FORMATION OF ROLL WAVES IN LAMINAR SHEET FLOW

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF SYMBOLS

Symbol	Description
A	cross-sectional area for uniform flow
A'	cross-sectional area for flow with a small perturbation
A_G	cross-sectional area for gradually varied flow
B	channel width for uniform flow
B'	channel width for flow with a small perturbation
B_G	channel width for gradually varied flow
c	wave celerity
c_o	velocity of the wave relative to the mean velocity $\bar{u}$
C	Chézy coefficient
D,E	empirical constants in Eqs. 29 and 34
F	Froude number
F_c	critical Froude number
g	gravitational acceleration
h	uniform flow depth
h'	flow depth for flow with a small perturbation
h_G	gradually varied flow depth
K	friction parameter
L	wavelength
P	wetted perimeter
q	unit discharge
Re	Reynolds number
R_h	hydraulic radius
S	channel slope
S_f	friction slope
S'_f	friction slope for flow with a small perturbation

Symbol	Description
S_{fG}	friction slope for gradually varied flow
t	time
T	wave period
$\bar{T}$	mean wave period
u	velocity
$\bar{u}$	mean velocity for uniform flow
$\bar{u}'$	mean velocity for flow with a small perturbation
$\bar{u}_G$	mean velocity for gradually varied flow
Ve	Vedernikov number
y	distance from the water surface
β,γ	functions of flow variables in Eqs. 22 and 23
β_m	momentum correction factor
ε	constant of integration
η	transformed distance moving with the wave
ν	kinematic viscosity of water
ξ	distance downslope
ξ_c	critical distance at which roll waves are formed
Φ,Ψ	functions of the Froude number

I. INTRODUCTION

Sheet flow is classified as "wide" open channel flow because channel walls do not affect the flow pattern. Wide open channel flow exists when the ratio of channel width to flow depth is larger than 10 (Chow, 1959). The hydraulic properties of sheet flows depend on the relative magnitude of inertia and viscous forces. The ratio of these two forces defines the Reynolds number, Re. For wide open channels the Reynolds number is equal to the ratio of the volumetric flow rate (unit discharge) to the kinematic viscosity of water. Laminar flow conditions prevail when Re < 500 for smooth surfaces. The corresponding unit discharge must be less than about 5 cm^2/s for the usual range of water temperatures. In laminar sheet flows the viscous forces damp the velocity fluctuations and the motion of fluid particles follow smooth paths. In turbulent flow (Re > 2000 for smooth surfaces) inertia forces overcome the friction forces and fluid particles move erratically, transferring mass and momentum between adjacent flow regions.

Under both laminar and turbulent conditions sheet flows can be unstable such that an initially small perturbation of the water surface amplifies with time and with distance downstream until a well-defined wave pattern is observed. These amplified perturbations are called roll waves.

Previous treatments of the formation of roll waves in laminar sheet flows were mainly confined to the definition of necessary conditions for the occurrence of surface instability. It became apparent with theoretical derivation for turbulent flow (Montuori, 1963 and Liggett, 1975) that conditions based on the Froude number are not sufficient since the length required for the formation of roll waves is not considered. In this study, previous theories relating to this distance are modified in the light of laminar sheet flow characteristics. An experimental study was conducted in order to verify the results of the theoretical analysis.

The characteristics of steady, uniform sheet flows are first described in Chapter II, followed by a theoretical analysis of free surface instability. Chapter III presents the results of the experimental study performed in the Hydraulics Laboratory of the Engineering Research Center at Colorado State University.

II. THEORY ON THE STABILITY OF LAMINAR SHEET FLOW

The analysis of the free surface stability of laminar sheet flows assumes that steady uniform flow conditions exist prior to the occurrence of a small perturbation of the water surface. This chapter discusses the characteristics of laminar, steady uniform sheet flow, and the criteria which have been used to determine its stability. Expressions for the length of roll wave formation are derived.

2.1 Steady uniform laminar sheet flow characteristics

The principal variables describing laminar, steady uniform sheet flows are: the slope S, the flow depth h, the mean velocity $\bar{u}$, the unit discharge q, the gravitational acceleration g, and the kinematic viscosity ν. Two nonlinear partial differential equations were derived by Saint-Venant to describe gradually varied unsteady flows. These are respectively the continuity and the momentum relationships. For steady uniform sheet flows, the continuity equation can be written as:

$$q = \bar{u}\, h \quad . \tag{1}$$

The momentum equation reduces to the so-called kinematic wave approximation for which the bed slope S is equal to the friction slope S_f. The friction slope in the laminar region is defined as follows from the Darcy-Weisbach equation (in Chow, 1969):

$$S_f = \left(\frac{K\nu}{8g} \right) \frac{\bar{u}}{h^2} \tag{2}$$

in which K is the friction coefficient. After combining Eqs. 1 and 2, the mean velocity and flow depth are:

$$\bar{u} = \left(\frac{8g}{K\nu} \right)^{1/3} S^{1/3}\, q^{2/3} \tag{3}$$

$$h = \left(\frac{Kv}{8g} \right)^{1/3} S^{-1/3} q^{1/3} \quad . \tag{4}$$

These relationships are valid for uniform or gradually varied laminar sheet flows only. The distribution of velocity u at a distance y from the water surface is expressed by the following relationship (see Chow, 1959):

$$u = \frac{12gS}{Kv} (h^2-y^2) \quad . \tag{5}$$

This velocity profile decreases parabolically from a maximum of 1.5 times the mean velocity at the free surface to zero at the boundary.

2.2 Critical Froude number and Vedernikov criteria

In deriving a fundamental stability criteria for the water surface, several approaches were used by different researchers. Early investigations by Thomas (1939) and Stoker (1957) suggested that the flow is unstable when $S > 4g/C^2$ in which C is the Chézy coefficient. The foremost criterion for instability published in the Russian literature was derived by Vedernikov (1945,1946). For laminar flows, the Vedernikov number Ve can be written as:

$$Ve = 2F \left(1 - R_h \frac{\partial P}{\partial A}\right) \tag{6}$$

in which R_h is the hydraulic radius; P is the wetted perimeter; and A is the cross-sectional area. The Froude number F equals the ratio $\bar{u}/\sqrt{gh}$ which represents the ratio of inertia to gravity forces. For an infinitely wide channel, the Vedernikov number is equal to twice the Froude number and the flow becomes unstable when the Froude number exceeds 0.5 (Ve > 1). This critical Froude number was also reported by Robertson and Rouse (1941) and Powell (1948). Mayer (1961) observed roll waves in subcritical laminar sheet flows but mistakenly concluded that roll waves can form only when the slope is larger than 3 percent. Yih (1954, 1963,1977) and Benjamin (1957) solved the problem of stability of sheet flows down an inclined plane using the Orr-Sommerfeld equation. For very long waves the flow is unstable when:

$$Re \geq \frac{5}{6S} \tag{7}$$

in which Re is the Reynolds number.

This criterion was also suggested by Taylor and Kennedy (1961). If Eq. 2 is substituted into Eq. 7 and a K value of 24 corresponding to a smooth channel is assumed, a critical Froude number of $F_c = 0.53$ results which is close to the Vedernikov criteria for wide rectangular channels. Ishihara et al. (1961) also suggested the critial value $F_c = 0.577$.

Unfortunately, these criteria based on the Froude number ignore the distance along the channel required for the formation of roll waves. This factor becomes extremely important for subcritical sheet flows since previous studies for turbulent flows (Montuori, 1963) demonstrate that the distance at which the waves are fully developed increases to infinity as the Froude number approaches the critical value.

2.3 Distance required for the formation of roll waves

When the flow is unstable (Ve > 1) a minor perturbation of the water surface will induce the formation of small waves. The amplitude of these waves will increase gradually as they move downstream until a bore is formed and the wave breaks. The distance travelled between the point at which the perturbation is initiated and the breaking point of the wave defines the distance required for the formation of roll waves.

This distance, ξ_c, is determined theoretically from the following procedure using the celerity of roll waves.

2.3.1 Celerity of roll waves

The total celerity, c, of a small gravity wave moving in a fluid with a uniform velocity distribution along the vertical is:

$$c = \bar{u} + \sqrt{gh} \quad . \tag{8}$$

In the more general case of a nonuniform vertical velocity distribution, the celerity can be theoretically derived from the momentum equation. After the momentum correction factor, β_m, is used instead of an empirical coefficient, the equation for celerity suggested by Arsenishvili (1965) becomes:

$$c = \bar{u} + c_o = \beta_m \bar{u} + \sqrt{gh + \beta_m(\beta_m - 1)\bar{u}^2} \tag{9}$$

in which c_o is the celerity of the wave relative to the mean velocity $\bar{u}$; and

$$\beta_m = \frac{1}{\bar{u}^2 h} \int_o^h u^2 \, dy \ . \tag{10}$$

When $\beta_m = 1$, Eq. 9 reduces to Eq. 8. For sheet flows, however, the momentum correction factor $\beta_m = 1.2$ is obtained from Eqs. 5 and 10. The ratio of celerities $c/\sqrt{gh}$ is:

$$\frac{c}{\sqrt{gh}} = \beta_m F + \sqrt{1 + \beta_m(\beta_m - 1)F^2} \quad , \tag{11}$$

Equations 9 and 11 are used to compute the celerity of roll waves.

2.3.2 Perturbation analysis

The following perturbation analysis of the shallow water equation has been used by Liggett (1975) to determine the distance ξ_c. As viewed from a fixed coordinate system the constant $\bar{u} + c_o$ defines the propagation speed of the wave. The flow appears to be steady to an observer moving downstream with the speed of the wave. In the derivation, the space and time coordinates x and t are replaced by $\xi = x$ and η, defined by:

$$\eta = (\bar{u} + c_o)t - \xi \tag{12}$$

in which ξ is the position relative to a fixed observer of a point on the wave while η defines its position relative to the moving coordinate system. This coordinate transformation allows the conservation of mass and momentum for a prismatic channel without lateral inflow to be written as follows (Dracos and Glenne, 1967):

$$B_G(\bar{u}+c_o)\frac{\partial h_G}{\partial \eta} + \bar{u}_G B_G\left(\frac{\partial h_G}{\partial \xi} - \frac{\partial h_G}{\partial \eta}\right) + A_G\left(\frac{\partial \bar{u}G}{\partial \xi} - \frac{\partial \bar{u}_G}{\partial \eta}\right) = 0 \tag{13}$$

and

$$(\bar{u}+c_o)\frac{\partial \bar{u}_G}{\partial \eta} + \bar{u}_G\left(\frac{\partial \bar{u}_G}{\partial \xi} - \frac{\partial \bar{u}_G}{\partial \eta}\right) + g\left(\frac{\partial h_G}{\partial \xi} - \frac{\partial h_G}{\partial \eta}\right) = g(S-S_{fG}). \tag{14}$$

in which the subscript G designates gradually varied flow variables.

A small perturbation of an initially steady uniform flow is then considered. The perturbed variables h', $\bar{u}$, B' and A' can be defined as follows:

$$h' = h + \frac{\partial h'}{\partial \eta}\eta + \frac{1}{2}\frac{\partial^2 h'}{\partial \eta^2}\eta^2 + \ldots \tag{15}$$

$$\bar{u}' = \bar{u} + \frac{\partial \bar{u}'}{\partial \eta}\eta + \frac{1}{2}\frac{\partial^2 \bar{u}'}{\partial \eta^2}\eta^2 + \ldots \tag{16}$$

$$B' = B + \frac{\partial B'}{\partial h'}\frac{\partial h'}{\partial \eta}\eta + \ldots \tag{17}$$

$$A' = A + \frac{\partial A'}{\partial h'}\frac{\partial h'}{\partial \eta}\eta + \ldots \tag{18}$$

in which the perturbed variables are primed while the uniform flow variables remain unprimed.

The truncated series are valid for small values of η and the solution is examined in the neighborhood of η=0.

The perturbed friction slope, S_f', can be approximated by substituting h' and $\bar{u}'$ for the depth and velocity in Eq. 2. After considering only the first order terms of the series:

$$S_f' = S \frac{(1 + \frac{1}{\bar{u}} \frac{\partial \bar{u}'}{\partial \eta} \eta)}{(1 + \frac{1}{h} \frac{\partial h'}{\partial \eta} \eta)^2} + \ldots \tag{19}$$

Reducing Eq. 19 to a first-order approximation results in:

$$S_f' = S + S\eta (\frac{1}{\bar{u}} \frac{\partial \bar{u}'}{\partial \eta} - \frac{2}{h} \frac{\partial h'}{\partial \eta}) + \ldots \tag{20}$$

The perturbed variables of Eqs. 15 through 18 and 20 replace the gradually varied flow varibles h_G, $\bar{u}_G$, B_G, A_G and S_{fG} in Eqs. 13 and 14 to describe fluid motion when a small perturbation is imposed. The terms of equivalent powers of η are set equal and after several elementary algebraic manipulations presented in Appendix I, the shallow water equations can be combined to give:

$$\frac{\partial^2 h'}{\partial \xi \partial \eta} - \beta \left(\frac{\partial h'}{\partial \eta} \right)^2 + \gamma \frac{\partial h'}{\partial \eta} = 0 \tag{21}$$

in which for rectangular channels ($B = B'$ and $\frac{\partial B'}{\partial h'} = 0$), the coefficients β and γ are respectively:

$$\beta = \frac{3g}{c_o^2 + 2\bar{u}c_o + gh} \tag{22}$$

$$\gamma = \frac{gS}{\bar{u}^2} \left(1 - \frac{2c_o F^2}{\bar{u}}\right) \left[\frac{1}{2 + \frac{c_o}{\bar{u}} + \frac{\bar{u}}{c_o F^2}} \right] \tag{23}$$

The derivation presented in Appendix I improves the one given by Liggett (1975) since the wave celerity defined by Eq. 9 is not restricted to the relationship $c_o = \sqrt{gh}$. The coefficients β and γ are a function of the variables S, $\bar{u}$, c_o, h, F and g and Eqs. 22 and 23 reduce to the coefficients proposed by Liggett for the particular case when $c_o = \sqrt{gh}$.

The solution of Eq. 21 is:

$$\frac{\partial h'}{\partial \eta} = \frac{\varepsilon}{\frac{\beta\varepsilon}{\gamma} + e^{\gamma\xi}} \tag{24}$$

in which ε is a constant of integration along the longitudinal distance ξ. The critical distance ξ_c at which the wave breaks is assumed to occur when the water surface is vertical. Mathematically, this condition is obtained when the denominator of Eq. 24 is set equal to zero, or when:

$$\xi_c = \frac{1}{\gamma}\,\ell n(-\frac{\beta}{\gamma}\,\varepsilon) \quad . \tag{25}$$

After combining Eqs. 22, 23 and 25, the distance ξ_c can be written as follows:

$$\xi_c = \frac{h}{S}\,\{\Psi[\Phi + \ell n(\frac{S}{3\varepsilon})]\} \tag{26}$$

in which, $$\Psi = \left[\frac{F^2}{\frac{2c_oF^2}{\bar{u}} - 1}\right](2 + \frac{c_o}{\bar{u}} + \frac{\bar{u}}{c_oF^2}) \tag{27}$$

and,

$$\Phi = \ell n\ (2\,\frac{c_o^2F^2}{\bar{u}^2} - \frac{c_o}{\bar{u}}) \quad . \tag{28}$$

From Eq. 10, $c_o/\bar{u}$ can be written as a function of the Froude number for a given value of β_m. Taking $\beta_m = 1.2$ for laminar sheet flows, the variables Ψ and Φ from Eqs. 27 and 28 are dimensionless and unique functions of the Froude number as plotted in Fig. 1. For supercritical flows, Ψ has a nearly constant value of 2.0 while Φ increases

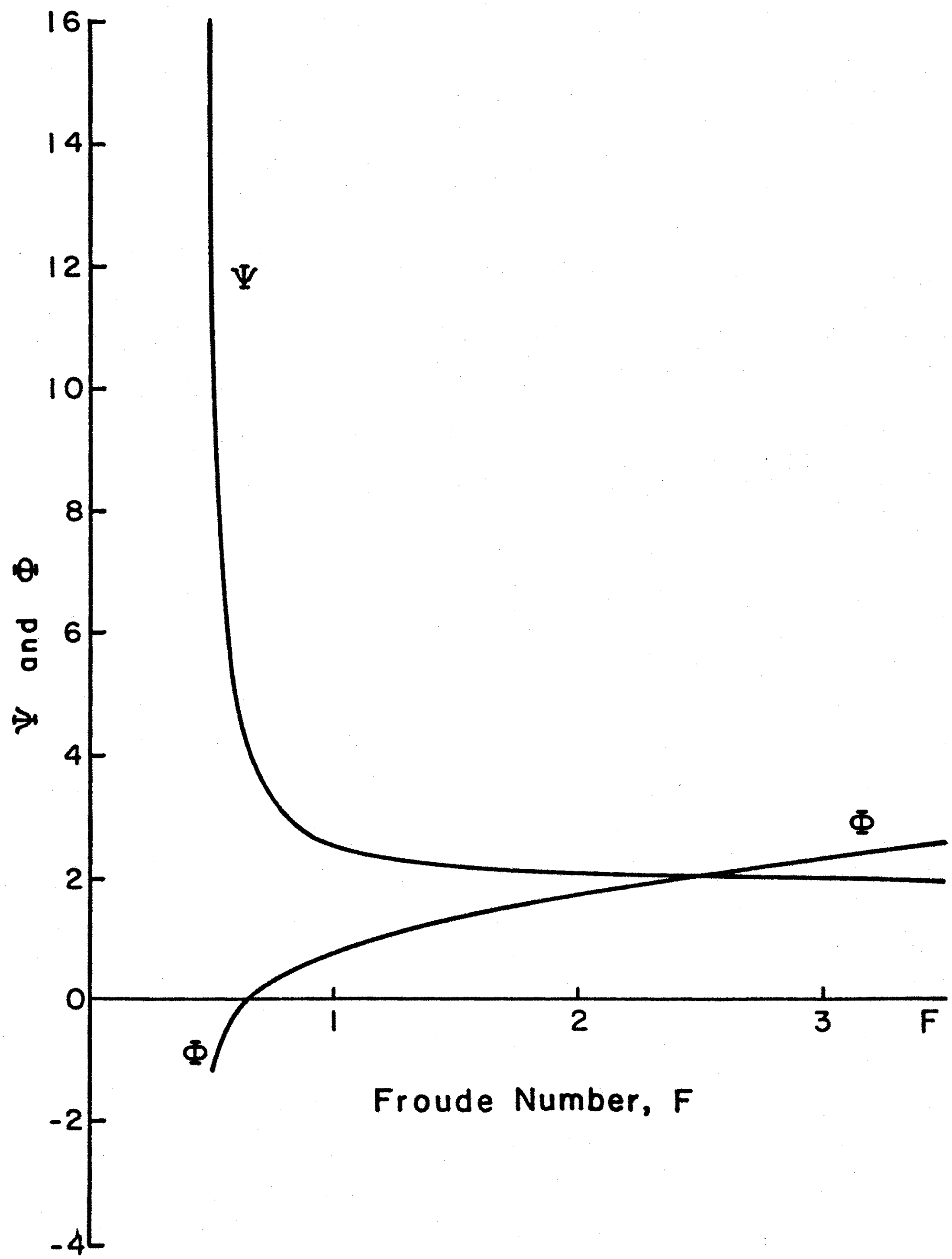

Fig. 1. Dimensionless variables Φ and Ψ as a function of Froude number.

gradually with the Froude number. It can also be demonstrated that over a fairly wide range of slopes the expression $\ln(\frac{S}{3\varepsilon})$ will be substantially constant. If Φ is small compared to $\ln(\frac{S}{3\varepsilon})$ then the following approximate relationship for ξ_c can be written:

$$\xi_c \cong D \frac{h}{S} \tag{29}$$

in which D is equivalent to the factor in braces in Eq. 26 and is approximately constant. Equations 25 and 29 represent alternate expressions for evaluating ξ_c, the latter expression being a simplified expression of the former for supercritical flows. The ability of Eqs. 25 and 29 to predict the distance ξ_c is evaluated with laboratory data described in the following section.

III. EXPERIMENTAL INVESTIGATION OF ROLL WAVES

Laboratory experiments were conducted in the Hydraulics Laboratory at the Engineering Research Center. The experiments determined laminar flow conditions which produced roll waves. Measured roll wave characteristics included the length required for their formation, wave frequency and wave celerity.

3.1 Laboratory Equipment and Experimental Procedure

A 0.21 m wide by 9.75 m long, rectangular flume constructed of plexiglass and supported by an aluminum I-beam was utilized for the experimental runs. A pump circulated water from a tailbox to the head end of the flume. The slope of the flume was adjusted with a screw jack. Discharge was controlled by a valve located on the discharge side of the pump. The range of flow conditions investigated were as follows:

Unit Discharge	6.5×10^{-5} to 5.5×10^{-4} m^2/sec
Channel Slope	1.5 to 4.0 percent
Water Temperature	20.0 to 24.0°C

Discharge was obtained using the volumetric method in which time and water volumes were measured with a stopwatch and a graduated cylinder. Channel slope was set using the screw jack and a slope scale which had been calibrated with a surveyor's level. Water temperature was measured using an electronic digital thermometer. Reynolds numbers were calculated using the unit discharge and the viscosity obtained from water temperature. The theoretical value of the friction parameter, K, was verified by measuring the surface velocity, u_s, of small buoyant particles (styrofoam and paper) under steady uniform sheet flow conditions. The friction factor, K, was calculated from measured u_s values using Eqs. 5 and 4:

$$K = \frac{8gq^2S}{\nu} \left(\frac{1.5}{u_s} \right)^3 \quad . \tag{30}$$

An average value of $K = 25.7$ was determined from the surface velocity measurements in the experimental flume. This result was considered sufficiently close to the theoretical value of $K = 24$ to justify its use in calculating the uniform flow depth, h, and the mean velocity, $\bar{u}$, from Eqs. 3 and 4.

For each of the 31 main experimental runs the flow conditions were given sufficient time to reach equilibrium before discharge measurements were made. Roll waves were noted by visual inspection when a well-defined breaking wave front could be observed across the entire width of the flume. The length ξ_c, for roll wave formation was estimated with the aid of reference marks at 0.61 m (2 ft) intervals along the transparent side walls of the flume. Consecutive reference marks which bounded the point where roll waves could first be observed were noted. The distance from the upstream end of the flume to the midpoint between the two noted reference marks was used to define the distance for roll wave formation ξ_c.

In addition to the formation length, roll wave period (frequency) and celerity were also measured. Wave period was determined by counting the number of flow surges over a given amount of time at the downstream end of the flume. Wave celerity was determined by timing the progress of 5 or more wave crests over a known distance and averaging the results.

All the data collected in this experimental investigation are presented in Appendix II. The main data base is composed of the first 31 experimental runs while the additional data (runs 32-57) were collected during a preliminary investigation.

3.2 Data Summary

A summary of the experimental data is presented in Table 1. The first 5 columns read as follows: slope, flow Reynolds number, wave celerity, wave period, and distance for roll wave formation. Wave celerity and period values represent the average of several measurements for each run. In Table 1, the parameters in columns 6 to 15 are calculated from columns 1 to 5 and will be discussed in the following section dealing with the analysis of experimental data.

3.3 Data analysis

The velocity, flow depth and Froude number were computed from the measured slope and Reynolds number using Eqs. 2, 3 and 4. These three variables are shown in Table 1 in columns 6, 7 and 8. The first part of this analysis of experimental data is focused on the evaluation of the wavelength, period and celerity.

3.3.1 Wavelength, period and celerity

In this section three important characteristics of roll waves are discussed: the wavelength, the period and the wave celerity. The wavelength can be evaluated from the wave celerity and the period. The observed values of the ratio $c/\sqrt{gh}$ have been plotted against the Froude number on Fig. 2a. The agreement with the theoretical relationship (Eq. 11 with $\beta_m = 1.2$) is excellent. Equation 11 can also be written as the ratio of the wave celerity to the mean flow velocity $\bar{u}$.

$$\frac{c}{\bar{u}} = \beta_m + \sqrt{\frac{1}{F^2} + \beta_m(\beta_m - 1)} \quad . \qquad (31)$$

For unstable flows ($F>0.5$), the ratio $c/\bar{u}$ calculated from Eq. 31 ($\beta_m = 1.2$) decreases from 3.26 to a minimum of 1.69 as shown in Fig. 2b.

Table 1. Data Summary

Run	S (1)	Re (2)	c m/s (3)	T s (4)	ξ_c m (5)	$\bar{u}$ m/s (6)	h mm (7)	F (8)	β m^{-1} (9)	γ m^{-1} (10)	$\gamma\xi_c$ (11)	$\ell n \varepsilon$ (12)	$\frac{gS\xi_c}{\bar{u}^2}$ (13)	D (14)	E mm (15)
1	0.040	335	0.46	1.33	0.91	0.24	1.37	2.11	176	-15.4	-14.0	-16.5	6.03	26.6	1.80
2	0.040	400	0.50	1.61	0.91	0.27	1.43	2.31	154	-14.2	-12.9	-15.3	4.74	25.4	1.71
3	0.040	500	0.57	1.96	0.91	0.32	1.55	2.58	124	-12.1	-11.0	-13.3	3.53	23.5	1.59
4	0.035	68	0.22	1.27	2.74	0.08	0.82	0.89	588	-21.3	-58.3	-61.3	147.30	-	-
5	0.035	95	0.26	1.28	1.52	0.10	0.94	1.05	441	-20.8	-31.7	-34.7	51.20	56.6	3.81
6	0.035	141	0.34	1.45	1.52	0.13	1.07	1.28	270	-21.8	-33.2	-35.7	30.50	49.7	3.35
7	0.035	188	0.34	1.32	2.13	0.16	1.19	1.48	289	-14.9	-31.8	-34.7	28.70	62.6	4.26
8	0.035	265	0.42	1.37	2.13	0.20	1.31	1.76	197	-14.7	-31.3	-33.9	18.40	56.9	3.81
9	0.035	380	0.43	1.19	2.74	0.26	1.49	2.11	222	-8.2	-22.4	-25.7	14.40	64.4	4.33
10	0.030	90	0.24	1.35	2.13	0.09	0.98	0.95	503	-16.8	-35.9	-39.3	72.70	-	-
11	0.030	122	0.25	1.43	2.13	0.11	1.07	1.10	487	-12.9	-27.5	-31.1	49.60	59.7	4.02
12	0.030	200	0.36	1.47	2.74	0.16	1.25	1.41	252	-12.5	-34.2	-37.2	33.10	65.8	4.36
13	0.030	260	0.41	1.25	2.13	0.19	1.37	1.61	202	-11.7	-24.9	-27.7	23.10	46.6	3.99
14	0.030	340	0.42	1.37	2.13	0.22	1.49	1.84	207	-9.7	-20.6	-23.7	16.30	42.9	3.66
15	0.030	460	0.48	1.35	1.52	0.28	1.68	2.15	174	-7.0	-10.6	-13.8	5.90	27.1	1.83
16	0.030	550	-	-	1.52	0.31	1.77	2.35	-	-	-	-	4.68	25.8	1.74

Table 1. (Continued)

Run	S (1)	Re (2)	c m/s (3)	T s (4)	ξ_c m (5)	$\bar{u}$ m/s (6)	h mm (7)	F (8)	β m^{-1} (9)	γ m^{-1} (10)	$\gamma\xi_c$ (11)	$\ell n \varepsilon$ (12)	$\frac{gS\xi_c}{\bar{u}^2}$ (13)	D (14)	E mm (15)
17	0.025	65	-	-	7.62	0.07	0.91	0.74	-	-	-	-	380.00	-	-
18	0.025	71	-	-	7.62	0.07	0.94	0.77	-	-	-	-	340.00	-	-
19	0.025	85	0.22	1.52	3.35	0.08	1.01	0.84	576	-12.9	-41.4	-45.2	118.00	-	-
20	0.025	104	0.24	1.52	3.35	0.10	1.07	0.93	497	-8.3	-27.6	-31.7	91.00	-	-
21	0.025	130	0.33	1.54	2.74	0.11	1.16	1.04	272	-15.1	-41.4	-44.3	54.70	59.1	3.96
22	0.025	200	0.34	1.61	2.74	0.15	1.34	1.29	276	-9.4	-25.7	-29.3	30.70	51.1	3.44
23	0.025	246	0.38	1.75	2.13	0.17	1.43	1.43	227	-9.5	-20.2	-23.3	18.20	37.2	2.50
24	0.025	320	0.44	1.52	1.52	0.20	1.55	1.63	175	-9.4	-14.3	-17.2	9.20	24.5	1.62
25	0.025	420	0.47	1.19	0.91	0.24	1.71	1.87	164	-7.5	-6.8	-9.9	3.83	13.3	0.88
26	0.025	530	0.50	1.15	1.52	0.28	1.86	2.10	155	-6.0	-9.2	-12.4	4.65	20.4	1.37
27	0.015	140	0.26	1.72	2.74	0.10	1.40	0.84	410	-4.1	-11.3	-15.9	41.60	-	-
28	0.015	173	0.27	1.33	2.13	0.11	1.52	0.93	393	-4.3	-9.2	-13.7	24.30	-	-
29	0.015	260	0.33	1.43	2.13	0.15	1.74	1.14	284	-3.6	-7.7	-12.0	14.20	18.4	1.25
30	0.015	320	0.40	1.23	2.13	0.17	1.86	1.26	197	-4.4	-9.4	-13.2	10.80	17.2	1.16
31	0.015	450	0.43	1.08	1.52	0.21	2.07	1.50	183	-3.6	-5.4	-9.4	5.90	11.0	0.88
Mean				1.41							-22.5	-25.7		38.5	2.67
Standard Deviation				0.20							13.1	13.0		18.5	1.28

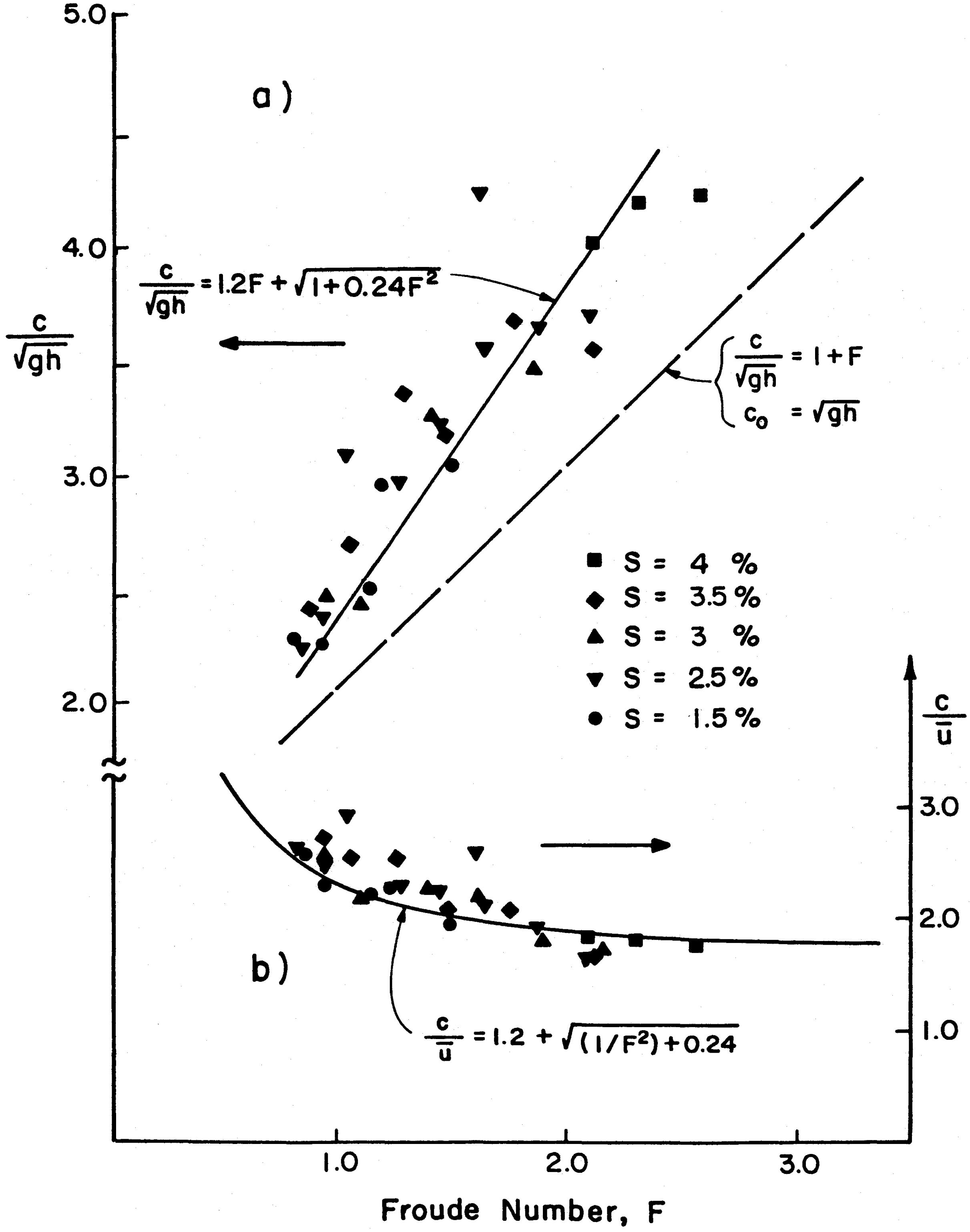

Fig. 2. Dimensionless celerity as a function of Froude number.

The measured wave periods T shown in Table 1 (column 4) were fairly constant with a mean value of $\bar{T} = 1.41$ seconds and a standard deviation of 0.20 seconds.

The wavelength L can be approximated by taking the product of the celerity c and the mean wave period. The following relationship for the wavelength is obtained from Eq. 8 with $\beta_m = 1.2$ and $\bar{T} = 1.41$:

$$L \cong c\bar{T} = 1.69\ \bar{u} + 1.41\ \sqrt{gh + 0.24\ \bar{u}^2} \tag{32}$$

The wavelength can also be written in the following dimensionless form, $L/\bar{u}T$. It is easily demonstrated from Eq. 31 with $\beta_m = 1.2$ that the dimensionless wavelength varies with the Froude number as follows:

$$\frac{L}{\bar{u}T} = 1.20 + \sqrt{\frac{1}{F^2} + 0.24} \tag{33}$$

This equation is more general than Eq. 32 since it depends on the wave period T as opposed to the mean value $\bar{T} = 1.41$ s used in Eq. 32.

3.3.2 Critical distance for the formation of roll waves

In Section 2.3.2, two equations were theoretically derived to define the critical distance ξ_c. Prediction of ξ_c from Eq. 25 requires evaluation of the parameter, ε, while Eq. 29, valid only for supercritical flows, requires the evaluation of the parameter D. In this section, both relationships are examined in the light of experimental data for laminar sheet flows.

In Eq. 25, the distance ξ_c is a function of β, γ, and ε. The parameters β and γ are computed from Eq. 22 and 23 and presented in Columns 9 and 10 in Table 1. The values of $\ell n\ \varepsilon$ calculated from Eq. 25 using measured values of ξ_c are presented in Column 12, Table 1. The values of $\ell n\ \varepsilon$ range from -61 to -9.4 with a mean value of -25.7. As suggested by Montuori (1963) and Liggett (1975), the values of $\ell n(\frac{-\beta\varepsilon}{\gamma})$ or $\gamma\xi_c$ were computed as shown in Table 1 (Col. 11).

Measured values of ξ_c were converted to the dimensionless parameter $gS\xi_c/\bar{u}^2$ in column 13 of Table 1 and plotted against the Froude number in Fig. 3. This figure clearly defines a region where roll waves were observed ($\xi > -35/\gamma$) and a region where roll waves were not completely developed ($\xi < -5/\gamma$). Between these limits exists a zone of uncertainty defined by $-35/\gamma < \xi_c < -5/\gamma$. This figure can be used to estimate the distance for the formation of roll waves from the parameter γ. The evaluation of γ from Eq. 23 is possible provided the variables S, $\bar{u}$, c_o and F are known.

If the flow is supercritical, the evaluation of ξ_c from Eq. 29 involves only the flow depth, slope and the coefficient D. It was demonstrated in Section 2.3.2 that D is substantially constant if ϕ is small compared to $\ln(S/3\varepsilon)$. This condition is satisfied for the range of data in this experimental study. The values of D tabulated in column 14 (Table 1) were computed from the experimental values of ξ_c, S and h using Eq. 29. The mean value for D is 38.5 with a standard deviation equal to 18.5. Equation 29 is therefore recommended to estimate ξ_c for supercritical flows, when depth and slope are known.

The flow depth in Eq. 29 can also be replaced by a function of the slope and the Reynolds number from Eq. 4:

$$\xi_c \cong E\,\frac{Re^{1/3}}{S^{4/3}} \tag{34}$$

in which, $$E = D\left(\frac{Kv^2}{8g}\right)^{1/3} \quad . \tag{35}$$

These relationships indicate that for the same slope and Reynolds number, the constant E, and therefore the critical distance, ξ_c, increases with increasing viscosity and surface roughness, K. The parameter E, has dimensions of length. Values of E from the experiments are tabulated in column 15 of Table 1. This parameter has a mean value of 2.67 mm and a coefficient of variation of 48 percent. Equation 34 is recommended for supercritical laminar sheet flows over smooth surfaces. It should be noted that the mean values of the coefficients, D = 38.5 and E = 2.67 mm, apply to the range of conditions used in this experimental study. These values may not be applicable beyond this range.

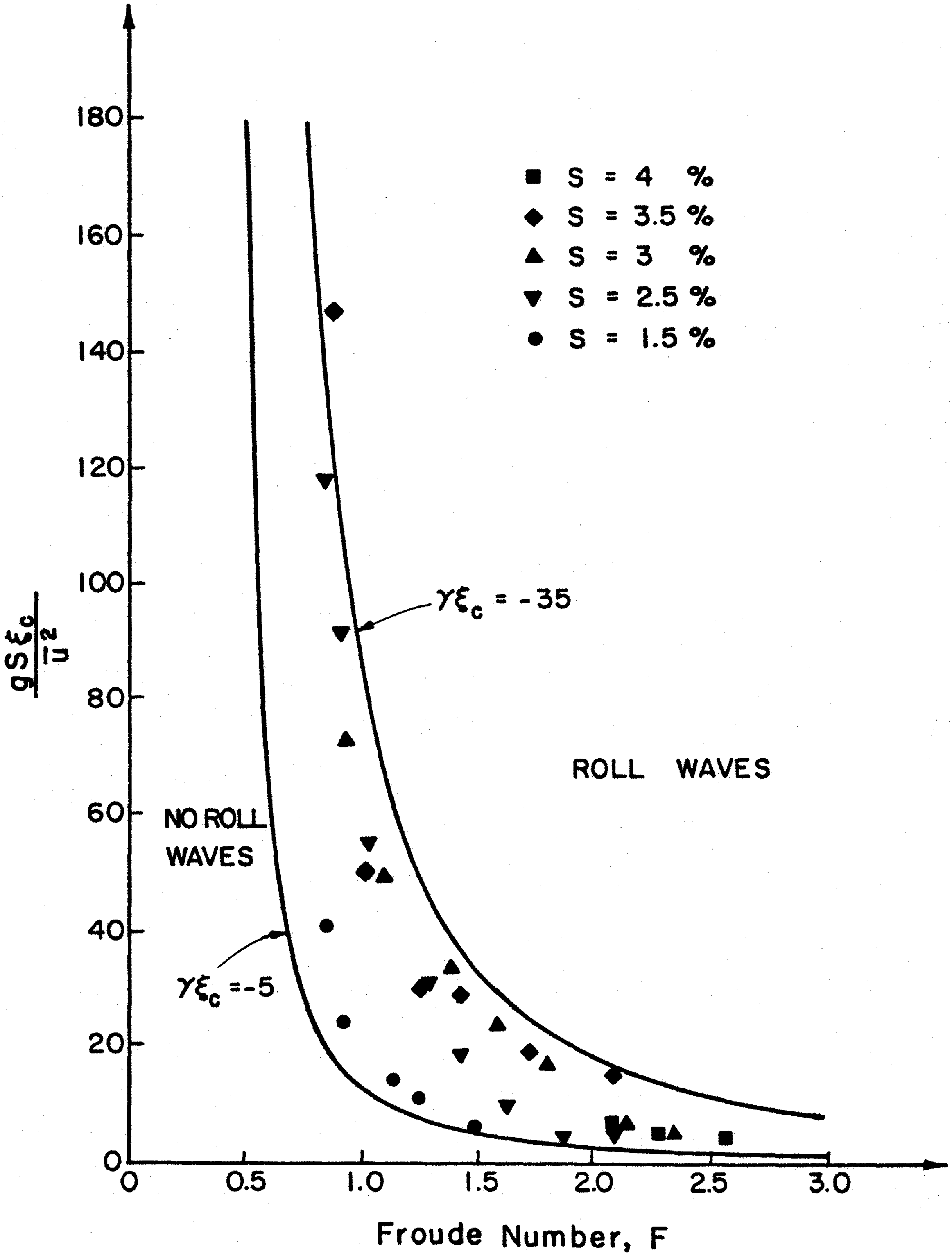

Fig. 3. Dimensionless critical distance as a function of Froude number.

IV. SUMMARY AND CONCLUSIONS

The formation of roll waves in laminar sheet flows is examined using a theoretical analysis supported by experimental data. Previous investigations indicate that roll waves are theoretically possible in laminar sheet flows at Froude numbers as low as 0.50. The existence of roll waves at Froude numbers near the lower limit is difficult to verify experimentally because of the extreme channel lengths required. However, in this study, roll waves were observed in laminar, subcritical flow at a Froude number as low as 0.74.

The parabolic velocity distribution in laminar sheet flows implies that the momentum correction factor is larger than unity ($\beta_m = 1.2$). Since the celerity of roll waves depends on the momentum correction factor, the relationship $c = \bar{u} + \sqrt{gh}$ suggested in previous studies is not applicable to laminar sheet flows. The recommended relationship for c uses the momentum correction factor, β_m, instead of an empirical coefficient proposed by Arsenishvili. The proposed relationship reduces to $c_o = \sqrt{gh}$ when $\beta_m = 1$ and is in good agreement with the measured celerities of roll waves when $\beta_m = 1.2$ as shown in Fig. 2. The measured periods of roll waves remained fairly constant in the experimental study at $\bar{T} = 1.41$ second. The wavelength is shown to vary between $1.69\ \bar{u}\bar{T} < L < 3.26\ \bar{u}\bar{T}$.

The linearized derivation by Liggett (1975) of the length, ξ_c, required for the formation of roll waves has been modified because experimental evidence demonstrates that the assumption $c_o = \sqrt{gh}$ does not hold true for laminar sheet flows. The modified derivation gives more general expressions for the coefficients β and γ which reduce to those proposed by Liggett when $\beta_m = 1$. The results indicate that the length

ξ_c is a function of several flow variables and a constant of integration ε which could be calculated from experiments. Though the parameter $\ln \varepsilon$ varies from -61 to -9.4, the dimensionless distance shown in Fig. 3 displays a relationship to the Froude number similar to the one found by Montuori (1963) for turbulent flows. These results show that in laminar flows, the distance ξ_c is inversely proportional to γ. For supercritical flows, ξ_c is proportional to the ratio of flow depth and slope. Alternatively, an equivalent function of Reynolds number and slope may be used.

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APPENDIX I

Derivation of the Coefficients β and γ for Laminar Sheet Flow

A. Coordinate Transformation of the Equations of Continuity and Momentum

The equations of continuity and momentum for shallow flow in a prismatic channel (Liggett, 1975a) are given by:

$$B_G \frac{\partial h_G}{\partial t} + \bar{u}_G B_G \frac{\partial h_G}{\partial x} + A_G \frac{\partial u_G}{\partial x} = 0 \tag{1.1}$$

and

$$\frac{\partial \bar{u}_G}{\partial t} + \frac{\bar{u}_G \partial \bar{u}_G}{\partial x} + g \frac{\partial h_G}{\partial x} = g(S - S_{fG}) \tag{1.2}$$

in which x is the downslope distance and the subscript G designates gradually varied flow. We now define an alternative set of coordinates:

$$\xi = x \tag{1.3}$$

$$\eta = (\bar{u} + c_o)t - x \tag{1.4}$$

in which $\bar{u}$ is the uniform flow velocity.

The variable, ξ, represents the distance between the origin of the fixed coordinate system and one point moving with characteristic speed, $(\bar{u} + c_o)$. The variable, η, defines position in the moving coordinate system.

Differential operators given by the chain rule are:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} \tag{1.5}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} \tag{1.6}$$

From 1.3 and 1.4:

$$\frac{\partial \xi}{\partial x} = 1 \tag{1.7}$$

$$\frac{\partial \eta}{\partial x} = -1 \tag{1.8}$$

$$\frac{\partial \eta}{\partial t} = (\bar{u} + c_o) \tag{1.9}$$

$$\frac{\partial \xi}{\partial t} = 0 \tag{1.10}$$

Substituting 1.7 and 1.8 into 1.5, and 1.9 and 1.10 into 1.6 results in:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} - \frac{\partial}{\partial \eta} \tag{1.11}$$

$$\frac{\partial}{\partial t} = (\bar{u} + c_o) \frac{\partial}{\partial \eta} \tag{1.12}$$

Equations 1.8 and 1.11 show that despite the dependence of η and ξ evident in Eq. 14, $\partial\eta/\partial\xi = 0$ in the transformed coordinate system.

Using 1.11 and 1.12, 1.1 and 1.2 are transformed into:

$$B_G(\bar{u} + c_o) \frac{\partial h_G}{\partial \eta} + \bar{u}_G B_G\left(\frac{\partial h_G}{\partial \xi} - \frac{\partial h_G}{\partial \eta}\right) + A_G\left(\frac{\partial \bar{u}_G}{\partial \xi} - \frac{\partial \bar{u}_G}{\partial \eta}\right) = 0 \tag{1.13}$$

and

$$(\bar{u} + c_o) \frac{\partial \bar{u}_G}{\partial \eta} + \bar{u}_G\left(\frac{\partial \bar{u}_G}{\partial \xi} - \frac{\partial \bar{u}_G}{\partial \eta}\right) + g\left(\frac{\partial h_G}{\partial \xi} - \frac{\partial h_G}{\partial \eta}\right) = g(S - S_{fG}) \tag{1.14}$$

in which ξ and η are the new independent variables. Note that Eqs. 1.13 and 1.14 are identical to Eqs. 13 and 14 in the text.

B. Perturbation Analysis of the Transformed Equations of Continuity and Momentum

Given a small perturbation, the flow variables h_G, $\bar{u}_G$, B_G and A_G in Eqs. 13 and 14 (1.13 and 1.14) are replaced by the perturbed variables, h', $\bar{u}'$, B' and A' as defined in Eqs. 15 through 18. For the sake of convenience, the partial derivatives of the perturbed variables are written using the following notation:

$$h^* = \frac{\partial h'}{\partial \eta} \tag{1.15}$$

$$h^{**} = \frac{\partial h^*}{\partial \eta} \tag{1.16}$$

$$\bar{u}^* = \frac{\partial \bar{u}'}{\partial \eta} \tag{1.17}$$

$$\bar{u}^{**} = \frac{\partial^2 \bar{u}'}{\partial \eta^2} \tag{1.18}$$

$$B^+ = \frac{\partial B'}{\partial h'} \tag{1.19}$$

$$B = \frac{\partial A'}{\partial h'} \tag{1.20}$$

Using Eqs. 1.15 through 1.20, and Eqs. 15 through 18, and 20 in the text, first order approximations for variables in the perturbed continuity and momentum equations can be written as follows:

$$\bar{u}' \cong \bar{u} + \bar{u}^*\eta \tag{1.21}$$

$$B' \cong B + B^+ h^*\eta \tag{1.23}$$

$$A' \cong A + Bh^*\eta \tag{1.24}$$

$$S_f' = S[1 + (\frac{\bar{u}^*}{\bar{u}} - \frac{2h^*}{h})\ \eta] \tag{1.25}$$

$$\frac{\partial \bar{u}}{\partial \eta} \cong \bar{u}^* + 2\bar{u}^{**}\eta \tag{1.26}$$

$$\frac{\partial h'}{\partial \eta} \cong h^* + 2h^{**}\eta \tag{1.27}$$

$$\frac{\partial h'}{\partial \xi} \cong \frac{\partial h^*}{\partial \xi}\ \eta \tag{1.28}$$

$$\frac{\partial \bar{u}'}{\partial \xi} \cong \frac{\partial \bar{u}^*}{\partial \xi}\ \eta \tag{1.29}$$

Assuming a perturbation, the continuity and momentum equations are given by:

$$B'(\bar{u} + c_o)\,\frac{\partial h'}{\partial \eta} + \bar{u}'B'\left(\frac{\partial h'}{\partial \xi} - \frac{\partial h'}{\partial \eta}\right) + A'\left(\frac{\partial \bar{u}'}{\partial \xi} - \frac{\partial \bar{u}'}{\partial \eta}\right) = 0 \qquad (1.30)$$

and

$$(\bar{u} + c_o)\,\frac{\partial \bar{u}'}{\partial \eta} + \bar{u}'\left(\frac{\partial \bar{u}'}{\partial \xi} - \frac{\partial \bar{u}'}{\partial \eta}\right) + g\left(\frac{\partial h'}{\partial \xi} - \frac{\partial h'}{\partial \eta}\right) = g(S - S_f') \qquad (1.31)$$

Note that the mean velocity in the term $(\bar{u} + c_o)$ remains unaffected by the perturbation because this term represents the characteristic speed which is assumed to be constant.

Substituting the right hand sides of Eqs. 1.21 through 1.29 into Eqs. 1.30 and 1.31, dropping second order and higher terms and simplifying results in:

$$c_o[B(h^* + 2h^{**}\eta) + B^+h^{*2}\eta] + \bar{u}B\,\frac{\partial h^*\eta}{\partial \xi}$$

$$- 2B\bar{u}^*h^*\eta + A\left(\frac{\partial \bar{u}^*\eta}{\partial \xi} - \bar{u}^* - 2\bar{u}^{**}\eta\right) \cong 0 \qquad (1.32)$$

$$c_o(\bar{u}^* + 2\bar{u}^{**}\eta) + \bar{u}\,\frac{\partial \bar{u}^*\eta}{\partial \xi} - \bar{u}^{*2}\eta + g\left(\frac{\partial h^*\eta}{\partial \xi} - h^* - 2h^{**}\eta\right)$$

$$= -S\ g\left(\frac{\bar{u}^*}{\bar{u}} - \frac{2h^*}{h}\right)\eta \qquad (1.33)$$

As η approaches zero, zero and first order terms in η on the left hand sides of Eqs. 1.32 and 1.33 can each independently be set equal to the respective right hand sides and rearranged to yield

$$Bc_oh^* \cong A\bar{u}^* \qquad (1.34)$$

$$c_o\bar{u}^* \cong gh^* \qquad (1.35)$$

$$2Bc_oh^{**} + B^+c_oh^{*2} + B\bar{u}\,\frac{\partial h^*}{\partial \xi}$$

$$-2B\bar{u}^*h^* + A\left(\frac{\partial \bar{u}^*}{\partial \xi} - 2\bar{u}^{**}\right) \cong 0 \qquad (1.36)$$

$$2c_o\bar{u}^{**} - \bar{u}^*\bar{u}^* + \bar{u}\,\frac{\partial\bar{u}^*}{\partial\xi} + g\left(\frac{\partial h^*}{\partial\xi} - 2h^{**}\right)$$

$$+ Sg\left(\frac{\bar{u}^*}{\bar{u}} - \frac{2h^*}{h}\right) \cong 0 \tag{1.37}$$

After assuming a rectangular channel ($B^+ = 0$ and $A = Bh$), Eq. 1.36 is divided by Bc_o and Eq. 1.37 by g to yield respectively:

$$2h^{**} + \frac{\bar{u}}{c_o}\frac{\partial h^*}{\partial\xi} - 2\,\frac{\bar{u}^*h^*}{c_o} + \frac{h}{c_o}\left(\frac{\partial\bar{u}^*}{\partial\xi} - 2\bar{u}^{**}\right) \cong 0 \tag{1.38}$$

and,

$$2\,\frac{c_o\bar{u}^{**}}{g} - \frac{\bar{u}^*\bar{u}^*}{g} + \frac{\bar{u}}{g}\frac{\partial\bar{u}^*}{\partial\xi} + \frac{\partial h^*}{\partial\xi} - 2h^{**} + S\left(\frac{\bar{u}^*}{\bar{u}} - \frac{2h^*}{h}\right) \cong 0 \quad . \tag{1.39}$$

Adding Eqs. 1.38 and 1.39 cancels the term h^{**} as follows:

$$\frac{\partial h^*}{\partial\xi}\left(1 + \frac{\bar{u}}{c_o}\right) + \frac{\partial\bar{u}^*}{\partial\xi}\left(\frac{\bar{u}}{g} + \frac{h}{c_o}\right) - \frac{2\bar{u}^*}{c_o}h^* - \frac{\bar{u}^*\bar{u}^*}{g}$$

$$+ \bar{u}^{**}\left(\frac{2c_o}{g} - \frac{2h}{c_o}\right) + S\left(\frac{\bar{u}^*}{\bar{u}} - \frac{2h^*}{h}\right) \cong 0 \tag{1.40}$$

Equation 1.35 is solved for u^* and differentiated with respect to η to obtain an expression for u^{**} as follows:

$$u^* \cong \frac{gh^*}{c_o} \tag{1.41}$$

$$u^{**} \cong \frac{g}{c_o}\frac{\partial h^*}{\partial\eta} \tag{1.42}$$

The right hand sides of Eqs. 1.41 and 1.42 are substituted into Eq. 1.40 to yield:

$$\frac{\partial h^*}{\partial \xi}\left(1 + \frac{2\bar{u}}{c_o} + \frac{gh}{c_o^2}\right) - \frac{3g}{c_o^2} h^*h^* + \frac{h^*gS}{\bar{u}c_o}\left(1 - \frac{2c_oF^2}{\bar{u}}\right)$$

$$\frac{\partial h^*}{\partial \eta}\left(2 - \frac{2gh}{c_o^2}\right) \cong 0 \tag{1.43}$$

Assuming that the partial derivatives in the first and last terms of Eq. 1.43 are of similar magnitude while noting that the coefficient of the last derivative is always much smaller than the first, the last term is dropped to give:

$$\frac{\partial h^*}{\partial \xi}\left(1 + \frac{2\bar{u}}{c_o} + \frac{gh}{c_o^2}\right) - \frac{3g}{c_o^2} h^*h^* + h^* \frac{gS}{\bar{u}c_o}\left(1 - \frac{2c_oF^2}{\bar{u}}\right) \cong 0 \tag{1.44}$$

This equation is in the form:

$$\frac{\partial h^*}{\partial \xi} - \beta\, h^*h^* + \gamma h^* = 0 \tag{1.45}$$

$$\beta = \frac{3g}{c_o^2 + 2\bar{u}c_o + gh} \tag{1.46}$$

$$\gamma = \frac{gS}{\bar{u}}\left(1 - \frac{2c_oF^2}{\bar{u}}\right)\left(\frac{c_o}{c_o^2+2c_o\bar{u}+gh}\right) . \tag{1.47}$$

These two relationships for β and γ reduce to those found by Liggett (1975) for the particular case where $c_o = \sqrt{gh}$.

APPENDIX II

Experimental Data

Date	Run	Slope	Water Temp. °C	Unit Discharge cm^2/sec	Roll Wave Formation m		ΔT_c[3] sec	ΔT_p[4] sec
10/19/84	1	0.040	20.0	3.33	0.61	No[1]	3.2	27.0
					1.22	Yes[2]	3.4	25.0
							3.2	28.0
							3.3	
							3.3	
10/19/84	2	0.040	20.0	3.97	0.61	No	3.0	31.8
					1.22	Yes	3.0	31.4
							3.2	33.6
							3.0	
							3.0	
10/19/84	3	0.040	20.0	4.97	0.61	No	2.6	40.6
					1.22	Yes	3.0	42.4
							2.6	35.8
							2.8	
							2.6	
							2.8	
							2.8	
							2.8	
							2.9	
							2.4	
							3.0	
							2.9	
10/19/84	4	0.035	20.0	0.68	2.44	No	7.0	25.2
					3.05	Yes	6.8	25.2
							7.4	23.6
							7.2	26.8
							6.9	
10/19/84	5	0.035	20.0	0.94	1.83	No	5.6	25.6
					2.44	Yes	6.6	24.6
							5.6	26.6
							6.6	
							5.4	
10/19/84	6	0.035	20.0	1.40	1.22	No	4.4	28.2
					1.83	Yes	4.2	32.0
							5.0	26.2
							4.2	
							4.5	
10/19/84	7	0.035	20.0	1.87	1.83	No	4.6	25.0
					2.44	Yes	4.2	27.8
							4.4	
							4.6	
							4.6	

[1]No roll waves observed at this length.
[2]Roll waves observed at this length.
[3]Time interval for wave crest to travel 1.52 m.
[4]Time interval for 20 wave crests to pass a fixed point.

Date	Run	Slope	Water Temp. °C	Unit Discharge cm^2/sec	Roll Wave Formation m		ΔT_c [3] sec	ΔT_p [4] sec
10/19/84	8	0.035	20.0	2.63	1.83	No	3.5	28.4
					2.44	Yes	3.1	27.8
							3.8	26.0
							4.0	
							3.8	
10/19/84	9	0.035	20.0	3.77	2.44	No	4.0	22.6
					3.05	Yes	3.3	25.1
							3.3	24.0
							3.5	
							3.6	
							3.8	
10/18/84	10	0.030	20.0	0.89	1.83	No	6.2	27.4
					2.4	Yes	6.4	25.4
							6.5	28.0
							6.4	
							6.4	
10/18/84	11	0.030	20.0	1.21	2.44	No	6.2	35.2
					3.05	Yes	5.6	25.4
							6.6	25.8
							6.4	26.6
							6.2	30.0
10/18/84	12	0.030	2.0	1.99	2.44	No	4.0	29.0
					3.05	Yes	4.5	30.2
							4.0	29.0
							4.6	
							4.1	
10/18/84	13	0.030	20.0	2.58	2.44	No	3.7	25.2
					3.05	Yes	3.5	24.4
							3.8	25.6
							4.0	
							3.5	
10/18/84	14	0.030	20.0	3.37	2.44	No	3.8	28.0
					3.05	Yes	3.8	26.6
							3.6	28.0
							3.6	
							3.4	
10/18/84	15	0.030	20.0	4.57	1.22	No	3.2	27.4
					1.83	Yes	3.2	28.0
							3.4	25.2
							3.0	
							3.1	
10/18/84	16	0.030	20.0	5.46	1.22	No	-	-
					1.83	Yes		
10/18/84	17	0.025	20.0	0.65	7.32	No	-	-
					7.93	Yes		
10/18/84	18	0.025	20.0	0.71	7.32	No		
					7.92	Yes		

Date	Run	Slope	Water Temp. °C	Unit Discharge cm^2/sec	Roll Wave Formation m		ΔT_c[3] sec	ΔT_p[4] sec
10/18/84	19	0.025	20.0	0.84	3.05	No	6.8	31.2
					3.66	Yes	6.8	29.2
							7.0	29.8
							6.8	
							7.2	
10/18/84	20	0.025	20.0	1.03	3.05	No	6.4	30.6
					3.66	Yes	6.4	28.6
							6.2	28.4
							6.2	34.2
							6.2	30.0
10/19/84	21	0.025	20.0	1.29	2.44	No	4.8	32.0
					3.05	Yes	4.6	32.4
							4.2	27.6
							5.0	31.6
							4.6	
10/19/84	22	0.025	20.0	1.99	2.44	No	4.8	34.4
					3.05	Yes	4.4	31.0
							4.4	31.4
							4.6	
							4.4	
10/19/84	23	0.025	20.0	2.44	1.83	No	3.8	34.4
					2.44	Yes	4.0	36.0
							4.2	34.6
							4.0	
							4.0	
10/19/84	24	0.025	20.0	3.17	1.22	No	3.7	30.0
					1.83	Yes	3.0	28.8
							3.4	31.7
							3.6	30.4
							3.6	
							3.6	
10/19/84	25	0.025	20.0	4.17	0.61	No	3.2	24.2
					1.22	Yes	3.8	23.4
							3.0	23.6
							3.0	
							2.8	
							2.8	
							3.4	
							3.4	
							3.0	
							3.4	
							3.4	
							3.4	
10/19/84	26	0.025	20.0	5.26	1.22	No	3.0	23.0
					1.83	Yes	3.0	23.0
							3.0	22.6
							3.2	
							3.0	

Date	Run	Slope	Water Temp. °C	Unit Discharge cm^2/sec	Roll Wave Formation m		ΔT_c[3] sec	ΔT_p[4] sec
10/19/84	27	0.015	20.0	1.39	2.44	No	5.2	36.4
					3.05	Yes	6.2	29.8
							5.6	36.4
							6.0	
							6.0	
							6.0	
10/19/84	28	0.015	20.0	1.72	1.83	No	5.6	27.6
					2.44	Yes	5.8	26.4
							5.4	21.8
							5.6	31.2
							5.6	
10/19/84	29	0.015	20.0	2.58	1.83	No	4.4	28.6
					2.44	Yes	4.6	29.0
							5.0	28.6
							4.4	
							4.8	
							4.8	
10/19/84	30	0.015	20.0	3.18	1.83	No	4.0	25.4
					2.44	Yes	3.8	25.0
							3.8	23.0
							3.8	
							3.8	
10/19/84	31	0.015	20.0	4.47	1.22	No	3.8	20.2
					1.83	Yes	3.4	22.2
							3.4	22.2
							3.4	
							3.6	
10/09/84	32	0.040	22.8	0.36	-	No	-	-
10/09/84	33	0.040	22.8	0.59	-	Yes	-	-
10/09/84	34	0.040	22.8	0.72	-	Yes	-	-
10/09/84	35	0.040	22.8	0.89	-	Yes	-	-
10/09/84	36	0.040	22.8	1.17	-	Yes	-	-
10/09/84	37	0.040	22.8	1.79	-	Yes	-	-
10/09/84	38	0.040	22.8	2.45	-	Yes	-	-
10/09/84	39	0.040	22.8	3.21	-	Yes	-	-
10/09/84	40	0.040	22.8	4.21	-	Yes	-	-
10/09/84	41	0.040	22.8	5.58	-	No	-	-
10/09/84	42	0.020	22.8	0.39	-	No	-	-
10/09/84	43	0.020	22.8	0.95	-	Yes	-	-
10/09/84	44	0.020	22.8	1.26	-	Yes	-	-
10/09/84	45	0.020	22.8	1.89	-	Yes	-	-
10/09/84	46	0.020	22.8	2.11	-	Yes	-	-
10/09/84	47	0.020	22.8	2.74	-	Yes	-	-

*Measurement not taken. For runs 32-57, "Yes" indicates presence of roll waves before end of flume. "No" indicates absence of roll waves along entire length of flume.

Date	Run	Slope	Water Temp. °C	Unit Discharge cm^2/sec	Roll Wave Formation m		ΔT_c [3] sec	ΔT_p [4] sec
10/09/84	48	0.020	22.8	3.84	-	Yes	-	-
10/09/84	49	0.020	22.8	5.26	-	Yes	-	-
10/09/84	50	0.010	22.8	0.41	-	No	-	-
10/09/84	51	0.010	22.8	0.55	-	No	-	-
10/09/84	52	0.010	22.8	1.25	-	No	-	-
10/09/84	53	0.010	22.8	1.36	-	No	-	-
10/09/84	54	0.010	22.8	2.69	-	No	-	-
10/09/84	55	0.010	22.8	3.15	-	No	-	-
10/09/84	56	0.010	22.8	4.21	-	Yes	-	-
10/09/84	57	0.010	22.8	4.89	-	Yes	-	-