Notes.
Symbols © and ® in Fig 4 represent approaching and receding shear force vectors. Bending and twisting moments in Fig 5 represented by right hand vectors.
### Load Formulas Required for Arch Computations

<table>
<thead>
<tr>
<th>FORMULA</th>
<th>RADIAL LOADS</th>
<th>TANGENTIAL LOADS</th>
<th>TWIST LOADS</th>
</tr>
</thead>
<tbody>
<tr>
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<td>TRIANGULAR</td>
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</tr>
<tr>
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<td>$\frac{d}{2} \phi , d\phi$</td>
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</table>

The following notations are used:

- $p$: Radial load applied to arch at extrados.
- $p_\phi$: Rate of change of arch thrust with respect to length along the arch centerline.
- $m$: A factor which when multiplied by $\phi$ gives the radial load at the extrados at $\phi$.
- $m_\phi$: A factor which when multiplied by $\phi$ gives the rate of change of the tangential load at $\phi$.
- $n$: Rate of change of the moment load with respect to length along the centerline.
- $n_\phi$: A factor which when multiplied by $\phi$ gives the rate of change of moment load with respect to length along the arch centerline at $\phi$.
- $h_\phi$, $v_\phi$, $m_\phi$: Thrust, shear and moment due to external loads acting alone.

**Arch Load**

**Physical Dimensions of Quantities**

- $p$: lb/ft$^2$
- $p_\phi$: lb/ft$^2$ per ft$^2$
- $m$: lb/ft$^2$
- $m_\phi$: lb/ft$^2$ per ft$^2$
- $n$: lb/ft$^2$
- $n_\phi$: lb/ft$^2$ per ft$^2$

**Arch at Extrados**

**Physical Dimensions**

- $h_\phi$, $v_\phi$, $m_\phi$: Thrust, shear and moment due to external loads acting alone.

**DEPARTMENT OF THE INTERIOR**

**BUREAU OF RECLAMATION**

**DENVER OFFICE**

**LOAD FORMULAS REQUIRED FOR ARCH COMPUTATIONS**

**Drawn, R.E.G., Submitted, Approved**

**DENVER, COLO., JAN. 11, 1950**

205-D-98
<table>
<thead>
<tr>
<th>FORMULA</th>
<th>RADIAL LOADS</th>
<th>TANGENTIAL LOADS</th>
<th>TWIST LOADS</th>
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</tr>
</tbody>
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**PHYSICAL DIMENSIONS OF QUANTITIES**

- \( p \): Radial load applied to arch at extrados.
- \( p_r \): Rate of change of arch thrust with respect to length along the centerline.
- \( m \): A factor which when multiplied by \( \phi \) gives the radial load at the extrados at \( \phi \).
- \( m_r \): A factor which when multiplied by \( \phi \) gives the rate of change of the tangential load at \( \phi \).
- \( n \): Rate of change of the moment load with respect to length along the centerline.
- \( n_r \): A factor which when multiplied by \( \phi \) gives the rate of change of moment load with respect to length along the arch centerline at \( \phi \).
- \( h, \psi, m \): Thrust, shear, and moment due to external loads acting alone.

**NOTATION**

- \( \phi \): Angle of load with respect to length along the arch centerline.
- \( \phi_0 \): Angle of load with respect to length along the arch centerline at \( \phi \).
- \( \phi_\phi \): Angle of load with respect to length along the arch centerline at \( \phi_0 \).
- \( \phi_\psi \): Angle of load with respect to length along the arch centerline at \( \phi_\phi \).
- \( \phi_\mu \): Angle of load with respect to length along the arch centerline at \( \phi_\psi \).

**DEPARTMENT OF THE INTERIOR**

**BUREAU OF RECLAMATION**

**DENVER OFFICE**

**LOAD FORMULAS REQUIRED FOR ARCH COMPUTATIONS**

**DRAWN:** R.G. **SUBMITTED:** Robert E. Johnson

**TRACED:** R.G. **RECOMMENDED:** James E. Nordyke

**CHECKED:** J.W. **APPROVED:** J.E. January 11, 1950

**DENVER, COLO., JAN. 11, 1950** 205-D-98
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### RADIAL LOADS

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### TANGENTIAL LOADS

#### UNIFORM TRIANGULAR

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#### TWIST LOADS

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</table>

### FORMULAE FOR LOAD FORCES AND MOVEMENTS FOR CIRCULAR ARCH ANALYSIS

- **NOTATION**
  - $M_L$ = Moment due to external load
  - $H_L$ = Thrust due to external load
  - $V_L$ = Shear due to external load
  - For radial load, $P$ = lb per sq. ft.
  - For tangential load, $P$ = lb per sq. ft.
  - For twist load, $P$ = ft-lb per sq. ft.

See also Drawings 205-D-1649 and 1650.

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This drawing is a reproduction of Denver Office Drawing 205-D-98 with certain additions, revisions and changes of notation. It does not supersede that drawing.

**NOTE:** Formula = Trigonometric part x multiplier.
Memorandum to Chief Designing Engineer

By: Robert E. Glover, Engineer

Subject: Application of Dr. Vogt's abutment constants to computation of the effect of abutment deformation on the stresses in an arch dam.

Introductory Note

In computations made in the past it has been customary to apply Dr. Vogt's formulas as though the arch terminated at a vertical abutment and the cantilever rested on a horizontal base. The conditions assumed for the arch and cantilever elements could not, of course, be fulfilled simultaneously, and while this simplification may be permissible for thin dams, its validity is open to question when the arches are short and thick and the abutment deformation becomes a large part of the total deflection. The purpose of the formulas and methods presented on the following pages is to provide a means for bringing the computations into closer agreement with the facts.

Rotations Due to Moments

The bending moment applied in a plane normal to the plane of the abutment may be computed by means of the following expression

\[ M = M_0 \cos^2 \theta + M_s \sin^2 \theta - 2 M_0 \sin \theta \cos \theta. \]
where the notations have the following significance:

\( \psi \) is the angle the plane of the abutment makes with the vertical.

\( M_T \), the twisting moment per unit of height, or unit of length along the center line of the arch.

\( M_a \), the bending moment in the arch per unit of height of dam.

\( M_c \), the bending moment in the cantilever per unit length of center line.

\( M_o \), the moment per unit length of abutment.

\( M_a \) and \( M_c \) are considered positive if they produce compression at the upstream face. \( M_T \) is positive if it tends to turn the part of the cantilever below the point of application in a clockwise direction. The cantilever is assumed to be viewed from above. The angle \( \psi \) is considered positive if measured in a counter-clockwise direction as viewed by an observer downstream from the dam. A positive \( M \) applied to the abutment tends to increase the compression against the abutment at the upstream face.

If \( \alpha \) is the rotation of the abutment produced by the application of unit moment per unit of length of abutment in a plane normal to the plane of the abutment, the following relations may be deduced:

1. Rotation of the arch in the plane of the arch due to abutment deformation by moment.

Rotation due to bending moment in the arch

\[
\frac{M_a \alpha \cos \psi}{2}
\]  

(2)
Rotation due to bending moment in the cantilever

\[ M_{\alpha 0} \sin^2 \psi \cos \psi. \quad (3) \]

Rotation due to twist

\[ -2 M_{\alpha 0} \sin \psi \cos \psi. \quad (4) \]

2. Rotation of the cantilever in the plane of the cantilever due to abutment deformation by moment.

Rotation due to bending moment in the arch

\[ M_{\alpha x} \cos^2 \psi \sin \psi. \quad (5) \]

Rotation due to bending moment in the cantilever

\[ M_{\alpha x} \sin^3 \psi. \quad (6) \]

Rotation due to twist

\[ -2 M_{\alpha x} \sin^2 \psi \cos \psi. \quad (7) \]

3. Rotation of the cantilever in the plane of the arch due to abutment deformation by moment must agree with the rotation of the arch in the plane of the arch.

Note that the rotation of the abutment in the plane of the abutment due to twist is neglected.
Deformations Due to Arch Thrust,
Tangential Shear, and Cantilever Weight.

The thrust applied normal to the surface of the abutment may be obtained from the expression

\[ H_n = H_a \cos \gamma + H_v \sin \gamma - 2V_T \sin \gamma \cos \gamma. \]  

(8)

where the notations have the following significance:

- \( H_a \) = the arch thrust at the abutment per unit height of dam.
- \( H_v \) = the vertical load applied at the abutment per unit of length of arch center line.
- \( V_T \) = the tangential shear per unit height or unit length at arch center line.
- \( H_n \) = the thrust applied normal to the surface of the abutment per unit of length measured along the surfaces of the abutment.

The positive directions in which the forces are assumed to act are shown in Fig. 1. These equations lead to expressions analogous to equations (2) to (7) inclusive. Since, however, the vertical loads have been applied at the time of grouting, only the part of \( H_v \) due to the vertical component of the water load should be used in equation (10).

If \( \beta \) is the deformation of the abutment in the direction of the normal due to unit force per unit length of abutment applied in the direction of the normal, the following equations may be deduced:
Tangential movement of the arch due to arch thrust is

\[ H_a B \cos \theta' \]  

(9)

due to vertical load is

\[ H_v B \sin \theta' \cos \theta' \]  

(10)

and due to tangential shear is

\[ -2H_f B \sin \theta' \cos^2 \theta' \]  

(11)

Since the vertical deformations are neglected, the equations analogous to (5), (6), and (7) are not needed. The tangential movement of the cantilever at the abutment must agree with that of the arch at the same point.

If \( \lambda \) is the shearing deformation produced by unit force per unit length of abutment applied along the intersection of the middle surface of the dam and the abutment, the deformation of the abutment due to shear acting in the plane of the abutment may be obtained as follows:

The shear \( \tau \) in the plane of the abutment is given by the formula

\[ \tau = H_v \sin \theta' \cos \theta' - H_a \sin \theta' \cos \theta' + \lambda (\sin^2 \theta' - \cos^2 \theta') \]  

(12)

The tangential movement of the arch because of the shearing deformation due to arch thrust is

\[ H_a \lambda \sin^2 \theta' \cos \theta' \]  

(13)
Due to vertical forces

$$\frac{-H \bar{A} \sin^2 \psi \cos \psi}{(14)}$$

Due to tangential shear

$$\frac{-T \bar{A} (\sin^2 \psi - \cos^2 \psi) \sin \psi}{(15)}$$

Table 1 contains values for $\bar{A}$ computed from Dr. Vogt's formulas.

**Deformations Due to Radial Shear**

If $Y$ is the deformation produced in the radial direction by unit force per unit length of abutment then the radial deformations due to arch and cantilever shear are the following:

Radial deformation due to shear in the arch

$$\frac{T_x \cos \psi}{(16)}$$

Radial deformation due to shear in the cantilever

$$\frac{T_c \cos \psi}{(17)}$$

The radial movement of the cantilever at the abutment must agree with the movement of the arch.

**Use of Abutment Loads**

The effects of abutment deformations due to arch moment, thrust and shear are accounted for in the arch computation. The
Table 1.*

Values of $\lambda$ computed for $m = 5$.

For $a = b$ the mean value of $\lambda = \frac{m^2 \sqrt{L}}{m^2 - 1}$.  

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*See "Uber die Berechnung der Fundament Deformationen," by Frederic Vogt, Oslo, Norway, 1925. (Translated for bureau by Peter Bier.). Equation 23.
effect on the arch stresses and deflections, of the abutment deformations due to cantilever moment, shear, and load, and to tangential shear must be taken care of by abutment loads. Their use may be explained as follows: the moment at the base of the cantilever produces a rotation of the arch in the plane of the arch whose effect upon the stresses in the arch may be ascertained by assuming the arch to be cut at the crown while the rotation takes place and computing the moments, thrusts and shear necessary to restore continuity. This may be easily accomplished by manipulation of the \( \phi \) constants. Similar computations may be made to evaluate the effect of cantilever shear, tangential shear and cantilever load. In order to economize on computations it is expedient to compute unit abutment loads before making the adjustments. These consist of computations of the arch deformation and stresses due to a moment of, usually, 1,000,000 ft.-lbs./ft., a tangential shear of 1,000,000 lb./ft., etc., at the base of the cantilever. When making the adjustments, the cantilevers are computed first in order to permit the arch deflections due to the abutment deformations by the cantilever to be included with those due to load.
Memorandum to Chief Designing Engineer

By Robert E. Glover, Engineer

Subject: Application of Dr. Vogt’s abutment constants to computation of the effect of abutment deformation on the stresses in an arch dam.

Introductory Note

In computations made in the past it has been customary to apply Dr. Vogt’s formulas as though the arch terminated at a vertical abutment and the cantilever rested on a horizontal base. The conditions assumed for the arch and cantilever elements could not, of course, be fulfilled simultaneously, and while this simplification may be permissible for thin dams, its validity is open to question when the arches are short and thick and the abutment deformation becomes a large part of the total deflection. The purpose of the formulas and methods presented on the following pages is to provide a means for bringing the computations into closer agreement with the facts.

Rotations Due to Moments

The bending moment applied in a plane normal to the plane of the abutment may be computed by means of the following expression

\[ M = M_a \cos^2 \phi + M_b \sin^2 \phi - 2M_c \sin \phi \cos \phi \]
where the notations have the following significance:

- $\gamma$ is the angle the plane of the abutment makes with the vertical.
- $M_T$, the twisting moment per unit of height, or unit of length along the center line of the arch.
- $M_a$, the bending moment in the arch per unit of height of dam.
- $M_c$, the bending moment in the cantilever per unit length of center line.
- $M$, the moment per unit length of abutment.

$M_a$ and $M_c$ are considered positive if they produce compression at the upstream face. $M_T$ is positive if it tends to turn the part of the cantilever below the point of application in a clockwise direction. The cantilever is assumed to be viewed from above. The angle $\gamma$ is considered positive if measured in a counter-clockwise direction as viewed by an observer downstream from the dam. A positive $M$ applied to the abutment tends to increase the compression against the abutment at the upstream face.

If $\alpha$ is the rotation of the abutment produced by the application of unit moment per unit of length of abutment in a plane normal to the plane of the abutment, the following relations may be deduced:

1. Rotation of the arch in the plane of the arch due to abutment deformation by moment.

Rotation due to bending moment in the arch

$$M_a \alpha \cos \gamma$$

(2)
Rotation due to bending moment in the cantilever

\[ M_{\text{max}} \sin^2 y \cos y. \]  

(3)

Rotation due to twist

\[ -2M_{\text{max}} \sin y \cos y. \]  

(4)

2. Rotation of the cantilever in the plane of the cantilever due to abutment deformation by moment.

Rotation due to bending moment in the arch

\[ M_{\text{max}} \cos^2 y \sin y. \]  

(5)

Rotation due to bending moment in the cantilever

\[ M_{\text{max}} \sin^3 y. \]  

(6)

Rotation due to twist

\[ -2M_{\text{max}} \sin^2 y \cos y. \]  

(7)

3. Rotation of the cantilever in the plane of the arch due to abutment deformation by moment must agree with the rotation of the arch in the plane of the arch.

Note that the rotation of the abutment in the plane of the abutment due to twist is neglected.
Deformations Due to Arch Thrust,
Tangential Shear, and Cantilever Weight.

The thrust applied normal to the surface of the abutment may be obtained from the expression

$$H_n = H_a \cos^2 \gamma + H_v \sin^2 \gamma - \frac{2}{T} \sin \gamma \cos \gamma.$$

where the notations have the following significance:

- $H_a$: the arch thrust at the abutment per unit height of dam.
- $H_v$: the vertical load applied at the abutment per unit of length of arch center line.
- $\frac{1}{T}$: the tangential shear per unit height or unit length at arch center line.
- $H_n$: the thrust applied normal to the surface of the abutment per unit of length measured along the surfaces of the abutment.

The positive directions in which the forces are assumed to act are shown in Fig. 1. These equations lead to expressions analogous to equations (2) to (7) inclusive. Since, however, the vertical loads have been applied at the time of grouting, only the part of $H_v$ due to the vertical component of the water load should be used in equation (10).

If $\beta$ is the deformation of the abutment in the direction of the normal due to unit force per unit length of abutment applied in the direction of the normal, the following equations may be deduced:
Tangential movement of the arch due to arch thrust is

\[ H_B \cos \gamma \cos \phi \]  \hspace{1cm} (9)

due to vertical load is

\[ H_B \sin \gamma \cos \phi \]  \hspace{1cm} (10)

and due to tangential shear is

\[-2TV_B \sin \gamma \cos^2 \phi \]  \hspace{1cm} (11)

Since the vertical deformations are neglected, the equations analogous to (5), (6), and (7) are not needed. The tangential movement of the cantilever at the abutment must agree with that of the arch at the same point.

If \( \lambda \) is the shearing deformation produced by unit force per unit length of abutment applied along the intersection of the middle surface of the dam and the abutment the deformation of the abutment due to shear acting in the plane of the abutment may be obtained as follows:

The shear \( V \) in the plane of the abutment is given by the formula

\[ V = H_B \sin \gamma \cos \phi - H_B \sin \gamma \cos \phi + T \left( \sin^2 \gamma - \cos^2 \phi \right) \]  \hspace{1cm} (12)

The tangential movement of the arch because of the shearing deformation due to arch thrust is

\[ H_B \lambda \sin \gamma \cos \phi \]  \hspace{1cm} (13)
Due to vertical forces

$$- \frac{1}{h} \lambda \sin^2 \psi \cos \psi.$$  \hspace{1cm} (14)

Due to tangential shear

$$- \frac{1}{h} \lambda (\sin^2 \psi - \cos^2 \psi) \sin \psi.$$  \hspace{1cm} (15)

Table 1 contains values for $\lambda$ computed from Dr. Vogt's formulas.

**Deformations Due to Radial Shear**

If $\lambda$ is the deformation produced in the radial direction by unit force per unit length of abutment then the radial deformations due to arch and cantilever shear are the following:

Radial deformation due to shear in the arch

$$\frac{1}{h} \lambda \cos \psi.$$  \hspace{1cm} (16)

Radial deformation due to shear in the cantilever

$$\frac{1}{h} \lambda \sin \psi.$$  \hspace{1cm} (17)

The radial movement of the cantilever at the abutment must agree with the movement of the arch.

**Use of Abutment Loads**

The effects of abutment deformations due to arch moment, thrust and shear are accounted for in the arch computation. The
Table 1.*

Values of \( \lambda \) computed for \( m = 5 \).

For \( a = b \) the mean value of

\[
\lambda = \frac{m^2 - 1}{m^2} \frac{\sqrt{V}}{z}
\]

is 1.065

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<td>0.426</td>
</tr>
<tr>
<td>( 6b )</td>
<td>0.378</td>
</tr>
<tr>
<td>( 7b )</td>
<td>0.340</td>
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<tr>
<td>( 8b )</td>
<td>0.313</td>
</tr>
<tr>
<td>( 9b )</td>
<td>0.286</td>
</tr>
<tr>
<td>( 10b )</td>
<td>0.265</td>
</tr>
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</table>

*See "Uber die Berechnung der Fundament Deformationen," by Frederic Vogt, Oslo, Norway, 1925. (Translated for bureau by Peter Bier.). Equation 23.
effect on the arch stresses and deflections, of the abutment deformations due to cantilever moment, shear, and load, and to tangential shear must be taken care of by abutment loads. Their use may be explained as follows: the moment at the base of the cantilever produces a rotation of the arch in the plane of the arch whose effect upon the stresses in the arch may be ascertained by assuming the arch to be cut at the crown while the rotation takes place and computing the moments, thrusts and shear necessary to restore continuity. This may be easily accomplished by manipulation of the $C$ constants. Similar computations may be made to evaluate the effect of cantilever shear, tangential shear and cantilever load. In order to economize on computations it is expedient to compute unit abutment loads before making the adjustments. These consist of computations of the arch deformation and stresses due to a moment of, usually, 1,000,000 ft.-lbs./ft., a tangential shear of 1,000,000 lb./ft., etc., at the base of the cantilever. When making the adjustments, the cantilevers are computed first in order to permit the arch deflections due to the abutment deformations by the cantilever to be included with those due to load.
Discussion of "Stresses in Gravity Dams by Principle of Least Work".

by

B. F. Jacobsen


In part IV* page 1617, Mr. Jacobsen says, "Instead of the simple equation (10) for Ny, the condition that equation (14) must be a minimum is now introduced. This is a problem in variational calculus, but instead of treating it as such, which would be very complicated, a method of approximation may be resorted to, . . . . ." The writer has observed that a certain amount of confusion exists among engineers as to what is to be expected of the variational calculus. This is not surprising since the subject is not included among the courses of study usually prescribed for students in engineering colleges.

It is believed, therefore, that it would be helpful to solve the variational problem presented, in order to have the results available for comparison with those obtained by other methods.

Mr. Jacobsen's equation (14) is the expression for the potential energy of deformation. If Z = unity it reduces to

\[ L = \int \int \left[ \frac{n_y^2 + n_x^2}{2} - \frac{(n_y + n_x)^2}{2(m+1)} + t z^2 \right] dy \, dx \]

It will be convenient to express the stresses in terms of an Airys' function F which is connected with the stresses by the relations

\[ n_y = \frac{\partial^2 F}{\partial x^2}, \quad n_x = \frac{\partial^2 F}{\partial y^2}, \quad t = -\frac{\partial^2 F}{\partial x \partial y} \]

*For discussion of the nature of this problem see paragraphs 117, 118 and 119 of Love's "Treatise on the Mathematical Theory of Elasticity, Fourth Edition."
This form of expression will be permissible providing it is consistent with the author's equilibrium equations (6) and (7). This may be ascertained by direct substitution as follows:

\[
\frac{\partial n_y}{\partial y} = \frac{\partial^3 F}{\partial x^2 dy} - \omega, \quad \frac{\partial n_x}{\partial x} = \frac{\partial^3 F}{\partial x \partial y^2}.
\]

\[
\frac{\partial t}{\partial x} = -\frac{\partial^3 F}{\partial x^2 dy}, \quad \frac{\partial t}{\partial y} = -\frac{\partial^3 F}{\partial x \partial y^2}.
\]

Then

\[
\frac{\partial n_y}{\partial y} + \frac{\partial t}{\partial x} + \omega = \frac{\partial^3 F}{\partial x^2 dy} - \omega - \frac{\partial^3 F}{\partial x^2 dy} + \omega = 0 \quad (6)
\]

\[
\frac{\partial n_x}{\partial x} - \frac{\partial t}{\partial y} = \frac{\partial^3 F}{\partial x \partial y^2} - \frac{\partial^3 F}{\partial x \partial y^2} = 0 \quad (7)
\]

and

\[
L = \sqrt{\int \left[ \left( \frac{\partial^2 F}{\partial x^2} - \omega y \right)^2 + \left( \frac{\partial^2 F}{\partial y^2} \right)^2 \right] - \left( \frac{\partial^2 F}{\partial x^2} - \omega y + \frac{\partial^2 F}{\partial y^2} \right)^2 - \frac{1}{2(m+1)} \left( \frac{\partial^2 F}{\partial x^2} \right) \right] \, dx \, dy.
\]
The writer's computations indicate that if

\[ L = \int \phi \left[ F, x, y, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x \partial y} \right] \, dx \, dy \]

the condition for a minimum is

\[ \frac{\partial \phi}{\partial F} - \frac{d}{dx} \left( \frac{\partial \phi}{\partial x} \right) - \frac{d}{dy} \left( \frac{\partial \phi}{\partial y} \right) + \frac{c_1^2}{d^2} \left( \frac{\partial \phi}{\partial x} \right)^2 + \frac{c_2^2}{d^2} \left( \frac{\partial \phi}{\partial y} \right)^2 \]

\[ + \frac{d^2}{dx \, dy} \left( \frac{\partial \phi}{\partial F} \right) = 0 \]

where

\[ p = \frac{\partial F}{\partial x}, \quad q = \frac{\partial F}{\partial y}, \quad r = \frac{\partial^2 F}{\partial x^2}, \quad s = \frac{\partial^2 F}{\partial y^2}, \quad t = \frac{\partial^2 F}{\partial x \partial y} \]

and \( \phi \) is a symbol used to denote that the integrand is a function of the variables in the brackets. In our case \( F, x, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \) and \( \frac{\partial^2 F}{\partial x \partial y} \) are absent and therefore

\[ \phi \left[ F, x, y, \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial^2 F}{\partial x^2}, \frac{\partial^2 F}{\partial y^2}, \frac{\partial^2 F}{\partial x \partial y} \right] = \]

\[ \left( \frac{\partial^2 F}{\partial x^2} - wy \right)^2 + \left( \frac{\partial^2 F}{\partial y^2} - \left( \frac{\partial^2 F}{\partial x^2} - wy + \frac{\partial^2 F}{\partial y^2} \right) \right)^2 \]

\[ + \frac{\left( \frac{\partial^2 F}{\partial x \partial y} \right)^2}{2(m+1)} + \left( \frac{\partial^2 F}{\partial x \partial y} \right)^2 \]
then

\[
\frac{\partial^2 \phi}{\partial r^2} = \frac{\partial^2 F}{\partial x^2} - wy - \frac{\left( \frac{\partial^2 F}{\partial x^2} - wy + \frac{\partial^2 F}{\partial y^2} \right)}{(m+1)}
\]

\[
\frac{\partial^2 \phi}{\partial s^2} = \frac{\partial^2 F}{\partial y^2} - \frac{\left( \frac{\partial^2 F}{\partial x^2} - wy + \frac{\partial^2 F}{\partial y^2} \right)}{(m+1)}
\]

\[
\frac{\partial \phi}{\partial t} = \frac{\partial^2 F}{\partial x \partial y}
\]

and

\[
\frac{\partial^2}{\partial x^2} \left( \frac{\partial \phi}{\partial r} \right) = \frac{\partial^4 F}{\partial x^4} - \frac{\left( \frac{\partial^4 F}{\partial x^2 \partial y^2} \right)}{(m+1)}
\]

\[
\frac{\partial^2}{\partial y^2} \left( \frac{\partial \phi}{\partial s} \right) = \frac{\partial^4 F}{\partial y^4} - \frac{\left( \frac{\partial^4 F}{\partial x^2 \partial y^2} \right)}{(m+1)}
\]

\[
\frac{\partial^2}{\partial x \partial y} \left( \frac{\partial \phi}{\partial t} \right) = \frac{\partial^4 F}{\partial x^2 \partial y^2}
\]

The condition for a minimum states that the sum of these three foregoing expressions must be equal to zero, therefore, the condition for a minimum of potential energy of deformation is

\[
\frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \quad (42)
\]
Note that the result of the solution of the variational problem is a condition, expressed in terms of a differential equation. This is generally the case.

The problem may be approached from another angle. We will determine what is the result of basing the argument upon the ideas of equilibrium and continuity. As in the previous case, it will be assumed that Hooke's law applies. Mr. Jacobsen has supplied the equations of equilibrium required. They are equations (6) and (7). The equation of continuity, or of compatibility, expresses the condition that if the \( x, y \) surface in its unstressed state is marked off into squares by closely spaced lines parallel to the \( x \) and \( y \) axes, each of the squares so defined will maintain contact with its neighbors on all four sides after deformation has taken place. The equation of continuity may be arrived at by expressing the strains in terms of the displacements and eliminating the displacements from the equations.

To accomplish this let,

\[
\begin{align*}
U & \text{, represent the displacement of points in the } x \text{ direction,} \\
V & \text{, the displacement of points in the } y \text{ direction,} \\
\varepsilon_x & \text{, the unit strain in the } x \text{ direction,} \\
\varepsilon_y & \text{, the unit strain in the } y \text{ direction,} \\
\gamma_{xy} & \text{, the unit shearing strain,}
\end{align*}
\]

then

\[
\begin{align*}
\varepsilon_x & = \frac{\partial U}{\partial x} \\
\varepsilon_y & = \frac{\partial V}{\partial y} \\
\gamma_{xy} & = \frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}
\end{align*}
\]
Differentiate as follows, and add
\[
\frac{\partial^2 \sigma_x}{\partial y^2} = \frac{\partial^3 u}{\partial x \partial y^2}.
\]
\[
\frac{\partial^2 \sigma_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2 \partial y}.
\]
\[
-\frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial^3 u}{\partial x \partial y^2} - \frac{\partial^3 v}{\partial x^2 \partial y}.
\]

then
\[
\frac{\partial^2 \sigma_y}{\partial x^2} - \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_x}{\partial y^2} = 0
\]
is the required equation.*


Introduce Hooke's law in terms of the ordinates to Airy's surface as follows
2 - 1
\[ \varepsilon_x = \frac{1}{E} (\sigma_{xx} - \mu \sigma_{yy}) = \frac{1}{E} \left( \frac{\partial^2 F}{\partial y^2} - \mu \frac{\partial^2 F}{\partial x^2} + \mu \sigma_{yy} \right) \]
\[ \varepsilon_y = \frac{1}{E} (\sigma_{yy} - \mu \sigma_{xx}) = \frac{1}{E} \left( \frac{\partial^2 F}{\partial x^2} - \omega y - \mu \frac{\partial^2 F}{\partial y^2} \right) \]
\[ \sigma_{xy} = \frac{2(1 + \mu) t}{E} = \frac{2(1 + \mu)}{E} \left( - \frac{\partial^2 F}{\partial x \partial y} \right) \]

Note: \[ \mu = \frac{1}{m} \]

If these expressions are substituted into the equation of continuity there is obtained

\[ \frac{\partial^4 F}{\partial x^4} + 2 \frac{\partial^4 F}{\partial x^2 \partial y^2} + \frac{\partial^4 F}{\partial y^4} = 0 \quad (42) \]

As a result of this work it may be seen that not only does equation (42) imply that the equations of compatibility, as stated by the author in Appendix II, are satisfied, but it also implies the satisfaction of the equations of equilibrium. Furthermore, the variational calculus tells us that if this equation is satisfied the potential energy of deformation will be less than that of any other distribution whatever which will hold the forces in equilibrium.

We are therefore in a position to write a specification for a solution of the problem of finding the stress distribution in a two
dimensional elastic structure. It is; solve equation (42) subject to the boundary conditions. Let us apply this test to the solutions in the author's paper. Equations (13) satisfy equation (42). That equations (27) do not, is not conclusive in this case since the author does not pretend that these equations represent an exact solution, but only an approximation to one. Neither equations (13) or (27), however, recognizes explicitly any elastic action of the foundation and, therefore, it must be concluded that neither is entitled to acceptance as a true solution of the problem. The objection may be stated specifically in the case of equations (27). The limits given for the integrals in equations (19) and (20) show that the integration was carried over the section of the dam only. This being the case we may conclude that Mr. Jacobsen has succeeded in solving, approximately, the problem: To find, among all possible stress distributions which satisfy the boundary conditions at the upstream and downstream faces of a triangular profile, that distribution which shall make the potential energy of deformation within the section of the dam a minimum. The problem to be solved in case the law of least work is to be complied with is the following: To find among all possible stress distributions which satisfy the boundary conditions at the surface of the dam and foundation, that distribution which shall make the total potential energy of deformation in the dam and foundation, a minimum.
The following comments apply to the arguments in Appendix III.

Mr. Westergaard has pointed out* that if lateral deformations are prevented, the effect of the restraint may be accounted for by replacing $E$ by $E' = \frac{E}{1-\mu^2}$, and $\mu$ by $\mu' = \frac{\mu}{1-\mu}$.

Since neither $E$ nor $\mu$ appears in equation (42) the stress distribution must be independent of both the elastic modulus and Poisson's ratio. This being the case, all the statements made previously concerning equation (42) hold without modification for the case of a body under complete lateral restraint.

Mr. Westergaard's observation is also important from the standpoint of the model builder since we may infer from it that stress distributions obtained from thin models, as was accomplished by Messrs Wilson and Gore,** apply without modification for the deeper sections of long gravity dams regardless of whether the dam is cut by contraction joints at close intervals, or joined together in such a way as to make it a continuous structure. Furthermore, the statement is not subject to restrictions regarding the value of the elastic modulus or of Poisson's ratio for either the material of the dam or the model.

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