky aquifers, tables of which are available.

Z = Vertical coordinate measured from top of aquifer, positive downward, L.

FLOW OF GROUND WATER IN SANDS OF NONUNIFORM THICKNESS





REFERENCE:

Hantush, M. S., 1962, Flow of ground water in sands of non-uniform thickness, 3, Flow to wells: Jour. Geophys. Research, v. 67, no. 4, p. 1527-34.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

The approximate differential equation for sands of nonuniform thickness is

$$\frac{\partial^2 h}{\partial x^2} + \frac{1}{b}\frac{\partial b}{\partial x}\frac{\partial h}{\partial x} + \frac{\partial^2 h}{\partial y^2} = \frac{1}{\nu}\frac{\partial h}{\partial t}$$

provided $\frac{\partial b}{\partial x} < 0.20$

For $b = b_o \exp\left[-(2)(x - x_o)/a\right]$ and in terms of $s = h_i - h$, the differential equation is

$$\frac{\partial^2 s}{\partial x^2} - \frac{2}{a}\frac{\partial s}{\partial x} + \frac{\partial^2 s}{\partial y^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS AND SOLUTIONS:

1. Well of constant discharge in an effectively infinite aquifer .

Boundary conditions:

$$s(x, y, 0) = 0 \qquad (1)$$

$$s(x \pm \infty, t) = 0 \qquad (2)$$

$$s(\pm\infty, y, t) = 0 \qquad (3)$$

$$\lim_{r \to 0} \int_0^{2\pi} r\,\frac{\partial s}{\partial r}\,d\theta = -\frac{Q}{Kb_o} \qquad (4)$$

Solution:

$$s = \frac{Q}{4\pi K b_o} \exp\left(\frac{r}{a}\cos\theta\right) W\left(u, \frac{r}{a}\right)$$

Limitation: The above equation is valid only for the period $t < r_o^2/20\nu$

and within the semi-infinite plane bounded by a line parallel to the y axis and located by $r = r_o$ and $\theta = \pi$ where

$$r_o = \frac{a}{2} \ln\left(\frac{a}{10 b_o}\right)$$

2. Well of constant discharge near a stream. (Stream coinciding with y-axis of sketch).

   Boundary conditions: Equations (1), (2), (4) and

$$s(\infty, y, t) = 0 \qquad (5)$$

$$s(0, y, t) = 0 \qquad (6)$$

   Solution:

$$s = \frac{Q}{4\pi K b_o} \exp\left(\frac{r}{a}\cos\theta\right)\left[W\left(u, \frac{r}{a}\right) - W\left(u', \frac{r'}{a}\right)\right]$$

3. Well of constant discharge near an impermeable boundary. (Impermeable boundary coinciding with y-axis of sketch).

   Boundary conditions: Equations (1), (2), (4), (5) and

$$\frac{\partial s}{\partial x}(0, y, t) = 0 \qquad (7)$$

   Solution:

$$s = \frac{Q}{4\pi K b_o}\left\{\exp\left(\frac{r}{a}\cos\theta\right)\left[W\left(u, \frac{r}{a}\right) + W\left(u', \frac{r'}{a}\right)\right] + \frac{2\sqrt{\pi}}{U_a}\exp\left(-\frac{2x_o}{a}\right) F\left(U_x, U_y, U_a\right)\right\}$$

   where

$$F\left(U_x, U_y, U_a\right) = \int_1^{\infty} e^{-\beta^2 U_y^2} \operatorname{erfc}\left(\beta U_x - \frac{1}{2\beta U_a}\right) \frac{d\beta}{\beta^2}$$

   is a function not available in tabular form. A possible approximation may be:

$$F \approx \frac{1}{2}\operatorname{erfc}\left(U_x - \frac{1}{2U_a}\right)\left[\frac{1}{U_y}\exp\left(-U_y^2\right) - \sqrt{\pi}\operatorname{erfc}\left(U_y\right)\right]$$

   In all the three previous cases the equivalent solutions for aquifers of uniform thickness can be used, provided

$$r/a \leq 0.01$$

   and

$$t \leq 2.5\, ra/\nu$$

4. Flowing well in an effectively infinite aquifer.

   Boundary c onditions: Equation (1), (2), (3) and

$$s(r_w, t) = s_w$$

   Solution:

$$s = s_w \exp\left(\frac{r - r_w}{a}\cos\theta\right)\left[\frac{K_o(r/a)}{K_o(r_w/a)} + \exp\left(-\tau r_w^2/a^2\right) \cdot E\left(\rho, \tau, \frac{r_w}{a}\right)\right]$$

where

$$E = \frac{2}{\pi} \int_0^{\infty} \frac{J_o(\mu\rho)\, Y_o(\mu) - Y_o(\mu\rho)\, J_o(\mu)}{\left[J_o^2(\mu) + Y_o^2(\mu)\right]\left[\mu^2 + \left(\frac{r_w}{a}\right)^2\right]} \; e^{-\tau\mu^2} \quad \mu d\mu$$

is a function not available in tabular form. However, the following approximations can be made:

(a) For $\tau/\rho^2 < 0.05$

$$s \approx s_w \exp\left(\frac{r - r_w}{a} \cos\theta\right)\left[\frac{1}{2\sqrt{\rho}} \left\{\exp\left[\frac{r_w}{a}(\rho - 1)\right] \text{erfc}\left[\frac{r_w\sqrt{\tau}}{a} + \frac{\rho - 1}{2\sqrt{\tau}}\right]\right.\right.$$

$$\left.\left. + \exp\left[-\frac{r_w}{a}(\rho - 1)\right] \text{erfc}\left[-\frac{r_w\sqrt{\tau}}{a} + \frac{\rho - 1}{2\sqrt{\tau}}\right]\right\}\right]$$

(b) For $\frac{\tau r_w^2}{a^2} > 1$

$$s \approx s_w \exp\left(\frac{r - r_w}{a} \cos\theta\right)\left[W\left(\frac{\rho^2}{4\tau}, \frac{r}{a}\right) \Big/ W\left(\frac{1}{4\tau}, \frac{r_w}{a}\right)\right]$$

(c) For $t = \infty$

$$E(\rho, \infty, \frac{r_w}{a}) = 0$$

(d) For $\frac{r_w}{a} = 0$

$$E(\rho, \tau, 0) = A(\rho, \tau) - 1$$

This last property of E can be used as an approximation for values of $\frac{r_w}{a} < 0.01$.

$$Q_t = 2\pi K b_o S_w G(\tau, \frac{r_w}{a})$$

where

$G(\tau, \frac{r_w}{a})$ is a tabulated function. (See "Nonsteady flow to flowing wells for leaky aquifers" by M. S. Hantush, JGR, V64, No. 8, 1959)

SYMBOLS:

ν	$= K / S_s$
ρ	$= r/r_w$
τ	$= \nu t/r_w^2$
θ	= The polar angle with the pole at the center of the well and the polar axis parallel to x-axis.
$A(\rho, \tau)$	= The flowing well function available in tabular form. (See Jaeger - "Numerical values for temperature in radial heat flow", J. Math. Phys., 34, 1956)
a	= A geometric parameter defining the exponential variation of aquifer thickness.
b	$= b_o \exp[-2(x - x_o)/a]$ = the aquifer thickness.
b_o	= Thickness of aquifer at the site of the well, (at points (x_o, y)).
E	= The function defined above.
erfc(x)	= The complementary error function.
$F(U_a, U_x, U_y)$	= The function defined above.

$G(\tau, r_w/a)$ = The flowing well discharge function, available in tabular form. (See previous page).

$h(x, y, t)$ = Average piezometric head in a vertical column of the aquifer at any point (x, y) and any time t.

h_i = Initial piezometric head.

J_o = Zero-order Bessel function of the first kind.

K = The hydraulic conductivity of the aquifer.

K_o = Zero-order Modified Bessel function of the second kind.

Q = Discharge of steady well.

Q_t = Discharge of a flowing well.

r' $= [(x + x_o)^2 + (y - y_o)^2]^{\frac{1}{2}}$ = Radial distance measured from the point $(-x_o, y_o)$.

r $= [(x - x_o)^2 + (y - y_o)^2]^{\frac{1}{2}}$ = Radial distance from center of well.

r_o $= (a/2)\ n\ (a/10b_o)$

r_w = Effective radius of the well.

S_s = Specific storage of the aquifer.

$s(x, y, t)$ $= h_i - h$ = Drawdown at any point at any time.

s_w = Constant drawdown of flowing well.

t = Time since initial condition of flow.

u $= r^2/4\nu t$

u' $= r'^2/4\nu t$

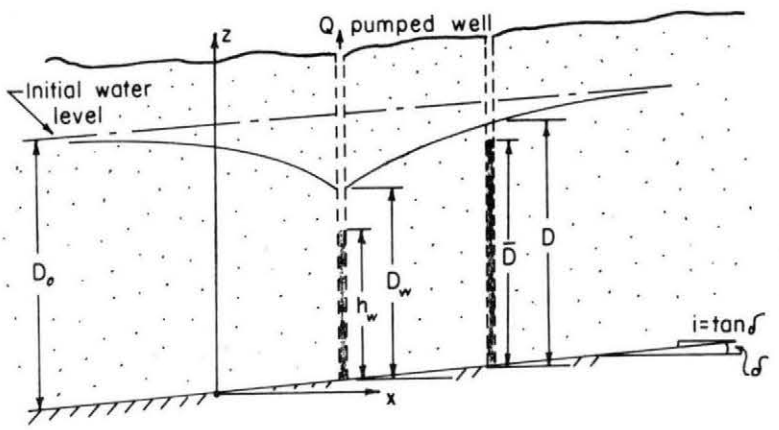
U_a, U_x, U_y $= a/\sqrt{4\nu t}$, $(x + x_o)/\sqrt{4\nu t}$, $(y - y_o)/\sqrt{4\nu t}$, respectively.

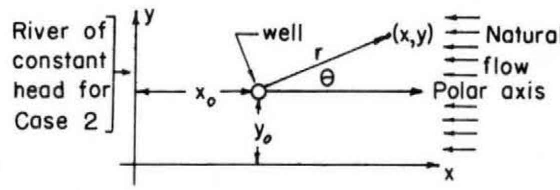
$W(u, \beta)$ = The well function for leaky aquifers available in tabular form.

(x, y) = Rectangular coordinates.

(x_o, y_o) = Location of the well.

Y_o = Zero-order Bessel function of the second kind.





REFERENCE:

Hantush, M. S., 1962, Hydraulics of gravity wells in sloping sands: Am. Soc. Civil Engineers Proc., v. 88, no. HY 4, p. 1-15.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION (Approximate):

$$\frac{\partial^2 \bar{D}^2}{\partial x^2} + \frac{2}{\beta}\frac{\partial \bar{D}^2}{\partial x} + \frac{\partial^2 \bar{D}^2}{\partial y^2} = \frac{1}{\nu}\frac{\partial \bar{D}^2}{\partial t} \qquad \text{(See reference for derivation)}$$

SOLUTIONS:

Case 1. Well of constant discharge in an infinitely sloping sand

$$D_o^2 - \bar{D}^2 = \frac{Q}{2\pi K} \exp\left(-\frac{r}{\beta}\cos\theta\right) W\left(u, \frac{r}{\beta}\right)$$

Case 2. Well of constant discharge upstream from a river and cutting across the natural flow.

$$D_o^2 - \bar{D}^2 = \frac{Q}{2\pi K} \exp\left(-\frac{r}{\beta}\cos\theta\right)\left\{ W\left(u, \frac{r}{\beta}\right) - W\left(u', \frac{r'}{\beta}\right)\right\}$$

Note: For wells downstream (i.e. aquifer dipping in + x direction) change β by $(-\beta)$ in the exponential.

Case 3. Well of constant head in an infinitely sloping sand.

$$D_o^2 - \bar{D}^2 = (D_o^2 - \bar{D}_w^2) \exp\left(-\frac{r - r_w}{\beta}\cos\theta\right)\left\{\frac{K_o(r/\beta)}{K_o(r_w/\beta)} - \left[1 - A(\rho,\tau)\right]\exp(-\tau\, r_w^2/\beta^2)\right\}$$

$$Q(t) = \pi K (D_o^2 - \bar{D}_w^2)\, G\left(\tau, \frac{r_w}{\beta}\right)$$

Limitations: The above solutions are limited to the cases where $i < 0.2$ and $\frac{D_o - h_w}{D_o} < 0.5$.

SYMBOLS:

β	$= 2 D_w / i$
ϵ	= Specific yield (effective porosity) of the aquifer.
θ	= The polar angle with the pole at the center of the well.
ν	$= K D_w / S_w$
ρ	$= r / r_w$
τ	$= \nu t / r_w^2$

$A(\rho, \tau)$	=	The flowing well function for nonleaky aquifers. For tabular values see "Numerical Values for the Temperature in Radial Heat Flow" by J. C. Jaeger, Journal of Math. and Phys., V34, 1956, pp. 316 -321.
$D(x, y, t)$	=	The height of the water table above the base of the sloping sand.
D_o	=	The distribution of the depth of flow above the base of the aquifer that would prevail if there was no pumping.
D_w	=	The height of the water table at the face of the well.
$\bar{D}(x, y, t)$	=	The depth of water in an observation well screened throughout and completely penetrating the sloping sand.
$\bar{D}_w$	=	$\bar{D}$ at the face of the well = $(h_w^2 + D_w^2)/2D_w$
$G(\tau, \frac{r_w}{\beta})$	=	The flowing well discharge function for leaky aquifers. For tabular values see "Non-steady flow to flowing wells in leaky aquifers", by M. S. Hantush, Journal of Geo. Res. V64, No. 8, 1959.
h_w	=	The depth of the water in the pumping well.
i	=	The slope of the base of the aquifer.
K	=	Hydraulic conductivity.
$K_o(\alpha)$	=	The zero-order Modified Bessel function of the second kind.
Q	=	The constant discharge of the well.
$Q(t)$	=	The discharge of a constant head well.
r	=	$\sqrt{(x - x_o)^2 + (y - y_o)^2}$
r'	=	$\sqrt{(x + x_o)^2 + (y - y_o)^2}$
r_w	=	Effective radius of the well.
S_s	=	Specific storage of the aquifer (see reference for definition).
S_w	=	$D_w S_s + \epsilon$ = Storage coefficient of a water table aquifer.
t	=	Time since the initial condition of flow.
u	=	$r^2/4\nu t$
u'	=	$r'^2/4\nu t$
$W(u, \alpha)$	=	The well function for leaky aquifers (see Professional Paper 104 by M. S. Hantush, NMIMT for tables of $W(u, \alpha)$.
x, y	=	Rectangular coordinates.
x_o, y_o	=	The point at which the center of the well is located.
x_o	=	The effective horizontal distance between the well and the stream bed; also the horizontal coordinate of the well location.

B - 18. DISCHARGE OF INTERFERING WELLS

REFERENCE:

Hantush, M. S., Discharge of interfering wells, unpublished notes.

SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

STATEMENT OF PROBLEM

If the location of each of N wells is known and the water levels in each of the N wells at the end of a given period of continuous pumping is preassigned, the discharge of each well can be obtained by solving the N linearly independent equations written for the water level in each of the wells, using the Theis formula for artesian aquifers or the modified Theis formula for water-table aquifers as the case may be. Thus, two artesian wells a distance m apart, discharging simultaneously over the same period of time t_o from a nonleaky aquifer, and having the same diameter $2r_w$ and drawdown, s_w, will have discharges Q_1 and Q_2 given by

$$Q_1 = Q_2 = 4\pi Ts_w / [\,W(r_w^2 / 4\nu t_o) + W(m^2/4\nu t_o)\,]$$

Similarly, for three wells forming an equilateral triangle a distance m on a side,

$$Q_1 = Q_2 = Q_3 = 4\pi Ts_w / [\,W(r_w^2 / 4\nu t_o) + 2W(m^2/4\nu t_o)\,]$$

If t_o is long enough that $m^2/4\nu t_o < 0.05$, the expressions for the preceding particular well patterns may be given, respectively by

$$Q_1 = Q_2 = 2\pi Ts_w / \ln(2.25\,\nu t_o / mr_w)$$

and

$$Q_1 = Q_2 = Q_3 = 2\pi Ts_w / \ln(R^3/r_w m^2)$$

where

$$R = 1.5\sqrt{\nu t_o}$$

The discharge of each of four wells forming a square of side m, provided $m^2/\nu t_o < 0.05$, is given by

$$Q_1 = Q_2 = Q_3 = Q_4 = 2\pi Ts_w / \ln(R^4 / r_w\, m^3 \sqrt{2}\,)$$

and for a line of three equally spaced wells a distance m apart, provided $m^2/4\nu t < 0.05$, the discharge of each of the outer wells is

$$Q_1 = Q_3 = [\,2\pi Ts_w \ln(m/r_w)\,] / f(R, m, r_w)$$

and the discharge of the middle well is

$$Q_2 = 2\pi Ts_w \ln(m/2r_w) / f(R, m, r_w)$$

where $f(R, m, r_w) = 2\ln(R/m)\ln(m/r_w)\ln(m/2r_w)\ln(R/r_w)$

The corresponding equations for a nonleaky, horizontal, water-table aquifer are obtained from the preceding expressions by merely replacing $(2\pi Ts_w)$ with $\pi K(D_o^2 - h_w^2)$.

SYMBOLS:

D_o	= Initial depth of flow in a water-table aquifer.
h_w	= Depth of water in the well.
K	= Hydraulic conductivity of the aquifer.
r_w	= Effective radius of a well.
S	= Storage coefficient of an artesian aquifer.
s_w	= Drawdown in a discharging well (neglecting well losses).
$T = Kb$	= Transmissivity of an artesian aquifer.
$T = KD_o$	= Transmissivity correspondent to initial depth of flow in a water-table aquifer.
t_o	= Length of period of continuous flow since the start.
$W(u) = \int_u^\infty \frac{e^{-y}}{y} dy$	= well function for nonleaky aquifers; tabular values are available.
ϵ	= specific yield of the aquifer.
ν	= $\frac{Kb}{\epsilon}$ for water-table aquifers.
ν	= T/S for artesian aquifers.

B - 19. FLOW OF GROUND WATER TO COLLECTOR WELLS

REFERENCE:

Hantush, M. S., and Papadopulos, I. S., 1962, Flow of ground water to collector wells: Am. Soc. Civil Engineers Proc., v. 88, no. HY 5, p. 221-44.

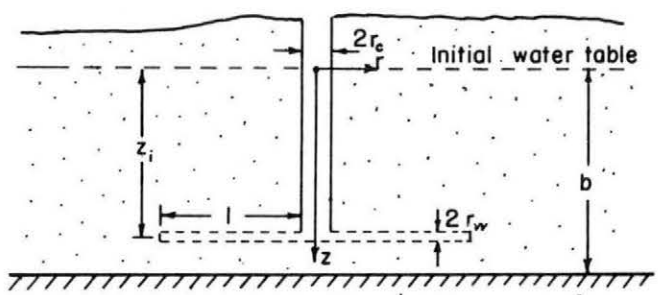
SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

Note: Only a few of the solutions obtained, those of the drawdown in the caisson of the collector well, are given below. For other solutions the reader is referred to the original paper. Only cases of symmetrically located laterals are considered here.

SOLUTIONS:

Collector well in an infinite water table aquifer.



For $t > 2.5\, b^2/\nu'$ and $> 5(r_c^2 + l^2)/\nu'$, $N \geq 4$ $l > \frac{1}{2} b$ and $r_w \leq \frac{b}{2\pi}$, the pumping level in the caisson can be approximated by:

$$s_c = \frac{Q/N}{4\pi Kb} \left\{ W\left(\frac{l^2}{4\nu' t}\right) + \frac{N-1}{l} \left[l' W\left(\frac{l'^2}{4\nu' t}\right) - r_c W\left(\frac{r_c^2}{4\nu' t}\right)\right] + 2N + \frac{b}{2l} \ln \frac{(b/\pi r_w)^2}{2\left[1 - \cos \frac{\pi}{b}(2Z_i + r_w)\right]} \right.$$

$$\left. + \frac{4b(N-1)}{\pi l} \sum_{1}^{M'} \frac{1}{n} \left[\frac{\pi}{2} - L\left(\frac{n\pi r_c}{b}, 0\right)\right] \cos \frac{n\pi Z_i}{b} \cos \frac{n\pi}{b}(Z_i + r_w) \right\} \qquad (1)$$

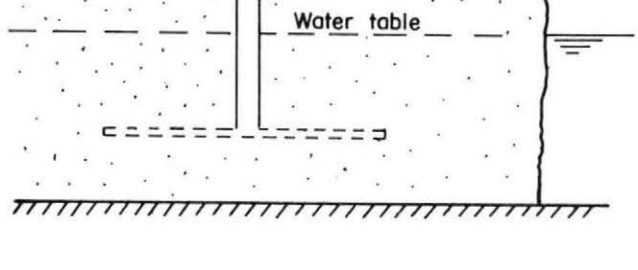
If $r_c \geq 0.5\, b$ the series can be neglected

For $t > 5\, l^2/\nu'$, $l > \frac{1}{2} b$ and $r_w \leq \frac{b}{2\pi}$ $\quad s_c \geq \frac{Q}{2\pi Kb} \ln \frac{R}{l}$

where

$$R = 4.08 \sqrt{\nu' t} \left\{ \frac{(b/\pi r_w)^2}{2\left[1 - \cos \frac{\pi}{b}(2Z_i + r_w)\right]} \right\}^{0.25\, b/l}$$

Collector well near a stream in a water table aquifer.



SOLUTIONS:

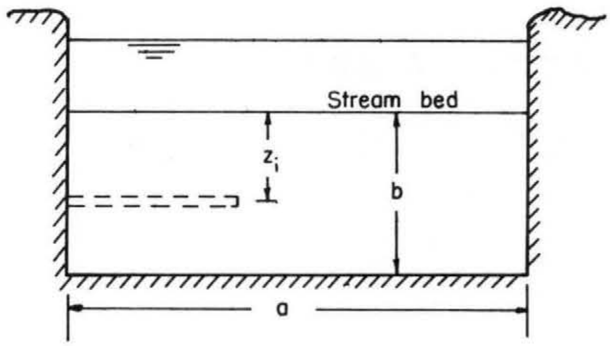
The drawdown during the steady state flow is given by

$$s_c = \frac{Q/N}{4\pi Kb} \left\{ 2 \ln \left[\frac{\gamma^\gamma}{\epsilon^\epsilon}\right] - 2(N-1) \ln \left[\frac{\epsilon^\epsilon f^f}{\gamma^\gamma \rho^\rho} \right] + (\text{the ln and the series terms of Equation (1)}) \right\}$$

Also the following holds if $1 > 0.5\,b$ and $r_w \leq b/2\pi$.

$$s_c = \frac{Q}{2\pi Kb} \ln \left\{ \frac{\gamma^\gamma}{\epsilon^\epsilon} \left[\frac{(b/\pi r_w)^2}{2\left[1 - \cos \frac{\pi}{b}(2Z_i + r_w)\right]} \right]^{b/41} \right\}$$

Collector wells under stream beds.



SOLUTIONS:

For $a > 0.5\,(b + r_c + 1')$, $1 > b$, $r_w \leq \frac{b}{\pi}$ and for $t > \frac{5\,b^2}{\nu}$ (steady state) the drawdown in the caisson is given by

$$s_c = \frac{Q/N}{8\pi K1} \left\{ \ln \left[\left(\frac{4b}{\pi r_w}\right)^2 \frac{1 - \cos \frac{\pi}{2b}(2Z_i + r_w)}{1 - \cos \frac{\pi}{2b}(2Z_i + r_w)} \right] + \frac{16}{\pi} \sum_{0}^{M'} \frac{1}{2n+1} \left[\frac{\pi}{2} - L\left(\frac{2n+1}{2b} 2\pi r_c, 0\right) \right.\right.$$

$$\left.\left. + 2(N-1)\left\{\frac{\pi}{2} - L\left(\frac{2n+1}{2b} \pi r_c, 0\right)\right\} \right] \sin \frac{2n+1}{2b} \pi(Z_i + r_w) \sin \frac{2n+1}{2b} \pi Z_i \right\}$$

The drawdown in the caisson of a collector well with a single lateral, perpendicular to the river bank and in an infinitely thick aquifer is given by

$$s_c = \frac{Q}{2\pi K1} \left\{ 0.5 \frac{\left(\frac{Z_i}{a}\right)^2 \left(\frac{1}{a}\right)}{\left[1 - \left(\frac{1}{2a}\right)^2\right]^2} + \ln \frac{2\left(1 + \frac{2Z_i}{r_w}\right)}{1 + \sqrt{1 + \left(\frac{2Z_i}{1}\right)^2}} \right\}$$

provided $1 > 10\,r_w$, $t \geq a^2 S_s / K$ and $Z_i \quad \frac{a}{2}$.

If the lateral extends from bank to bank, $1 = \varepsilon$ $\quad s_c = \frac{Q}{2\pi Ka} \ln\left(1 + \frac{2Z_i}{r_w}\right)$

for $t \geq 20 Z_i^2\ S_s / K$.

Also for a lateral extending from t nk to bank in an aquifer of finite thickness b , the steady state pumping level is given by

$$s_c = \frac{Q}{4\pi Ka} \ln \left\{ \frac{[1 - \cos\frac{\pi}{2b}(2Z_i + {}_w)](1 + \cos\frac{\pi}{2b} r_w)}{[1 + \cos\frac{\pi}{2b}(2Z_i + r_w)](1 - \cos\frac{\pi}{2b} r_w)} \right\}$$

SYMBOLS:

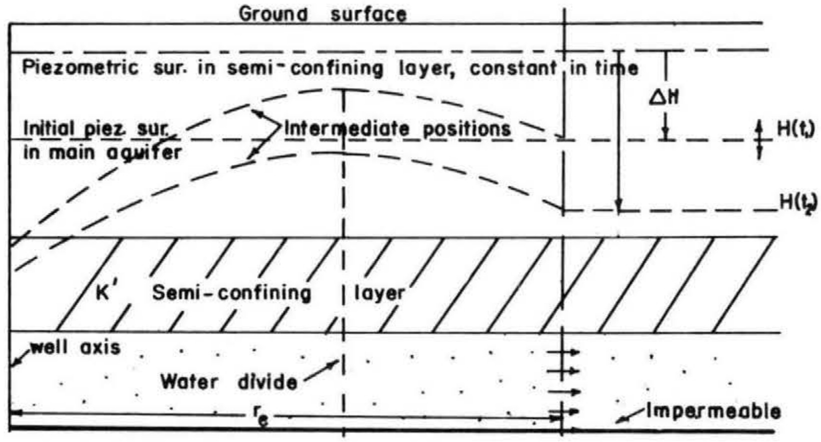
a	=	Effective distance between a collector well and a stream, also width of stream, L.
b	=	Initial depth of saturation of a water-table aquifer, L.
f	=	l'/l
K	=	Hydraulic conductivity of the aquifer, LT^{-1}
K_o	=	Zero-order Bessel function of the second kind.
$L(u, 0)$	=	$\int_0^u K_o(y)\,dy$, tabular values available.
l	=	Length of a lateral, L .
l'	=	$l + r_c$, L.
M'	=	An integer such that $M' > \frac{b}{2 r_c}$
N	=	Number of laterals of a collector well.
r_c	=	Radius of the caisson, L.
r_w	=	Effective radius of a lateral, L.
S_s	=	Specific storage.
S_y	=	Specific yield.
s_c	=	Drawdown in the caisson, L.
t	=	Time since pumping began, T.
$W(u)$	=	The well function for nonleaky aquifers.
Z_i	=	Vertical position of a lateral, L.
γ	=	$2(a - r_c)/l$
ϵ	=	$(2a - 2r_c - l)/l$.
ν'	=	Kb/S_y .
ρ	=	r_c/l .

APPENDIX C

SUMMARIES OF SOLVED CASES FOR LEAKY AQUIFERS

TABLE OF CONTENTS

C - 1. WELL OF CONSTANT DISCHARGE IN A LEAKY AQUIFER



REFERENCE:

De Wiest, R. J. M., 1961, On the theory of leaky aquifers: Jour. Geophys. Research, v. 66, no. 12, p. 4257-62.

SUMMARIZED BY:

R. J. M. De Wiest, Princeton University

DESCRIPTION OF CASE TREATED:

Potential distribution is found for two cases of unsteady flow in a finite leaky aquifer, where the boundary condition at the contour of influence is time dependent and where the initial condition is one of different head in the main aquifer and in the semi confining and quasi-impervious layer overlying it. The cases considered are a well pumped at a constant discharge and a flowing well at a constant drawdown.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} = \frac{S}{T}\frac{\partial s}{\partial t} + \frac{s}{B^2} + \frac{\Delta H}{B^2}$$

INITIAL CONDITION:

$$s(r, o) = \Delta H$$

BOUNDARY CONDITIONS:

$$s(r_e, t) = H(t)$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2rt}$$

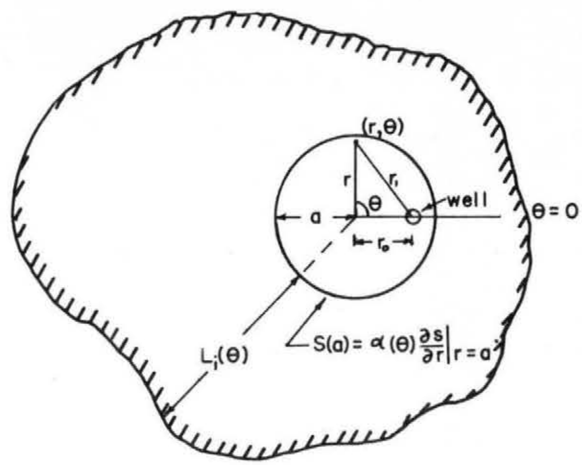
SOLUTION:

By method of variation of parameters.

SYMBOLS:

			Dimensions
B	=	Leakage factor.	L
r_e	=	Radius of influence.	L
T	=	Transmissivity.	$\frac{L^2}{T}$
S	=	Storage coefficient.	dimensionless
s	=	Drawdown .	L
t	=	Time.	T
r	=	Radial distance.	L
Q	=	Well yield.	$\frac{L^3}{T}$
ΔH	=	Head difference.	L

C - 2. ECCENTRIC WELL IN A LEAKY AQUIFER WITH VARIED LATERAL REPLENISHMENT



REFERENCE:

De Wiest, R. J. M., 1963 Flow to an eccentric well in a leaky circular aquifer with varied lateral replenishment: Geofisica Pura e Applicata, v. 54. no. 1, p. 87-102.

SUMMARIZED BY:

R. J. M. De Wiest, Princeton University

DESCRIPTION OF CASE TREATED:

An analytical solution is obtained for the flow to an eccentric well in a leaky circular aquifer with lateral replenishment both for steady and unsteady cases. The flows for external boundary conditions of constant head and zero flux, which were treated previously, follow in the limit from a more general boundary condition. Graphs are developed to show the influence of vertical leakage and lateral replenishment on the relationship between drawdown at the well and eccentricity.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t}$$

INITIAL CONDITION:

$$s(r,\theta,0) = \frac{\Delta Q}{T}\frac{1}{r}\delta(r - r_o)\,\delta(\theta)$$

BOUNDARY CONDITIONS :

$$s(r,0,t) = s(0,2r,t) \quad \frac{\partial s(r,0,t)}{\partial \theta} = \frac{\partial s(r,2\pi,t)}{\partial \theta}$$

$$s(0,0,t) = \text{finite}$$

$$s(a,\theta,t) = \alpha\,\frac{\partial s(a,\theta,t)}{\partial r}$$

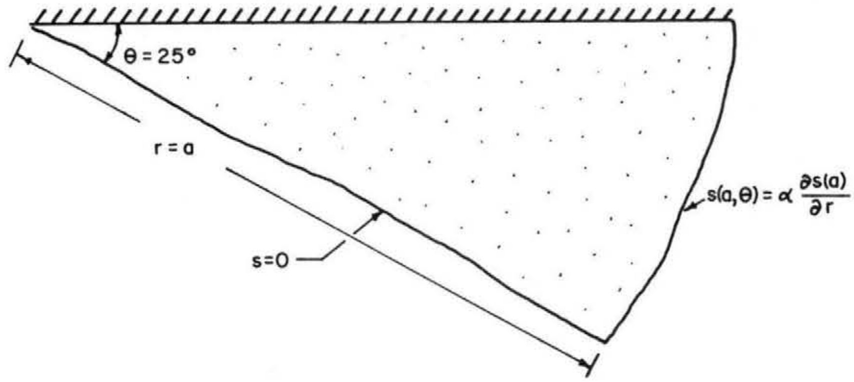
SOLUTIONS:

See original paper.

SYMBOLS:

s	= Drawdown	(L)	$\delta(x)$	= Dirac's delta function.	
S	= Storage coefficient of main aquifer (dimensionless)		r, θ	= Polar coordinates.	
T	= Transmissivity of main aquifer.	$(\frac{L^2}{T})$	α	= Constant, dependent on geometry of aquifer system.	(L)
B	= Leakage factor.	(L)			
ΔQ	= Increment in well yield.	$(\frac{L^3}{T})$	a	= Radius of circular aquifer.	(L)

C - 3. REPLENISHMENT OF LEAKY AQUIFERS INTERSECTED BY STREAMS



REFERENCE:

De Wiest, R. J. M., 1963, Replenishment of leaky aquifers intersected by streams: Am. Soc. Civil Engineers Proc., v. 89, no. HY 6, p. 165 - 191.

SUMMARIZED BY:

R. J. M. De Wiest, Princeton University

DESCRIPTION OF CASE TREATED:

The interrelationship between surface water and ground water is studied for certain confined and unconfined leaky aquifers, intersected by streams and subjected to water withdrawal by trenches or wells. The feasibility of a proposed ground-water recharge project in the vicinity of Princeton, N. J. is numerically evaluated. Use is made of Green's functions to find analytical solutions to the problems.

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t}$$

INITIAL CONDITION:

$$s(r,\theta,0) = \frac{\Delta Q}{T}\ \frac{1}{r}\ \delta(r - r_o)\,\delta(\theta)$$

BOUNDARY CONDITIONS:

$$s(r,0,t) = s(0,2r,t) \qquad \frac{\partial s(r,0,t)}{\partial \theta} = \frac{\partial s(r,2\pi,t)}{\partial \theta}$$

$$s(0,0,t) = \text{finite}$$

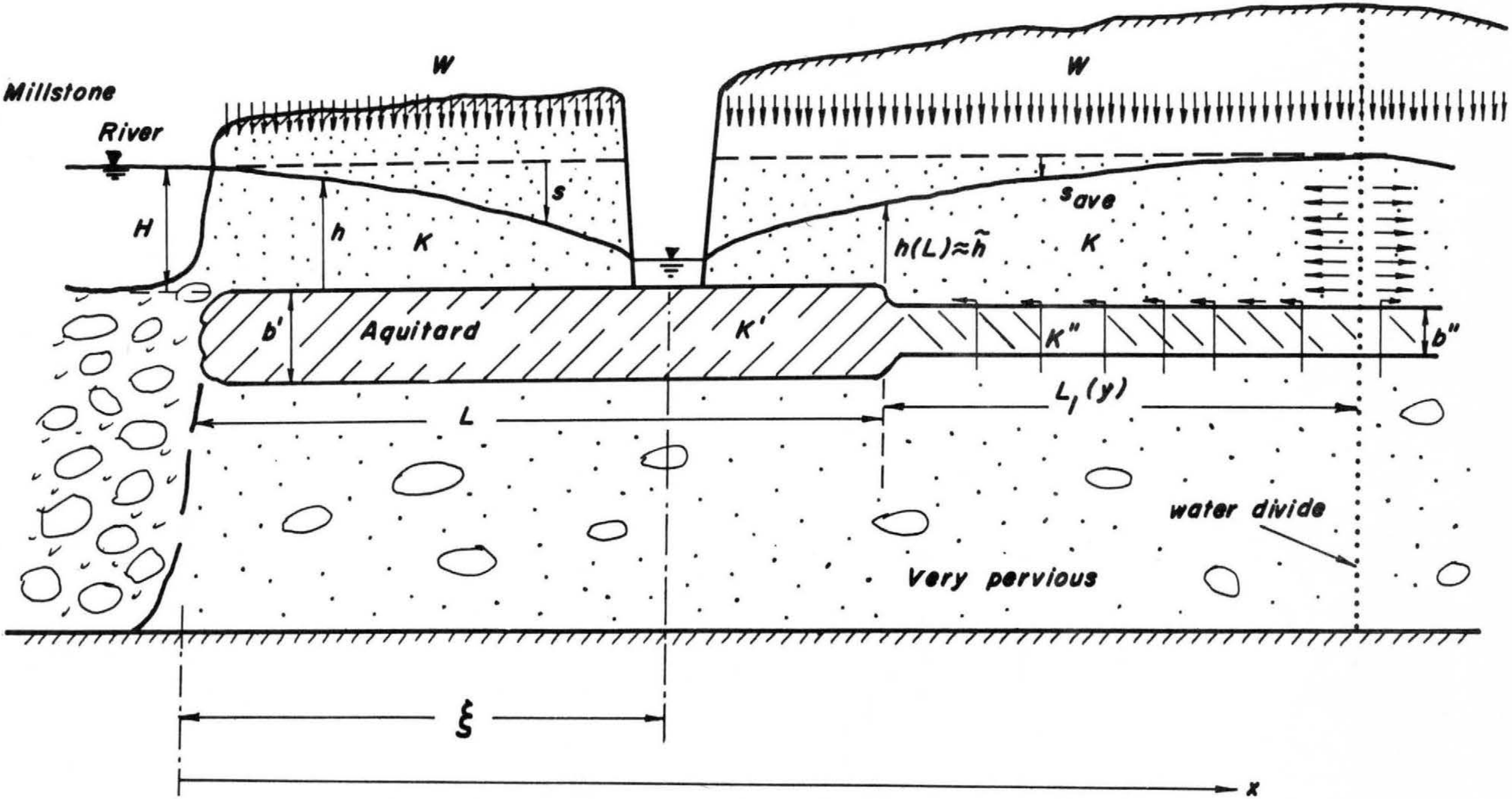
$$s(a,\theta,t) = \alpha\,\frac{\partial s(a,\theta,t)}{\partial r}$$

SOLUTIONS:

See original paper.

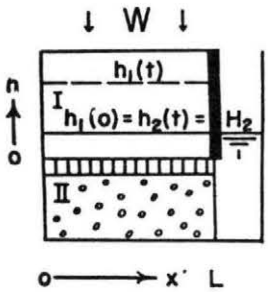
SYMBOLS:

See C-2 .



Idealized field conditions for flow into trench.

C - 4. NONSTEADY VERTICAL MOTION IN A STEADILY AND UNIFORMLY RECHARGED, LEAKY, SEMI-PERCHED AQUIFER OVERLYING A SEMICONFINED HIGHLY TRANSMISSIVE AQUIFER, AFTER INITIALLY ZERO HEAD DIFFERENCE; WATER LEVEL OF THE LOWER AQUIFER MAINTAINED STEADY.



REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATION:

$$(d/dt)\,h_1 + (K'/m'S_w)\,h_1 = (K'/m'S_w)\,h_2 + W/S_w$$

BOUNDARY CONDITION:

$$h_1(o) = H_2$$

NONSTEADY SOLUTION:

$$h_1(t) = H_2 + (Wm'/K')(1 - \exp[-(K'/m'S_w)t])$$

$$q_{1z}(t) = LW(1 - \exp[-(K'/m'S_w)t]) = q_2(L,t)$$

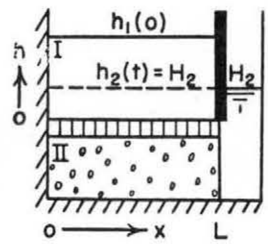
LIMITATIONS:

Transmissivity of aquifer II must be much larger than that of the semiconfining bed and aquifer I.

SYMBOLS:

See page 182

C - 5. NONSTEADY VERTICAL MOTION IN A LEAKY SEMIPERCHED AQUIFER OVERLYING A SEMICONFINED HIGHLY TRANSMISSIVE AQUIFER, AFTER CESSATION OF STEADY UNIFORM RECHARGE; WATER LEVEL OF LOWER AQUIFER MAINTAINED STEADY.



($h_1(t)$ declining)

REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATION:

$$d/dt)\, h_1 + (K'/m'S_w)(h_1 - h_2) = 0$$

BOUNDARY CONDITION:

$$h_1(0) = H_2 + W_o m'/K'$$

NONSTEADY SOLUTION:

$$h_1(t) = H_2 + (W_o m'/K') \exp[-(k'/m'S_w)t]$$

$$q_{1z}(t) = LW_o \cdot \exp[-(K'/m'S_w)t] = q_2(L, t)$$

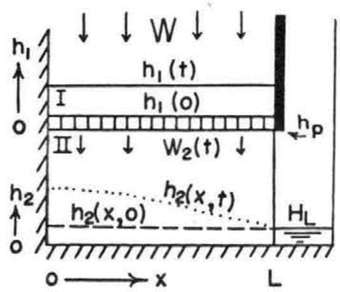
LIMITATIONS:

Transmissivity of aquifer II must be much larger than that of the semiconfining bed and aquifer I.

SYMBOLS:

See page 182

C - 6. NONSTEADY MOTION IN A PERCHED-AQUIFER SYSTEM CONSISTING OF A LEAKY, PERCHED, INFINITE CLOSED STRIP RECEIVING STEADY UNIFORM RECHARGE, AND A LOWER INFINITE HALF STRIP REPLENISHED BY LEAKAGE FROM THE PERCHED AQUIFER; LOWER AQUIFER CONNECTED TO A STREAM AT STEADY LEVEL, UPPER AQUIFER INITIALLY DRY



$W = W_o$ (steady)

$W_2(t) = W_o[1 - \exp(-Mt)]$

where $M = (K'/m'S_w)$;

REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATIONS:

Upper aquifer

$$(d/dt)\, h_1 + (K'/m'S_w) h_1 = W/S_w$$

Lower aquifer

$$(\partial^2/\partial x^2)\, h_2 = (1/k)(\partial/\partial t) h_2 - (W_2/T_2),\ W_2 = W_c\ [1 - \exp(-Mt)]\ \text{where } M = (K'/m'S_w)$$

BOUNDARY CONDITIONS:

Upper aquifer

$$h_1(0) = h_p = 0$$

Lower aquifer

$$h_2(x, 0) = H_L\ ; \quad h_2(L, t) = H_L\ ;\ (\partial/\partial x) h_2(0, t) = 0$$

SOLUTION:

Upper aquifer

$$h_1(t) = (Wm'/K')(1 - \exp[-(K'/m'S_w)t])$$

$$q_{1z}(t) = LW(1 - \exp[-(K'/m'S_w)t]) = q_2(L, t)$$

Lower aquifer

$$h_2(x, t) = H_L + (W_o/2T)/L^2 \left\{ 1 - \frac{x^2}{L^2} - \frac{32}{\pi^3} \sum_{n=0}^{\infty} \frac{(-1)^n \cos Nx \cdot \exp[-N^2 kt]}{(2n+1)^3 (1 - N^2 k/M)} \right\}$$

$$+ \frac{4W_o \exp[-Mt]}{MS\pi} \sum_{n=0}^{\infty} \frac{(-1)^n \cos Nx}{(1 - N^2 k/M)}$$

$$q_2(L, t) = W_o L \left\{ 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp[-N^2 kt]}{(2n+1)^2 (1 - N^2 k/M)} \right\} - \frac{W_o T \exp[-Mt]}{LMS} \sum_{n=0}^{\infty} \frac{(2n+1)}{(1 - N^2 k/M)}$$

Note: $N = \frac{(2n+1)\pi}{2L}$, $M = (K'/m'S_w)$

Aquifer outflow for large M

$$q_2(L, t) = W_o L \left\{ 1 - \frac{8}{\pi^2} \sum_{n=0}^{\infty} \frac{\exp[-N^2 kt]}{(2n+1)^2} \right\}$$

$$q_2(L, t) = 2 W_o (kt)^{\frac{1}{2}} \sum_{n=0}^{\infty} (-1)^n \left\{ \pi^{-\frac{1}{2}} - \text{ierfc}\left[\frac{(2n+1) L}{(kt)^{\frac{1}{2}}} \right] \right\}$$

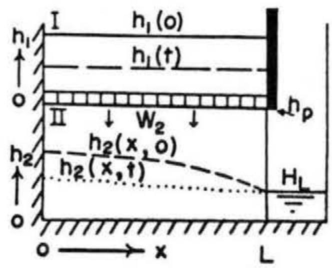
LIMITATIONS:

Rise in water level of aquifer II must be small with respect to its initial thickness. Validity near the bounding stream depends upon the percent of aquifer penetration by the stream.

SYMBOLS:

See page 182

C - 7. NONSTEADY MOTION IN A PERCHED-AQUIFER SYSTEM CONSISTING OF A LEAKY, PERCHED, INFINITE CLOSED STRIP (AFTER THE CESSATION OF STEADY UNIFORM RECHARGE) AND A LOWER INFINITE HALF STRIP REPLENISHED BY LEAKAGE FROM THE PERCHED AQUIFER AN CONNECTED TO A STREAM AT STEADY LEVEL; BOTH AQUIFERS INITIALLY AT A STEADY STATE IN EQUILIBRIUM WITH STEADY UNIFORM RECHARGE



REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATIONS:

Upper aquifer

$$(d/dt)h_1 + (K'/m'S_w)(h_1 - h_p) = 0$$

Lower aquifer

$$(\partial^2/\partial x^2)h_2 = (1/k)(\partial/\partial t)h - (W_o/T)\exp(-Mt)$$

BOUNDARY CONDITIONS:

Upper aquifer

$$h_1(0) = W_o(m'/K')$$

Lower aquifer

$$h_2(x,0) = h_s(x) = H_L + W_o(L^2 - x^2)/2T$$

$$h_2(L,t) = H_L \qquad (\partial/\partial K)h_2(0,t) = 0$$

SOLUTIONS:

Upper aquifer

$$h_1(t) = (W_o m'/K')\exp[-(k'/m'S_w)t]$$

$$q_{1z}(t) = LW_o \quad \exp[-(K'/m'S_w)t] = q_2(L,t)$$

Lower aquifer

$$q_2(L,t) = \frac{8W_oL}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n\exp(-N^2kt)}{(2n+1)^2} + W_o(k/M)^{\frac{1}{2}}\tan L(M/k)^{\frac{1}{2}}\exp(-Mt)$$

$$+\frac{2W_o}{TL}\sum_{n=0}^{\infty}\frac{(-1)^n\exp(-N^2kt)}{M/k - N^2} \quad \text{where } N=(2n+1)\pi/2L \text{ and } M=(K'/m'S)$$

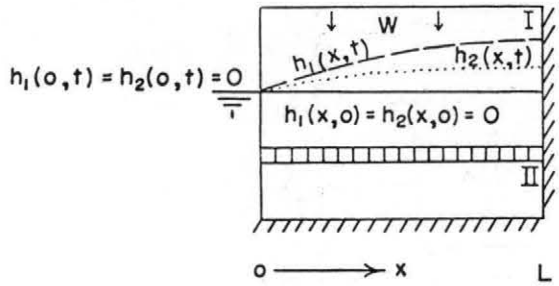
LIMITATIONS:

Rise in water level of aquifer II must be small with respect to its initial thickness. Validity near the bounding stream depends upon the percent of aquifer penetration by the stream.

NOTATION:

See page 182

C - 8. NONSTEADY MOTION IN A LEAKY INFINITE HALF-STRIP PARALLEL-AQUIFER SYSTEM CONSISTING OF A STEADILY AND UNIFORMLY RECHARGED AQUIFER OVERLYING A LEAKY ARTESIAN AQUIFER, THE BOUNDING STREAM AND BOTH AQUIFERS INITIALLY AT UNIFORM LEVEL



REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATION:

$$(\partial^2/\partial x^2)\,h_1 - (h_1 - h_2)/B_1^2 = (1/k_1)(\partial/\partial t)\,h_1 - W/T_1$$

$$(\partial^2/\partial x^2)\,h_2 + (h_1 - h_2)/B_2^2 \qquad (1/k_2)(\partial/\partial t)\,h_2$$

BOUNDARY CONDITIONS:

$$h_1(x, o) = 0; \quad h_2(x, o) = 0$$

$$(\partial/\partial x)\,h_1(L, t) = 0; \quad (\partial/\partial x)\,h_2(L, t) = 0$$

$$h_1(o, t) = 0; \quad h_2(o, t) = 0$$

SOLUTIONS:

Approximate outflow solution for small t .

$$q_1(o,t) = -\left(\frac{2W}{Lbk_3}\right)\Big\{[ck_1/k_2 - a]\sum_{n=0}^{\infty}\left[\frac{1 - \exp\,[-(a+N_1)\,t]}{(a+N_1)}\right]$$

$$- [(b-c)k_1/k_2 - (b-a)]\;e^{-bt}\sum_{n=0}^{\infty}\left[\frac{1 - \exp\,[-(a-b+N_1)t]}{(a-b+N_1)}\right]\Big\}$$

$$q_2(o,t) = \frac{-2T_2Wk_2}{LT_1B_2^2bk_3}\left\{\sum_{n=0}^{\infty}\left[\frac{1 - \exp\,[-(c+N_2)t]}{(c+N_2)}\right] - e^{-bt}\sum_{n=0}^{\infty}\left[\frac{1 - \exp\,[-(c-b+N_2)\,t]}{(c-b+N_2)}\right]\right\}$$

$$+ \frac{2\,T_2Wk_1}{LT_1B_2^2bk_3}\left\{\sum_{n=0}^{\infty}\left[\frac{1 - \exp[-(a+N_1)\,t]}{(a+N_1)}\right] - e^{-bt}\sum_{n=0}^{\infty}\left[\frac{1 - \exp\,[-(a-b+N_1)\,t]}{(a-b+N_1)}\right]\right\}$$

Approximate outflow solution for large t: .

$$q_1(o,t) = \frac{-WB_1B_2}{2Lk_2k_4}(2+k_2k_4)\left\{\exp(-gt)+\exp(-ft)\right\}\theta_1\left(\frac{1}{2}, i\pi t/2L^2k_4\right) - \frac{WB_1B_2}{Lk_1k_4}\left\{(2c+gk_2k_4)\cdot\right.$$

$$\left.\sum_{n=0}^{\infty}\left[\frac{1-\exp[-(g+N_4)t]}{(g+N_4)}\right] + (2c+fk_2k_4)\sum_{n=0}^{\infty}\left[\frac{1-\exp[-(f+N_4)t]}{(f+N_4)}\right]\right\}$$

$$q_2(o,t) = \frac{-2T_2WB_1}{LT_1B_2k_4}\left\{\sum_{n=0}^{\infty}\left[\frac{1-\exp[-(g+N_4)t]}{(g+N_4)}\right] - \sum_{n=0}^{\infty}\left[\frac{1-\exp[-(f+N_4)t]}{(f+N_4)}\right]\right\}$$

Note: $N_1 = (2n+1)^2\pi^2k_1/4L^2$

$N_2 = (2n+1)^2\pi^2k_2/4L^2$

$N_4 = (2n+1)^2\pi^2/2L^2k_4$

$a = k_1/B_1^2$, $b = (1/B_1^2 - 1/B_1^2)/k_3$, and $c = k_2/B_2^2$.

$g = (1/B_1 + 1/B_2)^2/k_4$, and $f = (1/B_1 - 1/B_2)^2k_4$.

where $k_3 = (1/k_1 - 1/k_2)$, $k_4 = (1/k_1 + 1/k_2)$

$\theta_1(\frac{1}{2})$ is the theta function (Whittaker and Watson, 1927, Ch. 21)

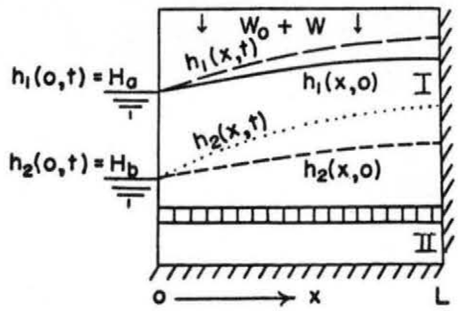
LIMITATIONS:

Maximum thickness of saturation in aquifer I must not be much larger than the minimum. Validity near the bounding stream depends upon the percent of aquifer penetration by the stream.

SYMBOLS:

See page 182

C - 9. NONSTEADY MOTION IN A LEAKY INFINITE HALF-STRIP PARALLEL-AQUIFER SYSTEM CONSISTING OF AN UNCONFINED AQUIFER SEMIPERCHED ON A LEAKY ARTESIAN AQUIFER. THE WATER LEVELS OF THE TWO AQUIFERS INITIALLY ARE AT A STEADY STATE IN EQUILIBRIUM WITH RECHARGE W_o, AND ARE MAINTAINED AT STEADY NON-EQUAL LEVELS AT THE STREAM BOUNDARY DURING STEADY UNIFORM RECHARGE AT THE RATE $(W_o + W)$.



REFERENCE:

Spiegel, Zane, 1962, Hydraulics of certain stream-connected aquifer systems: New Mexico State Engineer Spec. Rept., 105 p.

SUMMARIZED BY:

Zane Spiegel, New Mexico State Engineer Office

DIFFERENTIAL EQUATIONS:

$$(\partial^2/\partial x^2)h_1 - (h_1 - h_2)/B_1^2 = (1/k_1)(\partial/\partial t)h_1 - (W_o + W)/T_1$$

$$(\partial^2/\partial x^2)h_2 + (h_1 - h_2)/B_2^2 = (1/k_2)(\partial/\partial t)h_2$$

BOUNDARY CONDITIONS:

$$h_1(x, o) = h_{1s} \qquad h_2(x, o) = h_{2s}$$

$$(\partial/\partial x)h_1(L, t) = 0; \qquad (\partial/\partial x)h_2(L, t) = 0$$

$$h_1(o, t) = H_a; \qquad h_2(o, t) = H_b$$

SOLUTION:

For $q_1(o, t)$ add:

$$T_1\left[(H_2 - H_b) - \left(\frac{B_1^2 B_2^2}{B_1^2 + B_2^2}\right)\left(\frac{W_o}{T_1}\right)\right]\left(\frac{B_2^2}{B_1^2 + B_2^2}\right)\left(\frac{B_1^2 + B_2^2}{B_1^2\ B_2^2}\right)^{\frac{1}{2}} \tan h\left(\frac{B_1^2 + B_2^2}{B_1^2\ B_2^2}\right)^{\frac{1}{2}} L - \left(\frac{B_1^2}{B_1^2 + B_2^2}\right)W_o L$$

to the solutions found in C-8.

For $q_2(o,t)$ add:

$$T_2 \left[(H_b - H_a) + \left(\frac{B_1^2 B_2^2}{P_1^2 + B_2^2}\right)\left(\frac{W_o}{T_1}\right) \right] \left(\frac{B_1^2}{B_1^2 + B_2^2}\right)\left(\frac{B_1^2 + B_2^2}{B_1^2 \; B_2^2}\right)^{\frac{1}{2}} \tan h \left(\frac{B_1^2 + B_2^2}{B_1^2 \; B_2^2}\right)^{\frac{1}{2}} L - \left(\frac{B_2^2}{B_1^2 + B_2^2}\right) W_o L$$

to the solutions found in C-8.

LIMITATIONS:

Maximum thickness of saturation in aquifer I must not be much larger than the minimum. Validity near the bounding stream depends upon the percent of aquifer penetration by the stream.

SYMBOLS:

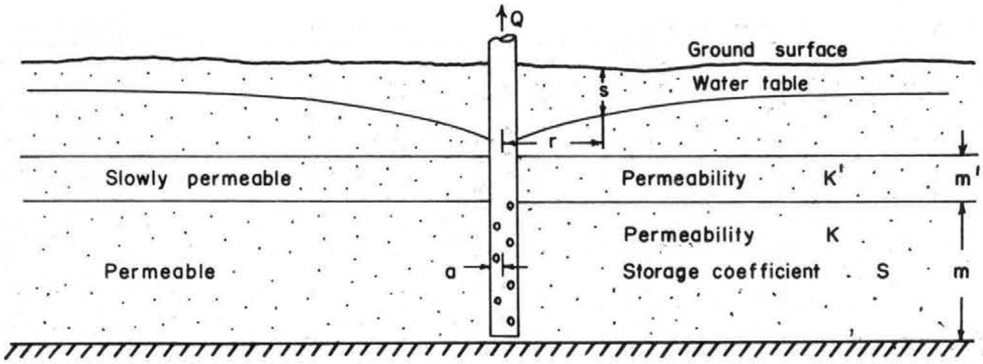
See page 182

SYMBOLS USED IN SUMMARIES BY SPIEGEL

Subscripts	Explanation
a, b	Denotes that the potential is measured at a boundary.
k = 1(1)n	Denotes the potential or aquifer coefficient pertaining to the kth of n parallel aquifers.
L, o	Value at a fixed point.
p	Denotes the potential or aquifer coefficient pertaining to the aquifer in parallel with the kth aquifer of a pair of mutually leaky aquifers; for a perched aquifer, denotes the atmospheric pressure (zero) at the lower face of the perching layer.
s	General space coordinate
t	Time.
x, y, z	Coordinate axes.

Symbols; units	
B_1, B_2; (L)	Leakage coefficient.
C	Constant.
H; (L)	Hydraulic potential at a stream boundary;
h; (L)	Hydraulic potential, head.
K, K'; (L/T)	Hydraulic conductivity of an aquifer and semi-confining bed, respectively.
k; (L^2/T)	Hydraulic diffusivity, (T/S) or (K/S_s).
L; (L)	Width of an aquifer.
M	Defined in C-7.
m, m'; (L)	Thickness of an aquifer and semiconfining bed, respectively.
Q, Q_t; (L^3/T)	Total aquifer inflow or outflow.
q; (L^2/T)	Aquifer inflow or outflow.
R; (L)	Point on the y-axis.
S, S_a, S_w	Aquifer storage coefficients (dimensionless). S, general coefficient; S_a, storage derived from expansion of aquifer and water; S_w, storage coefficient for unconfined aquifers.
S_s; (1/L)	Specific storage (storage coefficient for aquifer of unit thickness).
T; (L^2/T)	Aquifer transmissivity (transmissibility), equal to (Km).
t; (T)	Time.
t'; (T)	Current time variable of integration.
u, v; (L)	Hydraulic potentials.
v; (L/T)	Effective or bulk velocity of a fluid in a porous medium, defined as the volume of fluid passing a unit area of gross cross section per unit time.
v'_z; (L/T)	Vertical velocity in a semiconfining bed.
W; (L/T)	Areal recharge.
x, y, z	Coordinates in the rectangular Cartesian system.

C - 10. WELL PUMPING FROM AN AQUIFER OVERLAIN BY A SLOWLY PERMEABLE BED



REFERENCE:

Glover, R. E. , Florey, Q.L., and Balmer, G. G., 1952, Chart for analysis of test well data in cases where the water-bearing sand is overlain by beds of low permeability - Oahe Unit - Missouri River Basin Project, Bureau of Reclamation Technical Memorandum No. 657, Section C, p. 31-34; and Glover, R.E., Moody, W.T. and Tapp, W. N., 1954, Till permeabilities as estimated from the pump-test data obtained during irrigation wells investigations - Oahe Unit - Missouri River Basein Project, Bureau of Reclamation Technical Memorandum No. 657, Section T, p. 171-176.

SUMMARIZED BY:

R. E. Glover, U. S. Bureau of Reclamation

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} - u = \frac{\partial u}{\partial \eta}$$

BOUNDARY CONDITIONS:

$u = 0$ for $x > 0$ when $\eta = 0$

$u = 0$ at $x = x_e$ when $\eta > 0$.

SOLUTION:

$$u = \left[K_o(x) - \frac{K_o(x_e)}{I_o(x_e)} I_o(x)\right] - \sum_{n=1}^{\infty} \frac{2J_o(\alpha_n x)\, e^{-(1+\alpha_n^2)\eta}}{x_e^2 J_1^2(\alpha_n x_e)(1+\alpha_n^2)}$$

SYMBOLS (Consistent units):

s	=	ψu
x	=	θr
η	=	γt
t	=	Time
a	=	Well radius
b	=	An outer radius where $s = 0$
ψ	=	$\frac{Q}{2\pi K m}$
γ	=	$\frac{K'}{S m'}$
θ	=	$\frac{b}{a}$
Q	=	Well flow.
$J_o(\alpha_n x_e)$	=	0

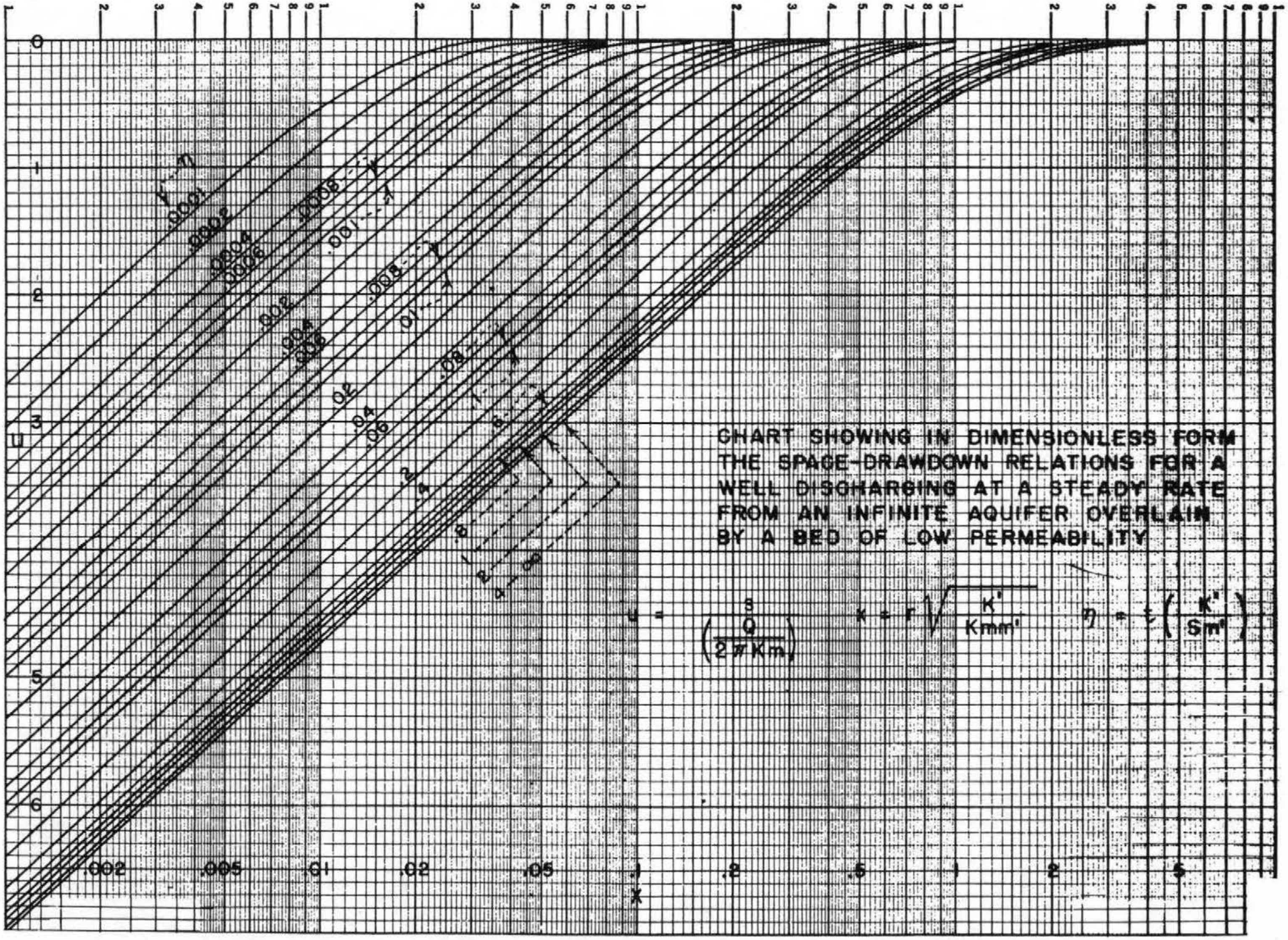


CHART SHOWING IN DIMENSIONLESS FORM THE SPACE-DRAWDOWN RELATIONS FOR A WELL DISCHARGING AT A STEADY RATE FROM AN INFINITE AQUIFER OVERLAIN BY A BED OF LOW PERMEABILITY
$u = \dfrac{s}{\left(\dfrac{Q}{2\pi Km}\right)}$
$x = r\sqrt{\dfrac{K'}{Kmm'}}$
$\eta = t\left(\dfrac{K'}{Sm'}\right)$
u
x
η
.0001
.0002
.0004
.0006
.0008
.001
.002
.004
.006
.008
.01
.02
.04
.06
.08
.1
.2
.4
.6
.8
1
2
4
.002
.005
.01
.02
.05
.1
.2
.5
1
2
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0
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4
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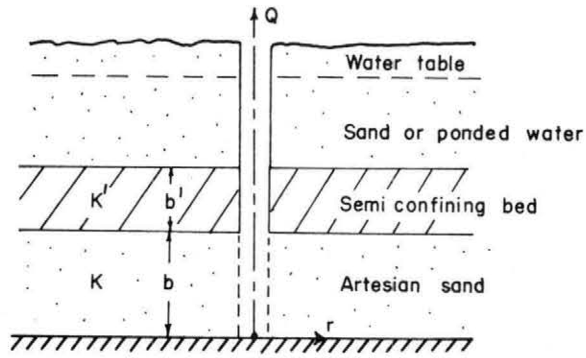
6

NON-STEADY RADIAL FLOW IN AN INFINITE LEAKY AQUIFER



REFERENCES:

Hantush, M. S., and Jacob, C. E., 1955, Non-steady radial flow in an infinite leaky aquifer: Am. Geophys. Union Trans., v. 36, no. 1, p. 95-100.

Hantush, M. S., 1956, Analysis of data from pumping tests in leaky aquifers: Am. Geophys. Union Trans., v. 37, no. 6, p. 702-14.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

$$s(r,o) = 0$$

$$s(\infty,t) = 0$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q}{2\pi T}$$

SOLUTION:

$$s = \frac{Q}{4\pi T}\int_u^\infty \frac{e^{-y-\frac{r^2}{4B^2 y}}}{y}\,dy = \frac{Q}{4\pi T}\,W(u, \frac{r}{B})$$

SYMBOLS:

ν = T/S, (L^2T^{-1})

B = $\sqrt{T/K'/b'}$, Leakage factor (L).

b = Thickness of artesian sand (L).

b' = Thickness of semiconfining bed (L).

K = Hydraulic conductivity of artesian sand, (LT^{-1})

K' = Hydraulic conductivity of semiconfining bed (LT^{-1})

Q = Constant discharge of well (L^3T^{-1})

r = Radial distance from pumped well (L).

S = Coefficient of storage.

s = Drawdown at any distance and at any time (L).

T = Kb = Transmissibility (L^2T^{-1}).

t = Time after pumping started (T).

u = $r^2S/4Tt$

$$W(u, \frac{r}{B}) = \int_u^\infty \frac{e^{-y-\frac{r^2}{4B^2 y}}}{y}\,dy$$

(See Table 2 of reference (2)).

C - 12 FLOW TO A WELL OF VARIABLE DISCHARGE

REFERENCE:

Hantush, M. S., Flow to a well of variable discharge, Unpublished notes.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

STATEMENT OF THE PROBLEM:

The problem is to find the non-steady drawdown distribution around a well completely penetrating an infinite leaky aquifer of limited thickness and pumping at a variable discharge defined by

$$Q_t = Q_c + [\, Q_i - Q_c \,]\, e^{-\left(\frac{t - t_i}{t^*}\right)}$$

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

$$s(r, o) = 0$$

$$s(\infty, t) = 0$$

$$\lim_{r \to 0} r\frac{\partial s}{\partial r} = -\frac{Q_t}{2\pi T}$$

SOLUTION:

If $\nu t^* < B^2$

$$s = \frac{Q_c}{4\pi T}\left\{ W(u, \frac{r}{B}) + (\frac{Q_i}{Q_c} - 1)\ \exp(-\frac{t-t_i}{t^*})\ I\left(u, \sqrt{\frac{r^2}{\nu t^*} - (\frac{r}{B})^2}\right)\right\}$$

and if $\nu t^* > B^2$

$$s = \frac{Q_c}{4\pi T}\left\{ W(u, \frac{r}{B}) + (\frac{Q_i}{Q_c} - 1)\ \exp(-\frac{t-t_i}{t^*})\ W\left(u, \sqrt{(\frac{r}{B})^2 - \frac{r^2}{\nu t^*}}\right)\right\}$$

SYMBOLS:

B = $\sqrt{T/K'/b'}$ = Leakage factor.

b' = Thickness of semipervious layer.

$I(u, \beta)$ = $\int_u^\infty \exp(-x + \frac{\beta^2}{4x})\, dx$ = A partly tabulated function.

K' = Hydraulic conductivity of semipervious layer.

Q_c = The ultimate constant discharge of the well.

Q_i = Discharge at time t_i generally taken within a few minutes after pumping begins.

Q_t = Discharge at any time t.

r = Radial distance from well.

S = Coefficient of storage.

s = Drawdown at any time at any distance r.

T = Transmissibility of the aquifer.

t = Time since pumping started.

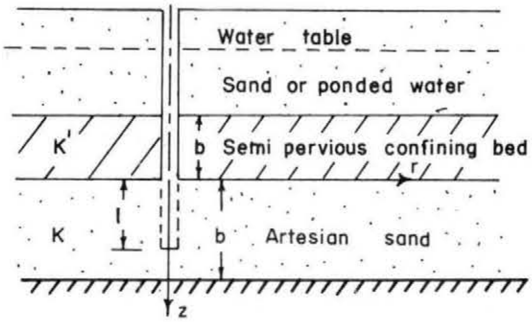
t_i = Time at which Q_i is measured.

t^* = A empirically obtained constant defining the variable discharge Q_t

$W(u, \beta)$ = The well function for leaky aquifers, available in tabular form.

ν = T/S.

C - 13 NON-STEADY FLOW TO A WELL PARTIALLY PENETRATING AN INFINITE LEAKY AQUIFER



REFERENCE:

Hantush, M. S., 1957, Non-steady flow to a well partially penetrating an infinite leaky aquifer: Iraqi Sci. Soc. Proc., v. 1, no. 1, p. 10-19.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{\partial^2 s}{\partial z^2} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

$$s(r, z, 0) = 0$$

$$s(\infty, z, t) = 0$$

$$\frac{\partial s}{\partial z}(r, 0, t) = \frac{\partial s}{\partial z}(r, b, t) = 0$$

$$\lim_{r \to 0} \left\{ 2\pi K r \int_0^b \frac{\partial s}{\partial r}\, dz \right\} = -Q$$

SOLUTION:

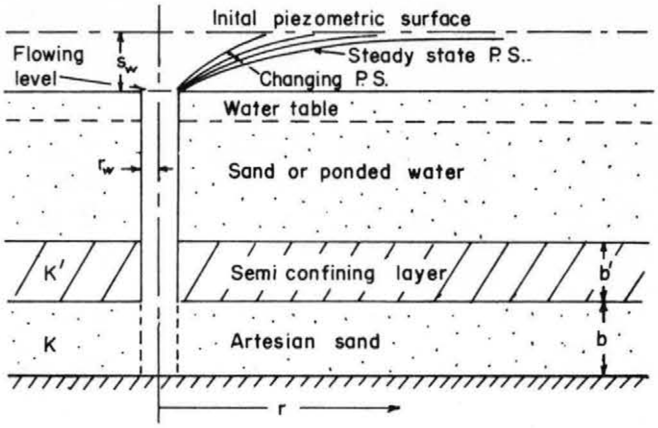
$$s = \frac{Q}{4\pi T}\left\{ W(u, \frac{r}{B}) + \frac{2b}{\pi l}\sum_{n=1}^{\infty} \frac{1}{n}\cos\frac{n\pi z}{b}\sin\frac{n\pi l}{b}\, W\left(u, \sqrt{(\frac{r}{B})^2 + (\frac{n\pi r}{b})^2}\right) \right\}$$

For $\frac{r}{b} > 1.5$ the series can be neglected, i.e. the drawdown at large distances $r > 1.5\,b$ is given by

$$s = \frac{Q}{4\pi T} W(u, \frac{r}{B}) .$$

SYMBOLS:

ν	$= T/S$ dimension (L^2T^{-1}).	l	= Length of penetration of well (L).
B	$= \sqrt{T/K'/b'}$ dimension (L) .	Q	= Discharge of well (L^3T^{-1}).
b	= Thickness of artesian sand (L).	r	= Radial distance from well (L).
b'	= Thickness of semiconfining bed (L).	S	= Coefficient of storage.
K	= Hydraulic conductivity of artesian sand (LT^{-1}).	s	= Drawdown at any time t, and any point (r, z) of the aquifer.
K'	= Hydraulic conductivity of semiconfining bed, (LT^{-1}).	T	= Transmissibility = Kb , (L^2T^{-1}).
		t	= Time after pumping started (T).
$W(u,\beta)$	= The well function for leaky aquifers, widely tabulated. See Prof. Paper 104, Hantush, NMIMT.	z	= Depth below top of artesian sand (L).



REFERENCE:

Hantush, M. S., 1959, Non-steady flow to flowing wells in leaky aquifers: Jour. Geophys. Research, v. 64, no. 8, p. 1043-52.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} - \frac{s}{B^2} = \frac{1}{\nu}\frac{\partial s}{\partial t}$$

CASE I. INFINITE LEAKY AQUIFER

BOUNDARY CONDITIONS:

$s(r, o) = 0$

$s(r_w, t) = s_w$

$s(\infty, t) = 0$

SOLUTIONS:

$$\frac{s}{s_w} = \frac{K_o(\frac{r}{B})}{K_o(\frac{r_w}{B})} + \frac{2}{\pi} e^{-\alpha r_w^2/B^2} \int_0^\infty \frac{e^{-\alpha\mu^2}}{\mu^2 + (\frac{r_w}{B})^2} \cdot \frac{J_o(\frac{\mu r}{r_w})\, Y_o(\mu) - Y_o(\frac{\mu r}{r_w})\, J_o(\mu)}{J_o^2(\mu) + Y_o^2(\mu)} \mu\, d\mu$$

The above infinite integral cannot be integrated directly; however it can be evaluated by numerical integration.

$$Q = 2\pi T s_w G(\alpha, \frac{r_w}{B})$$

For values of $G(\alpha, \frac{r_w}{B})$ see Table 1 and Figure 3 of reference.

ASYMPTOTIC SOLUTIONS:

(a) Small values of time, $t < r^2/25\nu$

$$s \approx \frac{s_w\sqrt{r_w/r}}{2} \left\{ \exp[\frac{r_w}{B}(1-\frac{r}{r_w})]\ \text{erfc}\,[-\frac{r_w}{B}\sqrt{\alpha} - \frac{(1-\frac{r}{r_w})}{2\sqrt{\alpha}}] \right.$$

$$\left. + \exp[-\frac{r_w}{B}(1-\frac{r}{r_w})]\ \text{erfc}\,[\frac{r_w}{B}\sqrt{\alpha} - \frac{(1-\frac{r}{r_w})}{2\sqrt{\alpha}}] \right\}$$

$$Q \approx 2\pi T s_w \left\{ \frac{1}{2} + \frac{1}{\sqrt{\pi \alpha}} \exp \left[-\left(\frac{r_w}{B}\right)^2 \alpha \right] \right\}$$

(b) Large values of time, $t > \dfrac{5B^2}{\nu}$

$$s \approx \frac{s_w}{2K_o\left(\frac{r_w}{B}\right)} W\left(u, \frac{r}{B}\right) \qquad Q \approx 4\pi T s_w \frac{1}{W\left(\frac{1}{4\alpha}, \frac{r_w}{B}\right)}$$

CASE II. CIRCULAR LEAKY AQUIFER WITH ZERO DRAWDOWN ON OUTER BOUNDARY

BOUNDARY CONDITIONS:

$s(r, o) = 0$

$s(r_w, t) = s_w$

$s(r_e, t) = 0$

SOLUTIONS:

$$\frac{s}{s_w} = \frac{R_o\left(\frac{r}{B}\right)}{R_o\left(\frac{r_w}{B}\right)} + \pi \sum_{n=1}^{\infty} \frac{A_n}{C_n} U_o\left(\frac{\beta_n r}{r_w}\right)$$

$$Q = 2\pi T s_w \left\{ \frac{r_w}{B} \frac{R_1\left(\frac{r_w}{B}\right)}{R_o\left(\frac{r_w}{B}\right)} + 2 \sum_{n=1}^{\infty} \frac{1}{C_n} \beta_n^2 J_o^2\left(\frac{\beta_n r_e}{r_w}\right) \exp\left[-\left(\beta_n^2 + \frac{r_w^2}{B^2}\right) \alpha \right] \right\}$$

CASE III . CLOSED CIRCULAR AQUIFER

BOUNDARY CONDITIONS:

$s(r, o) = 0$

$s(r_w, t) = s_w$

$\frac{\partial s}{\partial r}(r_e, t) = 0$

SOLUTIONS:

$$\frac{s}{s_w} = \frac{R_2\left(\frac{r}{B}\right)}{R_2\left(\frac{r_w}{B}\right)} + \pi \sum_{n=1}^{\infty} \frac{\epsilon_n^2}{E_n} J_1^2\left(\epsilon_n \frac{r_e}{r_w}\right) \exp\left[-\left(\epsilon_n^2 + \frac{r_w^2}{B^2}\right)\alpha \right] V_o\left(\epsilon_n \frac{r}{r_w}\right)$$

$$Q = 2\pi T s_w \left\{ \frac{r_w}{B} \frac{R_3\left(\frac{r_w}{B}\right)}{R_2\left(\frac{r_w}{B}\right)} + 2 \sum_{n=1}^{\infty} \frac{\epsilon_n^2}{E_n} J_1^2\left(\epsilon_n \frac{r_e}{r_w}\right) \exp\left[-\left(\epsilon_n^2 + \frac{r_w^2}{B^2}\right) \alpha \right] \right\}$$

SYMBOLS AND FUNCTIONS:

α = $\nu t / r_w^2$

β_n = Roots of $U_o(\beta_n) = 0$ (See Table 2 of reference.)

ϵ_n = Roots of $V_o'(\epsilon_n \frac{r_e}{r_w}) = 0$ (See Table 3 of reference.)

ν = T/S, dimension L^2T^{-1} .

A_n = $\beta_n^2 \; J_o(\beta_n) \; J_o(\beta_n \frac{r_e}{r_w}) \; \exp\left[-\left(\beta_n^2 + \frac{r_w^2}{B^2}\right)\alpha\right]$

B = $\sqrt{\frac{T}{K'/b'}}$, Leakage factor (L).

b = Thickness of the artesian sand (L).

b' = Thickness of semiconfining bed (L).

C_n = $\left[\beta_n^2 + \left(\frac{r_w}{B}\right)^2\right] \left[J_o^2(\beta_n) - J_o^2(\beta_n \frac{r_e}{r_w})\right]$

E_n = $\left[\epsilon_n^2 + \left(\frac{r_w}{B}\right)^2\right] \left[J_o^2(\epsilon_n) - J_1^2(\epsilon_n \frac{r_e}{r_w})\right]$

erfc(x) = $\frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-y^2} \, dy$ = Complementary error function.

= 1 - erf(x)

erf(x) = $\frac{2}{\sqrt{\pi}} \int_0^{x} e^{-y^2} \, dy$ = Error function, tables of which are available.

exp(x) = e^x = Exponential function.

$G(\alpha, r_w/B)$ = A function of α and r_w/B. See reference for definition and tabular values (Table 1).

$I_n(x)$ = The nth order Modified Bessel function of the first kind.

$J_n(x)$ = The nth order Bessel function of the first kind.

K = Hydraulic conductivity of the artesian sand (LT^{-1}).

K' = Hydraulic conductivity of semiconfining bed (LT^{-1}).

$K_n(x)$ = The nth order Modified Bessel function of the second kind.

Q = Discharge of the flowing well (L^3T^{-1}).

$R_o(\frac{x}{B})$ = $K_o(\frac{x}{B}) - \dfrac{K_o(\frac{r_e}{B}) I_o(\frac{x}{B})}{I_o(\frac{r_e}{B})} \approx K_o(\frac{x}{B})$

$R_1(\frac{r_w}{B})$ = $K_1(\frac{r_w}{B}) + \dfrac{K_o(\frac{r_e}{B}) I_1(\frac{r_w}{B})}{I_o(\frac{r_e}{B})} \approx K_1(\frac{r_w}{B})$

$$R_2\left(\frac{x}{B}\right) = K_o\left(\frac{x}{B}\right) + \frac{K_1\left(\frac{r_e}{B}\right) I_o\left(\frac{x}{B}\right)}{I_1\left(\frac{r_e}{B}\right)} \approx K_o\left(\frac{x}{B}\right)$$

$$R_3\left(\frac{r_w}{B}\right) = K_1\left(\frac{r_w}{B}\right) - \frac{K_1\left(\frac{r_e}{B}\right) I_1\left(\frac{r_w}{B}\right)}{I_1\left(\frac{r_e}{B}\right)} \approx K_1\left(\frac{r_w}{B}\right)$$

r = Radial distance from well (L).

r_e = Radius of influence of the well (L).

r_w = Effective radius of the well (L).

S = Storage coefficient.

s = Drawdown at any time t, and at any distance r from the well (L).

s_w = Drawdown at the face of the well.

T = Kb = Transmissibility (L^2T^{-1}).

t = Time since the well starts flowing (T).

$$U_o\left(\beta_n \frac{r}{r_w}\right) = J_o\left(\frac{\beta_n r}{r_w}\right) Y_o\left(\frac{\beta_n r_e}{r_w}\right) - J_o\left(\frac{\beta_n r_e}{r_w}\right) Y_o\left(\frac{\beta_n r}{r_w}\right)$$

$$U_o(\beta_n) = J_o(\beta_n) Y_o\left(\beta_n \frac{r_e}{r_w}\right) - J_o\left(\frac{\beta_n r_e}{r_w}\right) Y_o(\beta_n) = 0$$

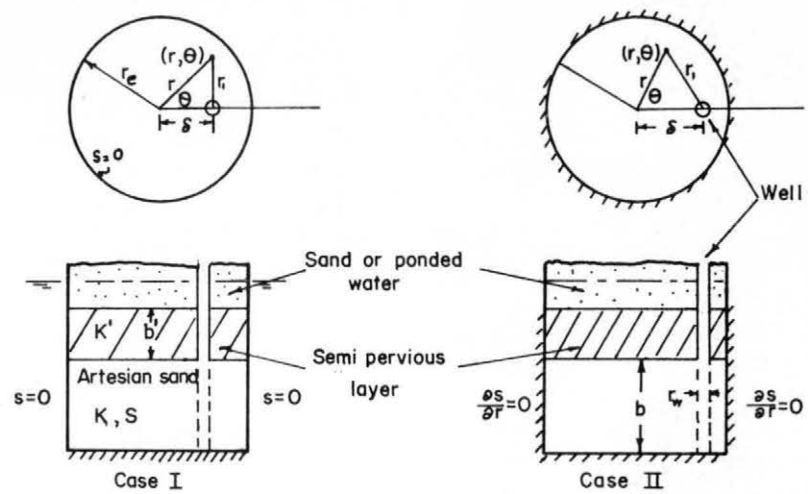
$$V_o\left(\frac{\epsilon_n r}{r_w}\right) = J_o\left(\frac{\epsilon_n r}{r_w}\right) Y_o(\epsilon_n) - Y_o\left(\frac{\epsilon_n r}{r_w}\right) J_o(\epsilon_n)$$

$$V_o'\left(\frac{\epsilon_n r}{r_w}\right) = J_o(\epsilon_n) Y_1\left(\frac{\epsilon_n r_e}{r_w}\right) - J_1\left(\frac{\epsilon_n r_e}{r_w}\right) Y_o(\epsilon_n) = 0$$

$$W(x, y) = \int_x^\infty e^{-\mu - \frac{y^2}{4\mu}} \frac{d\mu}{\mu}$$ = The well function for leaky aquifers, widely tabulated. See "Tables of the function $W(u,\beta)$" by M. S. Hantush, Professional paper 104, NMIMT.

$Y_n(x)$ = The nth order Bessel function of the second kind.

FLOW TO AN ECCENTRIC WELL IN A LEAKY CIRCULAR AQUIFER



REFERENCE:

Hantush, M. S. and Jacob, C. E., 1960, Flow to an eccentric well in a leaky circular aquifer: Jour. Geophys. Research, v. 65, no. 10, p. 3425-31.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} - \frac{s}{B^2} = \frac{S}{T}\frac{\partial s}{\partial t}$$

BOUNDARY CONDITIONS:

Case I

$$\lim_{r_1 \to 0} r_1 \frac{\partial s}{\partial r_1} = -\frac{Q}{2\pi T}$$

$$s(r_e, t) = 0$$

$$s(r, 0) = 0$$

Case II

$$\lim_{r_1 \to 0} r_1 \frac{\partial s}{\partial r_1} = -\frac{Q}{2\pi T}$$

$$s(r, o) = 0$$

$$\frac{\partial s}{\partial r}(r_e, t) = 0$$

SOLUTIONS:

Case I . Uniform head on the outer boundary

$$s = \frac{Q}{2\pi T}\left\{ K_o\left(\frac{r_1}{B}\right) - \frac{K_o\left(\frac{r_e}{B}\right) I_o\left(\frac{\delta}{B}\right) I_o\left(\frac{r}{B}\right)}{I_o\left(\frac{r_e}{B}\right)} - 2\sum_{n=1}^{\infty} \frac{K_n\left(\frac{r_e}{B}\right) I_n\left(\frac{\delta}{B}\right) I_n\left(\frac{r}{B}\right)\cos n\theta}{I_n\left(\frac{r_e}{B}\right)} \right.$$

$$+ \sum_{m=1}^{\infty} A_{om} J_o(\alpha_{om} r) \exp\left(\frac{-[(\alpha_{om} r_e)^2 + (\frac{r_e}{B})^2]}{4 u_e}\right)$$

$$\left. + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_n(\alpha_{nm} r) \cos n\theta \exp\left(\frac{-[(\alpha_{nm} r_e)^2 + (\frac{r_e}{B})^2]}{4 u_e}\right) \right\}$$

where α_{nm} are such that $J_n(\alpha_{nm} r_e) = 0$

and

$$A_{nm} = \frac{-4\, J_n(\alpha_{nm}\delta)}{[(\alpha_{nm} r_e)^2 + (\frac{r_e}{B})^2]\, J_{n+1}^2(\alpha_{nm} r_e)} \qquad A_{om} = \frac{-2\, J_o(\alpha_{om}\delta)}{[(\alpha_{om} r_e)^2 + (\frac{r_e}{B})^2]\, J_1^2(\alpha_{om} r_e)}$$

Case II. Zero flux across outer boundary.

$$s = \frac{Q}{2\pi T}\left\{ K_o(\frac{r_1}{B}) + \frac{K_1(\frac{r_e}{B})\, I_o(\frac{\delta}{B})\, I_o(\frac{r}{B})}{I_1(\frac{r_e}{B})} + 2\sum_{n=1}^{\infty} \frac{[K_{n+1}(\frac{r_e}{B}) + K_{n-1}(\frac{r_e}{B})]\, I_n(\frac{\delta}{B}) I_n(\frac{r}{B}) \cos n\theta}{I_{n+1}(\frac{r_e}{B}) + I_{n-1}(\frac{r_e}{B})}\right.$$

$$\left. + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} C_{nm}\, J_n(y_{nm}\frac{r}{r_e}) \cos n\theta \exp\left(\frac{-[y_{nm}^2 + (\frac{r_e}{B})^2]}{4\, u_e}\right)\right\}$$

where,

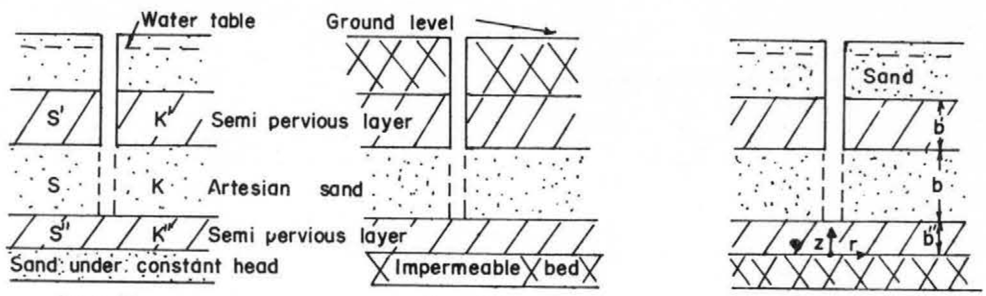
$$C_{nm} = \frac{4\, J_n(\frac{y_{nm}\delta}{r_e})}{(y_{nm}^2 - n^2)\,[1 + (\frac{r_e}{By_{nm}})^2] J_n^2(y_{nm})}$$

$$C_{om} = -\frac{2\, J_o(\frac{y_{om}\delta}{r_e})}{[y_{om}^2 + (\frac{r_e}{B})^2]\; J_o^2(y_{om})}$$

and where y_{nm} is the mth zero of J_n'.

SYMBOLS:

δ	=	Eccentricity or position of well from center of the circular aquifer (L).
B	=	$\sqrt{T/K'/b'}$ = Leakage factor (L).
b, b'	=	Thickness of the artesian sand and of the semipervious layer, respectively (L).
$I_\nu(x)$	=	The νth order Modified Bessel function of the first kind.
$J_\nu(x)$	=	The νth order Bessel function of the first kind.
K, K'	=	Hydraulic conductivity of the artesian sand and of the semipervious layer, respectively. (LT^{-1})
$K_\nu(x)$	=	The νth order Modified Bessel function of the second kind.
Q	=	Discharge of the well (L^3T^{-1}).
r	=	Radial distance from the center of the circular aquifer (L).
r_e	=	Radius of the circular aquifer (L).
r_1	=	$\sqrt{r^2 + \delta^2 - 2 r\delta \cos\theta}$ = Distance to any point from the center of the well (L).
S	=	Coefficient of storage of the artesian sand.
s	=	Drawdown at any time at any point (r, θ) of the aquifer (L).
T	=	Kb = Transmissibility (L^2T^{-1}).
t	=	Time since pumping started (T).
u_e	=	$r_e^2S/4Tt$.
θ	=	Polar angle measured from the pole at the center of the circular aquifer.



Case I Case II Case III

REFERENCE:

Hantush, M. S., 1960, Modification of the theory of leaky aquifers: Jour. Geophys. Research, v. 65, no. 11, p. 3713-25.

SUMMARIZED BY:

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DIFFERENTIAL EQUATION AND BOUNDARY CONDITIONS:

Case I

Upper Semipervious Layer (1)

$$\frac{\partial^2 s_1}{\partial z^2} = \frac{1}{\nu'}\frac{\partial s_1}{\partial t} \qquad (a)$$

$$s_1(r, z, 0) = 0 \qquad (b)$$

$$s_1(r, z', t) = 0 \qquad (c)$$

$$s_1(r, z_1, t) = s(r, t) \qquad (d)$$

Lower Semipervious Layer (2)

$$\frac{\partial^2 s_2}{\partial z^2} = \frac{1}{\nu''}\frac{\partial s_2}{\partial t} \qquad (a)$$

$$s_2(r, z, 0) = 0 \qquad (b)$$

$$s_2(r, 0, t) = 0 \qquad (c)$$

$$s_2(r, b'', t) = s(r, t) \qquad (d)$$

Main Aquifer (3)

$$\frac{\partial^2 s}{\partial r^2} + \frac{1}{r}\frac{\partial s}{\partial r} + \frac{K'}{T}\frac{\partial}{\partial z} s_1(r, z_1, t) - \frac{K''}{T}\frac{\partial}{\partial z} s_2(r, b'', t) = \frac{1}{\nu}\frac{\partial s}{\partial t} \qquad (a)$$

$$s(r, 0) = 0 \qquad (b)$$

$$s(\infty, t) = 0 \qquad (c)$$

$$\lim_{r \to 0} r\frac{\partial s(r, t)}{\partial r} = -\frac{Q}{2\pi T} \qquad (d)$$

Case II

Same as Case I, the conditions (1c) and (2c) being replaced respectively by

$$\frac{\partial}{\partial z} s_1(r, z', t) = 0$$

$$\frac{\partial}{\partial z} s_2(r, 0, t) = 0$$

Case III

Same as Case I, the condition (2c) being replaced by $\frac{\partial}{\partial z} s_2(r, 0, t) = 0$

SOLUTIONS:

Asymptotic solutions for relatively small and large values of time are obtained. These asymptotic solutions can be used to interpolate for the intermediate range of time.

(a) For $t < \frac{b'S'}{10K'}$ and $t < \frac{b''S''}{10K''}$

For all three cases

$$s = \frac{Q}{4\pi T} H(u, \beta)$$

$$q_L = Q \left[1 - e^{nt} \operatorname{erfc}\left(-\sqrt{nt}\right) \right]$$

$$V_L = V \left[1 - \frac{2}{-\sqrt{n\pi t}} + \frac{q_L}{Qnt} \right]$$

(b) For large values of time

Case I

For $t > \frac{5b'S'}{K'}$ and $t > \frac{5b''S''}{K''}$

$$s = \frac{Q}{4\pi T} W(u\delta_1, \alpha) \quad \text{and} \quad q_L = Q \left[1 - \frac{1}{\delta_1} \exp\left(-\nu t \alpha^2 / r^2 \delta_1\right)\right]$$

Case II

For $t > \frac{10b'S'}{K'}$ and $t > \frac{10b''S''}{K''}$

$$s = \frac{Q}{4\pi T} W(u\delta_2) \quad \text{and} \quad q_L = Q \frac{S' + S''}{S + S' + S''}$$

Case III

For $t > \frac{5b'S'}{K'}$ and $t > \frac{10b''S''}{K''}$

$$s = \frac{Q}{4\pi T} W\left(u\delta_3, r\sqrt{\frac{K'/b'}{T}}\right) \quad \text{and} \quad q_L = Q \left[1 - \frac{1}{\delta_3} \exp\left(-\nu K't/b'T\delta_3\right)\right]$$

The values of $\frac{s}{Q/4\pi T}$ in the intermediate range can be estimated by plotting $\frac{s}{Q/4\pi T}$ versus log u or log t for the two ranges of time given above and by joining the two branches by inspection.

SYMBOLS:

α	$= r\sqrt{K'/b'T + K''/b''T}$	
β	$= (1/4)r\lambda$	
δ_1	$= 1 + (S' + S'')/3S$	
δ_2	$= 1 + (S' + S'')/S$	
δ_3	$= 1 + (S'' + S'/3)/S$	
λ	$= \sqrt{\frac{K'/b'}{T} \cdot \frac{S'}{S}} + \sqrt{\frac{K''/b''}{T} \cdot \frac{S''}{S}}$	(L^{-1})
ν	$= T/S$	(L^2T^{-1})
ν'	$= K'b'/S'$	(L^2T^{-1})
ν''	$= K''b''/S''$	(L^2T^{-1})
b, b', b''	= Thicknesses of main aquifer and of upper and lower semipervious layers, respectively.	(L)

erf(x) $= \frac{2}{\sqrt{\pi}} \int_0^x e^{-\gamma^2} d\gamma$ = The error function, available in tabular form.

erfc(x) $= 1 - \mathrm{erf}(x)$ = The complement of the error function.

H(x, y) $= \int_x^\infty \frac{e^{-\gamma}}{\gamma} \mathrm{erfc}\left(y\sqrt{x} \,/\, \sqrt{\gamma(\gamma-x)} \right) d\gamma$ = A function, tables of which can be found in reference.

K, K', K'' = Hydraulic conductivities of the main aquifer and of upper and lower semipervious layer, respectively (LT^{-1}).

n $= T\lambda^2/S$ (T^{-1}).

Q = Constant well discharge (L^3T^{-1}).

q_L = Rate of leakage added to main aquifer (L^3T^{-1}).

r = Radial distance from center of well (L).

s, s_1, s_2 = Drawdowns at any time and any point in the main aquifer and the upper and lower semipervious layers respectively (L).

S, S', S'' = Storage coefficients of the main aquifer and of the upper and lower semipervious layers, respectively.

t = Time since pumping started (T).

T = Kb = Transmissibility of main aquifer (L^2T^{-1}).

u $= r^2/4\nu t$

V = Qt = Total volume withdrawn during pumping (L^3).

V_L = Total volume withdrawn from leakage (L^3).

W(x) $= \int_x^\infty \frac{e^{-\gamma}}{\gamma} d\gamma$ = The well function for nonleaky aquifers, available in tabular form.

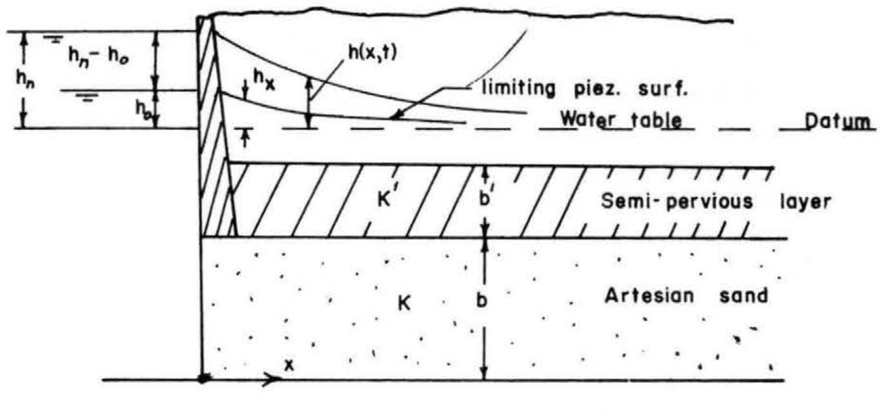
W(x, y) $= \int_x^\infty \exp\left(-\gamma - \frac{y^2}{4\gamma}\right) \frac{d\gamma}{\gamma}$ = The well function for leaky aquifers available in tabular form (see Prof. Paper 104 by M. S. Hantush, NMIMT).

z = Vertical coordinate (L).

z_1 $= b'' + b$ (L).

z' $= b'' + b + b'$ (L).

C - 17 INTERCEPTING DRAINAGE WELLS IN ARTESIAN AQUIFER



REFERENCE:

Hantush, M. S., 1961, Discussion of - Intercepting drainage wells in artesian aquifer by D. F. Peterson: Am. Soc. Civil Engineers Proc., v. 87, no. IR 4, p. 79-81.

SUMMARIZED BY:

I. S. Papadopulos, New Mexico Institute of Mining and Technology

DIFFERENTIAL EQUATION:

$$\frac{\partial^2 z}{\partial x^2} - \frac{z}{B^2} = \frac{1}{\nu}\frac{\partial z}{\partial t}$$

where $z = h(x,t) - h_x$ and $h_x = h_o e^{-x/B}$ is the initial steady head distribution.

BOUNDARY CONDITIONS:

$$z(x,0) = 0$$
$$z(\infty,t) = 0$$
$$z(0,t) = h_n - h_o$$

SOLUTION:

$$z = h(x,t) - h_x = \frac{h_n - h_o}{2}\left\{ e^{-x/B}\ \mathrm{erfc}\left(\frac{x/2B}{\tau} - \tau\right) + e^{x/B}\ \mathrm{erfc}\left(\frac{x/2B}{\tau} + \tau\right)\right\}$$

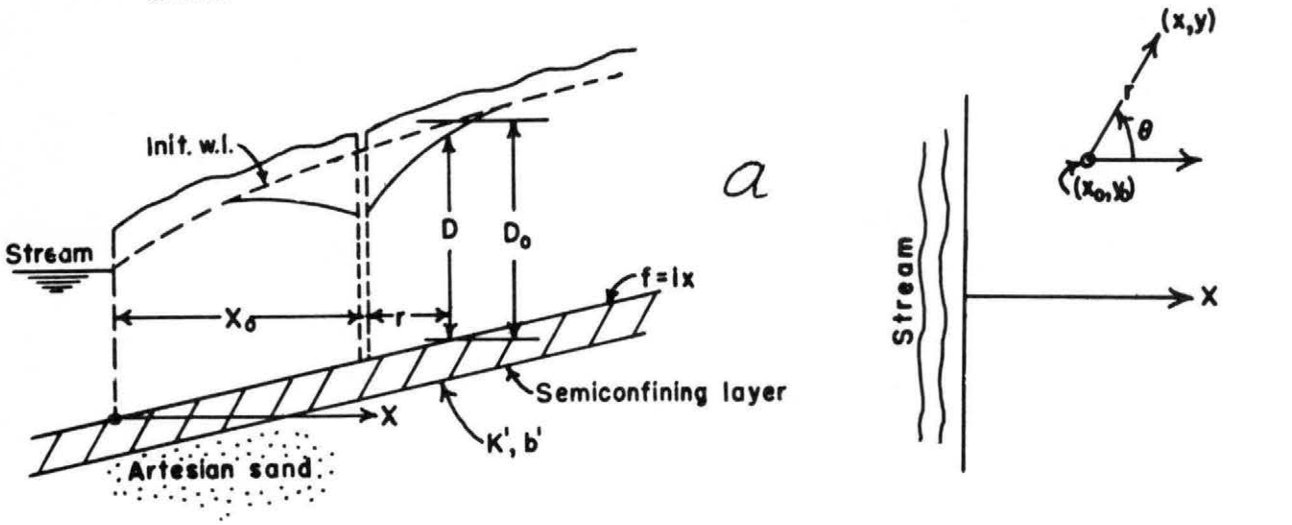
$$\Delta q = q(0,t) - q_o = \frac{T(h_n - h_o)}{B\sqrt{\pi}}\left\{\frac{e^{-\tau^2}}{\tau} + \sqrt{\pi}\ \mathrm{erf}(\tau)\right\} \quad \left(q_o = \frac{Th_o}{B}\right)$$

$$\Delta V = \int_0^t \Delta q\, dt = SB(h_n - h_o)\left[(0.5 + \tau^2)\,\mathrm{erf}(\tau) + \frac{\tau}{\sqrt{\pi}} e^{-\tau^2}\right]$$

SYMBOLS:

B	$= \sqrt{T/K'/b'}$ = Leakage factor (L).
b, b'	= Thickness of artesian sand and semipervious layer, respectively (L).
$\mathrm{erf}(x)$	= The error function.
$\mathrm{erfc}(x)$	$= 1 - \mathrm{erf}(x)$ = The complement of the error function.
$h(x,t)$	= Head at any point at any time t (L).
h_x	= Initial steady state head distribution (L).
h_o	= Initial head in channel (L).
h_n	= New head in channel (L).
K, K'	= Hydraulic conductivity of artesian sand and semipervious layer, respectively (LT^{-1}).
Δq	$= q(0,t) - q_o$.
q_o	= Initial rate of seepage or channel loss (L^3T^{-1}/L).
$q(0,t)$	= Rate of channel loss at time t (L^3T^{-1}/L).
T	$= Kb$ = Transmissibility of artesian sand (L^2T^{-1}).
t	= Time since sudden change from h_o to h_n in channel (T).
ΔV	= Total volume of seepage during a period t (L^3/L).
ν	$= T/S$ (L^2T^{-1}).
τ	$= \sqrt{\nu t/B^2}$

C - 18 DEPLETION OF STORAGE, LEAKAGE, AND RIVER FLOW BY GRAVITY WELLS IN SLOPING SANDS



REFERENCE:

Hantush, M. S., Depletion of storage, leakage and river flow by gravity wells in sloping sands, unpublished notes.

SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

STATEMENT OF THE PROBLEM:

To determine the depletion of storage, leakage, and river flow by gravity wells in sloping sands.

Case I

A well upstream from a river cutting across the natural flow.

SOLUTIONS:

1. Drawdown equation.

$$D_o^2 - D^2 = \frac{Q}{2\pi K} \exp\left(-\frac{x - x_o}{\beta}\right)\left\{W(u, \alpha r) - W(u', \alpha r')\right\} \tag{1}$$

2. Rate and total volume of river depletion.

$$q_r = 0.5\, Q \exp\left(\frac{x_o}{\beta}\right)\left\{\exp(\alpha x_o)\,\mathrm{erfc}(\delta') + \exp(-\alpha x_o)\,\mathrm{erfc}(\delta)\right\} \tag{2}$$

$$V_r = \frac{Q t_o}{2\alpha\sqrt{\nu t_o}} \exp\left(\frac{x_o}{\beta}\right)\left\{\delta_o' \exp(\alpha x_o)\,\mathrm{erfc}(\delta_o') - \delta_o \exp(-\alpha x_o)\,\mathrm{erfc}(\delta_o)\right\} \tag{3}$$

(a) Expressions for short times:

For $t_o < 0.09\ \nu/\alpha^2$: $$V_r \approx 4\, Q t_o\, i^2\,\mathrm{erfc}\left(\frac{x_o}{\sqrt{4\nu t_o}}\right) \tag{4}$$

(b) Expressions for long times:

For $\alpha\sqrt{\nu t} > 1 + \sqrt{1 + 0.5\,\alpha x_o}$: $$q_r \approx Q \exp[-(\alpha - 1/\beta)\, x_o] \tag{5}$$

$$V_r \approx Q t_o \left(1 - \frac{x_o}{2\alpha \nu t_o}\right) \exp[-(\alpha - 1/\beta)\, x_o] \tag{6}$$

3. Rate and total volume of storage depletion.

$$q_s = Q \exp\left(-\frac{\nu t}{B^2}\right)\left\{1 - 0.5\left[\exp\left(\frac{2x_o}{\beta}\right)\operatorname{erfc}(\gamma') + \operatorname{erfc}(\gamma)\right]\right\} \tag{7}$$

$$V_s = \frac{QB^2}{\nu}\left[1 - (q_s + q_r)/Q\right] \tag{8}$$

(a) Expressions for short times:

For $\sqrt{\nu t_o}\,/\beta < 0.3$: $V_s \approx Q t_o\left\{1 - \frac{\beta}{2\sqrt{\nu t_o}}\left[\gamma'_o \exp\left(\frac{2x_o}{\beta}\right)\operatorname{erfc}(\gamma'_o) - \gamma_o \operatorname{erfc}(\gamma_o)\right]\right\}$ (9)

For $\alpha\sqrt{\nu t_o} < 0.3$: $V_s \approx Q t_o\left[1 - 4\, i^2 \operatorname{erfc}\left(\frac{x_o}{\sqrt{4\nu t_o}}\right)\right]$ (10)

(b) Expressions for long times:

For $\alpha\sqrt{\nu t} > 1 + \sqrt{1 + 0.5\,\alpha x_o}$:

$$q_s = Q \exp\left(-\frac{\nu t}{B^2}\right)\left\{1 - \exp\left[x_o\left(\frac{1}{\beta} - \alpha + \frac{1}{2\alpha\beta^2}\right)\right]\right\} \tag{11}$$

$$V_s = \frac{QB^2}{\nu}\left\{\left[1 - \exp -\left(\frac{\nu t_o}{B^2}\right)\right] - \left[1 - \exp\left(-\frac{\nu t_o}{B^2} + \frac{x_o}{2\alpha B^2}\right)\right]\exp\left[-x_o(\alpha - 1/\beta)\right]\right\} \tag{12}$$

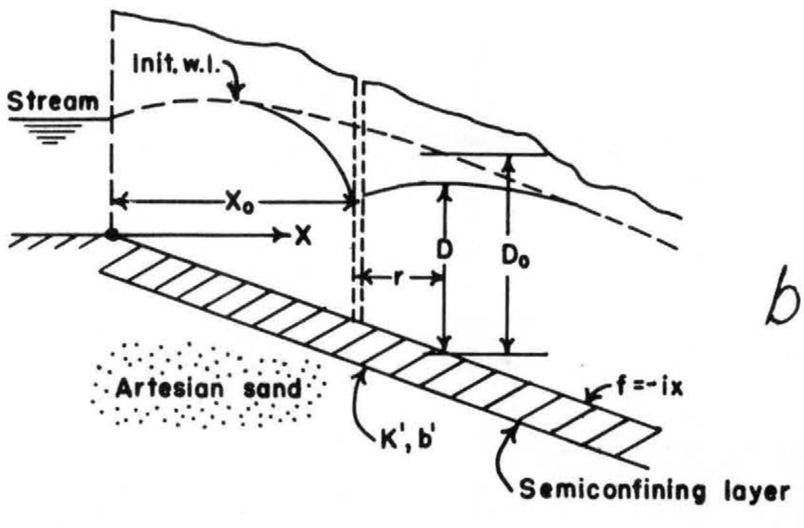
4. Rate and total volume of leakage depletion.

$$q_l = Q - (q_s + q_r) \tag{13}$$

$$V_l = Q t_o - (V_s + V_r) \tag{14}$$

Case II

A well downstream from a river cutting across the natural flow.

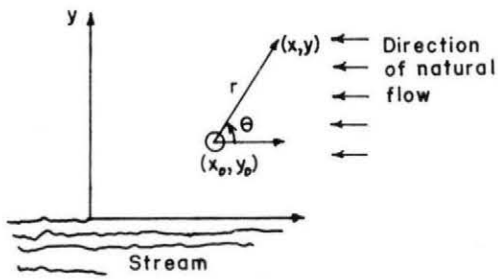


b

Figure b represents a sloping leaky water-table aquifer, with the aquifer sloping downward in the direction of positive x . If the sine of the angle of dip of the system of Figure a is reversed; that is, if the parameter β is replaced with $(-\beta)$, the present flow system results. Consequently, if β in the equations of the previous article is replaced with $(-\beta)$, the resulting equations will pertain to the present flow system.

Case III

A well near a stream cutting along the natural flow.



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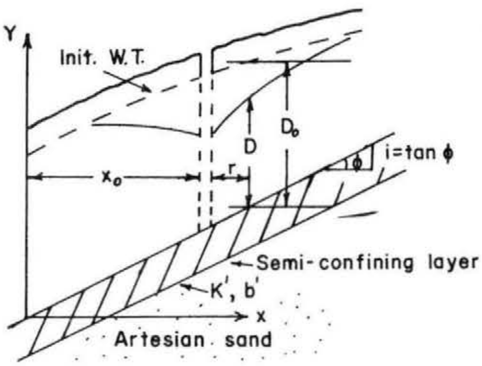


This flow system is shown schematically in Figure 2. The drawdown distribution around a well steadily discharging from such a system is given by

$$D_o^2 - D^2 = \frac{Q}{2\pi K} \exp\left(-\frac{x-x_o}{\beta}\right)\left\{W(u,\alpha r) - W(u'',\alpha r'')\right\}$$

Analysis similar to that followed in the first article shows that depletions of storage, of leakage, and of river flow are not affected by the slope of the sand as long as the stream is effectively infinite in length. In other words, in so far as the rate and total volume of depletion of storage, leakage, and river flow are concerned, the present flow system behaves as if the aquifer were horizontal. Consequently, the required expressions for the rate and total volume of depletions pertaining to the present flow situation are obtained from their counter-parts in article one by letting $i = 0$, that is $1/\beta = 0$, and replacing x_o with y_o, observing that the counterparts of (9) and (10) are the same and can be more readily obtained by using (10).

Special Cases

For a sloping non-leaky aquifer, the required expressions are obtained from their counterparts of the previous articles by letting $1/B \longrightarrow 0$; those for a horizontal leaky aquifer are obtained by letting $1/\beta \longrightarrow 0$; and those for a horizontal non-leaky aquifer are obtained by letting $\alpha \longrightarrow 0$ in which case, the expressions for V_r, V_s, and V_1 will be the same as those previously given for the range $\alpha\sqrt{\nu t_o} < 0.3$.

SYMBOLS:

B = $\sqrt{Kb/(K'/b')}$ = Leakage factor.

b = $0.5(D_o + D)$, a weight average of the depth of saturation, which, for $r > 1.5\,D_o$, may be taken equal to D_o and for $r < 1.5\,D_o$ equal to $0.5(D_o + D_w)$.

b' = Thickness of the semipervious layer.

$D(x,y,t)$ = Depth of saturation in the water-table aquifer ($\approx$ the height of the water table above the base of the sloping sand) at any time and at any point (x,y).

$D_o(x,y,t)$ = Depth of saturation that would prevail in the water-table aquifer if the well were not pumped.

$\mathrm{erf}(x)$ = $\frac{2}{\sqrt{\pi}}\int_0^x \exp(-\beta^2)\,d\beta$ = The error function of x, tabular values of which are available (Dwight, 1961; Carslaw and Jaeger, 1959).

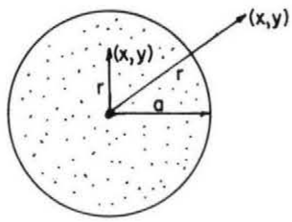
$\mathrm{erfc}(x)$ = $1 - \mathrm{erf}(x)$ = The complement of the error function, tabular values of which are available (Carslaw and Jaeger, 1959).

i = The tangent of the angle of dip of the sloping sand; it is positive for a bed sloping upward in the direction of positive x, and negative if the bed is sloping downward in the direction of positive x.

$i^n\,\mathrm{erfc}(x)$ = $\int_x^\infty i^{n-1}\,\mathrm{erfc}(\beta)\,d\beta$, with $n = 1, 2, \ldots$, and $i^o\,\mathrm{erfc}(x) = \mathrm{erfc}(x)$ = the nth repeated integral of the error function, which is available in tabular form (Kaye, 1955, Carslaw and Jaeger, 1959).

K, K'	= The hydraulic conductivities of the water-table aquifer and the semipervious layer respectively.
K'/b'	= The leakage coefficient (or leakance).
Q	= The constant discharge of the well.
q_l	= The rate of leakage depletion, or the rate of that part of the well discharge that is derived from induced leakage.
q_r	= The rate of river depletion, or the rate of that part of the well discharge that is derived from induced infiltration from the river and/or from the natural flow that would have discharged into the river if the well were not pumped.
q_s	= The rate of storage depletion, or the rate of that part of the well discharge that is derived from storage in the water-table aquifer.
r	= $\sqrt{(x-x_o)^2+(y-y_o)^2}$ = Radial distance to any point (x, y) measured from the center of a well located at (x_o, y_o).
r'	= $\sqrt{(x+x_o)^2+(y-y_o)^2}$, $r'' = \sqrt{(x-x_o)^2+(y+y_o)^2}$
t	= Time since pumping began.
t_o	= Period of continuous pumping.
u	= $r^2/4\nu t$
u'	= r'^2/ud_o $u'' = r''^2/4\nu t$
V_l, V_r, V_s	= Respectively, the total volumes by which leakage source, river flow, and aquifer storage are depleted during a period t_o of continuous pumping.
$W(u, w)$	= $\int_0^\infty \frac{1}{y} \exp\left(-y - \frac{w^2}{4y}\right) dy$ = The well function for leaky aquifers, tabular values of which are available (Hantush, 1956, 1963; Walton, 1962; Schoeller, 1962).
(x, y)	= Rectangular coordinates.
x_o and y_o	= The effective distance from the center of the well to the river site as indicated in Fig. 1 and Fig. 2 respectively.
x_o, y_o	= Location of the well center.
α	= $\sqrt{1/\beta^2 + 1/B^2}$
β	= $2b/i$
γ	= $(x_o/\sqrt{4\nu t}) - \sqrt{\nu t/\beta}$
γ'	= $(x_o/\sqrt{4\nu t}) + \sqrt{\nu t/\beta}$
γ_o	= The value of γ after replacing t with t_o.
γ_o'	= The value of γ' after replacing t with t_o.
δ	= $(x_o/\sqrt{4\nu t}) - \alpha\sqrt{\nu t}$
δ	= $(x_o/\sqrt{4\nu t}) + \alpha\sqrt{\nu t}$
δ_o	= The value of δ after replacing t with t_o.
δ_o'	= The value of δ' after replacing t with t_o.
ϵ	= Specific yield of sloping sand.
ν	= Kb/ϵ

C - 19 EFFECT OF WELL FIELD OPERATION OVER AN AREA



REFERENCE:

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SUMMARIZED BY:

M. A. Mariño, New Mexico Institute of Mining and Technology

An estimate of the effect of a large number of wells on the water levels within and outside the well field may be obtained by idealizing the problem by assuming that the pumping of the well field is uniformly distributed over a circular area of radius a within which most of the wells are located. Such a situation may be effectively realized if pumping for irrigation and drainage is accomplished by use of a large number of wells distributed throughout the area.

For leaky aquifers without storage in semipervious layers, the drawdown expressions during steady-state flow, are as follows:

For $r < a$:

$$4\pi Ts = 4V(B^2/a^2)\left[1 - (a/B)\, I_0(r/B)\, K_1(a/B)\right]$$

For $r > a$:

$$4\pi Ts = 4V(B/a)\, I_1(a/B)\, K_0(r/B)$$

For nonleaky aquifers, the corresponding expressions are:

For $r < a$ and $t > 0.4\, r^2/\nu$:

$$4\pi Ts = V\left\{W(u_a) + (1/u_a)\left[1 - \exp(-u_a)\right] - (r/a)^2 \exp(-u_a)\right\}$$

For $r > a$ and $t > 0.4\, a^2/\nu$:

$$4\pi Ts = V\left\{W(u) + (0.5\, u_a)\exp(-u)\right\}$$

in which $u_a = a^2/4\nu t$ and $u = r^2/4\nu t$.

If the well field taps a horizontal water-table aquifer, the corresponding expressions are obtained by replacing $(4\pi Ts)$ with $2\pi K(D_o^2 - \bar{D}^2)$.

SYMBOLS:

B	= Leakage factor = $\sqrt{Kb/(K'/b')}$ for artesian aquifers.
	= $\sqrt{K(D_o + \bar{D})/2(K'/b')}$ for water-table aquifers.
b	= Average thickness of leaky aquifers.
b'	= Thickness of a semipervious layer of leaky systems.
$\bar{D}$	= Height of water table above base of aquifer.
D_o	= Initial depth of flow in a water-table aquifer.
$I_o(x), I_1(x)$	= Zero and first order modified Bessel functions of the first kind.
K, K'	= Hydraulic conductivities in a main aquifer and in a semipervious layer respectively.
K'/b'	= Coefficient of leakage.
$K_o(x), K_1(x)$	= Zero and first order modified Bessel functions of the second kind.
S	= Storage coefficient of an artesian aquifer.
T	= Kb = Transmissivity of artesian aquifers.
T	= KD_o = Transmissivity corresponding to initial depth of flow in a water-table aquifer.
t	= Time
V	= Total discharge of the well field.
ϵ	= Specific yield of the aquifer.
ν	= $\frac{Kb}{\epsilon}$ for water-table aquifers.
ν	= T/S for artesian aquifers.

APPENDIX D

TRANSIENT FLOW THROUGH POROUS MEDIA - UNIQUENESS OF SOLUTIONS

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x, y	= Cartesian coordinates.
r	$= \sqrt{x^2 + y^2}$
t	= Time.
$\phi = y + \frac{p}{\gamma}$,	where $K\phi$ is the velocity potential.
$\vec{q} = u\underline{i} + v\underline{j}$	$= -K\nabla\phi$, velocity.
$\phi_x \equiv \partial\phi/\partial x$	
$\phi_y \equiv \partial\phi/\partial y$	
$\eta\,(x, t)$	= Free surface - two dimensional case.
$\zeta\,(t)$	= Free surface - one dimensional case.
ϵ	= Porosity.
K	= Hydraulic conductivity.

TRANSIENT FLOW THROUGH POROUS MEDIA - UNIQUENESS OF SOLUTIONS

In the study of engineering problems involving differential equations the question of uniqueness of solutions takes on special significance when solutions can be obtained only by approximate techniques. For when a problem is known to have a unique solution the possibility of the approximate method converging to an incorrect solution does not exist. On the other hand, when analytical solutions are available but not a uniqueness theorem, one can frequently employ arguments based on the physics of the problem to discard extraneous solutions.

For many of the problems of flow through porous media a uniqueness theorem is available, even for the transient cases. This is a consequence of the fact that the governing differential equation is the Laplace equation

$$\nabla^2 \phi = 0$$

So that Green's formula becomes

$$\int_A (\nabla \phi)^2 \, dA = \oint_c \phi \frac{\partial \phi}{\partial n} \, ds . \qquad (1)$$

(See [1] * for an application to a steady state problem.)

To show how this theorem is useful in a particular transient problem, consider the unsteady flow under a a barrier, e.g., row of sheetpiling (Fig. 1) where the unsteadiness is caused by a time (and possibly space) varying head both upstream and downstream of the barrier. The governing differential equation is $\nabla^2 \phi = 0$, and the boundary conditions are as follows: on the face of the impervious stratum and on either face of the barrier there is no normal flow, i.e., $\frac{\partial \phi}{\partial n} = 0$; at the surface of the sand $\phi = h_u(x, t)$ to the left of the barrier, and to the right of the barrier $\phi = h_d(x, t)$. It is clear, from physical considerations, that for large $|X|$ the direction of flow in the sand tends to be vertical.

To prove that there is not more than one solution satisfying the above boundary value problem, we assume the converse, namely that there are two solutions $\phi_1(x, y, t)$ and $\phi_2(x, y, t)$ satisfying all the conditions.

Define a new function equal to the difference,

$$\tilde{\phi} = \phi_1 - \phi_2 \qquad (2)$$

This new function also satisfies $\frac{\partial \tilde{\phi}}{\partial n} = 0$ on the impervious boundary and on the barrier, and since each of the functions ϕ_1 and ϕ_2 satisfies the given boundary conditions on the sand surface equation (2) gives $\tilde{\phi} = 0$ on the sand surface. Now consider vertical lines at large distances from the barrier. From what was said previously it is clear that both $\frac{\partial \phi_1}{\partial x}$ and $\frac{\partial \phi_2}{\partial x}$ must tend to zero on these lines as $x \longrightarrow \pm \infty$. Hence on these lines the difference must also go to zero.

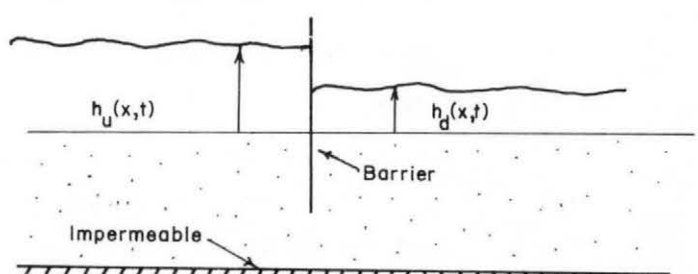

Fig. I

* Numbers in brackets [ ] refer to references at the end of this paper.

Equation (1) is now applied. Figure 2 shows the boundary conditions on $\widetilde{\phi}$ (or $\frac{\partial\widetilde{\phi}}{\partial n}$), and indicates the path of integration for the right hand side of Equation (1). Consider this line integral

$$\oint_C \widetilde{\phi}\,\frac{\partial\widetilde{\phi}}{\partial n}\,ds \tag{3}$$

As seen in Figure 2 on each segment of the boundary either $\widetilde{\phi} = 0$ or $\frac{\partial\widetilde{\phi}}{\partial n} = 0$. Hence (3) is identically equal to zero, leaving, from (1)

$$\int_A (\nabla\widetilde{\phi})^2\,dA = 0 \tag{4}$$

But $(\nabla\widetilde{\phi})^2 = \widetilde{\phi}_x^{\,2} + \widetilde{\phi}_y^{\,2}$, that is, a sum of squares, so that the only way (4) can be satisfied is if each term is zero; $\widetilde{\phi}_x = 0$, $\widetilde{\phi}_y = 0$, everywhere. Hence $\widetilde{\phi}$ can be at most a constant. But, since $\widetilde{\phi} = 0$ on the sand surface, the constant must be zero.

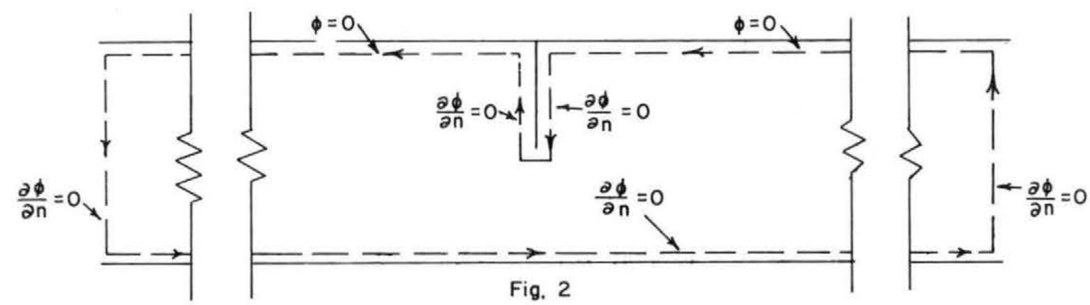

Fig. 2

It has thus been shown that for this transient problem only one solution can exist. This technique is quite general, however, and is applicable to many problems of flow through porous media, transient or steady state, provided the boundaries are known a priori, i. e., confined flow.

For problems involving a free surface there is, however, no general uniqueness theorem. This stems in part from the fact that since the location of the free surface is a priori unknown, an additional condition must be given for that part of the boundary which is the free surface, and these lead to a non-linear condition which makes for considerable mathematical difficulty. More important, however, is the fact that for the boundary value problem as customarily posed, there is frequently more than one solution so that additional physical conditions must be specified to make the mathematical problem unique.

Consider a problem with a free surface where the potential is time dependent, $\phi = \phi(x, y, t)$, and the free surface, $\eta = \eta(x, t)$ is also time dependent, Figure 3. The two boundary conditions on the free surface are (see [2, 3]

$$\phi(x,\ \eta(x),t) = Ky = K\,\eta(x,t) \tag{5}$$

since the pressure is zero, and a kinematical condition

$$\frac{\epsilon}{K}\,\eta_t = \phi_x\eta_x - \phi_y \tag{6}$$

These can be combined to give

$$\frac{\epsilon}{K}\,\phi_t = \phi_x^{\,2} + \phi_y^{\,2} - \phi_y$$

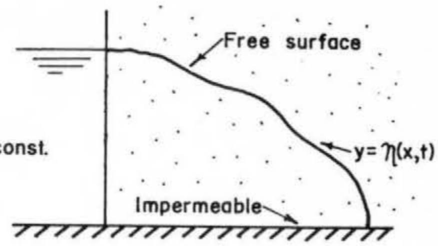

Fig. 3

To demonstrate non-uniqueness we consider a one dimensional problem, Figure 4, where Q cfs per square foot enter a basin of large lateral dimensions causing water to percolate through the soil and the height of water in the basin to rise. Let $\zeta(t)$ represent the distance from the top of the basin to the water surface in the soil. Utilizing (5) and (6), where $\zeta(t)$ replaces $\eta(x,t)$ and $\phi = \phi(y,t)$, and following Polubarinova - Kochina [4] one obtains the following ordinary differential equation for the free surface

$$\frac{\zeta}{\zeta(1-\epsilon) + Qt}\,\frac{d\zeta}{dt} = \frac{K}{\epsilon} \tag{7}$$

with the initial condition $\zeta(0) = 0$. The simplest way to attack this problem is to assume a solution of the form $\zeta = ct$ where c is to be determined. After substituting in the differential equation one finds that the above assumption is valid provided

$$c = \frac{K}{2\epsilon}(1-\epsilon) \pm \frac{1}{2}\sqrt{\frac{K^2}{\epsilon^2}(1-\epsilon)^2 + 4Q} \tag{8}$$

For either the plus or minus sign in (8) above, both the initial value and the differential equation are satisfied. Hence we see there are two possible solutions to the problem. For this relatively simple case the proper choice is obvious from physical conditions, since the minus sign would yield a solution where the free surfaces rises with increasing time. However, for more complicated problems, e.g., problems involving two or three space coordinates as well as time, where analytical solutions are not available and where a solution is sought by say a finite difference scheme it may, in fact, be quite difficult to ascertain which of several choices is the correct one for the physical problem at hand, or as is more likely, the investigator may be unaware that more than one solution satisfies the given conditions, and that his method may be producing results for a problem not physically correct.

An illustration of a case where the proper choice is not immediately obvious concerns the two-dimensional steady seepage out of a canal. This problem is treated in both Muskat [5] and Polubarinova-Kochina [6]. In these references, it is shown that for essentially the same canal shape there are two distinct solutions for the free surface, and thus the potential fields. A sketch indicating the free streamlines for each of these solutions is shown in Figure 5.

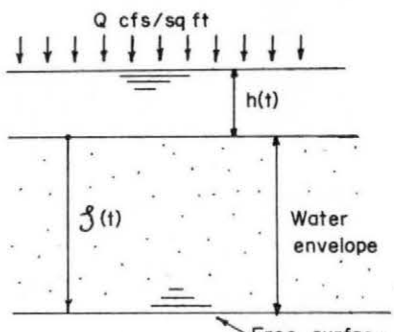

Fig. 4

To further study these solutions it is first noted that for the flow field between the free surfaces, Solution I satisfies the condition

$$\lim_{r \rightarrow \infty} \overrightarrow{q} = \text{const} = V_\infty \underline{j} \neq 0 \tag{9}$$

and that Solution II satisfies the condition

$$\lim_{r \rightarrow \infty} \overrightarrow{q} = 0 \tag{10}$$

Consider next the transient motion starting from an assumed initial condition. It seems plausible that since gravity is the cause of any fluid motion, initially the flow typified by Solution I will prevail, that is, the flow will be essentially vertical as given by (9). However, since there is no real semi-infinite permeable medium, eventually an impermeable stratum will be reached by the water envelope and the streamlines will be deflected laterally, ultimately (i.e., as $t \rightarrow \infty$) reaching Solution II.

Hence we see that Solution I is not in consonance with the real physical earth, and if we admit of a barrier to vertical flow, even one located at infinity, Solution II alone can be considered as the steady state solution to this problem. Nevertheless, Solution I, though a transitory stage in the development of the steady state flow field, may be taken as the "steady state" solution if the time under consideration is small as compared to the length of time it takes for the water envelope to reach the impermeable stratum and to be deflected appreciably.

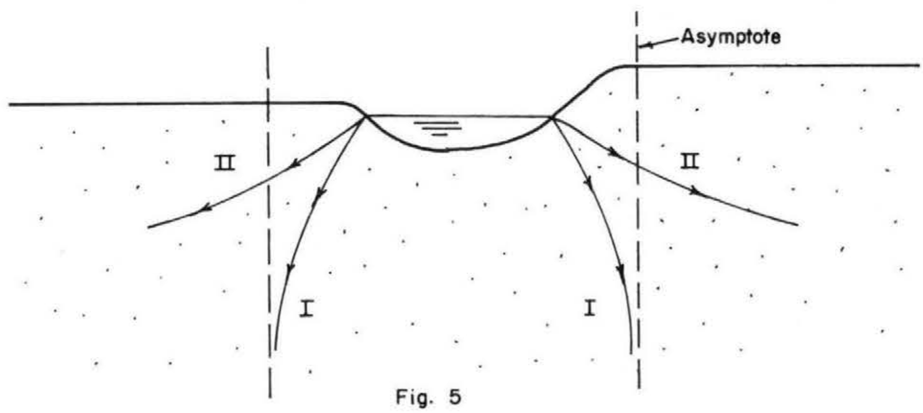

Fig. 5

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APPENDIX E

LIMITATIONS OF DUPUIT-FORCHHEIMER ASSUMPTION IN RECHARGE AND DRAINAGE

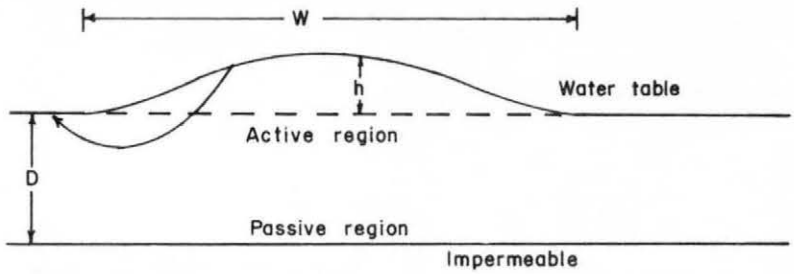

Fig. I

SUMMARIZED BY:

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DIFFERENTIAL EQUATIONS:

Using the Dupuit-Forchheimer assumption, the following equations have been presented in the literature to describe the flow in connection with ground-water mounds

a. circular mounds $$\frac{\partial h}{\partial t} = \frac{KD}{V} \left(\frac{\partial^2 h}{\partial r^2} + \frac{1}{r}\frac{\partial h}{\partial r} \right)$$

b. two-dimensional mounds $$\frac{\partial h}{\partial t} = \frac{KD}{V}\frac{\partial^2 h}{\partial x^2}$$

where t = time

V = fillable or drainable porosity.

The equation for the two-dimensional mounds applies also to the case of parallel drains in land drainage.

DISCUSSION:

Use of the Dupuit-Forchheimer assumption yields solutions that imply linearity between total flow rates, Q, and D . Because of the frequent application of the Dupuit-Forchheimer assumption (D-F assumption) in the literature pertaining to ground-water mounds and recharge, an examination of the validity of the D-F assumption is in order.

As with drainage of land by means of parallel drains, the direction of the streamlines under ground-water mounds is initially downward, changing to more horizontal, and finally more upward, if D is sufficiently large. The zone where the streamlines terminate is usually not much lower in elevation than where the streamlines originate. Thus, flow systems in connection with drainage by parallel drains and with ground-water mounds tend to exhibit an active zone in the upper region of the water-bearing material and a passive zone in the lower region. Therefore, starting with a small value of D , increasing D will increase Q until the active region is fully developed, after which Q will remain unaffected by further increases in D. This relationship is schematically shown in Figure 2, where Q initially increases linearly with D (D-F assumption valid) but becomes essentially constant at larger D-values (D-F assumption no longer valid). The point D_c where Q remains essentially unaffected by further increases in D can be estimated from graphs presented in references 1, 2, and 3, as follows

a. for circular mounds $D_c = 0.75\,W$

b. for two-dimensional mounds $D_c = W$

c. for parallel drains $D_c = 0.2S$ 1/

where W = diameter or width of mound or percolation zone

S = distance between parallel drains.

1/ Based on figure 3 in reference 3, which is constructed from data by S. B. Hooghoudt.

The difference between the straight line and the curve in figure 2 is the error due to use of the Dupuit-Forchheimer assumption, an error which can be quite large if D is not small compared to W or S.

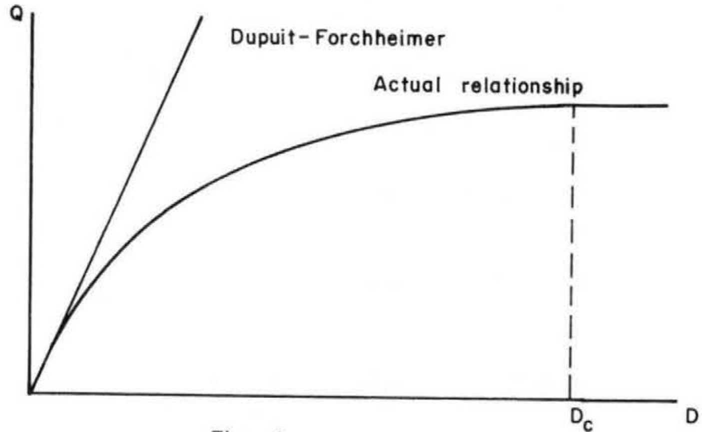

Fig. 2

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APPENDIX F

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By A. I. Johnson

Chief, Hydrologic Laboratory, U. S. Geological Survey, Denver, Colorado

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APPENDIX G

SELECTED REFERENCES ON MATHEMATICAL DEVELOPMENTS IN TRANSIENT GROUNDWATER HYDRAULICS

(Arranged in Chronological Order)

By Morton W. Bittinger and D. E. L. Maasland

Colorado State University

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