THESIS

ACOUSTIC

SCATTERING BY

FLUID SPHERES

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ii

ACKNOWLEDGEMENTS

TABLE OF CONTENTS

CHAPTER I
INTRODUCTION

Throughout physics a great variety of scattering problems is encountered. In classical physics, laws have been formulated for scattering by both microscopic and macroscopic objects. For example, Rutherford formulated a scattering law to describe the scattering of charged particles by nuclei. In the field of quantum mechanics, the scattering of electrons from atoms is still of considerable interest. In the last half century the scattering of electromagnetic energy has been extensively studied, and in the last twenty years has become increasingly important due to advances in radar technology. Beginning with the investigations of Lord Rayleigh in the 19th century, the field of acoustical scattering has also become increasingly important. The amount of research being reported in recent issues of acoustical journals indicates the extent of interest in scattering theory.

Upon investigation one is struck by the similarity in the methods of approach to scattering problems in quantum mechanics, nuclear physics, sound and electromagnetic theory. Thus, a contribution in one field generally extends the knowledge in all of them. This thesis is concerned with a problem in acoustic

scattering theory and will be confined to that field. Specifically, it is an attempt to find an approach which may be generalized to the problem of the scattering from spheres.

The scattering of sound from spheres was first investigated mathematically by Lord Rayleigh (1). Because of the complexity of the solution, he considered only the limiting case where the wavelength of sound was large compared to the radius of the sphere. Morse (2) calculated the solution for rigid immovable spheres, not necessarily small compared with the wavelength. Faran (3) calculated solutions for elastic spheres, considering them neither rigid nor immovable, and, hence, solutions for the inside of the sphere were also obtained. A few years prior to the publishing of Faran's result, Anderson (4) published results for scattering by a fluid sphere, which represents a slight simplification of the problem since the fluid sphere does not support a shear wave. The solutions he obtained both for the inside and outside of the sphere are complicated in that products and quotients of spherical Bessel and Neumann functions with different arguments occur. Because of this, numerical evaluations were made in computing the scattered acoustic field since little could be done analytically.

Since the exact solutions of Faran and Anderson

are difficult to deal with both analytically and numerically, it is useful to attempt to find methods for obtaining approximate results of a more convenient form. In a search for such methods, a problem even simpler than Anderson's may be considered. The purpose of this thesis is to consider the scattering of acoustic pulses and waves from a sphere which has acoustic properties nearly the same as those of the surrounding medium. This allows the expansion of the acoustic parameters inside the sphere by a Taylor series in terms of the acoustic parameters outside the sphere. This approach does permit more to be done analytically to low orders in the expansion than the above mentioned cases.

In Chapter II, the general problem of spherical scattering is considered and the results of Faran and Anderson are obtained for the steady state condition. In Chapter III, an approximation using the similarity of acoustical properties inside and outside the sphere is made, and the solution is obtained first for arbitrary angles and then for back and forward scattering. The results are compared with those obtained by the Born approximation. Chapter IV contains the back-scattered solutions for two types of acoustic pulses. In Chapter V, a discussion of the results is given as well as a description of how one would extend the theory.

CHAPTER II

GENERAL SOLUTION FOR SPHERICAL SCATTERING

It is well known from elastic theory (5) that the displacement $\vec{U}$ can be represented in terms of a scalar potential Φ and a vector potential $\vec{\psi}$ by the equation

$$\vec{U} = \nabla \Phi + \nabla \times \vec{\psi} \qquad (2.1)$$

where Φ and $\vec{\psi}$ satisfy the following relations

$$(2.2)$$

$$\frac{1}{C_L^2} \frac{\partial^2 \Phi}{\partial t^2} = \nabla^2 \Phi$$

$$\frac{1}{C_T^2} \frac{\partial^2 \vec{\psi}}{\partial t^2} = -\nabla \times \nabla \times \vec{\psi}$$

The velocities C_L and C_T are the velocities of propagation of the longitudinal and transverse waves, respectively, and are defined by the relations

$$C_L = \sqrt{\frac{\lambda + 2\mu}{\rho}} \qquad (2.3)$$

$$C_T = \sqrt{\frac{\mu}{\rho}}$$

Consider the solution for a plane wave of angular frequency ω incident on an elastic sphere. The sphere has a radius a, Lamé constants λ_1 and μ_1, and

has a density ρ_2. Consider the sphere to be immersed in an infinite ideal fluid with a Lamé constant λ_1 ($\mu_1 = 0$ for an ideal fluid) and a density ρ_1. The Lamé constant μ is the shear modulus and the constant λ may be written in terms of the bulk modulus β and the shear modulus μ as

$$\lambda = \beta - \frac{2\mu}{3}$$

Let the center of the sphere coincide with the origin of a rectangular coordinate system and let the plane wave approach the sphere along the negative z axis. The spherical coordinates used are defined in the usual manner. Due to the cylindrical symmetry about the z axis, there will be no φ dependence in the acoustic field. Also, since no displacements occur in the φ direction, the vector potential $\vec{\Psi}$ has only a component Ψ_φ in spherical coordinates.

The solutions of Eqns. (2.2) and (2.3) are well known in terms of spherical Bessel functions and Legendre polynomials. The incoming plane wave is expanded in spherical wave functions by

(2.4)

$$\Phi_i = e^{i(\omega t - kr\cos\theta)} = e^{i\omega t} \sum_{m=0}^{\infty} A_m(r)\, P_m(\cos\theta)$$

Upon multiplying by $P_m(\cos\theta)$ and then integrating with respect to $\cos\theta$ from -1 to $+1$,

$$\int_{-1}^{+1} P_m(\cos\theta)\, e^{-ikr\cos\theta}\, d(\cos\theta) = A_m(r) \int_{-1}^{+1}\left\{P_m(\cos\theta)\right\}^2 d(\cos\theta) \qquad (2.5)$$

where

$$\int_{-1}^{+1}\left\{P_m(\cos\theta)\right\}^2 d(\cos\theta) = \frac{2}{2m+1} \qquad (2.6)$$

Therefore,

$$A_m(r) = \frac{2m+1}{2}\int_{-1}^{+1} P_m(\cos\theta)\, e^{-ikr\cos\theta}\, d(\cos\theta) \qquad (2.7)$$

Carrying out the integration (6),

$$A_m(r) = \frac{2m+1}{2}(-i)^m \sqrt{\frac{\pi}{2kr}}\; J_{m+1/2}(kr) \qquad (2.8)$$

$$= \frac{2m+1}{2}(-i)^m j_m(kr)$$

The scalar potential for the incoming plane wave is then

$$\Phi_i = e^{iwt}\sum_{m=0}^{\infty}(-i)^m(2m+1)\, j_m(k_1 r)\, P_m(\cos\theta) \qquad (2.9)$$

where

$$k_1 = \frac{\omega}{c_{L_1}}$$

Hereafter, for convenience, the time dependence factor $e^{i\omega t}$ will be understood, but not written, in all expressions representing waves.

The scattered outgoing wave is of the form

(2.10)

$$\Phi_s = \sum_{m=0}^{\infty} B_m\, h_m^{(2)}(k_1 r)\, P_m(\cos\theta)$$

where the B_m are constants and $h_m^{(2)}(k_1 r)$ is the spherical Hankel function of second order. The Hankel function of second order appears here to assure that the scattered wave at great distances acts as an outgoing spherical wave, since

$$h_m^{(2)}(k_1 r) \xrightarrow{r \to \infty} \frac{e^{-i k_1 r}}{r}$$

The total scalar potential field outside the sphere is

(2.11)

$$\Phi^{1} = \sum_{m=0}^{\infty} \left\{ (-i)^m (2m+1)\, j_m(k_1 r) + B_m\, h_m^{(2)}(k_1 r) \right\} P_m(\cos\theta)$$

Since $\mu_1 = 0$ from the definition of an ideal fluid there is no vector potential outside the sphere.

Inside the sphere, the vector and scalar potentials are

(2.12)

$$\Phi^{II} = \sum_{m=0}^{\infty} C_m\, j_m(k_2 r)\, P_m(\cos\theta)$$

$$\psi_\varphi^{II} = \sum_{m=0}^{\infty} D_m \, j_m(k,r) \frac{d}{d\theta} P_m(\cos\theta)$$

where

$$k_2 = \frac{\omega}{c_{L_2}}$$

$$k_3 = \frac{\omega}{c_{T_2}}$$

Spherical Hankel functions do not appear here since they become singular at the origin.

Using the boundary conditions at the surface of the sphere, the coefficients B_m, C_m and D_m may be determined. These conditions are

1. The normal components of displacement must be continuous.
2. The normal components of stress must be continuous.
3. The tangential component of stress must vanish.

In spherical coordinates, in terms of Φ and ψ_φ, these three conditions become (3)

$$U_r^{I} = U_r^{II} \quad \text{at} \quad r = a \tag{2.13}$$

where

$$U_r = \frac{\partial \Phi}{\partial r} + \frac{1}{r}\left(\frac{\partial}{\partial\theta} + \frac{\cos\theta}{\sin\theta}\right)\psi_\varphi$$

$$\sigma_{rr}^{I} = \sigma_{rr}^{II} \quad \text{at} \quad r = a$$

where

$$\sigma_{rr} = \left(2\mu \frac{\partial^2}{\partial r^2} - \frac{\rho \omega^2 \lambda}{\lambda + 2\mu}\right)\Phi + \frac{2\mu}{r}\left\{\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\frac{\partial}{\partial \theta} + \frac{\cos\theta}{\sin\theta}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\right\}\psi_\phi$$

$$\sigma_{r\theta}^{II} = 0 \qquad at \qquad r = a$$

where

$$\sigma_{r\theta} = \mu\left\{\frac{2}{r}\left(\frac{\partial}{\partial r} - \frac{1}{r}\right)\frac{\partial \Phi}{\partial \theta} + \left(-\frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial \theta}\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\right)\psi_\phi\right\}$$

respectively.

Upon computing the expressions for the stresses and displacements and substituting them into the boundary requirements, the following equations are obtained which must be satisfied for all m .

(2.14)

$$k_1 B_m h_m'(k_1 a) - k_2 C_m j_m(k_2 a) + \frac{m}{a}(m+1) D_m j_m(k_3 a) = -k_1(-i)^m (2m+1) j_m'(k_1 a)$$

$$\lambda k_1^2 B_m h_m(k_1 a) - k_2^2 C_m \left\{\lambda_2 j_m(k_2 a) - 2\mu_2 j_m''(k_2 a)\right\} - \frac{2\mu_2 m(m+1)}{a^2} D_m \left\{k_3 a j_m'(k_3 a) - j_m(k_3 a)\right\}$$

$$= \lambda_1 k_1^2 (-i)^m (2m+1) j_m(k_1 a)$$

$$2 C_m \left\{\frac{k_2}{a} j_m'(k_2 a) - \frac{1}{a^2} j_m(k_2 a)\right\} + D_m \left\{\frac{2}{a^2} j_m(k_3 a) - k_3^2 j_m''(k_3 a) - \frac{1}{a^2} m(m+1) j_m(k_3 a)\right\} = 0$$

The prime on $h_m'(k_1 a)$ and $j_m'(k_2 a)$ refers to a derivative with respect to the argument and evaluated at $(k_1 a)$,

$(k_1 a)$ or $(k_3 a)$. The expression for B_m is of primary importance now and upon solving the above equations and may be written as

$$B_m = -(-i)^m (2m+1) \cdot \qquad (2.15)$$

$$\begin{vmatrix} k_1\, j'_m(k_1 a) & k_2\, j'_m(k_2 a) & \frac{m}{a}(m+1)\, j_m(k_3 a) \\[6pt] \lambda_1 k_1^2\, j_m(k_1 a) & -k_2^2\{\lambda_2\, j_m(k_2 a) - 2\mu_2\, j''_m(k_2 a)\} & -\frac{2}{a^2}\mu_2 m(m+1)\{k_3 a\, j'_m(k_3 a) - j_m(k_3 a)\} \\[6pt] 0 & 2\{\frac{k_2}{a}\, j'_m(k_2 a) - \frac{1}{a^2}\, j_m(k_2 a)\} & \frac{2}{a^2}\, j_m(k_3 a) - k_3^2\, j''_m(k_3 a) - \frac{1}{a^2}m(m+1)\, j_m(k_3 a) \end{vmatrix}$$

$$\begin{vmatrix} k_1\, h_m(k_1 a) & k_2\, j'_m(k_2 a) & \frac{m}{a}(m+1)\, j_m(k_3 a) \\[6pt] \lambda_1 k_1^2\, h_m(k_1 a) & -k_2^2\{\lambda_2\, j_m(k_2 a) - 2\mu_2\, j''_m(k_2 a)\} & -\frac{2}{a^2}\mu_2 m(m+1)\{k_3 a\, j'_m(k_3 a) - j_m(k_3 a)\} \\[6pt] 0 & 2\{\frac{k_2}{a}\, j'_m(k_2 a) - \frac{1}{a^2}\, j_m(k_2 a)\} & \frac{2}{a^2}\, j_m(k_3 a) - k_3^2\, j''_m(k_3 a) - \frac{1}{a^2}m(m+1)\, j_m(k_3 a) \end{vmatrix}$$

which is easily shown to be identical with Faran's result (3). The results of Anderson can be obtained from Eqn. (2.15) by taking the limit as $\mu_2 \rightarrow 0$, or more easily by substituting $\mu = 0$ into the boundary conditions. Upon doing the latter, the boundary conditions become

$$(2.16)$$

$$U_r^{\mathrm{I}} = U_r^{\mathrm{II}} \qquad at \qquad r = a$$

$$\sigma_{rr}^{\mathrm{I}} = \sigma_{rr}^{\mathrm{II}} \qquad at \qquad r = a$$

where

$$U_r = \frac{\partial \Phi}{\partial r} \tag{2.17}$$

$$\sigma_{rr} = -\rho \omega^2 \Phi$$

The third boundary condition in Eqn. (2.13) no longer applies since $\mu_2 = 0$. The two equations to be satisfied now for all m are

$$k_1 B_m h_m'(k_1 a) - k_2 C_m j_m'(k_2 a) = -k_1 (-i)^m (2m+1) j_m'(k_1 a) \tag{2.18}$$

$$\lambda_1 k_1^2 B_m h_m(k_1 a) - \lambda_2 k_2^2 C_m j_m(k_2 a) = -\lambda_1 k_1^2 (-i)^m (2m+1) j_m(k_1 a)$$

Solving for B_m,

$$B_m = -(-i)^m (2m+1) \frac{\begin{vmatrix} k_1 j_m'(k_1 a) & k_2 j_m'(k_2 a) \\ \lambda_1 k_1^2 j_m(k_1 a) & \lambda_2 k_2^2 j_m(k_2 a) \end{vmatrix}}{\begin{vmatrix} k_1 h_m'(k_1 a) & k_2 j_m'(k_2 a) \\ \lambda_1 k_1^2 h_m(k_1 a) & \lambda_2 k_2^2 j_m(k_2 a) \end{vmatrix}} \tag{2.19}$$

This expression may easily be shown to be identical with that obtained by Anderson (4).

CHAPTER III

AN APPROXIMATION METHOD FOR SCATTERING BY A SPHERE WITH ACOUSTIC PROPERTIES SIMILAR TO THE SURROUNDING FLUID

As calculated in Chapter II, the scattered scalar potential field for a fluid sphere immersed in a fluid medium is

$$\Phi_s = \sum_{m=0}^{\infty} B_m h_m^{(2)}(k_1 r) P_m(\cos\theta) \tag{3.1}$$

where

$$B_m = -(-i)^m (2m+1) \frac{\begin{vmatrix} k_1 j_m'(k_1 a) & k_2 j_m'(k_2 a) \\ \lambda_1 k_1^2 j_m(k_1 a) & \lambda_2 k_2^2 j_m(k_2 a) \end{vmatrix}}{\begin{vmatrix} k_1 h_m'(k_1 a) & k_2 j_m'(k_2 a) \\ \lambda_1 k_1^2 h_m(k_1 a) & \lambda_2 k_2^2 j_m(k_2 a) \end{vmatrix}} \tag{3.2}$$

For arbitrary k_1 and k_2, this expression is difficult to handle analytically, but numerical calculations may be made for specific values of k_1, k_2 and a. If the sphere is acoustically similar to the surrounding medium, $k_1 \approx k_2$ and the functions with the argument $k_2 a$ may be expanded in a Taylor series in terms of $k_1 a$

by letting

$$C_2 = C_1 (1 + \Delta) \qquad \text{3.3)}$$

where

$$\Delta \ll 1$$

and

$$\rho_2 = \rho_1 (1 + \delta) \qquad \text{(3.4)}$$

where $\delta \ll 1$.

The function $j_m (k_2 a)$ may then be expanded to first order in Δ as

$$j_m (k_2 a) = j_m (k_1 a) - k_1 a \Delta \, j_m' (k_1 a) . \qquad \text{(3.5)}$$

Similarly, $j_m' (k_2 a)$ may be expanded. For convenience, also introduce

$$\lambda_2 = \lambda_1 (1 + \epsilon). \qquad \text{(3.6)}$$

It is easily shown that

$$\Delta = \tfrac{1}{2} (\epsilon - \delta) \qquad \text{(3.7)}$$

Using the above expansions, Equation (2.2) may be written as

$$\text{(3.8)}$$

$$B_m = -(-i)^m (2m+1) \cdot$$

$$\left\{ \frac{(1+\tfrac{\epsilon}{2}) j_m'(k_1 a) \{ j_m(k_1 a) - k_1 a \Delta \, j_m'(k_1 a) \} - (1-\tfrac{\epsilon}{2}) j_m(k_1 a) \{ j_m'(k_1 a) - k_1 a \Delta \, j_m''(k_1 a) \}}{(1+\tfrac{\delta}{2}) h_m'(k_1 a) \{ j_m(k_1 a) - k_1 a \Delta \, j_m'(k_1 a) \} - (1-\tfrac{\epsilon}{2}) h_m(k_1 a) \{ j_m'(k_1 a) - k_1 a \Delta \, j_m''(k_1 a) \}} \right\}$$

Noting that the numerator contains no terms of zero order in ϵ and δ , the denominator needs only to be taken to zero order in these two quantities.

Rewriting Eqn. (3.8) and neglecting all terms with second or higher order in ϵ and δ gives

$$B_m = -(-i)^m (2m+1) \cdot \tag{3.9}$$

$$\left\{ \frac{\left\{ \frac{1}{2}(\epsilon+\delta) j_m(k_1 a) j_m'(k_1 a) - k_1 a \Delta j_m'(k_1 a) j_m'(k_1 a) + k_1 a \delta j_m(k_1 a) j_m''(k_1 a) \right\}}{h_m'(k_1 a) j_m(k_1 a) - h_m(k_1 a) j_m'(k_1 a)} \right\}.$$

Using the identity

$$h_m'(k_1 a) j_m(k_1 a) - h_m(k_1 a) j_m'(k_1 a) = \frac{-i}{(k_1 a)^2} \tag{3.10}$$

Eqn. (3.9) becomes

$$\tag{3.11}$$
$$B_m = -i(-i)^m (2m+1)(k_1 a)^2 \cdot$$
$$\left\{ \frac{1}{2}(\epsilon+\delta) j_m(k_1 a) j_m'(k_1 a) - k_1 a \Delta j_m'(k_1 a) j_m'(k_1 a) + k_1 a \delta j_m(k_1 a) j_m''(k_1 a) \right\}$$

and Φ_s may be written as

$$\tag{3.12}$$
$$\Phi_s = -i(k_1 a)^2 \frac{1}{2}(\epsilon+\delta) \sum_{m=0}^{\infty} (-i)^m (2m+1) j_m(k_1 a) j_m'(k_1 a) h_m^{(2)}(k_1 r) P_m(\cos\theta)$$
$$+ i(k_1 a)^2 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m (2m+1) j_m'(k_1 a) j_m'(k_1 a) h_m^{(2)}(k_1 r) P_m(\cos\theta)$$
$$- i(k_1 a)^2 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m (2m+1) j_m(k_1 a) j_m''(k_1 a) h_m^{(2)}(k_1 r) P_m(\cos\theta)$$

Using the recursion relation

$$j_m'(ka) = \frac{m}{2m+1} j_{m-1}(ka) - \frac{m+1}{2m+1} j_{m+1}(ka)$$

Eqn. (3.12) becomes

(3.13)

$$\Phi_s = -i(ka)^2 \frac{1}{2}(\epsilon+\delta) \sum_{m=0}^{\infty} (-i)^m \, m \, j_m(ka) \, j_{m-1}(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$+ i(ka)^2 \frac{1}{2}(\epsilon+\delta) \sum_{m=0}^{\infty} (-i)^m (m+1) \, j_m(ka) \, j_{m+1}(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$+ i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{m^2}{2m+1} \, j_{m-1}^2(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$- i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{2m(m+1)}{2m+1} \, j_{m+1}(ka) \, j_{m-1}(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$+ i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{(m+1)^2}{2m+1} \, j_{m+1}^2(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$- i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{m(m-1)}{2m-1} \, j_m(ka) \, j_{m-2}(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$+ i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{m^2}{2m-1} \, j_m^2(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$+ i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{(m+1)^2}{2m+3} \, j_m^2(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

$$- i(ka)^3 \frac{1}{2}(\epsilon-\delta) \sum_{m=0}^{\infty} (-i)^m \frac{(m+1)(m+2)}{2m+3} \, j_m(ka) \, j_{m+2}(ka) \, h_m^{(2)}(kr) \, P_m(\cos\theta)$$

This represents the scattered scalar potential field for arbitrary angles.

Perhaps the most useful and interesting result is for $\theta = \pi$, i.e., back-scattering which will now be considered.

For convenience the far field will be examined which allows the Hankel function $h_m^{(2)}(kr)$ to be replaced by its asymptotic form for large r

$$h_m^{(2)} \xrightarrow{r \to \infty} \frac{e^{-ikr}}{kr} \, i^{m+1}$$

For back-scattering,

$$P_m(\cos\theta) = (-1)^m$$

Therefore, Equation (3.14) becomes

(3.15)

$$\Phi_s = \frac{e^{-ikr}}{kr}(ka)^2 \frac{1}{2}(\epsilon+\delta)\sum_{m=0}^{\infty} m \, j_m(ka) \, j_{m-1}(ka)(-1)^m$$

$$- \frac{e^{-ikr}}{kr}(ka)^2 \frac{1}{2}(\epsilon+\delta)\sum_{m=0}^{\infty}(m+1)\, j_m(ka)\, j_{m+1}(ka)(-1)^m$$

$$- \frac{e^{-ikr}}{kr}(ka)^3 \frac{1}{2}(\epsilon-\delta)\sum_{m=0}^{\infty}\frac{m^2}{2m+1}\, j_{m-1}^2(ka)(-1)^m$$

$$+ \frac{e^{-ikr}}{kr}(ka)^3 \frac{1}{2}(\epsilon-\delta)\sum_{m=0}^{\infty}\frac{2m(m+1)}{2m+1}\, j_{m-1}(ka)\, j_{m+1}(ka)(-1)^m$$

$$- \frac{e^{-ikr}}{kr}(ka)^3 \frac{1}{2}(\epsilon-\delta)\sum_{m=0}^{\infty}\frac{(m+1)^2}{2m+1}\, j_{m+1}^2(ka)(-1)^m$$

$$+ \frac{e^{-ikr}}{kr} (ka)^3 \frac{1}{2}(\epsilon - \delta) \sum_{m=0}^{\infty} \frac{m(m-1)}{2m-1} j_m(ka) j_{m-2}(ka) (-1)^m$$

$$- \frac{e^{-ikr}}{kr} (ka)^3 \frac{1}{2}(\epsilon - \delta) \sum_{m=0}^{\infty} \frac{m^2}{2m-1} j_m^2(ka) (-1)^m$$

$$- \frac{e^{-ikr}}{kr} (ka)^3 \frac{1}{2}(\epsilon - \delta) \sum_{m=0}^{\infty} \frac{(m+1)^2}{2m+3} j_m^2(ka) (-1)^m$$

$$+ \frac{e^{-ikr}}{kr} (ka)^3 \frac{1}{2}(\epsilon - \delta) \sum_{m=0}^{\infty} \frac{(m+1)(m+2)}{2m+3} j_m(ka) j_{m+2}(ka) (-1)^m$$

It is seen immediately that when the fifth and seventh, third and eighth and, the fourth, sixth and ninth terms respectively are added together their sums are zero, leaving

$$\Phi_s = \frac{e^{-ikr}}{kr} (ka)^2 (\epsilon + \delta) \sum_{m=0}^{\infty} m \, j_m(ka) j_{m-1}(ka) \quad (-1)^m \tag{3.16}$$

The evaluation of this infinite sum is found in Appendix (A) and gives

$$\Phi_s = - \frac{e^{-ikr}}{kr} (ka)^2 \frac{1}{2}(\epsilon + \delta) j_1(2ka) \tag{3.17}$$

Using the asymptotic expression for $2ka \ll 1$ Eqn. (3.17) reduces to

$$\Phi_s = - \frac{k^2 a^3}{3r} (\epsilon + \delta) \tag{3.18}$$

which agrees with the Rayleigh limit for small spheres (1).

For forward scattering, it may be shown that

(3.19)

$$\Phi_s = - \frac{e^{-ikr}}{kr} (ka)^3 (\epsilon - \delta) \sum_{m=0}^{\infty} (2m+1) \dot{j}_m^{\,2} (ka)$$

This expression also agrees with the Rayleigh limit for forward scattering.

Since only two terms of the Taylor expansion of $j_m (k_1 a)$ are used, some amount of error is involved in the calculations, but since the spherical Bessel functions are well behaved, it is expected that the errors are small if the term $k_1 a \Delta$ is small. To determine the magnitude of these errors it is convenient to calculate a few numerical values. The expansion requires the term $k_1 a \Delta$ to be very much less than one, so a value of 0.1 was used for convenience and typical values of ka were chosen. Errors of the order of one percent of the quantity $k_1 a \Delta$ were found to exist but this quantity has already been assumed to be very small. Therefore, to a suitable degree of approximation, the errors may be assumed negligible by choosing $k_1 a \Delta$ appropriately small.

When the scattering can be considered weak the Born approximation (7) is another approach which may be used to compute the scattering of an acoustical wave.

The results of this approach, as calculated in Appendix (C) gives the scattered field

$$\Phi_s = \frac{e^{-ikr}}{8kr} (\delta - \epsilon) \left\{ \sin 2ka - 2ka \cos 2ka \right\} \tag{3.20}$$

The result of the approximation method used in this chapter gives

$$\Phi_s = - \frac{e^{-ikr}}{2rk} (ka)^2 (\epsilon + \delta) j_1 (2ka) \tag{3.21}$$

which may be written as

$$\Phi_s = - \frac{e^{-ikr}}{8kr} (\epsilon + \delta) \left\{ \sin 2ka - 2ka \cos 2ka \right\} \tag{3.22}$$

Comparing the two solutions, it is seen that they are identical with the exception of the sign of δ. In the Rayleigh limit, Eqn. (3.22) agrees with Rayleigh's result. However, taking the Rayleigh limit of the Born approximation still leaves the sign of δ different from Rayleigh's result. One possible explanation of the difference in the sign of δ is that the Born approximation tacitly assumes that the scattering region does not move during the time interval the wave is being scattered. However, an incoming oscillating wave will cause the sphere to vibrate about its rest position in a complicated vibrational pattern. This motion gives rise

to a sound field which is included in the exact
calculation but not in the Born approximation. Since
the difference in sign is associated with the inertial
rather than the compressibility term, this explanation
seems at least plausible. A detailed study of this
difference of results is now being carried out and will
be reported in a later publication.

CHAPTER IV
THE BACK-SCATTERING OF PULSES FROM FLUID SPHERES

The expansion of the acoustical properties inside the sphere in terms of the acoustical properties outside the sphere allows the steady state solution of the wave equation to be written in a very simple form. Having the back-scattered steady state solution in this form permits the back-scattered fields for acoustic pulses to be calculated quite easily. The types of pulses considered here are an exponential pulse and a sinusoidally varying pulse of short duration. The interest in these two types of pulses arises from the fact that they are the most frequently used in acoustic work.

The back-scattered field for an arbitrary incoming pulse $s(t)$ is

$$\phi = -\frac{a^2}{2\sqrt{2\pi}\, cr}(\epsilon+\delta)\int_{-\infty}^{+\infty} e^{i\omega t}\,\omega\, g(\omega)\, j_1(2ka)\, d\omega \qquad (4.1)$$

where $g(\omega)$ is the Fourier transform defined in Appendix B.

Exponential Pulse

The Fourier transform for this type of pulse is calculated in Appendix B and is

$$(4.2)$$

$$g(\omega) = \frac{-i}{\sqrt{2\pi}\,(\omega-i\sigma)}$$

Using the identity

$$j_1(2ka) = \frac{\sin 2ka}{(2ka)^2} - \frac{\cos 2ka}{2ka} \qquad (4.3)$$

and then expanding the trigonometric functions in terms of exponentials, Φ becomes the sum of four integrals

$$\Phi = I_1 + I_2 + I_3 + I_4 \qquad (4.4)$$

where

$$I_1 = -\frac{c}{32\pi r}(\epsilon+\delta)\int_{-\infty}^{+\infty}\frac{e^{i\omega(t'+\frac{2a}{c})}}{\omega(\omega-i\sigma)}d\omega \qquad (4.5)$$

$$I_2 = \frac{c}{32\pi r}(\epsilon+\delta)\int_{-\infty}^{+\infty}\frac{e^{i\omega(t'-\frac{2a}{c})}}{\omega(\omega-i\sigma)}d\omega$$

$$I_3 = \frac{ai}{16\pi r}(\epsilon+\delta)\int_{-\infty}^{+\infty}\frac{e^{i\omega(t'+\frac{2a}{c})}}{\omega-i\sigma}d\omega$$

$$I_4 = \frac{ai}{16\pi r}(\epsilon+\delta)\int_{-\infty}^{+\infty}\frac{e^{i\omega(t'-\frac{2a}{c})}}{\omega-i\sigma}d\omega$$

and where

$$t' = t - \frac{r}{c}$$

Evaluating these integrals by contour integration, the following results are obtained for the

time intervals indicated:

$$I_1 = -\frac{c}{16r\sigma}(\epsilon + \delta)\left\{e^{-\sigma\left(t' + \frac{2a}{c}\right)} - \frac{1}{2}\right\} \qquad t' + \frac{2a}{c} > 0$$

$$I_1 = -\frac{c}{32r\sigma}(\epsilon + \delta) \qquad t' + \frac{2a}{c} < 0$$

$$I_2 = \frac{c}{16r\sigma}(\epsilon + \delta)\left\{e^{-\sigma\left(t' - \frac{2a}{c}\right)} - \frac{1}{2}\right\} \qquad t' - \frac{2a}{c} > 0$$

$$I_2 = \frac{c}{32r\sigma}(\epsilon + \delta) \qquad t' - \frac{2a}{c} < 0$$

$$I_3 = -\frac{a}{8r}(\epsilon + \delta)\, e^{-\sigma\left(t' + \frac{2a}{c}\right)} \qquad t' + \frac{2a}{c} > 0$$

$$I_3 = 0 \qquad t' + \frac{2a}{c} < 0$$

$$I_4 = -\frac{a}{8r}(\epsilon + \delta)\, e^{-\sigma\left(t' - \frac{2a}{c}\right)} \qquad t' - \frac{2a}{c} > 0$$

$$I_4 = 0 \qquad t' - \frac{2a}{c} < 0$$

To clearly see the total scattered field for the different time intervals, it is convenient to construct the following table.

$$\phi =$$

$-\dfrac{c}{32r\sigma}(\epsilon+\delta)$	$-\dfrac{c}{16r\sigma}(\epsilon+\delta)\left\{e^{-\sigma\left(t'+\frac{2a}{c}\right)} - \dfrac{1}{2}\right\}$	$-\dfrac{c}{16r\sigma}(\epsilon+\delta)\left\{e^{-\sigma\left(t'+\frac{2a}{c}\right)} - \dfrac{1}{2}\right\}$
0	$-\dfrac{a}{8r}(\epsilon+\delta)\,e^{-\sigma\left(t'+\frac{2a}{c}\right)}$	$-\dfrac{a}{8r}(\epsilon+\delta)\,e^{-\sigma\left(t'+\frac{2a}{c}\right)}$
$\dfrac{c}{32r\sigma}(\epsilon+\delta)$	$\dfrac{c}{32r\sigma}(\epsilon+\delta)$	$\dfrac{c}{16r\sigma}(\epsilon+\delta)\left\{e^{-\sigma\left(t'+\frac{2a}{c}\right)} - \dfrac{1}{2}\right\}$
0	0	$-\dfrac{a}{8r}(\epsilon+\delta)\,e^{-\sigma\left(t'+\frac{2a}{c}\right)}$

Axis markers: $-\dfrac{2a}{c}$, $\dfrac{2a}{c}$, $t' \rightarrow$

$$\phi = 0$$

$$\phi = (\epsilon+\delta)\left\{\frac{c}{16r\sigma} - \frac{1}{8r}\left(\frac{c}{2\sigma}+a\right)e^{-\sigma\left(t'+\frac{2a}{c}\right)}\right\}$$

$$\phi = \frac{(\epsilon+\delta)\,e^{-\sigma t'}}{16r}\left\{\frac{c}{2\sigma}\operatorname{Sinh}\frac{2a\sigma}{c} - a\operatorname{Cosh}\frac{2a\sigma}{c}\right\}$$
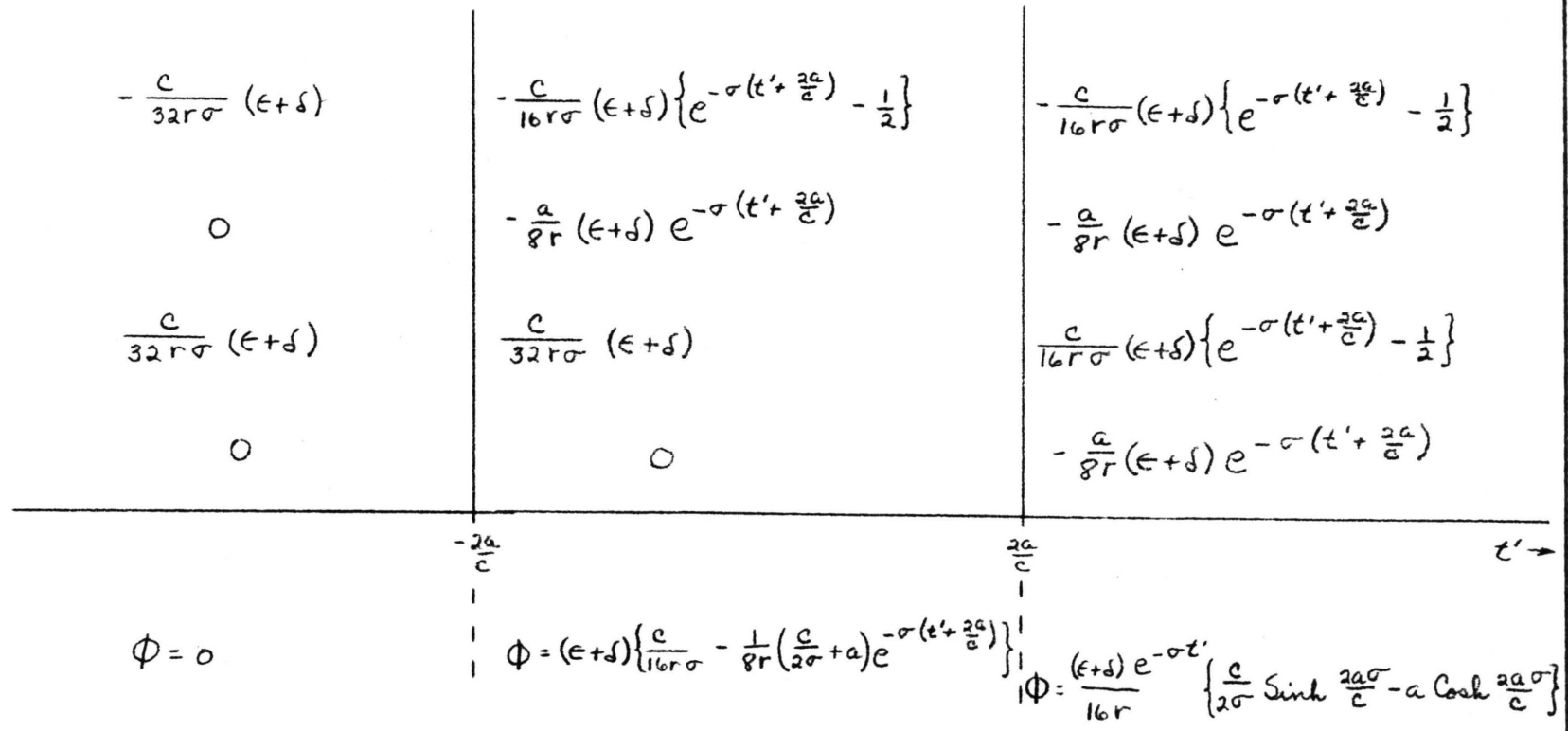
When the values of the integrals are summed in the respective time intervals the following results are obtained:

(a) $t' < -\frac{2a}{c}$. The total back-scattered field for this time interval is zero as would be expected.

(b) $-\frac{2a}{c} < t' < \frac{2a}{c}$ The total back-scattered field for this time interval is:

$$\phi = (\epsilon + \delta) \left\{ \frac{c}{16r\sigma} - \frac{1}{8r}\left(\frac{c}{2\sigma} + a\right) e^{-\sigma\left(t' + \frac{2a}{c}\right)} \right\}$$

This field has a time independent term and a term whose time dependence makes it appear as though the pulse is scattered from the front surface of the sphere.

(c) $t' > \frac{2a}{c}$ The total back-scattered field in this time interval is:

$$\phi = e^{-\sigma t'}\left(\frac{\delta + \epsilon}{16r}\right)\left\{ \frac{c}{2\sigma} \sinh\frac{2a\sigma}{c} - a\cosh\frac{2a\sigma}{c}\right\}$$

No time independent term occurs here and the time dependent term appears to have been scattered from the back surface of the sphere.

For the sinusoidal pulse, the acoustical disturbance is

$$S(t) = \sin \omega \frac{t}{t} = \sin \omega_0 t \qquad -\frac{T}{2} < t < \frac{T}{2} \qquad (4.7)$$

and

$$S(t) = 0 \qquad t > \frac{T}{2}, \quad t < -\frac{T}{2}.$$

The Fourier transform for this pulse as calculated in Appendix B is

$$g(\omega) = \frac{-2i\,\omega_0 \sin \omega \frac{T}{2}}{\sqrt{2\pi}\,(\omega - \omega_0)(\omega + \omega_0)} \qquad (4.8)$$

Expanding $\sin \omega \frac{T}{2}$ in exponentials and substituting Eqn. (4.8) into Eqn. (4.1) the back-scattered field ϕ becomes the sum of eight integrals

$$\phi = I_1 + I_2 + \cdots + I_8 \qquad (4.9)$$

where

$$(4.10)$$

$$I_1 = -\not\! y \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' + \frac{2a}{c} + \frac{T}{2}\right)}}{\omega(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_2 = \not\! y \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' + \frac{2a}{c} - \frac{T}{2}\right)}}{\omega(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_3 = \mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' - \frac{2a}{c} + \frac{T}{2}\right)}}{\omega(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_4 = -\mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' - \frac{2a}{c} - \frac{T}{2}\right)}}{\omega(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_5 = \frac{2a i}{c} \mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' + \frac{2a}{c} + \frac{T}{2}\right)}}{(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_6 = -\frac{2a i}{c} \mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' + \frac{2a}{c} - \frac{T}{2}\right)}}{(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_7 = \frac{2a i}{c} \mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' - \frac{2a}{c} + \frac{T}{2}\right)}}{(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

$$I_8 = -\frac{2a i}{c} \mathcal{H} \int_{-\infty}^{+\infty} \frac{e^{i\omega\left(t' - \frac{2a}{c} - \frac{T}{2}\right)}}{(\omega - \omega_0)(\omega + \omega_0)} d\omega$$

where

$$\mathcal{H} = \frac{c \omega_0}{32 \pi i r} (\epsilon + \delta)$$

Evaluating the integrals by the contour integration given in Appendix B, the integrals have the following values for the time intervals indicated:

$$I_1 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{1 - \cos\omega_0\left(t' + \frac{2a}{c} + \frac{T}{2}\right)\right\} \qquad t' + \frac{2a}{c} + \frac{T}{2} > 0$$

$$I_1 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{\cos\omega_0\left(t' + \frac{2a}{c} + \frac{T}{2}\right) - 1\right\} \qquad t' + \frac{2a}{c} + \frac{T}{2} < 0$$

$$I_2 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{\cos\omega_0\left(t' + \frac{2a}{c} - \frac{T}{2}\right) - 1\right\} \qquad t' + \frac{2a}{c} - \frac{T}{2} > 0$$

$$I_2 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{1 - \cos\omega_0\left(t' + \frac{2a}{c} - \frac{T}{2}\right)\right\} \qquad t' + \frac{2a}{c} - \frac{T}{2} < 0$$

$$I_3 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{\cos\omega_0\left(t' - \frac{2a}{c} + \frac{T}{2}\right) - 1\right\} \qquad t' - \frac{2a}{c} + \frac{T}{2} > 0$$

$$I_3 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{1 - \cos\omega_0\left(t' - \frac{2a}{c} + \frac{T}{2}\right)\right\} \qquad t' - \frac{2a}{c} + \frac{T}{2} < 0$$

$$I_4 = \frac{(\epsilon+\delta)c}{32\omega_0 r}\left\{1 - \cos\omega_0\left(t' - \frac{2a}{c} - \frac{T}{2}\right)\right\} \qquad t' - \frac{2a}{c} - \frac{T}{2} > 0$$

$$I_4 = \frac{(\epsilon+\delta)c}{32\omega r}\left\{\cos\omega_0\left(t' - \frac{2a}{c} - \frac{T}{2}\right) - 1\right\} \qquad t' - \frac{2a}{c} - \frac{T}{2} < 0$$

$$I_5 = -\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0\left(t' + \frac{2a}{c} + \frac{T}{2}\right) \qquad t' + \frac{2a}{c} + \frac{T}{2} > 0$$

$$I_5 = \frac{(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' + \frac{2a}{c} + \frac{T}{2} \right) \qquad t' + \frac{2a}{c} + \frac{T}{2} < 0$$

$$I_6 = \frac{(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' + \frac{2a}{c} - \frac{T}{2} \right) \qquad t' + \frac{2a}{c} - \frac{T}{2} > 0$$

$$I_6 = \frac{-(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' + \frac{2a}{c} - \frac{T}{2} \right) \qquad t' + \frac{2a}{c} - \frac{T}{2} < 0$$

$$I_7 = \frac{-(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' - \frac{2a}{c} + \frac{T}{2} \right) \qquad t' - \frac{2a}{c} + \frac{T}{2} > 0$$

$$I_7 = \frac{(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' - \frac{2a}{c} + \frac{T}{2} \right) \qquad t' - \frac{2a}{c} + \frac{T}{2} < 0$$

$$I_8 = \frac{(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' - \frac{2a}{c} - \frac{T}{2} \right) \qquad t' - \frac{2a}{c} - \frac{T}{2} > 0$$

$$I_8 = \frac{-(\epsilon + \delta) a}{16 r} \, \text{Sin} \, \omega_0 \left(t' - \frac{2a}{c} - \frac{T}{2} \right) \qquad t' - \frac{2a}{c} - \frac{T}{2} < 0$$

The results of this type of pulse are tabulated in the following table.

$$\phi =$$

$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'+\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$
$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'+\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'+\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'+\frac{2a}{c})-1\}$
$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'-\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'-\frac{2a}{c})\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$
$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{\cos\omega_0(t'-\frac{2a}{c})-1\}$	$\frac{(\epsilon+\delta)c}{32\omega_0 r}\{1-\cos\omega_0(t'-\frac{2a}{c})\}$
$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$
$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$	$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'+\frac{2a}{c})$
$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$
$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$-\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$	$\frac{(\epsilon+\delta)a}{16 r}\sin\omega_0(t'-\frac{2a}{c})$

Axis markers: $-\frac{2a}{c}-\frac{T}{2}$; $+\frac{2a}{c}-\frac{T}{2}$; $-\frac{2a}{c}+\frac{T}{2}$; $\frac{2a}{c}+\frac{T}{2}$; $t'\to$

$$\phi = 0$$

$$\phi = \frac{(\epsilon+\delta)c}{16\omega_0 r}\{1-\cos\omega_0(t'+\tfrac{2a}{c})\} - \frac{(\epsilon+\delta)a}{8r}\sin\omega_0(t'+\tfrac{2a}{c})$$

$$\phi = \frac{(\epsilon+\delta)c}{16\omega_0 r}\sin\omega_0 t\,\sin\omega_0\tfrac{2a}{c} - \frac{(\epsilon+\delta)a}{4r}\sin\omega_0 t\,\cos\omega_0\tfrac{2a}{c}$$

$$\phi = \frac{(\epsilon+\delta)c}{16\omega_0 r}\{\cos\omega_0(t'-\tfrac{2a}{c})-1\} - \frac{(\epsilon+\delta)a}{8r}\sin\omega_0(t'-\tfrac{2a}{c})$$

$$\phi = 0$$

When the values of the integrals are summed for each time interval, the following results are obtained:

(a) $t' < -\frac{2a}{c} - \frac{T}{2}$, As expected, the total back-scattered field for this time interval is zero.

(b) $-\frac{2a}{c} - \frac{T}{2} < t' < \frac{2a}{c} - \frac{T}{2}$, The total back-scattered field in this time interval is:

$$\Phi = (\epsilon + \delta)\left\{ \frac{c}{16\,\omega_0 r}\left(1 - \cos\omega_0\left(t' + \frac{2a}{c}\right)\right) - \frac{a}{8r}\sin\omega_0\left(t' + \frac{2a}{c}\right)\right\}$$

This field appears to be the first of the wave train being scattered by the front surface of the sphere, and, as in the exponential pulse case, a time independent term appears.

(c) $\frac{2a}{c} - \frac{T}{2} < t' < \frac{T}{2} - \frac{2a}{c}$, The total back-scattered field in this time interval is:

$$\Phi = (\epsilon + \delta)\frac{\sin\omega_0 t}{4r}\left\{ \frac{c}{4\omega_0}\sin\omega_0\frac{2a}{c} - a\cos\omega_0\frac{2a}{c}\right\}$$

This field appears to be the first of the pulse being scattered from the back surface of the sphere.

(d)
$$-\frac{2a}{c} + \frac{T}{2} < t < \frac{T}{2} + \frac{2a}{c} ,$$

The total back-scattered field in this time interval is:

$$\phi = \frac{(\epsilon + \delta)c}{16 w_0 r}\left\{ \cos w_0(t' - \frac{2a}{c}) - 1 \right\} - \frac{(\epsilon + \delta)a}{8r} \sin w_0(t' - \frac{2a}{c})$$

This field appears to be the last of the pulse being scattered from the front of the sphere and again a time independent term appears.

(e)
$$t' > \frac{T}{2} + \frac{2a}{c}$$

The total back-scattered field in this time interval is again zero as would be expected.

Some care must be taken in choosing a pulse since the method of analysis used in this thesis has some limitations. First, since the asymptotic limit of the spherical Hankel function becomes singular at $w = 0$, the pulse must be chosen such that its Fourier transform has a small contribution of frequencies near $w = 0$. It may also be shown that the integrand vanishes at $w = 0$ even when $h_m^{(2)}(k_r)$ is not used in its asymptotic form; thus the error due to representing $h^{(2)}(k_r)$ as $\frac{e^{-ik_r}}{k_r}$ is not large. Secondly, since the Taylor expansion requires $k_a \Delta \ll 1$, the Fourier transform of the pulse must not contain frequencies such that the inequality cannot be satisfied with an adjustment of a, Δ or c. Also,

since $k_1 a \Delta \ll 1$ and since frequencies near $\omega = 0$ are not allowed, pulses must be chosen such that with an adjustment of a, Δ and c, the inequality is satisfied without much contribution from frequencies near $\omega = 0$.

CHAPTER V
SUMMARY AND DISCUSSION

In this thesis it has been shown that the scattering of sound by fluid spheres immersed in an infinite fluid may be calculated easily by assuming that the acoustical properties of the sphere are very similar to the acoustical properties of the surrounding fluid. This allows the expansion of functions involving the acoustical parameters inside the sphere in terms of a Taylor series in which the acoustical parameters outside the sphere appear. This, in turn, allows the back-scattered field for arbitrary pulses to be easily calculated at least to first order in the expansion. The first order back-scattered field for the two types of pulses considered appears, from time of arrival considerations, to consist of a pulse being scattered from the front surface and one from the back surface of the sphere. For higher accuracy more terms would need to be used in the expansion. In order to see the result of including higher order terms the contribution due to second order terms may easily be determined. The second order terms in ϵ and δ that appear in the scattering coefficient B_m are

$$- \frac{\delta(\epsilon+\delta)}{4} j_m^2(ka)\, j_m''(ka)\, h_m''(ka) \; - \; \frac{\epsilon(\epsilon+\delta)}{4} j_m(ka)\, j_m''^2(ka)\, h_m(ka)$$

$$- \frac{ka}{4}(\epsilon-\delta)(\epsilon+\delta)\, j_m(ka)\, j_m''(ka) \left\{ j_m'''(ka)\, h_m(ka) - j_m'(ka)\, h_m''(ka) \right\}$$

$$- \frac{i(\epsilon+\delta)^2}{8} \left(j_m''(ka)\, j_m'(ka) - j_m(ka)\, j_m'''(ka) \right) + \frac{i}{4ka}(\epsilon+\delta)\left(\delta j_m'^2(k,a) + \epsilon\, j_m(ka)\, j_m''(k,a) \right)$$

$$- \frac{ka}{4}\delta(\epsilon-\delta)\, j_m(ka)\, h_m''(ka) \left\{ j_m(ka)\, j_m'''(ka) - j_m'^2(ka) \right\}$$

$$- \frac{ka}{4}\epsilon(\epsilon-\delta)\, h_m(ka)\, j_m''(ka) \left\{ j_m(ka)\, j_m'''(ka) - j_m'^2(ka) \right\}$$

$$- \frac{(ka)^2(\epsilon-\delta)^2}{4} \left\{ j_m(ka)\, j_m'''(ka) - j_m'^2(ka) \right\}\left\{ j_m'''(ka)\, h_m(ka) - j_m'(ka)\, h_m''(ka) \right\}$$

All these terms contain products of either two or four spherical Bessel functions. When the time dependences are investigated by means of contour integration similar to that appearing in Appendix B, it appears that some of the terms represent the pulse being scattered by the front and back surfaces of the sphere just as in the first order case. However, the other terms possess time dependences which seem to have no simple geometric interpretation. The origin of all these reflected

pulses is not adequately known and it is very doubtful
that any of them have such simple geometric origins when
one considers the ratio of wavelength to sphere size.
Further investigation of the dynamic behavior of the
sphere itself would be of interest. Due to the com-
plexity of this problem, time has not allowed for its
inclusion in this thesis.

The existence of time independent terms for the
back-scattered field produced by the two pulses is not
completely understood at this time and subsequent
investigation related to that mentioned above may provide
information on the origin of the terms. Since the time
between the arrivals of the two scattered pulses
discussed in Chapter IV is relatively short, one possible
explanation for the terms which appear to be time
independent is that they may represent the first term in
the expansion of a slowly varying time dependent function.

The assumption that the acoustical properties
of the sphere are very similar to the acoustical
properties of the surrounding fluid also allows the Born
approximation to be used to compute the scattering of a
plane wave. However, when the results of the exact
calculation and the Born approximation are compared, it
is seen that there exists a disagreement in the sign
of $\int$ for back-scattering and an even more complicated
discrepancy for other angles of scattering. In Chapter

III, it was suggested that this disagreement may be attributed to the fact that the Born approximation tacitly assumes that the scattering region does not move during the time interval the wave is being scattered. The exact solution, however, does take into account the movement of the sphere. Further investigation of this problem is under way.

The problem considered here has led to an easier way to obtain the approximate field due to scattering by spheres at least in the limit of small differences in the elastic constants as described above. It is hoped that the extension of this method to higher order terms will be of general use in scattering problems throughout physics.

APPENDIX A

EVALUATION OF INFINITE SUM

The infinite sum to be evaluated is

$$T = \sum_{m=0}^{\infty} (-1)^m \, m \, j_m(ka) \, j_{m-1}(ka) \qquad \text{(A-1)}$$

Expressing the spherical Bessel functions as cylindrical Bessel functions by

$$j_m(ka) = \sqrt{\frac{\pi}{2ka}} \, J_{m+1/2}(ka) \qquad \text{(A-2)}$$

and using the identity (8)

$$J_u(ka) \, J_v(ka) = \frac{2}{\pi} \int_0^{\pi/2} J_{u+v}(2ka \cos\theta) \cos(u-v)\theta \, d\theta \qquad \text{(A-3)}$$

Equation (A-1) becomes

$$T = \frac{1}{ka} \sum_{m=0}^{\infty} (-1)^m \, m \int_0^{\pi/2} J_{2m}(2ka \cos\theta) \cos\theta \, d\theta \qquad \text{(A-4)}$$

Since $J_{2m}(2ka \cos\theta)$ is a well behaved function, the interchange of the summation and integration operations presents no difficulty. Also, the first term of the summation vanishes so the summation may be started from one instead of zero.

Rewriting Equation (A-4)

$$T = \frac{1}{ka} \int_0^{\pi/2} \cos\theta \sum_{m=0}^{\infty} (-1)^m\, m\; J_{2m}(2ka\cos\theta)\, d\theta \qquad \text{(A-5)}$$

A recursion relationship of use at this point is (8)

$$J_m(z) = \frac{z}{2m} J_{m-1}(z) + \frac{z}{2m} J_{m+1}(z) \qquad \text{(A-6)}$$

and when used in Equation (A-5) gives

$$T = \frac{1}{2} \int_0^{\pi/2} \cos^2\theta \left\{ \sum_{m=0}^{\infty} (-1)^m\, J_{2m-1}(2ka\cos\theta) + \sum_{m=1}^{\infty} (-1)^m\, J_{2m+1}(2ka\cos\theta) \right\} d\theta \qquad \text{(A-7)}$$

Examining the terms in the bracket it is seen that the sum of the infinite series is equal to

$$\sum_{m=1}^{\infty} J_{2m-1}(2ka\cos\theta) + \sum_{m=1}^{\infty} (-1)^m\, J_{2m+1}(2ka\cos\theta) = - J_1(2ka\cos\theta)$$

Therefore,

$$T = - \frac{1}{2} \int_0^{\pi/2} J_1(2ka\cos\theta)\, \cos^2\theta\, d\theta \qquad \text{(A-8)}$$

Another identity of use here is (8)

$$J_{\nu+\mu+1}(2z) = \frac{(2z)^{\nu+1}}{2^\nu\, \Gamma(\nu+1)} \int_0^{\pi/2} J_\mu(2z\sin\theta)\, \sin^{\mu+1}\theta\, \cos^{2\nu+1}\theta\, d\theta \qquad \text{(A-9)}$$

By letting $\theta = \varphi + \tilde{\pi}/2$, $\mu = 1$ and $\nu = -\frac{1}{2}$

(A-10)

$$\int_0^{\tilde{\pi}/2} J_1(2ka\cos\varphi)\cos^2\varphi\, d\varphi = \frac{\Gamma(\frac{1}{2})\, J_{3/2}(2ka)}{\sqrt{2ka}\,\sqrt{2}}$$

and so

(A-11)

$$\int_0^{\tilde{\pi}/2} J_1(2ka\cos\varphi)\cos^2\varphi\, d\varphi = j_1(2ka)$$

Therefore,

(A-12)

$$T = -\frac{1}{2}\, j_1(2ka)$$

APPENDIX B

FOURIER TRANSFORMS AND CONTOURS OF INTEGRATION

Using the Fourier Integral theorem (9), a finite wave train or pulse may be represented mathematically as

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\omega t} \, d\omega \int_{-\infty}^{+\infty} S(t') \, e^{-i\omega t'} \, dt' \tag{B-1}$$

The quantity

$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} S(t) \, e^{-i\omega t} \, dt \tag{B-2}$$

is called the Fourier transform of $f(t)$.

1. Exponential Pulse:

   For the exponential pulse, the acoustical disturbance is given by

   $$S(t) = e^{-\sigma t} \qquad\qquad t > 0 \tag{B-3}$$

   $$S(t) = 0 \qquad\qquad t < 0$$

   The Fourier transform of this pulse is

   $$g(\omega) = \frac{-i}{\sqrt{2\pi} \, (\omega - i\sigma)} \tag{B-4}$$

2. Sinusoidal Pulse:

The acoustical disturbance for this type of pulse is

$$S(t) = \sin \omega \frac{t}{c} = \sin \omega_0 t \qquad -\frac{T}{2} < t < \frac{T}{2} \tag{B-5}$$

$$S(t) = 0 \qquad t > \frac{T}{2} \ , \ t < -\frac{T}{2}$$

where

$$\frac{T}{2} = 2n\pi \qquad , \qquad \frac{\omega}{c} = \omega_0$$

The Fourier transform becomes

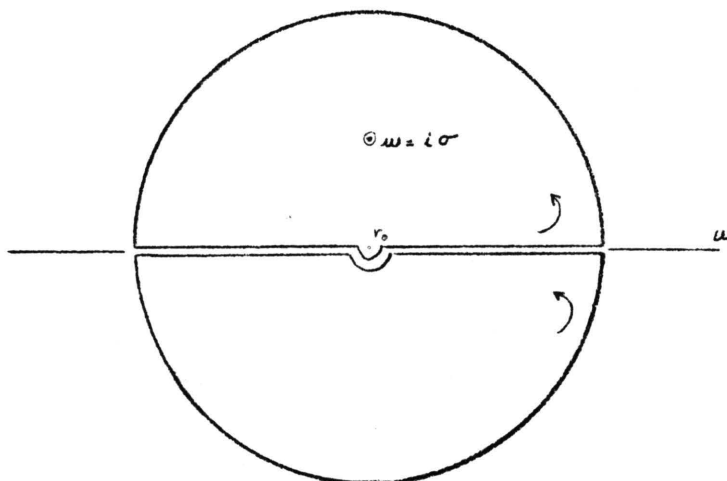$$g(\omega) = \frac{-2i\omega_0 \ \sin \omega \frac{T}{2}}{\sqrt{2\pi} \ (\omega - \omega_0)(\omega + \omega_0)} \tag{B-6}$$

3. Note on Contour Integration:

The first integral in Equation (4.5) will be calculated as an example to demonstrate the contours used for integration in Chapter IV.

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{i\omega(t' + \frac{2a}{c})}}{\omega(\omega - i\sigma)} d\omega \tag{B-7}$$

The following contour in the complex ω plane has been chosen:

By observing the term $e^{i\omega(t' + \frac{2a}{c})}$ it is evident that for $t' + \frac{2a}{c} > 0$, the upper contour must be used and for $t' + \frac{2a}{c} < 0$ the bottom contour must be used, since the function is singular at $\omega = i\infty$ for $t' + \frac{2a}{c} < 0$ and at $\omega = -i\infty$ for $t' + \frac{2a}{c} > 0$. This ensures that the value of the integral along the semicircle will vanish in the limit as the radius of the semicircle becomes infinite. The poles on the real axis may be included in either contour but in this thesis they are always included in the upper contour.

From the theory of complex integration (10)

$$(B-8)$$

$$\int_{G_c} f(z)\, dz = 2\pi i \sum Residues$$

For I_1, Equation (B-8) yields for $t' + \frac{2a}{c} > 0$

$$(B-9)$$

$$\int_{-\infty}^{+\infty} \frac{e^{i\omega(t'+\frac{2a}{c})}}{\omega(\omega - i\sigma)} d\omega = 2\pi i \left\{ \text{Residue } (\omega=0) + \text{Residue } (\omega=i\sigma) \right\}$$

$$- \int_{\pi}^{2\pi} \frac{e^{ir_0 e^{i\varphi}(t'+\frac{2a}{c})}}{r_0 e^{i\varphi}(r_0 e^{i\varphi} - i\sigma)} i r_0 e^{i\varphi} d\varphi$$

where $\omega = r_0 e^{i\varphi}$ around the pole at the origin.

Thus,

$$I_1 = \int_{-\infty}^{+\infty} \frac{e^{i\omega(t'+\frac{2a}{c})}}{\omega(\omega - i\sigma)} d\omega = 2\pi i \left\{ \frac{1}{-i\sigma} + \frac{e^{-\sigma(t'+\frac{2a}{c})}}{i\sigma} \right\} \qquad (B-10)$$

$$+ \int_{\pi}^{2\pi} \frac{d\varphi}{\sigma}$$

$$I_1 = \frac{2\pi}{\sigma} \left\{ e^{-\sigma(t'+\frac{2a}{c})} - \frac{1}{2} \right\} \qquad t' + \frac{2a}{c} > 0$$

and it clearly has the value

$$(B-11)$$

$$I_1 = \frac{\pi}{\sigma} \qquad t' + \frac{2a}{c} < 0$$

APPENDIX C

THE BORN APPROXIMATION FOR SPHERICAL SCATTERING

Since the acoustical properties of the sphere are very similar to those of the surrounding medium, the sphere may be considered a perturbation in an otherwise unbounded medium and the Born approximation may be used to calculate an approximate solution to the wave equation.

For the infinite medium, the scalar potential satisfies the three dimensional wave equation

$$\nabla^2 \phi + k_1^2 \phi = 0$$

where k is the wave number in that medium. When the sphere is introduced into the medium, the potential in the vicinity of the sphere changes slightly and the wave equation may be written

$$\nabla^2 \phi_1 + (k_1^2 + \epsilon_0) \phi_1 = 0 \qquad (C-1)$$

where

$$\epsilon_0 = k_2^2 - k_1^2 \qquad (C-2)$$

and

$$\frac{\epsilon_0}{k_1^2} \ll 1$$

This problem is one which may well be approximated by the method due to Born for which an excellent presentation

is given by Schiff (7). Letting

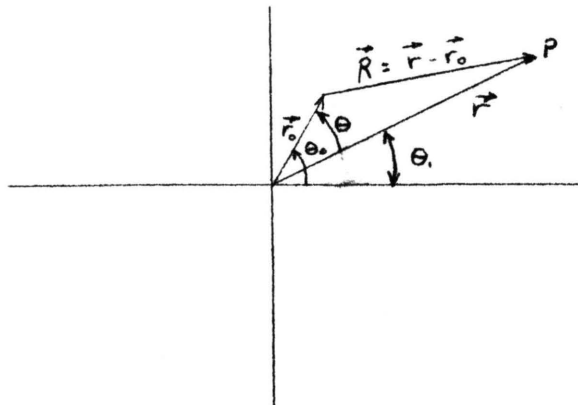$$\Phi_1 = \Phi_i + \Phi_s = e^{-ik,r\cos\theta} + \Phi_s$$

and neglecting the term $\epsilon_0 \Phi_s$, Equation (C-1) becomes

$$\nabla^2 \Phi_s + k^2 \Phi_s = -\epsilon_0 e^{-ikr\cos\theta}$$

The solution to this inhomogeneous equation is (7)

$$\Phi_s = \frac{\epsilon_0}{4\pi} \int_0^a \int_0^\pi \int_0^{2\pi} d\Phi_0 \, r_0^2 dr_0 \, \sin\theta_0 \, d\theta_0 \, \frac{e^{-ik\{|\vec{r}-\vec{r}_0| + r_0 \cos\theta\}}}{|\vec{r}-\vec{r}_0|}$$

The diagram defines all distances and angles.

The distance $R = |\vec{r}-\vec{r}_0| = r - r_0 \cos\theta$ and for large $\vec{r}$
$|\vec{r}-\vec{r}_0| \sim r$. For back-scattering $\theta_1 = \pi$ and
$\theta = \theta_0 - \pi$, and

$$\Phi_s = -\frac{e^{-ikr}}{2kr} \epsilon_0 \int_0^a \int_0^\pi r_0^2 dr_0 \, \sin\theta_0 \, e^{-2ikr_0 \cos\theta_0'} \, d\theta_0$$

Upon evaluating the integral, the Born approximation
for the scattered field is

$$\phi_s = - \frac{e^{-ikr}}{2kr} \, \omega^2 (\epsilon - \delta) \left\{ \frac{1}{4k^2} \sin 2ka - \frac{a}{k} \cos 2ka \right\}$$

This expression for ϕ_s is essentially the same as found
by Pekeris (11). The integration may be extended easily
to general angles of scattering.

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ABSTRACT OF THESIS

ACOUSTIC SCATTERING BY FLUID SPHERES

The problem of scattering of acoustic waves and
pulses by an elastic sphere embedded in an infinite
elastic medium is investigated for the case where the two
media are very similar acoustically. This physical
situation allows functions with arguments involving the
acoustic parameters inside the sphere to be expanded in a
Taylor series involving the acoustic parameters outside
the sphere. Using only the first order terms in this
expansion, the solution for plane wave conditions in the
back-scattered direction is much simpler than the exact
solution. This allows the solutions for the scattering
of acoustic pulses to be calculated.

The steady state solutions are compared with
those obtained using the Born approximation, and are
found to differ only in the algebraic sign of the
difference in density of the two media; although they
agree with the results obtained by Rayleigh in the
proper limit. It is also found that the Born approxi-
mation differs from the results obtained by Rayleigh,
again only in the algebraic sign of the difference in

density of the two media.

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