of the parameters internal to the structure implementing $A_0(z)$ or $A_1(z)$ does not increase the passband level of $H'(z)$. In the FIR case, however, this situation is not true because the functions $A(z)$ and $B(z)$ are not *structurally allpass* (however assuming that $P(z)$ and $Q(z)$ are designed to have "good" passbands and stopbands, this is approximately so). The size of the ripples in the arbitrary-level filters depend on the ripples of $A(z)$ and $B(z)$, which in turn depend on the stopband attenuations of the original filter and its power complementary filter. It must be noted that we have the freedom to choose only one of the two magnitude levels and that the accuracy of the spectral factorization technique directly affects the design.

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Reduced Order Strip Kalman Filtering Using Singular Perturbation Method

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Abstract —**Strip Kalman filtering for restoration of images degraded by linear shift invariant (LSI) blur and additive white Gaussian (WG) noise is considered. The image process is modeled by a 1-D vector autoregressive (AR) model in each strip. It is shown that the composite dynamic model that is obtained by combining the image model and the blur model takes the form of a singularly perturbed system owing to the strong-weak correlation effects within a window. The time scale property of the singularly perturbed system is then utilized to decompose the original system into reduced order subsystems which closely capture the behavior of the full order system. For these subsystems the relevant Kalman filtering equations are given which provide the suboptimal filtered estimates of the image and the one-step prediction estimates of the blur needed for the next stage. Simulation results are also provided.**

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I. Introduction

Parametric representation of digital images have found numerous applications in image restoration [1]–[3], image data compression [4], and texture analysis [5]. An image is modeled by a finite-order autoregressive (AR) or autoregressive moving average (ARMA) representation which closely match the autocorrelation function or equivalently the spectral density function (SDF) of the image field. Even though high-order models may take more correlations into account, in general they are not necessarily capable of better image representation. Additionally, the principle of parsimony precludes the use of a large order model for this representation.

Suresh and Shenoi [2] proposed a strip Kalman filtering process which makes use of a vector scanning scheme. The image process is modeled by a finite-order vector AR model which relates a column of pixels to the past columns in a certain region within the strip. The effect of an LSI blur is modeled by a 2-D state-space structure [6] implemented by a 1-D structure with intrastrip and interstrip recursion characteristics [2]. The size of the state-vector in the composite dynamic model and hence the computational effort of the filtering process are dependent on (i) the size of the blur model and (ii) the order of the AR model used to generate the image. If the width of each strip and the order of the vector AR model are denoted by W and M, respectively, the size of the state-vector in the composite dynamic model is shown [2] to be $WM+W+1$ which may be large even for moderate values of W and M.

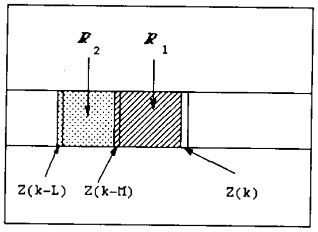
The contribution of this paper is to employ singular perturbation methodology for decomposing a given image model into reduced order models whereby the image restoration can be performed effectively. To this effect, a singularly perturbed model of the original system in [2] is obtained by expressing the state variables into a set of slow and fast variables. Using the model reduction capabilities of the singular perturbation technique the full order model is decomposed into reduced order sub-models corresponding to strong-weak correlation areas. The utility of the singular perturbation method lies in the fact that the aggregated effects of the weakly correlated states are taken into account in the reduced order models. These models can be used in the strip Kalman filtering process without losing significant accuracy in estimation.

II. Modeling the Image Process

Consider an $N \times N$ image which is vector scanned horizontally in strips of size $W \times N$. The direction of scanning is assumed to be from left-to-right and top-to-bottom. Each strip is processed independently with an overlap between the adjacent strips to reduce the edge effects. The image is assumed to be represented by a vector Markovian field and modeled, within each strip by an Lth-order vector AR model with causal quarter-plane region of support. If the support region of this model is denoted by $\mathbf{R}$, the following AR model of order L can be written for the process

$$Z(k)=\hat{\phi}_1^t Z(k-1)+\hat{\phi}_2^t Z(k-2)+\cdots+\hat{\phi}_L^t Z(k-L)+U(k) \tag{1}$$

where the superscript t denotes matrix transposition; $\hat{\phi}_1,\cdots,\hat{\phi}_L$ are constant $W\times W$ matrices which constitute the autoregressive parameters; and $Z(k)$ represents a $W\times 1$ vector with elements that are the pixels intensity values in the kth column of a given



$R = R_1 \cup R_2$

R_1 : Strong Correlation Area

R_2 : Weak Correlation Area

Fig. 1. Region of support for the vector AR model.

(say ith) strip in the image (See Fig. 1) i.e.,

$$Z(k) = [z_{(i-1)W,k} z_{(i-1)W+1,k} \cdots z_{iW-1,k}]' \tag{2}$$

where $z_{m,n}$ denotes the intensity of the pixel at location (m,n). The image process is assumed to be column wide-sense stationary within each strip. Vector $U(k)$ which is defined similar to $Z(k)$ represents a white vector sequence which drives the AR process. The statistics of this error sequence are

$$E[U(k)] = 0 \quad \text{and} \quad E[U(k)U'(k-l)] = Q_U \delta(l) \tag{3}$$

where Q_U is the covariance matrix of the error vector $U(k)$ and $\delta(l)$ represents the Kronecker delta function. It is interesting to note that although $U(k)$ is vectorially an uncorrelated process, the elements within each vector are mutually correlated. This fact has been shown in [11].

A state space model for the image process can be obtained by defining a vector $X(k)$ as

$$X(k) = [Z(k-L+1)' \quad Z(k-L+2)' \quad \cdots \quad Z(k)']'. \tag{4}$$

Then the state equation becomes

$$X(k) = AX(k-1) + BU(k) \tag{5}$$

where

$$A = \begin{bmatrix} 0 & I_W & 0 & \cdots & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & & I_W \\ \hat{\phi}_L' & \hat{\phi}_{L-1}' & \cdots & & \hat{\phi}_1' \end{bmatrix}; \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ I_W \end{bmatrix} \tag{6}$$

and I_W represents an identity matrix of order W. Due to strong-weak coupling effects in the vector AR model (1), the above system can be represented in a singularly perturbed form.

If the support region R of this model is partitioned into two subsets R_1 and R_2 (see Fig. 1) associated with the strong and weak correlation areas, respectively, the following perturbed AR model can be written for the process.

$$\begin{aligned} Z(k) &= \phi_1' Z(k-1) + \phi_2' Z(k-2) + \cdots + \phi_M' Z(k-M) \\ &\quad + \epsilon \phi_{M+1}' Z(k-M-1) + \epsilon^2 \phi_{M+2}' Z(k-M-2) \\ &\quad + \cdots + \epsilon^{L-M} \phi_L' Z(k-L) + U(k) \end{aligned} \tag{7}$$

where

$$\hat{\phi}_i = \phi_i, \qquad 1 \leqslant i \leqslant M$$

$$\hat{\phi}_i = \epsilon^{i-M} \phi_i, \qquad M+1 \leqslant i \leqslant L.$$

That is the model coefficient matrices, $\hat{\phi}_m$'s, that are of the size $W \times W$ have norms which can either be of order 1, $(1 \leqslant m \leqslant M)$, if the corresponding vector is located in the strong correlation area or of order ϵ, $(M+1 \leqslant m \leqslant L)$, if the relevant vector is inside the weak correlation area. The parameter ϵ known as the "singular perturbation parameter" is normally obtained by clustering the eigenvalues of matrix A into two isolated groups of sizes M and $L-M$. The eigenvalues that are located in the cluster close to the origin of the unit circle in the z-plane are referred to as "fast eigenvalues" and the ones that are located in the cluster in the vicinity of the unit circle are called "slow eigenvalues" [8]. In our case due to the vector (multichannel) nature of the process the Frobenious norm (strong norm), that is, $\|A\| = [\mathrm{trace}(A'A)]^{1/2}$, of each coefficient matrix $\hat{\phi}_m$ is computed and then ϵ is evaluated using

$$\epsilon = \max \left[\frac{1 - \|\hat{\phi}_m\|}{1 - \|\hat{\phi}_n\|} \right], \qquad m \epsilon [1, M];\; n \epsilon [M+1, L-M]. \tag{8}$$

Now, the system described by (7) is in singularly perturbed form in the sense that by formally setting $\epsilon = 0$, the order of the system is reduced from L to M. That is, the reduced model becomes

$$\begin{aligned} Z(k) &= \phi_1' Z(k-1) + \phi_2' Z(k-2) \\ &\quad + \cdots + \phi_M' Z(k-M) + U(k). \end{aligned}$$

This model, although provides computation simplicity, in general, will not be able to yield satisfactory filtering results. This reduced model corresponds to the truncated AR model with the region of support of R_1. The main advantage of the singular perturbation analysis is to utilize effectively the simplicity offered by the order reduction and simultaneously provide a satisfactory filtering process. This method is developed in this section and later in Section III.

Let us define the following states in region R_2 and R_1,

$$X_2(k-1): \begin{cases} \xi_1(k-1) = \epsilon^{L-M-1} Z(k-L) \\ \xi_2(k-1) = \epsilon^{L-M-2} Z(k+1-L) \\ \vdots \\ \xi_{L-M}(k-1) = Z(k-M-1) \end{cases} \quad \textit{fast states}$$

$$X_1(k-1): \begin{cases} X_1(k-1) = Z(k-M) \\ \vdots \\ X_M(k-1) = Z(k-1). \end{cases} \quad \textit{slow states} \tag{9}$$

Now the AR model (7) can be arranged in a state-space form as

$$\begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} = \begin{bmatrix} A_{11} & \epsilon A_{12} \\ A_{21} & \epsilon A_{22} \end{bmatrix} \begin{bmatrix} X_1(k-1) \\ X_2(k-1) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} U(k) \tag{10a}$$

The vector of interest $Z(k)$ can be extracted using

$$Z(k) = [C_1 \quad 0] \begin{bmatrix} X_1(k) \\ X_2(k) \end{bmatrix} \tag{10b}$$

where

$$A_{11} = \begin{bmatrix} 0 & I_W & 0 & \cdots & 0 \\ & & \ddots & & \\ & & & & I_W \\ \phi'_M & \phi'_{M-1} & \cdots & \cdots & \phi'_1 \end{bmatrix}$$

$$A_{12} = \begin{bmatrix} 0 \\ \phi'_L \cdots \phi'_{M+1} \end{bmatrix} \quad A_{21} = \begin{bmatrix} & 0 \\ I_W & \end{bmatrix},$$

$$A_{22} = \begin{bmatrix} 0 & I_W & 0 & \cdots & 0 \\ & & \ddots & & I_W \\ & 0 & & & 0 \end{bmatrix}$$

$$B_1 = [0 \ \ 0 \ \ \cdots \ \ I_W]' \quad \text{and} \quad C_1 = [0 \ \ 0 \ \ \cdots \ \ I_W].$$

Note that the states associated with the strong correlation area are named as "slow states" as they lead to the slow dynamics of the system whereas those corresponding to the weak correlation area are named as "fast states" since they relate to the fast dynamics of the system [8]. The state-space model in (10) is now in the standard singularly perturbed form. Using the singular perturbation technique this dynamic system can be decomposed into reduced order subsystems. The main crux of the singular perturbation analysis lies in the fact that the reduced order subsystems incorporate the effects of those weakly correlated pixels in region $\boldsymbol{R}_2$ that are normally ignored to avoid having a large order model. In Section III we shall describe how these subsystems can be derived.

The image generated by the process (10) passes through a degradation process which blurs the image with an LSI blur having an infinite extent point spread function (PSF) and also adds a WG noise to the blurred image. In [2] the LSI blur is modeled in each strip by a 1-D state-space equation of form:

$$\begin{aligned} X_3(k+1) &= A_3 X_3(k) + B_3 Z(k) \\ Y(k) &= C_3 X_3(k) + D_3 Z(k) + V(k) \end{aligned} \tag{11}$$

where the state vector $X_3(k)$ consists of two parts which propagate vertically during the interstrip recursion and horizontally during the intrastrip recursion [2], respectively; $Y(k)$ represents a column of the degraded image and $V(k)$ is a $W \times 1$ vector consisting of WG additive noise elements with zero mean and variance σ_v^2. The composite dynamic model can then be obtained by combining (10) and (11) to give

$$\begin{aligned} \boldsymbol{X}(k) &= \boldsymbol{A}\boldsymbol{X}(k-1) + \boldsymbol{B}U(k) \\ Y(k) &= \boldsymbol{C}\boldsymbol{X}_c(k) + V(k) \end{aligned} \tag{12}$$

where

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & \epsilon A_{12} & 0 \\ A_{21} & \epsilon A_{22} & 0 \\ \tilde{A}_{31} & \epsilon \tilde{A}_{32} & A_3 \end{bmatrix}; \qquad \boldsymbol{B} = \begin{bmatrix} B_1 \\ 0 \\ \tilde{B}_3 \end{bmatrix}$$

$$\boldsymbol{C} = [\,D_3 C_1 \quad 0 \quad C_3\,] \tag{13}$$

$$\boldsymbol{X}(k) \triangleq \begin{bmatrix} X_1(k) \\ X_2(k) \\ X_3(k+1) \end{bmatrix}; \qquad \boldsymbol{X}_c(k) \triangleq \begin{bmatrix} X_1(k) \\ X_2(k) \\ X_3(k) \end{bmatrix} \tag{14}$$

and

$$\begin{aligned} &\tilde{A}_{31} \triangleq B_3 C_1 A_{11}; \qquad \tilde{A}_{32} \triangleq B_3 C_1 A_{12} \\ &\tilde{B}_3 \triangleq B_3 C_1 B_1. \end{aligned} \tag{15}$$

Note that the above composite dynamic equation clearly indicates that the state vector associated with the blur, i.e., X_3 is evaluated one step ahead in each stage. In the Kalman filtering equations this part leads to one-step prediction estimates whereas for the other state vectors, i.e., X_1 and X_2 the filtered estimates will be computed. In what follows we make use of the singular perturbation technique to obtain reduced order subsystems for the above composite dynamic model.

III. Model Reduction using Singular Perturbation

As pointed out earlier the dynamic equation for the image process is already in singularly perturbed form due to the strong-weak coupling in the vector AR model. The perturbation parameter ϵ is estimated using (8). Now let us assume that the matrix A_3 in the blur model can also be decomposed into

$$A_3 = \begin{bmatrix} A_{31} & \epsilon A_{32} \\ A_{33} & \epsilon A_{34} \end{bmatrix}$$

and further C_3 is given by

$$C_3 = [\,C_{31} \quad C_{32}\,]$$

where the dominant slow eigenvalues are approximated by those of A_{31} and the fast ones by ϵA_{34}. Note that this partitioning can always be accomplished by separating the blur dynamics into slow and fast groups with a time scale separation proportional to that of the image process dynamics. If the blur dynamics is all slow (or fast) then matrix A_3 will only contain block A_{31} (or ϵA_{34}). Having decomposed A_3 as such, X_3 will also be partitioned into slow and fast parts, i.e., $[X_{3s}' \quad X_{3f}']'$ resulting in the following composite dynamic model:

$$\begin{bmatrix} X_1(k) \\ X_{3s}(k+1) \\ X_2(k) \\ X_{3f}(k+1) \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & \epsilon A_{12} & 0 \\ \tilde{A}_{311} & A_{31} & \epsilon \tilde{A}_{321} & \epsilon A_{32} \\ A_{21} & 0 & \epsilon A_{22} & 0 \\ \tilde{A}_{312} & A_{33} & \epsilon \tilde{A}_{322} & \epsilon A_{34} \end{bmatrix} \begin{bmatrix} X_1(k-1) \\ X_{3s}(k) \\ X_2(k-1) \\ X_{3f}(k) \end{bmatrix} + \begin{bmatrix} B_1 \\ \tilde{B}_{31} \\ 0 \\ \tilde{B}_{32} \end{bmatrix} U(k) \tag{16a}$$

where

$$\tilde{A}_{31} = \begin{bmatrix} \tilde{A}_{311} \\ \tilde{A}_{312} \end{bmatrix}; \qquad \tilde{A}_{32} = \begin{bmatrix} \tilde{A}_{321} \\ \tilde{A}_{322} \end{bmatrix} \quad \text{and} \quad \tilde{B}_3 = \begin{bmatrix} \tilde{B}_{31} \\ \tilde{B}_{32} \end{bmatrix}.$$

The output equation is given by

$$Y(k) = [\,D_3 C_1 \quad C_{31} \quad 0 \quad C_{32}\,] \begin{bmatrix} X_1(k) \\ X_{3s}(k) \\ X_2(k) \\ X_{3f}(k) \end{bmatrix} + V(k) \tag{16b}$$

Taking into account (16a), this output equation can be rewritten as

$$Y(k) = [\,\overline{C}_{11} \quad \overline{C}_{22}\,] \begin{bmatrix} X_1(k-1) \\ X_{3s}(k) \\ X_2(k-1) \\ X_{3f}(k) \end{bmatrix} + \overline{D}U(k) + V(k) \tag{17a}$$

where

$$\bar{C}_{11} \triangleq [D_3C_1A_{11} \quad C_{31}] \tag{17b}$$

$$\bar{C}_{22} \triangleq [\epsilon D_3C_1A_{12} \quad C_{32}] \tag{17c}$$

$$\bar{D} \triangleq D_3C_1B_1. \tag{17d}$$

Let us denote

$$\bar{A}_{11} \triangleq \begin{bmatrix} A_{11} & 0 \\ \tilde{A}_{311} & A_{31} \end{bmatrix}; \qquad \bar{A}_{21} \triangleq \begin{bmatrix} A_{21} & 0 \\ \tilde{A}_{312} & A_{33} \end{bmatrix}$$

$$\bar{A}_{12} \triangleq \begin{bmatrix} \epsilon A_{12} & 0 \\ \epsilon\tilde{A}_{321} & \epsilon A_{32} \end{bmatrix}; \qquad \bar{A}_{22} = \begin{bmatrix} \epsilon A_{22} & 0 \\ \epsilon\tilde{A}_{322} & \epsilon A_{34} \end{bmatrix}$$

$$\bar{B}_1 \triangleq \begin{bmatrix} B_1 \\ \tilde{B}_{31} \end{bmatrix}; \qquad \bar{B}_2 \triangleq \begin{bmatrix} 0 \\ \tilde{B}_{32} \end{bmatrix}$$

$$X_s(k) \triangleq \begin{bmatrix} X_1(k) \\ X_{3s}(k+1) \end{bmatrix}; \qquad X_f(k) \triangleq \begin{bmatrix} X_2(k) \\ X_{3f}(k+1) \end{bmatrix}. \tag{18}$$

Then (16a) and (17a) become

$$\begin{bmatrix} X_s(k) \\ X_f(k) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} X_s(k-1) \\ X_f(k-1) \end{bmatrix} + \begin{bmatrix} \bar{B}_1 \\ \bar{B}_2 \end{bmatrix} U(k) \tag{19a}$$

$$Y(k) = [\bar{C}_{11} \quad \bar{C}_{22}] \begin{bmatrix} X_s(k-1) \\ X_f(k-1) \end{bmatrix} + \bar{D}U(k) + V(k). \tag{19b}$$

It is easily seen that the eigenvalues of $\bar{A}_{11}$ are the union of those of A_{11} and A_{31} which are slow and the eigenvalues of $\bar{A}_{22}$ are the union of those of ϵA_{22} and ϵA_{34} which are fast, i.e., the composite dynamic model is also in the standard singularly perturbed form. To obtain the decoupled slow and fast subsystems, let us define the following block diagonalizing transformation [9].

$$\begin{bmatrix} \bar{X}_s(k) \\ \bar{X}_f(k) \end{bmatrix} = \begin{bmatrix} I-ML & -M \\ L & I \end{bmatrix} \begin{bmatrix} X_s(k) \\ X_f(k) \end{bmatrix} \tag{20a}$$

or

$$\begin{bmatrix} X_s(k) \\ X_f(k) \end{bmatrix} = \begin{bmatrix} I & M \\ -L & I-LM \end{bmatrix} \begin{bmatrix} \bar{X}_s(k) \\ \bar{X}_f(k) \end{bmatrix} \tag{20b}$$

where matrices L and M satisfy the following [9]:

$$L\bar{A}_{11} + \bar{A}_{21} - L\bar{A}_{12}L - \bar{A}_{22}L = 0 \quad \text{(Riccati-type equation)} \tag{21}$$

$$\bar{A}_{11}M + \bar{A}_{12} - ML\bar{A}_{12} - M\bar{A}_{22} - \bar{A}_{12}LM = 0 \quad \text{(Lyapunov-type equation)}. \tag{22}$$

Using the above transformation, system (19) is transformed into

$$\begin{bmatrix} \bar{X}_s(k) \\ \bar{X}_f(k) \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} - \bar{A}_{12}L & 0 \\ 0 & \bar{A}_{22} + L\bar{A}_{12} \end{bmatrix} \begin{bmatrix} \bar{X}_s(k-1) \\ \bar{X}_f(k-1) \end{bmatrix} + \begin{bmatrix} (I-ML)\bar{B}_1 - M\bar{B}_2 \\ L\bar{B}_1 + \bar{B}_2 \end{bmatrix} U(k) \tag{23}$$

$$Y(k) = [\bar{C}_{11} - \bar{C}_{22}L \quad \bar{C}_{11}M + \bar{C}_{22}(I-LM)] \begin{bmatrix} \bar{X}_s(k-1) \\ \bar{X}_f(k-1) \end{bmatrix} + \bar{D}U(k) + V(k). \tag{24}$$

A computationally efficient iterative technique for obtaining the solutions of (21) and (22) is suggested in [10] which gives the following results. (It is assumed that $\bar{A}_{11}$ is non-singular.)

$$L_0 = -\bar{A}_{21}\bar{A}_{11}^{-1}; \qquad L_{k+1} = (\bar{A}_{22}L_k + L_k\bar{A}_{12}L_k - \bar{A}_{21})\bar{A}_{11}^{-1} \tag{25a}$$

$$M_0 = 0; \qquad M_{k+1} = \bar{A}_{11}^{-1}(\bar{A}_{12}L_kM_k + M_k(\bar{A}_{22} + L_k\bar{A}_{12}) - \bar{A}_{12}). \tag{25b}$$

To obtain low order models it is assumed that the magnitude of the stable fast eigenvalues go to zero, thus all the fast modes are assumed to decay instantaneously. This is analogous to approximating all the fast eigenvalues as "deadbeat." For system (23), this means approximating the groups of fast eigenvalues that are clustered within an $0(\epsilon)$ radius of the origin of the unit circle as zero eigenvalues. Thus $\bar{X}_f(k) = 0$, which reduces (20b) to

$$X_f(k) = -L\bar{X}_s(k) + (I-ML)\bar{X}_f(k) = -L\bar{X}_s(k)$$

$$X_s(k) = \bar{X}_s(k) + M\bar{X}_f(k) = \bar{X}_s(k) \qquad \forall k > 0. \tag{26}$$

Applying the above expressions for all k, our reduced order model in the original coordinate system becomes

$$X_s^0(k) = (\bar{A}_{11} - \bar{A}_{12}L)X_s^0(k-1) + ((I-ML)\bar{B}_1 - M\bar{B}_2)U(k)$$

$$X_s^0(0) = \bar{X}_s(0) = X_s(0) \tag{27a}$$

and the fast states in the original coordinate system appear only as quasi-steady-state function of $\bar{X}_s$, i.e.,

$$X_f^0(k) = -LX_s^0(k). \tag{27b}$$

Note that (27b) may differ from the actual $X_f(k)$ states for some values of $k = 0$ to $k =$ (say k^*). However, since all the fast modes are stable in this analysis $X_f^0(k)$ will indeed converge to $X_f(k)$ after some $k > 0$.

The uncorrected slow subsystem is obtained from (27a) by formally setting $\epsilon = 0$, that is

$$X_s^0(k) = \bar{A}_{11}X_s^0(k-1) + \bar{B}_1U(k)$$

$$X_s^0(0) = X_s(0) \tag{28a}$$

$$Y(k) = (\bar{C}_{11} - \bar{C}_{22}L_0)X_s^0(k-1) + \bar{D}U(k) + V(k). \tag{28b}$$

More improved result is obtained by keeping $0(\epsilon)$ terms only, i.e., setting ϵ^2 and higher order terms in ϵ to zero and by letting $L = L_0$ and $M = M_1$ in the state equation and $L = L_1$ in the output equation. This yields the first-order "corrected slow" subsystem as

$$\begin{aligned} X_s^0(k) &= (\bar{A}_{11} - \bar{A}_{12}L_0)X_s^0(k-1) + ((I - M_1L_0)\bar{B}_1 - M_1\bar{B}_2)U(k) \\ &= (\bar{A}_{11} + \bar{A}_{12}\bar{A}_{21}\bar{A}_{11}^{-1})X_s^0(k-1) + ((I - \bar{A}_{11}^{-1}\bar{A}_{12}\bar{A}_{21}\bar{A}_{11}^{-1})\bar{B}_1 + \bar{A}_{11}^{-1}\bar{A}_{12}\bar{B}_2)U(k) \end{aligned} \tag{29a}$$

$$Y(k) = (\bar{C}_{11} - \bar{C}_{22}L_1)X_s^0(k-1) + \bar{D}U(k) + V(k) \tag{29b}$$

Fig. 2. Original "Mona Lisa" image.

Fig. 3. Blurred and noisy image ($\tau = 0.8$ and SNR $= 10$ dB).

This process can be continued to obtain more accurate slow subsystems. For most practical applications prescribed error bounds are obtained after 3 or 4 iterations of L and M. In general, let us assume that the relevant accuracy is achieved after "l" iterations, thus we have

$$X_s^0(k) = (\bar{A}_{11} - \bar{A}_{12}L_{l-1})X_s^0(k-1) + ((I - M_l L_{l-1})\bar{B}_1 - M_l\bar{B}_2)U(k) \quad (30a)$$

$$X_s^0(0) = X_s(0)$$

$$Y(k) = (\bar{C}_{11} - \bar{C}_{22}L_l)X_s^0(k-1) + \bar{D}U(k) + V(k). \quad (30b)$$

The fast states are obtained from (27b). For the uncorrected case $X_f^0(k) = -L_0X_s^0(k)$, and for the corrected case $X_f^0(k) = -L_lX_s^0(k)$.

IV. Kalman Filter Equations

Having derived the corrected slow subsystem, the problem of constructing Kalman filter for the full order system in (19) has now been reduced to that of constructing a Kalman filter for this subsystem.

Note that the slow state $X_s^0(k)$ in (30) consists of two parts associated with the image, i.e., $X_1^0(k)$, and the blur, i.e., $X_{3s}^0(k+1)$. Thus, the conditional mean estimate of $X_1^0(k)$ given the observation vectors up to $Y(k)$ gives the "filtered estimates" of the image whereas the conditional mean estimate of $X_{3s}^0(k+1)$ based upon these observation vectors leads to "one-step prediction estimates." Now taking into account the above facts and considering the presence of the feedthrough term with gain $\bar{D}$, the Kalman filter equations for the $\bar{D}$ model in (30) are derived to be

$$\hat{X}_s^0(k) = \tilde{A}\hat{\hat{X}}_s^0(k-1) \quad (31a)$$

$$K(k) = [\tilde{A}P(k-1)\tilde{C}^t + \tilde{B}Q_U\bar{D}^t] \cdot [\tilde{C}P(k-1)\tilde{C}^t + \bar{D}Q_U\bar{D}^t + \sigma_v^2 I]^{-1} \quad (31b)$$

$$\hat{\hat{X}}_s^0(k) = \hat{X}_s^0(k) + K(k)[Y(k) - \tilde{C}\hat{\hat{X}}_s^0(k-1)] \quad (31c)$$

$$P(k) = \tilde{A}P(k-1)\tilde{A}^t + \tilde{B}Q_U\tilde{B}^t - K(k) \times [\tilde{C}P(k-1)\tilde{A}^t + \bar{D}Q_U\tilde{B}^t] \quad (31d)$$

where $\hat{X}_s^0(k)$ and $\hat{\hat{X}}_s^0(k)$ are, respectively, the *a priori* (before updating) and the *a posteriori* (after updating) estimates of $X_s^0(k)$, $P(k)$ is the error covariance matrix, $K(k)$ is the Kalman gain matrix, and the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ are defined by

$$\tilde{A} \triangleq (\bar{A}_{11} - \bar{A}_{12}L_{l-1}) \quad (32a)$$

$$\tilde{B} \triangleq ((I - M_lL_l)\bar{B}_1 - M_l\bar{B}_2) \quad (32b)$$

$$\tilde{C} \triangleq (\bar{C}_{11} - \bar{C}_{22}L_{l-1}). \quad (32c)$$

V. Implementation and Results

The proposed scheme is implemented on a VAX 11/780 computer to restore the "Mona Lisa" image corrupted by both additive WG noise and LSI blur. Fig. 2 shows the original Mona Lisa image having a resolution of 512×512 pixels and 256 grey levels. The image is first blurred and then corrupted by additive WG noise to achieve a SNR of 10 dB. Fig. 3 shows the corrupted image which is used as the input to our restoration filter. LSI blur with separable PSF of form

$$h(m,n) = e^{-\tau m}e^{-\tau n}, \qquad (m,n) > 0 \quad (33)$$

is used. The blur factor τ is chosen to be 0.8. The PSF has been realized by the following separable 2-D state-space model [6].

$$R(m+1,n) = e^{-\tau}R(m,n) + e^{-\tau}S(m,n) + z(m,n)$$

$$S(m,n+1) = e^{-\tau}S(m,n) + z(m,n)$$

$$y(m,n) = e^{-\tau}R(m,n) + e^{-\tau}S(m,n) + z(m,n) \quad (34)$$

where $R(m,n)$, $S(m,n)$, $z(m,n)$, and $y(m,n)$ are respectively the vertical state vector, the horizontal state vector, the original uncorrupted image and the degraded image. The model in (34) can be arranged into a 1-D strip state-space structure like that in (11) with intrastrip and interstrip propagation characteristics [2]. The blur model matrices A_3, B_3, C_3, and D_3 in (11) can be formed using [2, eq. (57)].

The image model can be generated by computing the correlation matrices in each strip of width $W = 8$ and then solving a vector Yule–Walker equation. Since the image is assumed to be column wide-sense stationary in every strip, using the ergodicity property reasonable estimates of the correlation matrices can be evaluated using

$$\rho_i \triangleq E[Z(k-i)Z^t(k)] = \frac{1}{(N-i)}\sum_{k=i}^{N-1} Z(k-i)Z^t(k). \quad (35)$$

Fig. 4. Processed image using the reduced order strip Kalman filter (SNR = 16.6 dB).

Fig. 5. Processed image using the full order strip Kalman filter (SNR = 17.7 dB).

A fourth-order ($L = 4$) vector AR model is fitted to the image process in every strip and the relevant state-space equation (13) has been formed. This model is then combined with that of the blur to yield the full order composite dynamic structure. The order of this model is 41 ($= 4\times 8+8+1$). The norms of the coefficient matrices $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$, and $\hat{\phi}_4$ are evaluated in all the strips and the range of variations of the perturbation parameter ϵ is measured (for $M=1$) using (8) to be $0.573325 \leqslant \epsilon \leqslant 0.663886$. However, in order to set up the singularly perturbed model in (16), the value of ϵ should be determined in each strip. The norms of matrices $\hat{\phi}_1$, $\hat{\phi}_2$, $\hat{\phi}_3$, and $\hat{\phi}_4$ in a typical strip, say strip 33 where spatial variations exist, are evaluated to be

$$\|\hat{\phi}_1\| = 14.0965; \qquad \|\hat{\phi}_2\| = 0.490766$$

$$\|\hat{\phi}_3\| = 1.03670\times 10^{-2}; \qquad \|\hat{\phi}_4\| = 4.59258\times 10^{-2}.$$

These values dictate the decomposition of the system into an eight-order slow and 33rd-order fast subsystems. The value of the perturbation parameter ϵ based on norm criterion is

$$\epsilon = \max \frac{1-\|\hat{\phi}_1\|}{1-\|\hat{\phi}_i\|}, \qquad i = 2,3,4$$
$$= 0.58164.$$

The eigenvalues of the state matrix in the full order system (16) are also computed in this strip which confirm the validity of our method for calculating ϵ. Note that since $\epsilon > e^{-\tau}$ ($\tau = 0.8$), all the blur state variables are associated with the fast dynamics. Thus, the reduced order model which contains only the slow state variables, will be of size 8 corresponding to image states. Interestingly, this reduced order model closely captures the dynamics of the full order system. The Riccati and Lyapunov equations (25a) and (25b) are solved iteratively and the prescribed error bound (ϵ^5) is achieved in each strip after less than 10 iterations. The relevant values of $\boldsymbol{L}_l$ and $\boldsymbol{M}_l$ are then used in (30) to yield the corrected reduced order subsystem which is used in (31) for the filtering process. Note that (31b) and (31d) can be implemented off-line in each strip to compute the steady-state Kalman gain matrix needed for filtering process in (31a) and (31c). Fig. 4 shows the result of applying the reduced order strip Kalman filtering in (31) to the degraded image in Fig. 3. The SNR of this image is measured to be 16.6 dB. The CPU time for the entire process is approximately 8 min. Fig. 5 shows the result of apply direct strip Kalman filtering (full order) to the degraded image in Fig. 3. The SNR of image is found to be 17.7 dB and the CPU time is approximately 34 min. The comparison between the processed images indicates that the reduced order model that is of size 8, with the new matrices obtained through solving the Riccati and the Lyapunov equations in (25a) and (25b) iteratively is capable of performing satisfactory filtering and deconvolution. Note that, in spite of the fact that the blur states do not appear in the reduced order model, their effects and the aggregated effects of the other weakly correlated vectors, i.e., $Z(k-2)$, $Z(k-3)$, and $\underline{Z}(k-4)$ have been taken into account in matrices $\tilde{\boldsymbol{A}}$, $\tilde{\boldsymbol{B}}$, $\tilde{\boldsymbol{C}}$, and $\tilde{D}$ in the reduced order strip Kalman filter equations in (31). The uncorrected reduced subsystem leads to unsatisfactory result as shown in Fig. 6. The lines in this image appear as the state variables associated with the blur (both horizontal and vertical) are totally ignored in the uncorrected reduced order subsystems. The edge effects resulted from ignoring the blur states (mainly the horizontal ones) are disappeared in the image in Fig. 4, since the corrected reduced order subsystem contains the aggregated effects of these states as explained before.

Fig. 6. Processed image using uncorrected reduced order model (SNR = 14.3 dB).

VI. Conclusions

The time-scale property of an image processing system is effectively utilized for the decomposition–aggregation of the system into reduced order models. Although the results are stated for a separation in two time-scales, they can be extended to image processing systems with multiple time-scales using a nested application of decomposition–aggregation transformations. For the relevant reduced order models, new Kalman filtering equations are given which provide the suboptimal filtered estimates of the image states and the one-step prediction estimates of the blur states. The effectiveness of the proposed decomposition–aggregation method has been demonstrated on a real world image.

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A Unifying Frame for Stability-Test Algorithms for Continuous-Time Systems

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***Abstract*—A number of stability-test algorithms for linear continuous-time systems are presented in a unified manner, based on a general recursive relation for generating a sequence of polynomials of descending degree. Precisely, each polynomial in the sequence is expressed as a suitable combination of the "preceding" polynomial of higher degree and the polynomial with opposite zeros. It is shown how the general recursion may give rise to the known stability-test algorithms as well as to some families of new algorithms. The corresponding criteria are proved with the aid of simple geometrical considerations based on root loci. In this way an insight is gained into the nature of the different procedures.**

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I. Introduction

Recently, some notable attempts have been made to give a common interpretation to the algorithms for testing the stability of continuous- and discrete-time linear systems by adopting an approach based on the theory of pseudo-lossless functions [1] or an "interpolation" approach [2] or a "synthesis" approach [3].

In this paper, we show that the known stability-test algorithms for linear systems can be regarded as particular cases of a unique type of procedure that is capable of generating new stability tests too. The following treatment refers to the two-term recursive forms of the classical algorithms and to continuous-time systems, but the same approach could be used for discrete-time systems as well. Precisely, by generalizing both the Åström form of Routh's algorithm [4] and the s-domain Schur–Cohn type algorithm, we derive a recursion for generating a polynomial $P_{i-1}(s)$ of degree $i-1$ from a polynomial $P_i(s)$ of degree i and from $P_i(-s)$. It is then shown that $P_i(s)$ is Hurwitz if and only if $P_{i-1}(s)$ is Hurwitz and the values of the relevant parameters belong to suitable ranges. A geometrical interpretation of this property, based on the well-known root locus method [5], is given, which provides some insight into the similarities and differences between the considered procedures.

II. General Recursions

Let us consider the ith degree polynomial:

$$P_i(s) := \sum_{j=0}^{i} a_{i,j}s^j \tag{1}$$

and decompose it into the sum of its even and odd parts as

$$P_i(s) = Q_{i,i}(s) + Q_{i,i-1}(s) \tag{2}$$

where

$$Q_{i,i}(s) := \frac{1}{2}\left[P_i(s) + (-1)^i P_i(-s)\right] = a_{i,i}s^i + a_{i,i-2}s^{i-2} + \cdots \tag{3}$$

and

$$Q_{i,i-1}(s) := \frac{1}{2}\left[P_i(s) - (-1)^i P_i(-s)\right] = a_{i,i-1}s^{i-1} + a_{i,i-3}s^{i-3} + \cdots . \tag{4}$$

According to the standard Routh algorithm, a polynomial $P_{i-1}(s)$ of degree $i-1$ is formed as follows:

$$P_{i-1}(s) = Q_{i-1,i-1}(s) + Q_{i-1,i-2}(s) \tag{5}$$

whose even and odd parts are related to those of $P_i(s)$ via

$$Q_{i-1,i-1}(s) = Q_{i,i-1}(s) \tag{6a}$$

$$Q_{i-1,i-2}(s) = Q_{i,i}(s) - q_i s Q_{i,i-1}(s) \tag{6b}$$

where

$$q_i = a_{i,i}/a_{i,i-1}. \tag{7}$$

Relation (6b) is nothing but a particular case of the classical Euclid algorithm and corresponds to dividing $Q_{i,i}(s)$ by $Q_{i,i-1}(s)$ [6]. As shown, e.g., in [4], polynomial $P_{i-1}(s)$ may directly be expressed in terms of $P_i(s)$ as

$$P_{i-1}(s) = -\frac{q_i}{2}\left[\left(s - \frac{2}{q_i}\right)P_i(s) - (-1)^i s P_i(-s)\right] \tag{8}$$