TABLE II
ORDER N OF THE DIGITAL DIFFERENTIATORS FOR HIGH FREQUENCIES. N1 FOR MINIMAX
RELATIVE ERROR DIFFERENTIATORS [1]; N2 FOR MAXIMALLY LINEAR
DIFFERENTIATORS (PROPOSED)

| Frequency/\r
| Relative Error | N1 | N2 | N3 | N4 | N5 | N6 | N7 | N8 | N9 | N10 | Remarks |
|-------------|--------------|----|----|----|----|----|----|----|----|----|------|---------|
| 0.50 to 1.0 | -40 dB (1%)  | 22 | 7  | >158| 4  | >158| 4  | >158| 4  | >158| 6    | 1. The minimax relative error differentiators give uniform performance from \r
| 0.60 to 1.0 | -60 dB (0.1%)| *  | *  | *  | *  | *  | *  | *  | *  | *  | 6    | 2. Multiplications required per input sample for the minimax DD is \r
| 0.70 to 1.0 | -80 dB (0.001%)| *  | *  | *  | *  | *  | *  | *  | *  | *  | 10   | DD is \r
| 0.80 to 1.0 | -100 dB (0.0001%)| *  | *  | *  | *  | *  | *  | *  | *  | *  | 14   | DD is \r
| 0.90 to 1.0 | *  | *  | *  | *  | *  | *  | *  | *  | *  | *  | 10   | DD is \r

order N of the minimax DD's with twice the order N2 of the proposed differentiators since the proposed design contains twice as many coefficients as the minimax design of equal order. It may be seen that the maximally linear DD's have an edge over their minimax counterparts. If high accuracies are desired the maximally linear DD's are much superior to the minimax ones; for example, for the frequency coverage of 0.5\pi \leq w \leq \pi and a \|RE\| < 0.1 percent, the proposed DD requires only 16 multiplications per input sample as compared to 64 in the case of minimax DD. The designed digital differentiators are specially suitable for high frequencies (upto \pi/2) and for achieving extremely low RE's.

The values of the coefficients c_i and d_j, required in the proposed design, are computed by using the mathematical formulas (8), as against the lengthy optimization algorithms needed to realize the minimax differentiators. As in the minimax design, [1], \[ z^{-1/2} \] is imperative for the maximally linear differentiators; otherwise the proposed design is canonic. Since in multirate systems half-sample delay can be easily realized, the suggested design would be particularly suitable for such systems.

V. CONCLUSION

An efficient FIR digital differentiator structure, especially suitable for high frequency ranges, has been proposed. Mathematical relations for calculating the exact values of the weighting coefficients, required in the design, have been derived.

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REFERENCES


New Results in Strip Kalman Filtering

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Abstract — The Strip Kalman filtering proposed in [1] for image restoration is reconsidered. The procedure given in this reference for parameter estimation of the image model does not take into account the vector nature of the image process, and as a result can lead to incorrect identification. It

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is also shown that for the composite dynamic model derived in this reference the standard Kalman filtering equations cannot be applied, as the blur states in this model should be estimated one step ahead. These issues are addressed in this paper.

I. INTRODUCTION

In [1] Suresh and Shenoi proposed a strip Kalman filtering process which makes use of a vector scanning scheme. The image process is modeled by a finite-order vector autoregressive (AR) model which relates a column of pixels to the past columns in a certain region within a strip. It is assumed that the image process is wide sense stationarity within each strip. Based upon this assumption and ergodicity property the parameters of the vector AR model are evaluated using a Yule–Walker system of equations. This model is then arranged into a state-space form. The blur need to be estimated one step ahead of those associated with the image.

In this paper we have shown that the assumption of wide sense stationarity within each strip is not valid since the image is modeled by a vector or multichannel AR process. The assumption of column wide sense stationarity within each strip is more appropriate for these processes [2], [3]. In this connection, new procedures for estimating the parameters of the vector AR model are suggested. In addition, new Kalman filtering equations are derived which account for the combination of filtering for the image state and one-step prediction for the blur state.

II. A VECTOR AR MODEL FOR THE IMAGE PROCESS

Consider an \( N \times N \) image which is vector scanned horizontally in strips of width \( W \). The direction of scanning is assumed to be from left-to-right to top-to-bottom. Each strip is processed independently. The image is assumed to be represented by a vector (or multichannel) Markovian process and modeled within each strip by a \( P \)-th order vector AR process with a causal quar­tane plane region of support, \( R \) (see Fig. 1). This AR model is given by

\[
Z(k) = \phi_1 Z(k-1) + \cdots + \phi_p Z(k-p) + U(k)
\]

(1)

and \( Z(k) \) represents a \( W \times 1 \) vector with elements that are the pixel intensity values in the \( k \) th column of a given (say \( i \) th) strip in the image, i.e.,

\[
Z(k) = [z_{(i-1)W+1,k} \cdots z_{iW-1,k}]^T
\]

(2)

where \( z_{m,n} \) denotes the intensity of the pixel at location \((m,n)\). Vector \( U(k) \) which is defined similar to \( Z(k) \), represents a white noise vector process which drives the autoregression. The statistics of this process are

\[
E[U(k)] = 0 \quad \text{and} \quad E[U(k)U'(k-i)] = Q_U \delta(i)
\]

(3)

where \( Q_U \) is the covariance matrix of the error vector \( U(k) \); \( \delta(i) \) represents the Kronecker delta function and \( E \) denotes the expectation operator. Matrices \( \phi_1, \cdots, \phi_p \) are \( W \times W \) matrices that have to be identified in each strip. It is interesting to note that although \( U(k) \) is vectorially an uncorrelated process, this does not imply that the elements within each vector are also mutually uncorrelated. This fact can be evident when for each pixel in \( Z(k) \) a scalar relationship is derived from (1). This relationship which represents a particular pixel in \( Z(k) \) in terms of all the pixels in the support region \( R \) and the corresponding scalar error term, is a semicausal representation. The noncausality in this scalar model occurs along the vertical direction. It has been shown in [4]-[5] that semicausal finite order models are driven by a noise process which is white along the causal direction and color along the noncausal direction. Thus the elements of \( U(k) \) are mutually correlated as they are along the noncausal (vertical) direction, whereas the elements of \( U(k) \) and \( U(l) \), \( k \neq l \), are uncorrelated since these are along the causal (horizontal) direction. To elaborate on this issue, let us consider the minimum variance vector representation

\[
Z(k) = \hat{Z}(k) + U(k)
\]

(4)

where

\[
\hat{Z}(k) = \sum_{i=0}^{P} \phi_i Z(k-i) \quad \text{is the minimum variance estimate of } Z(k).
\]

The orthogonal properties of this estimator give

\[
E[Z(k-1)U'(k)] = Q_{U} \delta(i)
\]

and hence

\[
E[\hat{Z}(k)U'(k)] = 0.
\]

(5)

The scalar representation for each pixel \( z_{j+m,k} \) in \( Z(k) \) is given by

\[
z_{j+m,k} = \sum_{p,q \in S} a_{m,p,q} z_{j+p,k-q} + u_{j+m,k}, \quad m \in [0,W-1], \quad j = (i-1)W, i = 1,2,\cdots,N/W
\]

(7a)

where

\[
S = \{(p,q), 0 \leq p \leq W-1, 1 \leq q \leq P\}
\]

(7b)

and

\[
a_{m,p,q} = \phi_q(p,m)
\]

(7c)

where \( \phi_q(i,j) \) is the \((i,j)\)th entry of matrix \( \phi_p \) and \( u_{j+m,k} \) is the \( m \)th element of \( U(k) \). To show that \( u_{j+m,k} \)'s for \( m \in [0,W-1] \) are correlated, let us form

\[
r_{m,j+n,0} = E[u_{j+m,k}u_{j+n,k}]
\]

(8)
Now using the orthogonality property of (6) the covariance function \( r_z(m-n,0) = E[Z_{m,n} U_{m,n}] \neq 0 \) (9) since a vector model is used. Thus, in this case \( U_{m,n} \) cannot be modeled as a white noise process. In other words, for any multichannel process described by a vector AR model, it is not generally true to assume that each channel is also individually an AR process. Furthermore, considering the properties of the vector scan, the assumption of wide sense stationarity cannot be valid and thus must be changed to column wide sense stationarity within each strip which is more suitable for multichannel processes [2], [3]. Consequently, the procedure given in [1] for obtaining the AR model parameters is not valid since the correlation matrices are obtained as if the process is wide sense stationary and described by a series of decoupled single channel processes. This inconsistency in modeling and the parameter identification may result in a vector AR model with instability problems [2], [3].

III. Parameter Identification for the Vector AR Model

To estimate the parameter matrices \( \phi_1, \phi_2, \ldots, \phi_p \) of the model, transpose (1), premultiply both sides by \( Z(k-r) \) and then take the expectation. This yields

\[
\rho_r = E[Z(k-r)Z'(k)]
\]

\[
= \rho_r - \phi_1 \rho_r + \phi_2 \rho_{r-1} + \cdots + \phi_p \rho_{r-p} + E[Z(k-r)U'(k)]
\] (10)

where \( \rho_r \) is the covariance matrix of \( Z(k) \). Putting \( r = 0,1,\cdots, P \) in this normal equations gives the following vector Yule–Walker system of equations which must be solved for \( \phi_1, \phi_2, \ldots, \phi_p \) and \( Q_U \):

\[
\begin{bmatrix}
\rho_0 & \phi_1 & \cdots & \phi_p \\
\rho_1 & \rho_0 & \cdots & \phi_{p-1} \\
\vdots & \vdots & \ddots & \vdots \\
\rho_p & \rho_{p-1} & \cdots & \rho_0
\end{bmatrix}
\begin{bmatrix}
I \\
-\phi_1 \\
\vdots \\
-\phi_p
\end{bmatrix}
= Q_U
\] (11a)

where \( \rho_{-r} = \rho_r \) and \( I \) is an identity matrix of appropriate order. This equation in compact form is

\[
T \Phi = d
\] (11b)

From (11a) one gets

\[
I - T^{-1} \Phi = Q_U
\] (12a)

Thus we can write

\[
I = [T^{-1}]_{1,1} Q_U
\] (12b)

or

\[
Q_U = [T^{-1}]_{1,1}^{-1}
\]

and

\[
\Phi = - [T^{-1}]_{1,1} Q_U, \quad i=1,2,\ldots, P.
\] (12c)

Note that the definition of \( \rho_r \) given in [1, eq. (25)] does not apply here since it relates to the wide sense stationary case. In order to solve (11) estimates of \( \hat{\rho}_r \)'s are required. Invoking column wide sense stationarity assumption, reasonable estimates of these matrices can be obtained by ergodicity property and using

\[
\hat{\rho}_r = \frac{1}{N-r} \sum_{k=r}^{N-1} Z(k-r)Z'(k).
\] (13)

The parameter identification algorithm can be implemented online to update the model parameters at each stage of the process. Let us denote the estimate of \( \rho_r \) based upon \( n \) vectors in a given strip by \( \hat{\rho}_r(n) \). Thus we can write

\[
\hat{\rho}_r(n) = \frac{1}{n-r} \sum_{k=r}^{n-1} Z(k-r)Z'(k), \quad r \in [0, P].
\] (14)

Then the estimate based upon \( n+1 \) vectors is

\[
\hat{\rho}_r(n+1) = \frac{1}{(n+1-r)} \sum_{k=r}^{n} Z(k-r)Z'(k) = \hat{\rho}_r(n) + \frac{1}{n+1-r} [Z(n-r)Z'(n) - \hat{\rho}_r(n)]
\] (15)

Writing (15) for each constituent block in \( T \) yields

\[
\hat{\Phi}(n+1) = \hat{\Phi}(n) + \delta \hat{\Phi}(n)
\] (16a)

where

\[
\delta \hat{\Phi}(n) = \begin{bmatrix}
\delta \rho_0(n) & \delta \rho_1(n) & \cdots & \delta \rho_p(n) \\
\delta \rho_p(n) & \delta \rho_{p-1}(n) & \cdots & \delta \rho_0(n)
\end{bmatrix}
\] (16b)

Similarly \( \hat{d}(n+1) = \hat{d}(n) + \delta \hat{d}(n) \). Now using the matrix inversion lemma [1]

\[
\hat{T}^{-1}(n+1) = \hat{T}^{-1}(n) [I - \delta \hat{T}(n) \hat{T}^{-1}(n)]
\] (17)

The solution of the vector Yule–Walker equation (11) at iteration \( (n+1) \) can then be written [1] as

\[
\hat{\Phi}(n+1) = \hat{T}^{-1}(n+1) \hat{d}(n+1)
\]

\[
= \hat{T}^{-1}(n) [I - \delta \hat{T}(n) \hat{T}^{-1}(n)] \hat{d}(n+1)
\]

\[
= \hat{\Phi}(n) + \delta \hat{T}(n+1) \delta \hat{d}(n).
\] (18)

This equation provides a simple recursive scheme for estimating the vector AR model parameters when the original image is assumed to be given. The identification of the parameters from the corrupted image for multivariable stochastic systems is complicated due to lack of unique minimal realization for these systems [6].

IV. Kalman Filtering Equations

In [1] the vector AR model for the image is arranged into a 1-D state-space form, namely

\[
x_i(k) = A_i x_i(k-1) + B_i U(k)
\] (19a)

where

\[
x_i(k) = \begin{bmatrix} Z'(k) & Z'(k-1) & \cdots & Z'(k-P+1) \end{bmatrix}'
\] (19b)

and

\[
A_i = \begin{bmatrix}
\phi_1 & \phi_2 & \cdots & \phi_p \\
I & 0 & 0 & 0 \\
0 & I & \ddots & 0 \\
0 & 0 & \cdots & 0
\end{bmatrix}, \quad B_i = \begin{bmatrix}
I \\
0 \\
\vdots \\
0
\end{bmatrix}
\] (19c)
The vector of interest, \( Z(k) \), can be extracted from \( x_1(k) \) by projection lemma [7], i.e.,

\[
E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \left| \eta(k) \right. \right] = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(25)

Defining the Kalman gain matrix by

\[
K(k) = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(26)

The second term in (23) can be expressed using the orthogonal projection properties [7], i.e.,

\[
E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \left| \eta(k) \right. \right] = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(25)

Since \( V(k) \) is uncorrelated with the image and the blur states, the term \( E \left[ \eta(k) \eta(k) \right] \) in (26) can be written as

\[
K(k) = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(26)

\[
E \left[ \eta(k) \eta(k) \right] = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(25)

Defining the Kalman gain matrix by

\[
K(k) = E \left[ \begin{bmatrix} x_1(k) \\ x_1(k+1) \end{bmatrix} \eta(k) \right] \left[ E \left[ \eta(k) \eta(k) \right] \right]^{-1} \eta(k).
\]

(26)

Thus the expression for the Kalman gain becomes

\[
K(k) = A'\hat{P}(k)\tilde{C} + \sigma_I^2 I.
\]

(32)

Now, in order to compute the Kalman gain matrix, \( \hat{P}(k) \) must be evaluated recursively at every stage. Considering (19) we can
write
\[ \hat{P}(k) = (A_k \otimes I) \hat{P}(k-1)(A_k \otimes I)' + \begin{bmatrix} B_k \ 0 \end{bmatrix} Q_k \begin{bmatrix} B_k' \ 0 \end{bmatrix} \]  
where
\[ \hat{P}(k-1) \triangleq \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \] and \( \otimes \) denotes the direct sum operation. \( \hat{P}(k-1) \) is the a posteriori error covariance matrix at stage \( k - 1 \). To find the expression for \( \hat{P}(k) \), let us subtract (27) from the state vector at present stage \( k \), i.e.,
\[ \begin{bmatrix} x_1(k) \\ x_2(k+1) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} x_1(k) \\ x_2(k+1) - \hat{x}_2(k+1) \end{bmatrix} - \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} = K(k) \eta(k) \] or
\[ \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 \ \hat{x}_1(k) \\ A_2 \ \hat{x}_2(k) \end{bmatrix} - K(k) \eta(k). \] Now, transposing both sides, postmultiplying by (34) and taking expectation yields
\[ \hat{P}(k) = [A' - K(k) C] \hat{P}(k) A''. \] As a result, the Kalman filtering equations for computing the filtered estimate of the image and the one-step prediction estimate of the blur, are given in order by
\[ \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} = \begin{bmatrix} A_1 \ \hat{x}_1(k) \\ A_2 \ \hat{x}_2(k) \end{bmatrix} \] \[ \hat{P}(k) = (A_k \otimes I) \hat{P}(k-1)(A_k \otimes I)' + \begin{bmatrix} B_k \ 0 \end{bmatrix} Q_k \begin{bmatrix} B_k' \ 0 \end{bmatrix} \] \[ K(k) = \begin{bmatrix} A' \hat{P}(k) C[\hat{P}(k) C + \sigma^2 I]^{-1} \end{bmatrix} \] \[ \hat{\eta}(k) = \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k+1) \end{bmatrix} + K(k) \begin{bmatrix} \eta(k) \ - C \begin{bmatrix} \hat{x}_1(k) \\ \hat{x}_2(k) \end{bmatrix} \end{bmatrix} \] \[ \hat{P}(k) = [A' - K(k) C] \hat{P}(k) A''. \] The form of the above equations [7] and the presence of matrix \( A' \) with its special structure clearly reveal the fact that these equations represent the combination of filtering and one-step prediction.

V. Conclusion

New procedures for parameter identification of the image model in strip Kalman filtering is suggested. This method which takes into account the vector nature of the image model, leads to a vector Yule–Walker system of equations. An algorithm for on-line adaptation of the image model parameters is also given. In the composite dynamic model for the image and degradation processes, the blur states are computed one-step ahead of those of the image. Thus a new set of Kalman filtering equations which accounts for both one-step prediction for the blur states and filtering for the image states was required. The derivations of these equations are presented.