DISSERTATION

# New Constructions of Strongly Regular Graphs

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## Abstract


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There are many open problems concerning strongly regular graphs: proving non-existence for parameters where none are known; proving existence for parameters where none are known; constructing more parameters where examples are already known. The work addressed in this dissertation falls into the last two categories. The methods used involve symmetry, geometry, and experimentation in computer algebra systems.

In order to construct new strongly regular graphs, we rely heavily on objects found in finite geometry, specifically two intersection sets and generalized quadrangles, in which six independent successes occur. New two intersection sets are constructed in finite Desarguesian projective planes whose strongly regular graph parameters correspond to previously unknown and known ones. An infinite family of new two intersection sets is also constructed in finite projective spaces in 5 dimensions. The infinite family of strongly regular graphs have the same parameters as Paley graphs. Next, using the point graph of the classical GQ $H(3, q^2)$, $q$ even, a new infinite family of strongly regular graphs is constructed. Then we generalize three infinite families of strongly regular graphs from large arcs in Desarguesian projective planes to the non-Desarguesian case. Finally, a construction of strongly regular graphs from ovoids of generalized quadrangles of Godsil and Hensel is applied to non-classical generalized quadrangles to obtain new families of strongly regular graphs.

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CHAPTER 1

# Introduction

The main aim of this thesis is the construction of strongly regular graphs, a concept introduced by Bose in 1963 [14]. Interestingly, Bose was a former professor at CSU in the 1970's.

A graph is regular if every vertex has a constant number of neighbors. It is strongly regular if, in addition, any pair of adjacent vertices have a constant number of neighbors and any pair of non-adjacent vertices have a constant number of neighbors.

The study of strongly regular graphs is part of algebraic graph theory. One illustration of this is that amongst regular graphs that are neither complete nor null, the strongly regular graphs are precisely those graphs with the algebra generated by the adjacency matrix of minimum dimension (namely, two).

The methods used in this thesis to construct strongly regular graphs are a novel mix of computer algebra, group theory and finite geometry. The computer algebra is used to perform experiments to discern the truth, and leans on the finite group theory to do this. The finite geometry is a constant companion.

The thesis has six independent successes. In Chapter 4, two of them occur. First, new two intersection sets are constructed in finite Desarguesian projective planes giving strongly regular graphs, some with new parameters, and some new ones with known parameters. Second, an infinite family of new two intersection sets are constructed in finite projective spaces of five dimensions giving an infinite family of strongly regular graphs. In Chapter 5, the whole mix of techniques is applied to construct new infinite families of strongly regular graphs with the parameters of the collinearity graphs of some classical generalized quadrangles. In Chapter 6, two of the successes occur. The first is tripartite: three different constructions of infinite families of strongly regular graphs from large arcs in Desarguesian projective planes are generalized to the non-Desarguesian case. The second is a generalization of one of these three to maximal arcs in non-Desarguesian planes. In Chapter 7, a construction of strongly regular graphs from ovoids of generalized quadrangles of Godsil and Hensel is applied to non-classical generalized quadrangles to obtain new infinite families of strongly regular graphs.

There are recurrent themes. Finite projective geometries are used explicitly in all but Chapter 7, and there the non-classical generalized quadrangles are constructed in finite projective spaces of dimension 5. Two intersection sets occur explicitly in the results of Chapters 4 and 6. Generalized quadrangles occur explicitly in the results of Chapters 5 and 7, and implicitly in Chapter 6. Classical simple groups underly all the successes: $PSL(3, q)$ in Chapters 4 and 6, $P\Omega(3, q)$ in Chapter 6, $PSp(6, q)$ in Chapter 7, $P\Omega^{-}(6, q)$ in Chapter 1, and in its other guise as $PSU(4, q^2)$ in Chapter 5.

Put more broadly, our mix of techniques involves experimental mathematics enabled by computer algebra, geometry and symmetry.

Finite geometry is the study of an incidence structure with a finite number of elements. Projective geometry as an independent subject was introduced by Poncelet's treatise in 1822 [86]; affine geometry by Möbius in 1827 [72]. Finite projective geometries first attracted serious interest in the 1890's in the work of Peano and Fano. In dimension at least three, all projective and affine geometries arise from vector spaces; but in dimension two, there are other geometries, for which the theorem of Desargues (1648) fails. (Consequently, finite projective spaces of dimension at least three and finite Desarguesian projective planes are operands for the simple projective special linear groups.) Finite projective spaces occur throughout the thesis; non-Desarguesian finite affine planes occur in Chapter 6 in order to obtain a more general result for large arcs.

One of the sources of great beauty in projective geometry is the principle of duality. If the distinction between a projective plane and its dual is abandoned, then projective planes can be considered as being bipartite graphs with girth six and diameter three, such that every vertex has at least three neighbors. Bipartite graphs with girth double their diameter, such that every vertex has at least three neighbors are called generalized polygons, a concept introduced by Jacques Tits in 1959 [107]. The case of diameter four-generalized quadrangles- occurs in many places throught the thesis. Both the collinearity and concurrency graphs of finite generalized quadrangles are strongly regular. Partial geometries, a further generalization of generalized quadrangles, were introduced by Bose in 1963, and they occur in Chapter 6. (They also have collinearity and concurrency graphs of finite generalized quadrangles that are strongly regular.)

Configurations in finite projective spaces also play a role. Two intersection sets (sets of points such that every hyperplane meets them in one of two numbers) were connected to two weight codes and strongly regular graphs by Delsarte in the 1970's [22]. Special two intersection sets include quadrics and Hermitian varieties, which are associated with the classical orthogonal and unitary simple groups, respectively. Maximal arcs and hyperovals are also special instances of two intersection sets. Spreads of projective spaces have been used to construct projective planes since the work of Andre in 1954, independently discovered by Bruck and Bose in 1964. The fact that some of these spreads consist of totally isotropic subspaces with respect to a non-degenerate alternating form is one motivation for studying spreads in generalized quadrangles, and the dual concept, ovoids. (Another motivation is given by the family of simple groups discovered by Suzuki in 1959, see [106].) Families of (classical) generalized quadrangles are associated with the classical groups of Lie rank two.

Hyperovals (and related arcs) were used to construct generalized quadrangles by Tits in 1968, Aherns and Szekeres in 1969, Hall in 1971 and Payne in 1985. Maximal arcs were used to construct partial geometries in two ways by Wallis and Thas in the 1970's. Ovoids of generalized quadrangles were used to construct strongly regular grpahs by Godsil, building on work of his student Hensel, all of which will be discussed in further detail.

Thus, despite the focus of this paper being on strongly regular graphs, much of this work can be related to many other areas within finite geometry.

## 1.1. Symmetry

In addition to these objects, symmetry is a central component when constructing new strongly regular graphs. Not only is symmetry useful for constructing, the use of groups is necessary for the characterization, classification, and calculation of objects. Objects are distinguished via their automorphism groups.

If one chooses a permutation group at random and calculates all the strongly regular graphs on which it acts, then, with probability one, you will get none. Some permutations are not automorphisms of strongly regular graphs. One way to avoid these bad permutations is to choose a subgroup of the automoprhism group of a known strongly regular graph. Unlike in group theory, a small subgroup is better, for then there are more likely to be new graphs found. If the known strongly regular graph has a small group, then this technique is less likely to be effective (or possible to be implemented). When it can be implemented, it allows the consideration of paramter sets beyond the range of complete enumeration techniques. This is one explanation for the recurring role of classical groups in the thesis- two others being their beauty and the ease of understanding their subgroups.

## 1.2. Graphs

While symmetry is an underlying and unifying theme, so are graphs. Graph theory is the study of graphs, which are structures used to depict pairwise relations between objects. Graphs are used in both mathematics and computer science. Graphs are comprised of **vertices** and lines or **edges** that connect vertices. More specifically, a **graph** is an ordered pair $G = (V, E)$ consisting of a set $V$ of **vertices** together with a set $E$ of **edges**, which are two-element subsets of $V$.

### 1.2.1. History.

The study of graphs did not begin until 1736 when Leonhard Euler published a paper on the *Seven Bridges of Konigsberg* [50]. The ideas that Euler used (relating edges and vertices) were picked up by Cauchy and L'Huiller in order to study convex polyhedrons [25, 70], which was the start of topology. More than a century after the Konigsberg result, Cayley discovered connections between trees (a specific type of graph) and calculus [26]. These results lead to the branch of graph theory known as enumerative graph theory, which has had numerous implications in the field of chemistry. (The other branches of graph theory are algebraic, geometric, and algorithmic.) The first text on graph theory was not published until 1936 and it was written by Denes Konig [67]. It is believed that it wasn't until Frank Harary's book [63], published in 1969, that mathematicians, chemists, and engineers could speak to one another about graph theory. Since then, various texts have been published on the subject.

1.2.2. Definitions. There are numerous adjectives associated with graphs. The definitions provided here are the only ones necessary for the remainder of this paper. A graph is **undirected** whenever each edge has no orientation. That is, edge $\{a, b\}$ is identical to edge $\{b, a\}$. We will assume that our graphs are undirected. In addition to that, the graphs will be simple. A **simple** graph is an undirected graph containing no loops or multiple edges. In order for a graph to be loopless, there can be no edges of the form $\{a, a\}$. A **complete** graph is a simple undirected graph in which every pair of distinct vertices is connected by a unique edge. A graph is **regular** if each vertex has the same number of neighbors; i.e. every vertex has the same degree or valency. A **distance-regular graph** is a regular graph such that for any two vertices $v$ and $w$, the number of vertices at distance $j$ from $v$ and at distance $k$ from $w$ depends only on $j$, $k$, and the distance $i$ between $v$ to $w$. We can also discuss the diameter of a graph, which deals with the distance between two vertices. The **distance** between two vertices in a graph is the number of edges in the shortest path connecting them. There can be more than one shortest path between two vertices. The **diameter** of a graph is the maximum distance between all pairs of vertices in that graph. A distance-regular graph with diameter 2 is strongly regular. Lastly, a **bipartite graph** is a graph whose vertices can be divided into two disjoint sets $U$ and $V$ such that every edge connects a vertex in $U$ to one in $V$. Bipartite graphs will be discussed in further detail in Chapter 7. The **girth** of a graph is the length of its shortest cycle.

1.2.3. Adjacency Matrix. For each graph, there is an associated adjacency matrix. The **adjacency matrix** for a finite graph on $n$ vertices is an $n \times n$ matrix where the non-diagonal entry $a_{ij}$ equals the number of edges from vertex $i$ to vertex $j$. Entry $a_{ii} = 0$, as the graphs we will be dealing with are loopless. Furthermore, $a_{ij}$ will equal 0 or 1, as we will not allow for multiple edges between vertex $i$ and vertex $j$. If the graph is undirected, which we are assuming, then the adjacency matrix is symmetric. That is, $a_{ij} = a_{ji}$ for all $i, j \in \{1, 2, ..., n\}$.

An adjacency matrix can embody multiple characteristics of a graph. Since the adjacency matrix of an undirected simple graph is symmetric, it must have real eigenvalues and an orthogonal eigenvector. Furthermore, suppose two graphs $\Gamma_1$ and $\Gamma_2$ have adjacency matrices $A_1$ and $A_2$, respectively. Then $\Gamma_1$ and $\Gamma_2$ are isomorphic if and only if there exists a permutation matrix $P$ such that

$$PA_1P^{-1} = A_2.$$

In particular, if $\Gamma_1$ and $\Gamma_2$ are isomorphic, then by definition, $A_1$ and $A_2$ are similar, which implies that they have the same minimal polynomial, characteristic polynomial, eigenvalues, determinant, and trace. These five objects can then serve as isomorphism invariants of graphs. However, it should be noted that two graphs can have the same set of eigenvalues but not be isomorphic.

1.2.4. Automorphism Group. As previously mentioned, the automorphism group of a graph is a distiguishing characteristic between like graphs. An **automorphism** of a graph $G = (V, E)$ is a permutation $\sigma$ of the vertex set $V$, such that the pair of vertices $(u, v)$ form an edge of $G$ if and only if $(\sigma(u), \sigma(v))$ also form an edge. Less formally, an automorphism of a graph $G$ is a graph isomorphism from $G$ to itself. The **automorphism group** of a graph is the set of automorphisms of that graph, under the composition function.

The Petersen graph is an undirected graph with 10 vertices and 15 edges. It is a useful graph in the sense that it is small and can be used as an example or a counterexample for many graph theory problems. The automorphism group of the Petersen graph is $S_5$, the symmetric group on 5 points. To illuminate this fact, it is important to note that the Petersen graph is symmetric, meaning it is edge and vertex transitive. An even stronger condition is that it is 3-arc transitive, meaning that every directed 3-edge path can be transformed into every other such path by a symmetry of the graph. Every homomorphism of the Petersen graph to itself that doesn't identify adjacent vertices is an automorphism. There are various ways of drawing the Petersen graph, each of which can exhibit 5-way or 3-way symmetry, but it is impossible to draw the graph in the plane in such a way as to exhibit the full symmetry group, which helps to emphasize the fact that computing a graph automorphism is NP hard.

1.2.5. Applications. Graphs are used to model various relations and processes; they are used to represent many practical problems, ranging from biology to information systems. Biochemists use them in order to study genomics; electrical engineers use them to represent communication networks and for coding theory. Graphs are also used in operations research for scheduling purposes. While the previously mentioned applications are very important for their specific users, graphs can also be used for less practical purposes, like playing chess. Besides the practical application side of graph theory, the methods used to study such objects have proven to be useful when proving results in various arenas of mathematics, like Fermat's Little Theorem. The coding-theoretic duals of two-weight codes- uniformly packed codes- have found application in communication over noisy channels- the theory of error-correcting codes.

## 1.3. A Case Study

The result in Chapter 5 illustrates beautifully the mixture of ideas. The result began with a known graph which had the same parameters as the collinearity graph of the classical generalized quadrangle $H(3,4)$ associated with the unitary group $PSU(4,4)$. The automorphism group of this graph suggested using the stabilizer of a point in $P\Gamma U(4, q^2)$. Computer algebra experiments in Magma showed that similar graphs could be constructed from subgroups of $P\Gamma U(4, q^2)$, for $q$=4 and 8, but not for $q$ odd. Proper subgroups of the stabilizer of a point in $P\Gamma U(4, q^2)$ were needed. Moreover, the pre-existing knowledge of the subgroup structure of $P\Gamma U(4, q^2)$ identified the appropriate group as being the stabilizer of two orthogonal tangent lines to the Hermitian variety at the point (which equalled the stabiliser of the point when $q = 2$). This suggested that there is a construction of a new family of strongly regular graphs from the classical generalized quadrangle $H(3, q^2)$ and that this construction is in terms of a point $P$ of the generalized quadrangle and two lines $t_1, t_2$ of $PG(3, q^2)$ on $P$, with only totally isotropic point $P$, which are interchanges by the unitary polarity. Further computer algebra experiments in Magma suggested what the new adjacency should be. Then a proof was sought and found. It should be noted that exerperimentation was used throughout this process, including during the write up of the proof, with finite geometry providing the language necessary to describe what was occuring.

CHAPTER 2

# Background

Strongly regular graphs originate from the study of association schemes. This section will begin by looking at the evolution of strongly regular graphs, ending with how they are related to finite projective spaces.

## 2.1. Association Schemes

The history of strongly regular graphs is a short one, with the term being coined in 1963. The driving force behind such graphs are association schemes. While Bose and Shimamoto first defined association schemes in 1952 in terms of partially balanced designs [16], we will present the standard definition. An $n$-class **association scheme** consists of a set $X$ together with a partition $S$ of $X \times X$ into $n+1$ binary relations, $R_0$, $R_1, ..., R_n$ which satisfy the following:

(i) $R_0 = \{(x, x)|x \in X\}$,

(ii) If $R \in S$, then $R^* := \{(x, y)|(y, x) \in R\} \in S$, and

(iii) If $(x, y) \in R_k$, then the number $z \in X$ such that $(x, z) \in R_i$ and $(z, y) \in R_j$ is a constant $p_{ij}^k$ depending on $i, j, k$.

With Bose and Shimamoto first coining the term association scheme, appropriately enough its primary use was in statistics. And despite the term being coined so late, the concept actually arose in 1939 when Bose and Nair published a paper on Partially Balanced Incomplete Block Designs, as these are categorically association schemes [15]. Nevertheless, association schemes provide a unified approach to combinatorics, coding theory, and graph theory, with specific uses in statistics as well.

Bose first introduced the concept of strongly regular graphs in his 1963 paper [14], stating that strongly regular graphs are isomorphic to Partially Balanced Incomplete Block Designs with 2 symmetric associate classes. However, this is not the standard definition. Rather, a graph $\Gamma$, that is simple, undirected, and loopless of order $v$ is a **strongly regular graph** with parameters $v, k, \lambda$, and $\mu$ whenever $\Gamma$ is not complete or edgeless and

(i) each vertex is adjacent to $k$ vertices,

(ii) for each pair of adjacent vertices, there are $\lambda$ vertices adjacent to both, and

(iii) for each pair of non-adjacent vertices, there are $\mu$ vertices adjacent to both.

The parameters of a strongly regular graph are denoted as $(v, k, \lambda, \mu)$. Per the original definition, a strongly regular graph is a symmetric association scheme with two non-identity classes, where by one relation is the adjacency relation and the parameters are given in terms of the constants $p_{ij}^k$.

2.1.1. Examples of Strongly Regular Graphs. Two simple well-known examples of strongly regular graphs include the pentagon, whose parameters are $(5, 2, 0, 1)$, and the Peterson graph, whose parameters are $(10, 3, 0, 1)$. Interestingly enough, each of these graphs are uniquely determined by their parameters. This is not always the case though, as will be seen later on.

## 2.2. Finite Projective Spaces

While the definition of a strongly regular graph and an association scheme give no immediate insight as to how they are related to finite projective spaces, these three areas are, in fact, interrelated. These relationships were not exposed until 1968 by Delsarte, who made specific connections between certain strongly regular graphs and other combinatorial objects [22]. The connecting objects are projective two-weight codes, $t$-sets of type $(m, n)$, and difference sets. The connection between $t$-sets of type $(m, n)$ and strongly regular graphs is a driving force behind understanding finite projective spaces. (Generalized quadrangles are another reason to study projective spaces. This will be discussed extensively in Chapter 5.)

2.2.1. Definitions and Properties. In order to define a projective space, we must first understand incidence geometry. An **incidence structure** is a triple $(X, f, I)$, where $X$ is a set, $f$ is a function whose domain is $X$, and $I$ is a symmetric relation on $X$ such that for any $x, y \in X$, if $xIy$ and $f(x) = f(y)$, then $x = y$. The set $X$ is viewed as the objects of the incidence structure, $f(x)$ as the type of $x$, and $I$ as the incidence relation on $X$. A finite projective space is an example of an incidence geometry.

Let $V$ be a finite-dimensional vector space of dimension $d+1$ over the finite field $\mathrm{GF}(q)$. Let $X$ be a set of proper, non-trivial subspaces of $V$, $f(U) = \dim(U)$, with incidence defined by symmetrized inclusion. The resulting structure is the **projective geometry** $\mathcal{PV} = \mathrm{PG}(d, q)$. The one-dimensional subspaces are called points, the two-dimensional subspaces are called lines, and the $d$-dimensional subspaces are hyperplanes. When $d = 2$, we obtain the projective plane $\mathrm{PG}(2, q)$, which is of particular interest to us, as will be seen later on.

In addition to the definition given here, projective spaces can also be defined synthetically. Defining a projective space synthetically can be beneficial to understanding $t$-sets of type $(m, n)$, so we will present that definition as well. A **projective space** is a collection of points and lines such that

(i) there is a unique line through two distinct points;

(ii) every line contains at least three points;

(iii) there is at least one line;

(iv) if a line meets two sides of a triangle, not at a vertex, then the line meets the third side of the triangle.

The projective space $\mathrm{PG}(d,q)$ is the collection of subspaces of $\mathrm{GF}(q)^{d+1}$, and it is common to denote the point $<(x_1,...,x_{d+1})>\in \mathrm{PG}(d,q)$ by its **homogeneous coordinates** $(x_1,...,x_{d+1})$. This is because a point is defined up to multiplication by an element of $\mathrm{GF}(q)^*$.

THEOREM 2.2.1. *Let* $\mathbf{P}$ *be a finite projective space of dimension* $d$ *and order* $q$*; that is,* $\mathbf{P}=\mathrm{PG}(d,q)$. *Let* $\mathbf{U}$ *be a* $t$*-dimensional subspace of* $\mathbf{P}$ $(1\leq t\leq d)$. *Then the following statements are true:*

*(i) The number of points of* $\mathbf{U}$ *is*

$$q^t+q^{t-1}+...+q+1=\tfrac{q^{t+1}-1}{q-1}.$$

*In particular,* $\mathbf{P}$ *has exactly* $q^d+...+q+1$ *points.*

*(ii) The number of lines of* $\mathbf{U}$ *through a fixed point of* $\mathbf{U}$ *equals*

$$q^{t-1}+...+q+1.$$

*(iii) The total number of lines of* $\mathbf{U}$ *equals*

$$\tfrac{(q^t+q^{t-1}+...+q+1)(q^{t-1}+...+q+1)}{q+1}.$$

The proof of parts (i) and (ii) rely on induction, while the proof of part (iii) relies solely on a counting argument.

THEOREM 2.2.2. *Let* $\mathbf{P}$ *be a finite projective space of dimension* $d$ *and order* $q$*. Then:*

*(i) The number of hyperplanes of* $\mathbf{P}$ *is*

$$q^d + ... + q + 1;$$

*(ii) The number of hyperplanes of* $\mathbf{P}$ *through a fixed point of* $\mathbf{P}$ *equals*

$$q^{d-1} + ... + q + 1.$$

PROOF. Part (i) will be proven by induction on $d$. For $d = 1$, the theorem says that any line contains $q + 1$ points. For the case when $d = 2$, the assertion follows from the previous theorem. Suppose (i) is true for all projective spaces of dimension $d - 1 \geq 1$. Let $\mathbf{H}$ be a hyperplane of $\mathbf{P}$. Since $\dim(<\mathbf{M}, \mathbf{N}>) = \dim(\mathbf{M}) + \dim(\mathbf{N}) - \dim(\mathbf{M} \cap \mathbf{N})$ for any subspaces $\mathbf{M}$ and $\mathbf{N}$ of $\mathbf{P}$, every hyperplane different than $\mathbf{H}$ intersects $\mathbf{H}$ in a subspace of dimension $d-2$. Thus, any hyperplane not equal to $\mathbf{H}$ of $\mathbf{P}$ is spanned by a $(d-2)$-dimensional subspace of $\mathbf{H}$ and a point outside of $\mathbf{H}$.

For each $(d-2)$-dimensional subspace $\mathbf{U}$ of $\mathbf{H}$ and each point $P \in \mathbf{P} \setminus \mathbf{H}$, the subspace $< \mathbf{U}, P >$ is a hyperplane, which contains $(q^{d-1} + ... + q + 1) - (q^{d-2} + ... + q + 1) = q^{d-1}$ points outside of $\mathbf{H}$. Since there exist $q^d$ points of $\mathbf{P}$ outside of $\mathbf{H}$, there are $q$ hyperplanes not equal to $\mathbf{H}$ through $\mathbf{U}$. By induction, there are exactly $q^{d-1} + ... + q + 1$ hyperplanes of $\mathbf{H}$, which are subspaces of dimension $d - 1$. Thus, the total number of hyperplanes of $\mathbf{P}$ is $q(q^{d-1} + ... + q + 1) + 1 = q^d + ... + q + 1$.

(ii) Let $P$ be a point of $\mathbf{P}$ and let $\mathbf{H}$ be a hyperplane not containing $P$. Every hyperplane of $\mathbf{P}$ through $P$ intersects $\mathbf{H}$ in a hyperplane of $\mathbf{H}$. Thus, by (i), there are exactly $q^{d-1} + ... + 1$ such hyperplanes. $\square$

Since finite projective planes are a major component of this thesis, we present the following Corollary.

COROLLARY 2.2.3. *Let* $\mathbf{P}$ *be a finite projective plane. Then there exists an integer* $q \geq 2$ *such that every line of* $\mathbf{P}$ *has exactly* $q+1$ *points, and the total number of points is* $q^2+q+1$.

2.2.2. DESARGUESIAN PLANES. All projective planes of the form $\mathrm{PG}(2,q)$ satisfy Desargues' Theorem, named after Girard Desargues. Let $\mathbf{P}$ be a projective space. Then the **theorem of Desargues** holds if the following statement is valid:

For any choice $A_1$, $A_2$, $A_3$, $B_1$, $B_2$, and $B_3$ of points with the properties

- $A_i$, $B_i$ are collinear with a point $C \neq A_i \neq B_i$ for all $i$,
- no three of the points $A_1$, $A_2$, $A_3$, and $C$ and no three points $C$, $B_1$, $B_2$, and $B_3$ are collinear,

we have that the points

$$P_{12} := A_1A_2 \cap B_1B_2,\ P_{23} := A_2A_3 \cap B_2B_3, \text{ and } P_{31} := A_3A_1 \cap B_3B_1$$

lie on a common line.

THEOREM 2.2.4. *Desargues' Theorem holds in* $\mathrm{PG}(d,q)$, $d \geq 3$.

PROOF. Let $v_1$, $v_2$, and $v_3$ be vectors such that $A_1 =< v_1 >$, $A_2 =< v_2 >$, and $A_3 =< v_3 >$. Since $A_1$, $A_2$, and $A_3$ are not collinear, $v_1$, $v_2$, and $v_3$ are linearly independent. Therefore they form a basis of the 3-dimensional vector space $V :=< v_1, v_2, v_3 >$.

**Case 1:** The point $C$ lines in the plane that is spanned by $A_1$, $A_2$, and $A_3$.

Then there are $a_1, a_2, a_3 \in \mathrm{GF}(q)$ with $C =< a_1v_1 + a_2v_2 + a_3v_3 >$. Since no three of the points $A_1$, $A_2$, $A_3$, and $C$ are collinear, we have $a_1, a_2, a_3 \neq 0$. Therefore, replacing $v_i$ by $a_iv_i$, if necessary, we may assume without loss of generality that $C =< v_1 + v_2 + v_3 >$.

Since $C$, $A_i$, and $B_i$ are collinear, there exists an $a_1, a_2, a_3 \in \mathrm{GF}(q)$ such that

$$B_1 =< v_1 + v_2 + v_3 + a_1v_1 >=< (a_1+1)v_1 + v_2 + v_3 >,$$

$$B_2 =< v_1 + (a_2 + 1)v_2 + v_3 >,$$

$$B_3 =< v_1 + v_2 + (a_3 + 1)v_3 >.$$

Now we can determine $P_{ij}$.

$$P_{12} = A_1A_2 \cap B_1B_2 =< a_1v_1 - a_2v_2 >,$$

$$P_{23} =< a_2v_2 - a_3v_3 >, \text{ and}$$

$$P_{31} =< a_3v_3 - a_1v_1 >.$$

Thus, all three points are collinear and lie on the line

$$< a_1v_1 - a_2v_2, a_3v_3 - a_1v_1 >.$$

**Case 3:** The points $C =< v >$, $A_1 =< v_1 >$, $A_2 =< v_2 >$, and $A_3 =< v_3 >$ do not lie on a common plane.

The vectors $v$, $v_1$, $v_2$, and $v_3$ are linearly independent. Therefore, without loss of generality, we assume that

$$B_1 =< v + v_1 >,$$

$$B_2 =< v + v_2 >, \text{ and}$$

$$B_3 =< v + v_3 >.$$

It then follows that

$$P_{12} =< v_1 - v_2 >,$$

$$P_{23} =< v_2 - v_3 >, \text{ and}$$

$$P_{31} =< v_3 - v_1 >.$$

These three points lie on the line $< v_1 - v_2, v_2 - v_3 >$.

Therefore, Desargues' Theorem holds for $\mathrm{PG}(d, q)$, $d \geq 3$. $\square$

More generally, Desargues' Theorem holds for the real projective plane, for any projective space defined from a division ring, and for any projective space in which Pappus's Theorem holds.

Let **P** be a projective space. We say that in **P** the **theorem of Pappus** holds if any two intersecting lines $g$ and $h$ with $g \neq h$ satisfy the following condition. If $A_1$, $A_2$, and $A_3$ are distinct points on $g$ and $B_1$, $B_2$, and $B_3$ are distinct points on $h$ all different from $g \cap h$, then the points

$$Q_{12} = A_1B_2 \cap B_1A_2,$$

$$Q_{23} = A_2B_3 \cap B_2A_3, \text{ and}$$

$$Q_{31} = A_3B_1 \cap B_3A_1$$

lie on a common line.

On the other hand, a **non-Desarguesian plane** is a projective plane that does not satisfy Desargues' Theorem. Examples of non-Desarguesian planes include

- the Moulton plane,
- 3 projective planes of order 9, each with 91 points and 91 lines,
- Hughes planes,
- Moufang planes,
- André planes.

2.2.3. THE GROUP. Additionally, these objects are highly symmetric, and in order to determine how much symmetry they actually possess, we need to define an automorphism. An **automorphism** (or **collineation**) of $\mathrm{PG}(d, q)$ is an incidence preserving bijection from the point set onto itself. We can then describe the automorphisms by considering invertible matrices $A$. Multiplication by $A$ is a linear map for the underlying vector space $\mathrm{GF}(q)^{d+1}$

that preserves inclusion of subspaces. We can also think of $A$ as acting on the projective space by multiplication on the homogeneous coordinates of points in $\mathrm{PG}(d, q)$. But this means that $A$ is only defined up to multiplication by a scalar, in particular, an element of $\mathrm{GF}(q)^*$. This tells us that the group $\mathrm{PGL}(d+1, q) = \mathrm{GL}(d+1, q)/\{\lambda I : \lambda \in \mathrm{GF}(q)^*\}$ is a subgroup of $\mathrm{Aut}(\mathrm{PG}(d, q))$. Unfortunately, the use of the matrix $A$ does not lead to all symmetries. A map of the form $x \mapsto x^\alpha$ for $\alpha \in \mathrm{Aut}(\mathrm{GF}(q))$ is also an automorphism of $\mathrm{PG}(d, q)$. In addition, we can compose both types of automorphisms in order to obtain other automorphisms. Thus, the automorphism group of $\mathrm{PG}(d, q)$ is $\mathrm{P\Gamma L}(d+1, q) = \{x \mapsto Ax^\alpha : A \in \mathrm{PGL}(d+1, q), \alpha \in \mathrm{Aut}(\mathrm{GF}(q))\}$, and any symmetry of $\mathrm{PG}(d, q)$ has this form.

## 2.3. Affine Spaces

Given a projective space, one can construct an affine space. Let $\mathbf{P}$ be a projective space of dimension $d \geq 2$, and let $\mathbf{H}_\infty$ be a hyperplane of $\mathbf{P}$. Then the geoemtry $\mathbf{A} = \mathbf{P} \setminus \mathbf{H}_\infty$ is defined as follows:

- The **points** of $\mathbf{A}$ are the points of $\mathbf{P}$ that are not in $\mathbf{H}_\infty$.
- The **lines** of $\mathbf{A}$ are the lines of $\mathbf{P}$ that are not contained in $\mathbf{H}_\infty$.
- The $t$**-dimensional subspaces** of $\mathbf{A}$ are those $t$-dimensional subspaces of $\mathbf{P}$ that are not contained in $\mathbf{H}_\infty$.
- The **incidence** of $\mathbf{A}$ is induced by the incidence of $\mathbf{P}$.

The rank 2 geometry consisting of the points and lines of $\mathbf{A}$ is called an **affine space** of dimension $d$. An affine space of dimension 2 is an **affine plane**. Furthermore, the set of all subspaces of $\mathbf{A}$ is called an **affine geometry**. The hyperplane $\mathbf{H}_\infty$ is called the **hyperplane at infinity** and the points of $\mathbf{H}_\infty$ are called the **points at infinity** of $\mathbf{A}$.

It seems counterintuitive to define an affine space in terms of a projective space, as the latter seems much more theoretical in nature. (That is, an affine geometry has the concept of parallel lines, which seems natural.) However, these structures are basically the same, as witnessed by their definitions. The distinguishing factor is only that of differing view points. Interestingly enough, one may prefer focusing on projective geometries (over an affine one) because of the homogeneous properties and the fact that there is no need to distinguish many special cases (like one must do for affine geometries). Nevertheless, various results of an affine space will be presented here.

Let $\mathbf{G} = (\mathcal{P}, \mathcal{B}, I)$ be a rank 2 geometry. A **parallelism** of $\mathbf{G}$ is an equivalence relation $||$ on the block set $\mathcal{B}$ satisfying the **parallel axiom**:

If $P \in \mathcal{P}$ is a point and $B \in \mathcal{B}$ is a block of $\mathbf{G}$, then there is precisely one block $C \in \mathcal{B}$ through $P$ such that $C||B$. (Blocks $B$ and $C$ are said to be **parallel**.)

THEOREM 2.3.1. *Every affine geometry has a parallelism.*

PROOF. Let $\mathbf{P}$ be a projective space belonging to $\mathbf{A}$, and let $\mathbf{H}_\infty$ be the hyperplane at infinity. Consider two $t$-dimensional subspaces $U$ and $W$ of $\mathbf{A}$. By definition, $U$ and $W$ are $t$-dimensional subspaces of $\mathbf{P}$ not contained in $\mathbf{H}_\infty$. Thus, $U \cap \mathbf{H}_\infty$ and $W \cap \mathbf{H}_\infty$ are $t-1$-dimensional subspaces of $\mathbf{H}_\infty$. Now define

$$U||W :\Leftrightarrow U \cap \mathbf{H}_\infty = W \cap \mathbf{H}_\infty.$$

We will show that $||$ is a parallelism. Clearly $||$ is an equivalence relation as it is defined via an equality relation. Let $P$ be a point of $\mathbf{A}$. Then if $W$ through $P$ is parallel to $U$, then $U$ must contain $P$ and $U \cap \mathbf{H}_\infty$. Since $< U \cap \mathbf{H}_\infty, P >$ is a $t$-dimensional subspace of $P$, which is contained in $W$, it follows that $W =< U \cap \mathbf{H}_\infty, P >$. Therefore, $||$ is a parallelism. $\square$

Lemma 2.3.2. *Let $\mathbf{A} = \mathbf{P} \setminus \mathbf{H}_\infty$, where $\mathbf{P}$ is a $d$-dimensional projective space and $\mathbf{H}_\infty$ is the hyperplane at infinity of $\mathbf{A}$.*

*(i.) Each line that is not parallel to a hyperplane $H$ intersects $H$ in precisely one point of $\mathbf{A}$.*

*(ii.) If $d = 2$, then any two non-parallel lines intersect in a point of $\mathbf{A}$.*

Our primary focus is with the affine plane, so a few results will be presented.

Corollary 2.3.3. *Any affine plane $\mathbf{A}$ has the following properties:*

1. *Through any two distinct points, there is exactly one line.*
2. *(Playfair's parallel axiom) If $g$ is a line and $P$ is a point outside $g$ then there is precisely one line through $P$ that has no point in common with $g$.*
3. *There exist three points that are not on a common line.*

Theorem 2.3.4. *Let $\mathbf{S} = (\mathcal{P}, \mathcal{G}, I)$ be a geometry that satisfies conditions (1), (2), and (3). Then $\mathbf{S}$ is an affine plane.*

This theorem implies that the Euclidean plane is also an affine plane. The smallest affine plane is $K_4$, the complete graph on four vertices. Furthermore, it is the Fano plane with the removal of one line (and the 3 points on that line). If $\mathbf{P}$ is a finite projective space of order $q$ and $\mathbf{H}$ a hyperplane of $\mathbf{P}$, then the affine space $\mathbf{A} = \mathbf{P} \setminus \mathbf{H}$ is said to have order $q$.

Theorem 2.3.5. *Let $\mathbf{A}$ be an affine space of order $q$.*

*(i) There exists a positive integer $q \geq 2$ such that any line of $\mathbf{A}$ is incident with exactly $q$ points.*

*(ii) If $U$ is a $t$-dimensional subspace of $\mathbf{A}$, then $U$ has $q^t$ points.*

Proof. Let $\mathbf{P}$ be a projective space belonging to $\mathbf{A}$. Let $\mathbf{H}_\infty$ be the hyperplane at infinity of $\mathbf{A}$.

(i) Let $g$ be a line of $\mathbf{A}$, and let $q$ be the number of points (in $\mathbf{A}$). Then $g$ has $q+1$ points in $\mathbf{P}$. Hence $\mathbf{P}$ has order $q$. Therefore any line of $\mathbf{A}$ has exactly $q$ points of $\mathbf{A}$.

(ii) Any $t$-dimensional subspace $U$ of $\mathbf{A}$ intersects, when considered as a subspace of $\mathbf{P}$, the hyperplane at infinity in a $(t-1)$-dimensional subspace $U \cap \mathbf{H}_\infty$. Thus we have

$$\begin{aligned} &\text{the number of points of } \mathbf{A} \text{ in } U \\ &= \text{the number of points of } U \text{ in } \mathbf{P} - \text{the number of points in } U \cap \mathbf{H}_\infty \\ &= q^t + ... + q + 1 - (q^{t-1} + ... + q + 1) = q^t. \end{aligned}$$

□

Thus, every affine plane of order $q$ has $q^2$ points, every line contains $q$ points, every point is contained in $q+1$ lines, and there is a total of $q^2+q$ lines.

CHAPTER 3

# Strongly Regular Graphs

As previously noted, the pentagon and the Petersen graph are both examples of strongly regular graphs that are determined uniquely by their parameters. This is not always the case though. In this chapter, we will examine various properties of strongly regular graphs.

## 3.1. Families of Graphs

3.1.1. Kneser Graph. The Petersen graph not only provides us with a unique example of a strongly regular graph whose parameters entirely define it, it also leads us to a family of graphs, not all necessarily strongly regular. (This is also true of the pentagon.) The Kneser graph $KG_{m,k}$ is a graph whose vertices correspond to $k$-element subsets of a set of $m$ elements. Two vertices are adjacent if and only if the two corresponding sets are disjoint. The Petersen graph can be denoted by $KG_{5,2}$. Thus, we can define the Petersen graph as a graph whose vertices correspond to 2-element subsets of the set $\{1, ..., 5\}$, and two vertices $\{a, b\}$ and $\{c, d\}$ for $a, b, c, d \in \{1, ..., 5\}$ are adjacent if and only if $|\{a, b\} \cap \{c, d\}| = 0$. While not all Kneser graphs are strongly regular, we do get that $KG_{m,2}$ is always a strongly regular graph.

Furthermore, we can discuss the complement of the Petersen graph, $T(5)$. $T(5)$ is also a strongly regular graph with parameters $(10, 6, 3, 4)$. More generally, $T(5)$ falls into the family of triangular graphs, $T(m)$, whose vertices correspond to 2-element subsets of a set of $m$-elements, and two vertices are adjacent whenever they meet in one point. Unlike the Kneser graphs, all triangular graphs are strongly regular, with parameters $(\binom{m}{2}, 2(m-2), m-2, 4)$.

It is not a coincidence that the complement of the Petersen graph is also a strongly regular graph.

THEOREM 3.1.1. *The complement of a graph* $\Gamma$ *is the graph* $\Gamma'$ *with the same vertex set as* $\Gamma$*, where two vertices are adjacent if and only if they are not adjacent in* $\Gamma$*. If the parameters of* $\Gamma$ *are* $(v, k, \lambda, \mu)$*, then the parameters of* $\Gamma'$ *are* $(v, v-k-1, v-2k+\mu-2, v-2k+\lambda)$*.*

PROOF. Clearly $\Gamma'$ is a regular graph of valency $v-k-1$. The other two parameters can easily be calculated using Inclusion-Exclusion. In $\Gamma'$, the number of common neighbors of adjacent vertices is $(v-2)-2k+\mu = v-2k+\mu-2$, and for non-adjacent vertices, the number of common neighbors is $v-2k+\lambda$. □

Since all parameters must be non-negative, if follows that

$$v \geq 2k - \mu + 2,$$

$$v \geq 2k - \lambda.$$

Later on, we will discuss further parameter restrictions as well. At this time, we will introduce another family of strongly regular graphs. This family will play a critical role later on, so while not all properties will be addressed at this stage, they will be addressed later on, as needed.

3.1.2. GENERALIZED QUADRANGLES. Generalized quadrangles were first introduced in 1959 by Tits in the study of finite geometry. These objects are related to numerous combinatorial objects, such as partial geometries. But, they also happen to be related to strongly regular graphs. A **generalized quadrangle** is an incidence structure $(P, B, I)$, with $P$ being the set of points, $B$ the set of lines, and $I \subset P \times B$ an incidence relation satisfying the following axioms:

- There is an $s$ ($s \geq 1$) such that every line contains exactly $s+1$ points. There is at most one point on two distinct lines.
- There is a $t$ ($t \geq 1$) such that through every point there are exactly $t+1$ lines. There is at most one line through two distinct points.
- For every point $p$ not on a line $L$, there is a unique line $M$ and a unique points $q$ such that $p$ is on $M$ and $q$ is on $M$ and $L$.

The parameters of a generalized quadrangle are denoted by $(s, t)$; if $s = t$, then we say the generalized quadrangle has order $s$. The number of points of a generalized quadrangle is $(st+1)(s+1)$, while the number of lines is $(st+1)(t+1)$. The **collinearity** (or **point**) graph of a generalized quadrangle is a strongly regular graph, whose points are those of the generalized quadrangle with adjacency defined by collinearity in the generalized quadrangle. The parameters for its strongly regular graph are $((st+1)(s+1), s(t+1), s-1, t+1)$.

Now that two families of strongly regular graphs have been introduced, we will proceed with properties of strongly regular graphs.

## 3.2. Properties

By considering the complement of a strongly regular graph, parameter restrictions became apparent. More restrictions can be obtained by considering the adjacency matrix of a strongly regular graph. Let $\Gamma$ be a strongly regular graph whose vertices are distinct. If $\Gamma$ has $v$ vertices, then there exists an $v \times v$ **adjacency matrix** $A$ whose entries consist of zeros and ones. Entry $A_{i,j} := 1$ if vertices $i$ and $j$ are adjacent in $\Gamma$ for $i, j \in \{1, ..., v\}$ and $i \neq j$; otherwise $A_{i,j} := 0$. Matrix $A$ consists of three eigenvalues: $k$, $r$, and $s$. We will assume that $r > s$. It can easily be shown that the multiplicity of $k$ is 1. Additionally, $r$

and $s$ are **restricted eigenvalues** of $A$. That is, the eigenvectors corresponding to $r$ and $s$ are perpendicular to the all-ones vector, the eigenvector of $k$. This information can be summarized in the following theorem.

THEOREM 3.2.1. *For a simple graph $\Gamma$ of order $v$, not complete or edgeless, with adjacency matrix $A$, the following are equivalent:*

*(i) $\Gamma$ is a strongly regular graph with parameters $(v, k, \lambda, \mu)$ for certain integers $k, \lambda, \mu$.*

*(ii) $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$ for certain real numbers $k, \lambda, \mu$ and where $I$ is the $v \times v$ identity matrix and $J$ is the $v \times v$ all-ones matrix.*

*(iii) $A$ has precisely two distinct restricted eigenvalues.*

PROOF. Note that the equation in (ii) can be rewritten as

$$A^2 = kI + \lambda A + \mu(J - I - A).$$

Thus, (i)⇔(ii) is obvious, as the entry $A^2_{i,j}$ counts the number of common neighbors of vertices $i$ and $j$ and $A^2_{i,i}$ is the valency of vertex $i$. Furthermore, the complement of $\Gamma$ has adjacency matrix $J - I - A$.

(ii)⇔(iii): For every vector $w$ orthogonal to $\mathbf{1}$,

$$A^2 w = (\lambda - \mu)Aw + (k - \mu)w,$$

as $Jw = 0$. Thus, every eigenvalue in $\Theta$ of $A$ other than $k$ satisfies

$$\Theta^2 = (\lambda - \mu)\Theta + k - \mu.$$

□

Strongly regular graphs can be classified as primitive and imprimitive. The focus here is on the primitive type. This implies that $k > r > 0$ and $s < -1$. But further restrictions are also placed on the parameters. For example, there are the *rationality conditions*.

THEOREM 3.2.2. *Let* $\Gamma$ *be a strongly regular graph with adjacency matrix* $A$ *and parameters* $(v, k, \lambda, \mu)$. *Let* $r$ *and* $s$ *(*$r > s$*) be the restricted eigenvalues of* $A$ *and let* $f, g$ *be their respective multiplicities. Then*

*(i)* $k(k-1-\lambda) = \mu(v-k-1)$,

*(ii)* $rs = \mu - k$, $r + s = \lambda - \mu$,

*(iii)* $f, g = \frac{1}{2}(v - 1 \mp \frac{(r+s)(v-1)+2k}{r-s})$.

PROOF. (i): Fix a vector $x$ of $\Gamma$. Let $A(x)$ and $N(x)$ be the sets of vertices adjacent and non-adjacent to $x$, respectively. By counting the number of edges between $A(x)$ and $N(x)$ in two different ways yields (i).
(ii): Follows directly from the fact that $A^2 = (\lambda - \mu)A + (k - \mu)I + \mu J$.
(iii): Since the multiplicity of $k$ is 1, $f + g = v - 1$. Since $\Gamma$ is loopless and by definition, $0 =$trace$(A)$$= k + fr + gs = k + \frac{1}{2}(r+s)(f+g) + \frac{1}{2}(r-s)(f-g)$. These two facts yield equation (iii). ☐

Few other parameter restrictions are known, besides the rationality conditions. The *Krein conditions* are given as follows:

THEOREM 3.2.3. *Let* $\Gamma$ *be a strongly regular graph with adjacency matrix* $A$ *and parameters* $(v, k, \lambda, \mu)$. *Let* $r$ *and* $s$ *be the restricted eigenvalues of* $A$. *Then*

$$(r+1)(k+r+2rs) \leq (k+r)(s+1)^2,$$

$$(s+1)(k+s+2rs) \leq (k+s)(r+1)^2.$$

When equality holds in either of these two inequalities, the induced subgraphs on the neighbors and non-neighbors of a given point in $\Gamma$ are both strongly regular graphs as well. In addition to these restrictions, we have two additional ones. The *absolute bound* for the number of vertices is

$$v \leq f(f+3)/2,$$

$$v \leq g(g+3)/2.$$

Another useful identity is the *Frame quotient*,

$$fg(r-s)^2 = vk(v-k-1).$$

From the Frame quotient, it can be concluded that if $v$ is prime, then $r - s = \sqrt{v}$.

As previously mentioned, strongly regular graphs originated from association schemes. The Krein conditions, absolute bound, and Frame quotient are all just special cases of general (in)equalities for association schemes.

3.2.1. Group constructions. While strongly regular graphs are defined in terms of parameters and spectra, the way a group acts on a set of points can also determine a strongly regular graph. Let $G$ be a permutation group acting on a set $\Omega$. The **rank** of an action is the number of orbits of $G$ on $\Omega \times \Omega$. The orbits of $G$ on $\Omega \times \Omega$ are called **orbitals**. If $G$ is transitive, of rank 3, and its orbitals are symmetric (for all $x, y \in \Omega$, $(x, y)$ and $(y, x)$ are in the same orbital), where $I$, $R$, and $S$ are the three orbitals, then $(\Omega, R)$ and $(\Omega, S)$ is a pair of complementary strongly regular graphs. In the case when $G = S_5$, we obtain the Petersen Graph and its complement, $T(5)$.

3.2.2. Automorphisms. As mentioned in the first chapter, computing automorphism groups, no matter the type of graph, can be NP hard. But, once again, knowing the automorphism group of a strongly regular graph can be extremely beneficial, if not necessary, in determining whether the graph is new or not. For a complete list of strongly regular graphs and their automorphism groups, see [19]. This list also contains restricted parameters as well as parameters for which nothing is known.

If $A$ is the adjacency matrix of a graph $\Gamma$ and $P$ the permutation matrix that describes an automorphism $\omega$ of $\Gamma$, then $PA = AP$. Furthermore, if $\omega$ has order $j$, then $P^j = I$, the identity matrix. Therefore, the eigenvalues of $AP$ are $j$-th roots of unity times the eigenvalues of $A$. If we further restrict $\Gamma$ so that it is a strongly regular graph, more information can be obtained. Suppose that $\omega$ has $l$ fixed points and moves $m$ points to a neighbor. Then we get that the trace (the sum of the diagonal entries) of $P$ is $l$ and the trace of $AP$ is $m$. Next, define $M := A - sI$. Then the eigenvalues of $M$ are $(k-s)$ (with multiplicity 1), $(r-s)$ (with multiplicity $l$), and 0 (with multiplicity $m$). (Note that the definitions of $k$, $r$, and $s$ are the same as those given in the previous section.) Thus, $MP$ has eigenvalues $k - s$, $(r - s)\zeta$ for certain $j$-th roots of unity $\zeta$, and 0. Then it follows that $m - sl = \mathrm{tr}(MP) \equiv k - s(\mathrm{mod}(r - s))$. As for the Petersen graph, every automorphism must satisfy $f \equiv g + 1(\mathrm{mod}3)$.

There do exist some tools to help determine what the automorphism graph is [2].

Theorem 3.2.4. *Let $\Gamma$ be a non-trivial strongly regular graph of degree $k \leq (v - 1)/2$. Then any nontrivial automorphism of $\Gamma$ fixes fewer than $v - k/2$ vertices.*

Before presenting the next result, the following definition is needed. A graph $\Gamma$ is **graphic** if $\Gamma$ or its complement is the line graph of a graph. Testing for isomorphisms of graphic graphs is straightforward.

Theorem 3.2.5. *Let $\Gamma$ be a non-trivial and non-graphic strongly regular graph with $v$ vertices. Let $\sigma$ be a non-identity automorphism of $\Gamma$. Then*

*(i) $\sigma$ has at most $7v/8$ fixed points;*

*(ii) $\sigma$ has order less than or equal to $v^8$.*

To reiterate, calculating the automorphisms of a graph can be difficult. But in some cases, it is unnecessary. For example, if the collinearity graph of a generalized quadrangle is the only known strongly regular graph, and a new graph with the same parameters is constructed, automorphisms are not necessary.

Theorem 3.2.6. *The point graph of a finite generalized quadrangle does not contain $K_4 \setminus \{e\}$, for some edge $e$, as an induced subgraph.*

Corollary 3.2.7. *If a graph contains $K_4 \setminus \{e\}$ as an induced subgraph, it is not the point graph of a generalized quadrangle.*

Proof. Let $P, Q, R, S$ be distinct points on a generalized quadrangle with $R$ not collinear with $S$, but all other pairs collinear. Then since $P, Q, R$ are all collinear, $R$ is collinear with more than one point on the line $l$ joining $P$ and $Q$, so $R$ is on $l$. Similarly, $S$ is on $l$. But then $R$ and $S$ are collinear, a contradiction. □

Based on this result, showing that a graph has this particular property can be used in order to verify that the graph is new. Similarly, a clique search can be used to determine whether a graph is new or not as well.

With the primary background information in place, the next chapter will proceed with the direct connection between finite projective spaces and strongly regular graphs, as surveyed in the 1986 Calderbank and Kantor paper [22].

CHAPTER 4

# Two Intersection Sets

An object of particular interest in finite projective spaces is a $t$-set of type $(m, n)$ (or a **two intersection set**). A **$t$-set $K$ of type** $(m, n)$ in $\mathrm{PG}(d, q)$ is a set $K$ consisting of $t$ points such that every hyperplane of the projective space $\mathrm{PG}(d, q)$ contains either $m$ or $n$ points of $K$, $m < n$. These sets are also called two intersection sets as they consist of two intersection numbers (with respect to hyperplanes), $m$ and $n$. We will begin by discussing some properties of two intersection sets and their relationship to various combinatorial objects, followed by a survey of known two intersection sets. Finally, new results will be presented.

## 4.1. Properties

If $K$ is a $t$-set of type $(m, n)$ in $\mathrm{PG}(d, q)$, we define an $m$-secant ($n$-secant) to be a hyperplane of $\mathrm{PG}(d, q)$ that intersects $K$ in exactly $m$ ($n$) points of $K$. Let $t_m$ and $t_n$ denote the number of $m$-secants and $n$-secants. Then elementary counting leads to the fundamental equations

(1) $$t_m + t_n = \frac{q^{d+1} - 1}{q - 1},$$

(2) $$mt_m + nt_n = t\frac{q^d - 1}{q - 1},$$

(3) $$m(m-1)t_m + n(n-1)t_n = t(t-1)\frac{q^{d-1} - 1}{q - 1}.$$

By taking the linear combination $mn(1) + (1 - m - n)(2) + (3)$, we obtain a quadratic in $t$:

$$t^2 \frac{q^{d-1} - 1}{q - 1} + t(1 - m - n)\frac{q^d - 1}{q - 1} - t\frac{q^{d-1} - 1}{q - 1} + mn\frac{q^{d+1} - 1}{q - 1} = 0. \tag{4}$$

Now, let $r_n$ and $r_m$ denote the number of $n$-secants and $m$-secants, respectively, through a points $P \in K$. Let $s_n$ and $s_m$ be the number of $n$-secants and $m$-secants, respectively, through a point $Q \notin K$. Then simple counting yields

$$s_n + s_m = \tfrac{q^d-1}{q-1},$$

$$ns_n + ms_m = t\tfrac{q^{d-1}-1}{q-1},$$

$$r_n + r_m = \tfrac{q^d-1}{q-1},$$

$$(n-1)r_n + (m-1)r_m = (t-1)\tfrac{q^{d-1}-1}{q-1}.$$

Using these equations, we can then solve for $r_m$, $r_n$, $s_n$, and $s_m$, obtaining

$$r_m = \frac{(n-1)\frac{q^d-1}{q-1} - (t-1)\frac{q^{d-1}-1}{q-1}}{n-m},$$

$$r_n = \frac{(t-1)\frac{q^{d-1}-1}{q-1} - (m-1)\frac{q^d-1}{q-1}}{n-m},$$

$$s_m = \frac{n\frac{q^d-1}{q-1} - t\frac{q^{d-1}-1}{q-1}}{n-m},$$

$$s_m = \frac{t\frac{q^{d-1}-1}{q-1} - m\frac{q^d-1}{q-1}}{n-m}.$$

An important consequence of these equations is that $n-m$ is a divisor of $q^{d-1}$ (as $r_m - s_m \in \mathbb{Z}$).

In addition, equation (4) is useful in obtaining the following result.

LEMMA 4.1.1. *Let $K$ be a $t$-set of type $(m, n)$ in* $\mathrm{PG}(d, q)$. *Then:*

*a.) The complement of $K$ is a $(\frac{q^{d+1}-1}{q-1} - t)$-set of type $(\frac{q^d-1}{q-1} - n, \frac{q^d-1}{q-1} - m)$ in* $\mathrm{PG}(d, q)$.

*b.) The $m$-secants of $K$ form a $t_m$-set of type $(r_m, s_m)$ in* $\mathrm{PG}(d, q)$.

*c.) The $n$-secants of $K$ form a $t_n$-set of type $(s_n, r_n)$ in* $\mathrm{PG}(d, q)$.

When constructing a two intersection set with the same parameters as a previously known one, applying group theoretic techniques can be useful in determining set equivalence. Let $K$ be a two intersection set. $K$ is equivalent to another two intersection set $L$ if there exists an automorphism taking $K$ to $L$.

Theorem 4.1.2. *Let $K$ and $L$ be two intersection sets. Then $K \equiv L$ if and only if there exists a $f \in \mathrm{P\Gamma L}(d+1,q)$ such that $K = fL$.*

Based on the above theorem, without loss of generality, $\mathrm{P\Gamma L}(d+1,q)_K = \mathrm{P\Gamma L}(d+1,q)_{fL}$.

Theorem 4.1.3. *Let $G = \mathrm{P\Gamma L}(d+1,q)$. If $\mathrm{P\Gamma L}(d+1,q)_K = \mathrm{P\Gamma L}(d+1,q)_L = H$, then $K \equiv L$ if and only if $K$ and $L$ are in the same orbit of $N_G(H)$.*

Therefore, the subgroup of the full automorphism group that stabilizes a set can be used to determine if two sets are equivalent or not. Using this information, as well as the fundamental equations, we can construct new two intersection sets.

## 4.2. Two Intersection Sets and Strongly Regular Graphs

Due to notational constraints, let $K$ be a $t$-set of type $(m,n)$ in $\mathrm{PG}(d-1,q)$ that spans $\mathrm{PG}(d-1,q)$. Based on Delsarte's work, using $K$, we can construct a strongly regular graph. Embed $\mathrm{PG}(d-1,q)$ in $\mathrm{PG}(d,q)$ as a hyperplane $H$, and define $\Gamma(K)$ to be the graph with vertices the points of $\mathrm{PG}(d,q) \setminus H$. Two vertices $P$ and $Q$ are adjacent in $\Gamma(K)$ if and only if $PQ \cap H \in K$. Then by [22, Theorem 3.2], $\Gamma(K)$ is a strongly regular graph with parameters

$$v = q^d,$$

$$k = t(q-1),$$

$$\lambda = t^2 + tq - 3t + qm + qn - tqm - tqn + mnq^2,$$

$$\mu = \frac{q^2}{q^d}(t-m)(t-n).$$

The fact that two intersection sets give rise to strongly regular graphs is a motivating factor in studying such objects. They also give rise to projective two-weight codes and difference sets.

4.2.1. Projective two-weight codes. Given a two intersection set $K$ in $\mathrm{PG}(d,q)$, if we define each point in $K$ by its corresponding ordered tuple in $\mathrm{GF}(q)^{d+1}$ a generating matrix of dimension $(d+1)\times t$ exists, with the columns of the matrix being the ordered tuples of $K$. Such a matrix corresponds to a projective two-weight code.

A **linear** $[t,d+1]_q$ **code** $C$ is a $(d+1)$-dimensional subspace of $\mathrm{GF}(q)^t$. The vectors of the subspace are called **codewords**. The easiest means of constructing a generating matrix is to arbitrarily choose a $(d+1)$-dimensional basis and let the vectors of the basis form the rows, which will correspond to a $(d+1)\times t$ generating matrix. By using a generating matrix, we are creating a compact form in which all the codewords can be computed by taking linear combinations of the rows of that matrix.

Some interesting facts arise from the use of a generating matrix. Let $M$ be the generating matrix for a code $C$, and let $c$ be a codeword of $C$. Then there exists a unique vector $v$ of $\mathrm{GF}(q)^{d+1}$ such that $vM = c$. Furthermore, by using a generating matrix, it is easy to determine if two codewords are equivalent. This is due to the fact that the generating matrix stores a basis for a $(d+1)$-dimensional subspace, and the ordering of the rows is unimportant. Similarly, the code will remain unchanged if we permute the columns, scale the rows or columns, or add additional rows. These facts hold in general for any code $C$.

The **weight** of a codeword $c$ of $C$ is defined to be the number of nonzero elements of the vector $c$, denoted by $wt(c)$. As the name suggests, for two-weight codes we only consider codes which consist of two weights, $w_1$ and $w_2$. As perfect codes and uniformly packed codes are completely classified, two-weight codes are the most interesting types of codes to study.

Additional restrictions can be placed on two-weight codes. For example, we can require the code to be projective. A linear code $C$ is **projective** if the $i^{th}$ and $j^{th}$ columns of the generating matrix are linearly independent for any pair $(i, j)$, $i \neq j$.

Before proceeding with the statement of the relationship between two intersection sets and projective two-weight codes, an interesting fact should be noted. As was seen in Lemma 4.1.1, finding one two intersection set led to three additional two intersection sets, with one of them being the dual of the original. The same can be said for codes. The dual of a $[t, d+1]_q$ code is a $[t, t-d-1]_q$ code, whereby the dual is defined to be the space orthogonal to the $(d+1)$-dimensional space of the original code. A projective linear code will have a dual code with minimum weight at least 3.

Assume that the points of the two intersection set span the entire space. This implies that there exist $d+1$ columns which form a basis for the entire projective space. Thus, $d+1$ linearly independent rows exist.

Recall that points and hyperplanes of a projective space can be denoted by vectors of length $d+1$, and a point lies on a hyperplane if their dot product is zero. This last condition results in a code consisting of two-weights. Furthermore, for any codeword $c$ of $C$, there exists a unique $v$ such that $c = vM$, where $M$ is the generating matrix for $C$. The unique vectors $v$ correspond to the hyperplanes. Thus, the resulting entries of $c$ will either have $m$ or $n$ zeros depending on whether the hyperplane was an $m$ or $n$-secant. We also get that the resulting weights are $t - m$ and $t - n$. Thus, the code must be a two-weight code. Lastly, we know that all the columns of the generating matrix are pairwise linearly independent, and thus, the code is also projective. By a similar argument, it can be shown that given a projective two-weight code, we can obtain a two intersection set. Thus, the two objects are equivalent.

4.2.2. Difference Sets. Like projective two-weight codes, there is an equivalence relationship between two intersection sets and $\{\alpha, \beta\}$-difference sets (which can also be extended to projective two-weight codes and strongly regular graphs). Let $\Omega$ be a proper set of non-zero vectors in a vector space $V$ over GF$(q)$. Then $\Omega$ is a $\{\alpha, \beta\}$-**difference set** over GF$(q)$ if GF$(q) * \Omega = \Omega$ and if, for $v \in V$, $v \neq 0$,

$$|\{(x, y)|x, y \in \Omega, x - y = v\}| = \begin{cases} \alpha & : v \in \Omega \\ \beta & : v \notin \Omega \end{cases}$$

As surveyed by [22], if $K$ is a proper non-empty set of points of PG$(d-1, q)$ and if $\Omega = \{v \in V| < v >\in K\}$, then $\Omega$ is a difference set if and only if $K$ is a two intersection set.

## 4.3. Survey of Known Two Intersection Sets

The primary focus of this section will be on the projective plane. Beginning in the projective plane PG$(2, q)$, there are two trivial sets: the 1-sets of type $(0, 1)$ and the $q+1$-sets of type $(1, q+1)$. The first set consists of a single point and its intersection numbers follow from the definition of a projective plane. Similarly, the second set consists of a line and its intersection numbers follow from the definition. More generally, a subspace of PG$(d, q)$ of dimension $f$ is a $\frac{q^{f+1}-1}{q-1}$-set of type $(\frac{q^f-1}{q-1}, \frac{q^{f+1}-1}{q-1})$.

There are a few families of two intersection sets that are notable. A $q+2$-set of type $(0,2)$ in $\mathrm{PG}(2,q)$, for $q$ even, is called a **hyperoval**. Hyperovals are related to maximal arcs, another combinatorial object. **Maximal arcs** are also two intersection sets. They are sets of type $(0,n)$ in $\mathrm{PG}(2,q)$ where $n$ is the degree of the set $K$. Numerous families of maximal arcs have been studied [42, 60, 100, 101]. Currently, all known examples of maximal arcs are in planes of even order. (Ball-Blokhuis-Mazzocca [5] showed that maximal arcs in *Desarguesian* planes of odd order do not exist.) Two other sets with names occur in $\mathrm{PG}(2,q^2)$. A **Baer subplane** is a $q^2+q+1$-set of type $(1,q+1)$, while a $q^3+1$-set of type $(1,q+1)$ in $\mathrm{PG}(2,q^2)$ is a **unital**. Lastly, a $q^2+1$-set of type $(1,q+1)$ in $\mathrm{PG}(3,q)$ is an **ovoid**. Furthermore, we can obtain a family of two intersection sets by taking the union of disjoint Baer subplanes.

THEOREM 4.3.1. *The union of $u$ disjoint Baer subplanes in $\mathrm{PG}(2,q^2)$ is a $u(q+1)$-set of type $(u,q+u)$ in $\mathrm{PG}(2,q^2)$.*

PROOF. Let $u$ be the number of disjoint Baer subplanes in $\mathrm{PG}(2,q^2)$. Then all $u$ Baer suplanes are $q+1$-sets of type $(1,q+1)$. Thus, the union of these will have $u(q+1)$ points with possible intersection numbers $u$, $q+u$, and $u(q+1)$. Consider the fundamental equations (1), (2), and (3). These are three linear equation in the unknowns $t_u$, $t_{q+u}$, and $t_{u(q+1)}$. It is easy to see that the determinant of the coefficient matrix is non-zero. Thus, it is non-singular, and hence, these equations have a unique solution. As a $u(q+1)$-set of type $(u,q+u)$ is arithmetically feasible, we can find a solution to these equations with $t_{u(q+1)}=0$. Hence, this must be the unique solution. □

The history of two intersection sets in finite projective planes goes back to at least two 1966 papers by Tallini Scafati [97, 98], with precursors dealing with unitals [90], maximal arcs [6], and hyperovals [13]. Calderbank and Kantor surveyed these sets, as well as their higher-dimensional counterparts, in 1986 [22]. Postdating their survey, a number of constructions have been given for two intersection sets in finite projective planes, which will now be presented.

Payne (1985) [79], O'Keefe-Penttila (1992) [77], Cherowitzo-Penttila-Pinneri-Royle (1996) [33, 77], Cherowitzo (1998) [28], and Cherowitzo-O'Keefe-Penttila (2003) [32] construct hyperovals in $\mathrm{PG}(2, 2^n)$. Mathon (2002) [71], Hamilton (2002) [59], Hamilton-Mathon (2003, 2004) [60, 61], and De Clerck-De Winter-Maes (2011,2012, 2012) [37, 38, 39] construct maximal arcs in $\mathrm{PG}(2, 2^h)$. Blokhuis-Lavrauw (2000) [21] constructs $\frac{(q^t-1)(q^{2t}+q^t+1)}{q-1}$-sets of type $(\frac{q^t-1}{q-1}, \frac{q^t-1}{q-1} + q^t)$ in $\mathrm{PG}(2, q^{2t})$. de Resmini-Migliori (1986) [48] constructs a 78-set of type $(2, 6)$ in $\mathrm{PG}(2, 16)$; de Resmini (1987)[43] constructs a 35-set of type $(2, 5)$ in $\mathrm{PG}(2, 9)$; Penttila-Royle (1995) [84] classify all two intersection sets in $\mathrm{PG}(2, 9)$; Hamilton (1995) [58] constructs 104-sets of type $(4, 8)$ in $\mathrm{PG}(2, 16)$ and 912-sets of type $(8, 16)$ in $\mathrm{PG}(2, 64)$, and Batten-Dover (1999) [8] constructs a 829-set of type $(4, 9)$ in $\mathrm{PG}(2, 125)$ and a 3189-set of type $(4, 11)$ in $\mathrm{PG}(2, 343)$. Further constructions, which are hard to summarize, appear in [62] and [88].

## 4.4. New Two Intersection Sets

4.4.1. Sets in the plane. There is also a family of $\frac{(q^2-q+1)(q^2+1)}{2}$-sets of type $(\frac{(q+1)^2}{2}, \frac{q^2+1}{2})$ in $\mathrm{PG}(2, q^2)$ for $q$ odd in Casse-Jackson-Penttila-Royle (in preparation), and these are parameters not previously obtained (for $q > 3$). To these, we add examples with new parameters in $\mathrm{PG}(2, 25)$ and $\mathrm{PG}(2, 81)$. To the best of our knowledge, the only examples of two intersection sets in finite Desarguesian planes admitting insoluble groups known before this paper are the regular hyperovals, the classical unitals, and Baer subplanes (and their complements). Thus, it is of some interest that four of our examples (and their complements) have insoluble groups. We also contribute further examples admitting insoluble groups (but some without new parameters). All two intersection sets in $\mathrm{PG}(2,25)$ were found using Magma [18], while the two intersection set in $\mathrm{PG}(2,81)$ was found using GAP [53].

Inspired by an example found by Penttila and Royle [84] of a 35-set of type $(2,5)$ in $\mathrm{PG}(2,9)$ admitting $\mathrm{P}\Omega(3,3)$, we explore the possibility of existence of two intersection sets in $\mathrm{PG}(2,q^2)$, $q$ odd, admitting $\mathrm{P}\Omega(3,q)$. Before stating the new theorems, for the remainder of this section, we assume that points of $B$, a Baer subplane, are **real**, and points not in $B$ are **imaginary**. In addition, lines meeting $B$ in more than 1 point are **real**.

THEOREM 4.4.1. *Let $C$ be a conic in* $\mathrm{PG}(2,25)$ *and let $B$ be a Baer subplane meeting $C$ in* 6 *points. The derived group of the stabilizer of $B \cap C$ in* $\mathrm{P\Gamma L}(3,25)$ *has two orbits on imaginary points on a real tangent line. Arbitrarily choose one of them and call it $O$. Let $S$ be the union of :*

*the real points internal to the conic $B \cap C$ of* $\mathrm{PG}(2,5)$*;*

*the points of $C$ not in $B$;*

*$O$;*

*the imaginary internal points $P$ on a real secant line (to $B \cap C$) such that $P^{\perp} \cap O$ is not empty (where $P^{\perp}$ is the polar line of $P$ with respect to $C$).*

*Then $S$ is a* 210*-set of type* $(5,10)$ *in* $\mathrm{PG}(2,25)$ *with group* $\mathrm{PSL}(2,5) \times C_2$ *(of order* 120*).*

PROOF. This is a straightforward computation in Magma. $\square$

Note that the stabilizer of $S$ has index two in its normalizer, which is isomorphic to $\mathrm{PGL}(2,5) \times C_2$ and maps $S$ to the 210-set of type $(5,10)$ arising from the other choice of $O$. This set gives rise to a strongly regular graph with parameters $(15625, 5040, 1595, 1640)$. No strongly regular graph with these parameters was previously known. In addition to the strongly regular graph, the weights for the projective two-weight code are $200(= 210 - 10)$ and $205(= 210 - 5)$.

THEOREM 4.4.2. *Let $C$ be a conic in* $\mathrm{PG}(2, 25)$ *and let $B$ be a Baer subplane meeting $C$ in 6 points. The derived group $G$ of the stabilizer of $B \cap C$ has two orbits on points on a real tangent line. Arbitrarily choose one of them and call it $O$. $G$ has two orbits on internal imaginary points on a real external line. Arbitrarily choose one of them and call it $O'$. Let $S$ be the union of:*

*$O$;*

*$O'$;*

*the imaginary external points on a real external line to $B \cap C$;*

*the two orbits of imaginary internal points $P$ on a real secant line (to $B \cap C$) such that $|P^{\perp} \cap O| = 2$ (call the union of these two orbits $O''$);*

*the orbit of imaginary external points $P$ on a real secant line such that $|P^{\perp} \cap O| = 4$;*

*the orbit of imaginary external points $P$ on a real secant line such that $P^{\perp} \cap O''$ is non-empty.*

*Then $S$ is a 315-set of type* $(10, 15)$ *in* $\mathrm{PG}(2, 25)$ *with group* $\mathrm{PSL}(2, 5)$ *(of order 60).*

PROOF. This is a straightforward computation in Magma. □

The stabilizer of $S$ has index four in its normalizer, which is isomorphic to $\mathrm{PGL}(2, 5) \times C_2$ and maps $S$ to the other three 315-sets of type $(10, 15)$ arising from the other choices of $O$ and $O'$. This set gives rise to a strongly regular graph with parameters $(15625, 7560, 3655, 3660)$. No strongly regular graph with these parameters was previously known. The two-weight code parameters are 300 and 305.

THEOREM 4.4.3. *Let $G$ be the stabilizer of* $\{(1,t,t^2) : t \in GF(5)\}$ *in* $\mathrm{P\Gamma L}(3,25)$, $\eta \in GF(25)$ *with* $\eta^2 - \eta + 2 = 0$. *Then the union $S$ of the orbits of*

$$(0,0,1),(1,0,0),(0,1,0),(1,4,\eta^{11}),(1,1,\eta^2),$$

$$(1,\eta^5,\eta^{10}),(1,\eta^3,0),(1,\eta^{20},\eta^2),(1,\eta^5,\eta^{16}),(1,\eta^{22},2),(1,\eta^{15},\eta^{13})$$

*is a* $231$*-set of type* $(6,11)$ *with full stabilizer $G$, which is isomorphic to* $\mathrm{AGL}(1,5) \times C_2$ *(of order 40).*

PROOF. This is a straightforward computation in Magma. □

This two intersection set is also self-polar and was constructed by fixing a Baer-subconic, as well as a point in the Baer-subconic. This set gives rise to a strongly regular graph with parameters $(15625, 5544, 1943, 1980)$. No strongly regular graph with these parameters was previously known. The weights for the code are 220 and 225.

We were also able to construct a 155-set $S$ of type $(5,10)$. It should be noted that $S$ in $\mathrm{PG}(2,25)$ has stabilizer $\mathrm{PGL}(2,5)$ (of order 120). Sets with these parameters were known previously (unions of five disjoint Baer subplanes), but $S$ is not a union of disjoint subplanes and $S$ has an insoluble group. The code consist weights are 145 and 150, while the strongly regular graph parameters are $(15625, 3720, 935, 870)$.

The next construction includes the use of a block-tactical decomposition matrix, so we will first define this before proceeding. Given a collineation group $G$ of $\mathrm{PG}(2,q)$, the number $n$ of point orbits equals the number $n$ of line orbits [[41] 2.3.1(b)]. Ordering the pint orbits $P_1,...,P_n$ and the line orbits $L_1,...,L_n$, and choosing representative $l_j$ of $L_j$, for $j=1,...,n$, the $n \times n$ matrix with entry in row $i$ and column $j$ equal to $|l_j \cap P_i|$ is the **block-tactical decomposition matrix** (of $G$).

The embedding of $A_6$ as a subgroup of PSU$(3,5)$ (and so of PSL$(3,25)$) goes back to Mitchel (1911), Section 15 and Theorem 15. From the proof of Theorem 15, the following orbits of $A_6$ can be read off:

$O_1$: 36 points fixed by an element of order 5 (and on the axis of a homology of order 2);

$O_2$: 45 points that are the centers of homolgies of order 2 in $A_6$;

two orbits $O_3, O_4$ of 60 points that are fixed by elements of order 3 (and on the axis of a homology of order 2);

$O_5$: 90 points that are fixed by elements of order 4;

two orbits $O_6, O_7$ of 180 points that are on the axis of a homology of order 2, but fixed by no elements of orders 3, 4, or 5.

The union of $O_1$ and $O_5$ is a classical unital. Thus $A_6$ commutes with the corresponding unitary polarity $\pi$, and thus the orbit lengths of lines are the same as those on pionts. The line orbits are thus $O_i^\pi$, $i = 1, ..., 7$.

The block-tactical decomposition matrix is

$$\begin{pmatrix} 1 & 5 & 0 & 0 & 0 & 10 & 10 \\ 4 & 4 & 4 & 4 & 2 & 4 & 4 \\ 0 & 3 & 5 & 0 & 6 & 6 & 6 \\ 0 & 3 & 0 & 5 & 6 & 6 & 6 \\ 0 & 1 & 4 & 4 & 1 & 8 & 8 \\ 2 & 1 & 2 & 2 & 4 & 10 & 5 \end{pmatrix}$$

where the orbits are in the order above.

Then the union of $O_2$, $O_5$, and either $O_6$ or $O_7$ is a 315-set of type $(10, 15)$, as the sums of the entries in the 2nd, 5th, and 6th (or 2nd, 5th, and 7th) columns are all equal to 10 or to 15. Thus we have:

THEOREM 4.4.4. *There exists a* 315*-set* $S$ *of type* $(10, 15)$ *in* $\mathrm{PG}(2, 25)$ *with the full stabilizer of* $S$ *in* $\mathrm{P\Gamma L}(3, 25)$*, which is isomorphic to* $\mathrm{PGL}(2, 9)$ *(of order 720).*

PROOF. $\mathrm{PSL}(2, 9)$ is isomorphic to $A_6$, and $\mathrm{Aut}(A_6)$ is isomorphic to $\mathrm{P\Gamma L}(2, 9)$, which is a maximal subgroup $N$ of $\mathrm{P\Gamma L}(3, 5)$, but Section 15 of Mitchell (1911). Indeed, the rest of the results of Mitchell (1911) establish that an overgroup of $A_6$ in $\mathrm{P\Gamma L}(3, 25)$ contains $\mathrm{PSU}(3, 5)$ or normalizes $A_6$ (and so is contained in $\mathrm{P\Gamma L}(2, 9)$). Since $\mathrm{PSU}(3, 5)$ does not stabilize $S$, it follows that the full stabilizer of $S$ in $\mathrm{P\Gamma L}(3, 25)$ is isomorphic to a subgroup of $\mathrm{P\Gamma L}(2, 9)$. Since $N$ has interchanges the two choices for $S$, the full stabilizer of $S$ in $\mathrm{P\Gamma L}(3, 25)$ is isomorphic to a subgroup of index 2 in $\mathrm{P\Gamma L}(2, 9)$. A calculation show that it is $\mathrm{PGL}(2, 9)$. $\square$

This set also gives rise to a strongly regular graph with parameters $(15625, 7560, 3655, 3660)$. We were able to construct another, yet different set with the same parameters.

THEOREM 4.4.5. *Let* $G$ *be the full stabilizer of* $\mathrm{P\Gamma L}(3, 25)$*. Then* $G$ *has two orbits of length 36 with groups of orders 72. Arbitrarily choose one of them and call it* $O$*. Based on the choice of* $O$*, there are two choices out of six for selecting 2 orbits of length 72 with groups of order 72. These two possibilities give different (non-isomorphic) sets. Denote one choice at* $O'$ *and the other as* $O''$*. Let* $S$ *be the union of:*

*$O$;*

*all three orbits of length 9;*

*the one orbit of length 36 with a group of order 144;*

*the one orbit of length 72 with a group of order 432;*

*either $O'$ or $O''$.*

*Then $S$ is a* 315*-set of type* $(10, 15)$ *with full stabilizer $G$, which is isomorphic to* $\mathrm{AGL}(1, 9)$ *(of order 72).*

PROOF. This is a straightforward computation in Magma. □

This set gives rise to a strongly regular graph with parameters $(15625, 7560, 3655, 3660)$.

We were able to construct two different 252-sets of type $(7, 12)$ in $\mathrm{PG}(2, 25)$.

THEOREM 4.4.6. *The full stabilizer of* $\mathrm{P\Gamma L}(3, 25)$ *has two orbits of length 72 whose group order is 216. Arbitrarily choose one of them and call it $O$. Let $S$ be the union of:*

*$O$;*

*the only orbit of length 108;*

*the only orbit of length 72 whose group has order 432.*

*Then $S$ is a* 252*-set of type* $(7, 12)$ *with full stabilizer in* $\mathrm{P\Gamma L}(3, 25)$ *isomorphic to* $\mathrm{AGL}(2, 3)$ *(of order 216).*

PROOF. This is a straightforward computation in Magma. □

THEOREM 4.4.7. *Let $G$ be the full stabilizer of* $\mathrm{P\Gamma L}(3, 25)$*. Then $G$ has two orbits of length 18 with group order 54. Arbitrarily choose one of them and call it $O$. Based on the choice of $O$, there are two choices for selecting 2 orbits of length 54; one with group order 54 and the other with group order 108. These two possibilities give different (non-isomorphic) sets. Denote one choice as $O'$ and the other as $O''$. Let $S$ be the union of:*

*$O$;*

*$O'$ or $O''$;*

*the only orbit of length 18 with group order 432;*

*the only two orbits of length 27 with group order 108;*

*the orbits of $(0, 1, \omega^3)$ and $(1, \omega^4, 1)$, where $\omega \in GF(25)$ with $\omega^2 - \omega + 2 = 0$.*

*Then $S$ is a 252-set of type $(7, 12)$ in $\mathrm{PG}(2, 25)$ with group $G$, which is isomorphic to $(C_3 \times C_3) \rtimes C_6$.*

PROOF. This is a straightforward computation in Magma. □

Both of these sets give rise to strongly regular graphs with parameters $(15625, 6048, 2323, 2352)$. No strongly regular graph with these parameters was previously known. The weights of the projective two-weight code are 240 and 245.

We now present the last set constructed in $\mathrm{PG}(2, 25)$.

THEOREM 4.4.8. *Let $G$ be the full stabilizer of $\mathrm{P\Gamma L}(3, 25)$, $\nu \in GF(25)$ with $\nu^2 - \nu + 2 = 0$. Then the union $S$ of orbits of*

$$(1, \nu^9, 2), (1, \nu^{22}, \nu), (1, \nu^{13}, 4),$$
$$(1, \nu^{16}, \nu^7), (1, \nu^3, 0), (1, \nu^{22}, \nu^{16})$$

*is a 189-set of type $(4, 9)$ in $\mathrm{PG}(2, 25)$ with full stabilizer in $G$, which is isomorphic to $(C_3 \times C_3) \rtimes C_6$ (of order 54).*

PROOF. This is a straightforward computation in Magma. □

This set gives rise to a strongly regular graph with parameters $(15625, 4536, 1279, 1332)$. No strongly regular graph with these parameters was previously known. The projective two-weight code weights are 180 and 185. However, in $\mathrm{PG}(2, 81)$, we were able to construct one two intersection set.

THEOREM 4.4.9. *Let $G$ be the full stabilizer of* $\mathrm{P\Gamma L}(3,81)$, $\pi \in GF(81)$ *with* $\pi^4+2\pi^3+2=0$. *Then the union $S$ of orbits of*

$$(0,1,0),(1,\pi^{65},\pi^{45}),(1,\pi^{53},2),$$

$$(1,\pi^{68},\pi^{64}),(1,\pi^{70},\pi^{69}),(1,\pi^{51},\pi^{4}),$$

$$(1,\pi^{5},\pi^{60}),(1,\pi^{50},\pi^{12}),(1,0,\pi^{12}),$$

$$(1,\pi^{78},\pi^{36}),(1,\pi^{31},0),(1,\pi^{42},\pi^{61}),$$

$$(1,\pi^{62},\pi^{19}),(1,\pi^{76},\pi^{4})$$

*is a 3285-set of type* $(36,45)$ *in* $\mathrm{PG}(2,81)$ *with full stabilizer in $G$ isomorphic to* $\mathrm{PSL}(2,9)$ *(of order 720).*

PROOF. This is a straightforward computation in Magma. □

It should be noted that this set was constructed by fixing a Baer-subconic. This set gives rise to a strongly regular graph with parameters $(531441, 262800, 129951, 129960)$. No strongly regular graph with these parameters was previously known. The weights of the projective two-weight code are 3240 and 3249.

Based on two of the newly constructed sets, as well as previous results, we have $\frac{(q^2-q+1)(q^2+q)}{2}$-sets of type $(\frac{q^2-q}{2},\frac{q^2+q}{2})$ in $\mathrm{PG}(2,q^2)$ admitting $\mathrm{P\Omega}(3,q)$, for $q=3,5,9$. Unfortunately, no such set exists for $q=7$. Perhaps there is some connection with the fact that, for these $q$, $\mathrm{P\Omega}(3,q)$ is isomorphic to an alternating group.

4.4.2. HIGHER DIMENSIONAL SETS. All of the sets thus far have been constructed in the plane. We will now consider higher dimensional two intersection sets. In order to do so, we must first state a few results. The first is due to de Finis in 1981 [40].

THEOREM 4.4.10. *In* PG$(2, q^2)$, $q = p^h$, $p$ *is a prime,* $h \geq 1$ *an integer, there exists* $k$*-sets of type* $(m, n)$*, where*

$$k = s(q^2 + q + 1),\ m = s,\ n = q + s$$

*for* $s = 2, 3, ..., q^2 - q - 2$.

The main idea behind this theorem is that we can take the union of two disjoint two intersection sets in PG$(2, q^2)$ and obtain a new two intersection set. We can extend this technique to higher dimensions as well.

THEOREM 4.4.11. *The union of a* $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1+q^{(d-1)/2})$*-set* $K_1$ *of type* $(m_1, m_1+q^{(d-1)/2})$ *in* PG$(d, q)$ *and a* $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_2 + q^{(d-1)/2})$*-set* $K_2$ *of type* $(m_2, m_2 + q^{(d-1)/2})$ *in* PG$(d, q)$*, disjoint from* $K_1$*, is a* $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1 + m_2 + 2q^{(d-1)/2})$*-set* $K$ *of type* $(m_1 + m_2 + q^{(d-1)/2}, m_1 + m_2 + 2q^{(d-1)/2})$ *in* PG$(d, q)$*,* $q$ *odd.*

PROOF. Any hyperplane intersects $K$ in either $m_1+m_2$, $m_1+m_2+q^{(d-1)/2}$, or $m_1+m_2+2q^{(d-1)/2}$ points. We must show that there are no hyperplanes in the first class. Consider the fundamental equations (1), (2), and (3) for the set $K$. There are three linear equations in the unknowns $t_{m_1+m_2}$, $t_{m_1+m_2+q^{(d-1)/2}}$, and $t_{m_1+m_2+2q^{(d-1)/2}}$. It is easy to see that the coefficient matrix has determinant $2q^{(3/2)(q-1)}$. Thus, it is non-singular, and hence these equations have a unique solution. As an $\frac{q^{(d+1)/2}+1}{q^{(d-1)/2}+1}(m_1 + m_2 + 2q^{(d-1)/2})$-set of type $(m_1 + m_2 + q^{(d-1)/2}, m_1 + m_2 + 2q^{(d-1)/2})$ is arithmetically feasible, we can find a solution to these equations with $t_{m_1+m_2} = 0$. Hence, this must be the unique solution. $\square$

Let us consider two infinite families of two intersection sets in $\mathrm{PG}(5,q)$. In [22, Example FE1, page 112], Calderbank and Kantor constructed the following family.

THEOREM 4.4.12. *There exists a* $(q^5+q^2)/2$*-set* $K_1$ *of type* $((q^4-q^2)/2,(q^4+q^2)/2)$ *in* $\mathrm{PG}(5,q)$ *admitting* $\mathrm{P}\Omega^-(5,q)$ *for* $q$ *odd. Namely, if* $Q$ *is a non-degenerate quadratic form on* $GF(q)^6$ *of minus type, then* $K_1=\{<v>: Q(v) \text{ is a non-square}\}$.

In [35, Remark 3.3(4)], Cossidente and Penttila constructed another infinite family of two intersection sets in $\mathrm{PG}(5,q)$.

THEOREM 4.4.13. *There exists a* $(q^4+q^3+q+1)/2$*-set* $K_2$ *of type* $((q^3-q^2+q+1)/2,(q^3+q^2+q+1)/2)$ *in* $\mathrm{PG}(5,q)$ *admitting* $\mathrm{PSL}(2,q^2)$*, for* $q$ *odd. Moverover, there is a non-degenerate quadratic form* $Q$ *on* $GF(q)^6$ *of minus type, with* $Q(v)=0$ *for all* $<v>\in K_2$.

It should be noted that $K_1$ is disjoint from $K_2$ if we fix the quadratic form $Q$ on $GF(q)^6$ of minus type. Thus

COROLLARY 4.4.14. *There exists a* $(q^5+q^4+q^3+q^2+q+1)/2$*-set* $K$ *of type* $((q^4+q^3+q+1)/2,(q^4+q^3+2q^2+q+1)/2)$ *in* $\mathrm{PG}(5,q)$ *admitting the group* $\mathrm{PSL}(2,q^2)$ *for* $q$ *odd.*

PROOF. This follows from the observation immediately before this theorem and from Theorem 4.4.11, with $K=K_1\cup K_2$. $\square$

4.4.3. PALEY SETS. The set constructed in Theorem 4.4.14 has the same parameters as some previously constructed sets. First we consider the Paley set [78], which is an orbit of the group generated by the square of a Singer cycle.

Theorem 4.4.15. *The sets $K$ from Theorem 4.4.14 are inequivalent to Paley sets.*

Proof. Let $K$ be a set from Theorem 4.4.14, and let $P$ be a Paley set in $\mathrm{PG}(5,q)$ for $q$ odd. If $K$ and $P$ are equivalent under an element of $\mathrm{P\Gamma L}(6,q)$, then $\Gamma(K)$ and $\Gamma(P)$ are isomorphic, and thus have isomorphic automorphism groups. By [65, Corollary 8.2] or [73], the automorphism group of $\Gamma(P)$ (the Paley graph) is a subgroup of $\mathrm{A\Gamma L}(1,q^6)$, and so is solvable. But $\mathrm{PSL}(2,q^2)$ is a subgroup of $\mathrm{Aut}(\Gamma(K))$, so $\mathrm{Aut}(\Gamma(K))$ is not solvable. Therefore, $K$ and $P$ are not equivalent. $\square$

As mentioned before, given a two intersection set, we can construct a strongly regular graph. **Paley graphs**, named after Raymond Paley, are dense (the number of edges is close to the maximal number of edges) undirected graphs constructed from the elements of a finite field by connecting pairs of elements that differ in a quadratic residue. Paley graphs are strongly regular graphs. More specifically, they form an infinite family of conference graphs, a type of strongly regular graph with parameters $(v, \frac{v-1}{2}, \frac{v-5}{4}, \frac{v-1}{4})$. A conference graph is associated with a symmetric conference matrix, and must then have the order of $v$ being 1 (modulo 4) and a sum of two squares.

Corollary 4.4.16. *The strongly regular graphs arising from the sets of Theorem 4.4.14 have the same parameters as, but are not isomorphic to, Paley graphs. Namely, the parameters are* $(q^6, \frac{q^6-1}{2}, \frac{q^6-5}{4}, \frac{q^6-1}{4})$.

The same proof shows that $\Gamma(K)$ is not isomorphic to the strongly regular graphs constructed from commutative semifields of order $q^6$ in Theorem 1.3 of Chen and Polhill (2011) [27], by [27, Theorem 1.4]. Also, $K$ contains no plane, so cannot be equivalent to the union of a partial 2-spread of size $\frac{q^3+1}{2}$.

CHAPTER 5

# Generalized Quadrangles

Generalized quadrangles play a prominent role in finite geometry, as they are connected to various other combinatorial structures, sometimes in multiple ways. Generalized quadrangles also play a central role in this thesis, so we will flesh-out some of these relationships.

Generalized quadrangles were first introduced by Tits in 1959 [107]. A (finite) **generalized quadrangle** ($GQ$) is an incidence structure, like that of a projective space. The incidence structure will be denoted by $S = (\mathcal{P}, \mathcal{B}, I)$ such that $\mathcal{P}$ is a non-empty set of points and $\mathcal{B}$ is a non-empty set of lines. $I$ is a symmetric point-line incidence relation that satisfies the following axioms:

(i) Each point is incident with $1+t$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $1+s$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If $x$ is a point and $L$ a line not incident with $x$, then there is a unique pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $xIMIyIL$.

## 5.1. Properties

The parameters of a generalized quadrangle are integers $s$ and $t$ and $S$ is said to have **order** $(s, t)$. If $s = t$, then we just say that $S$ has *order* $s$. The dual of a generalized quadrangle is also a generalized quadrangle in which points and lines are reversed, as well as $s$ and $t$.

Given two points $x, y$ of $S$, we say that $x$ and $y$ are **collinear** provided that there exists a line $L$ such that $xILIy$. We will denote that $x$ is collinear with $y$ by $x \sim y$. Dually, two lines $L$ and $M$ are **concurrent** if they intersect, denoted by $L \sim M$. For $x \in \mathcal{P}$, $x^{\perp} = \{y \in \mathcal{P} | y \sim x\}$. Per the definition of collinearity, we get that $x \in x^{\perp}$. We also define $\{x,y\}^{\perp} = x^{\perp} \cap y^{\perp}$ and $\{x,y\}^{\perp\perp} = \cap_{z \in \{x,y\}^{\perp}} z^{\perp}$. If $x \not\sim y$, we call $\{x,y\}^{\perp\perp}$ a **hyperbolic line** of $S$. Lastly, a point $x \in \mathcal{P}$ is **regular** if $|\{x,y\}^{\perp\perp}| = t+1$ for all $y \not\sim x$.

Based on the definition of a generalized quadrangle, we can extract some parameter information.

THEOREM 5.1.1. *Let $S = (\mathcal{P}, \mathcal{B}, I)$ be a generalized quadrangle of order $(s,t)$, then*

*(i)* $|\mathcal{P}| = (s+1)(st+1)$,

*(ii)* $|\mathcal{B}| = (t+1)(st+1)$.

PROOF. Fix a line $L$ of $S$ and count the number of ordered pairs $(x, M) \in \mathcal{P} \times \mathcal{B}$ such that $x \not{I} L$, $xIM$, and $L \sim M$. Then $v - s - 1 = (s+1)ts$, and solving for $v$ yields $v = (s+1)(st+1)$. Part (ii) follows from duality. □

Furthermore, for a generalized quadrangle of order $(s,t)$ we get the following restrictions:

1. $s + t$ divides $st(s+1)(t+1)$.
2. If $s > 1$ and $t > 1$, then $t \leq s^2$, and dually, $s \leq t^2$.
3. If $s \neq 1$, $t \neq 1$, $s \neq t^2$, and $t \neq s^2$, then $t \leq s^2 - s$, and dually, $s \leq t^2 - t$.

The proofs of these facts can be found in [82].

5.1.1. CLASSICAL GENERALIZED QUADRANGLES. There are three known families of **classical generalized quadrangles**, all associated with classical groups. All classical generalized quadrangles embed in $PG(d,q)$ for $3 \leq d \leq 5$.

The first family arises from a nonsigular quadric $Q$ of projective index 1 of the projective space $PG(d,q)$. The points and lines (hyperplanes) of $Q$ form a generalized quadrangle, denoted by $Q(d,q)$, with parameters

$$s = q,\ t = 1,\ |\mathcal{P}| = (q+1)^2,\ |\mathcal{B}| = 2(q+1) \text{ for } d = 3,$$

$$s = t = 1,\ |\mathcal{P}| = |\mathcal{B}| = (q+1)(q^2+1) \text{ for } d = 4, \text{ and}$$

$$s = q,\ t = q^2,\ |\mathcal{P}| = (q+1)(q^3+1),\ |\mathcal{B}| = (q^2+1)(q^3+1) \text{ for } d = 5.$$

The canonical equations for the quadric $Q$ are as follows:

$$x_0x_1 + x_2x_3 = 0 \text{ for } d = 3,$$

$$x_0^2 + x_1x_2 + x_3x_4 = 0 \text{ for } d = 4, \text{ and}$$

$$f(x_0, x_1) + x_2x_3 + x_4x_5 = 0 \text{ for } d = 5,$$

where $f$ is an irreducible binary quadric form. The hyperbolic quadric $Q(3,q)$ may also be denoted as $Q^+(3,q)$, and the elliptic quadric $Q(5,q)$ is often denoted as $Q^-(5,q)$. The parabolic quadric, $Q(4,q)$, is always denoted as such. It should be noted that $Q^+(3,q)$ is considered a trivial point-line incidence structure, as it is just the structure of a grid. However, as a quadric, it is associated with numerous interesting combinatorial objects.

The second classical generalized quadrangle arises from a nonsignular hermitian variety $\mathcal{H}$ of the projective space $PG(d,q^2)$ for $d = 3$ or 4. The points and lines of $\mathcal{H}$ form a generalized quadrangle $H(d,q^2)$ with parameters

$$s = q^2,\ t = q,\ |\mathcal{P}| = (q^2+1)(q^3+1),\ |\mathcal{B}| = (q+1)(q^3+1) \text{ for } d = 3 \text{ and}$$

$$s = q^2,\ t = q^3,\ |\mathcal{P}| = (q^2+1)(q^5+1),\ |\mathcal{B}| = (q^3+1)(q^5+1) \text{ for } d = 4.$$

The canonical equation for $\mathcal{H}$ is $x_0^{q+1} + x_1^{q+1} + ... + x_d^{q+1} = 0$.

The final classical generalized quadrangle arises, specifically, from $PG(3,q)$. The points of $PG(3,q)$, along with the totally isotropic lines with respect to a symplectic polarity form a generalized quadrangle, denoted as $W(q)$. The parameters of $W(q)$ are

$$s = t = q,\ |\mathcal{P}| = |\mathcal{B}| = (q+1)(q^2+1).$$

The canonical bilinear form for the symplectic polarity of $PG(3,q)$ is

$$x_0y_1 - x_1y_0 + x_2y_3 - x_3y_2 = 0.$$

Several theorems relate the three classical generalized quadrangles. A few will be presented at this time.

THEOREM 5.1.2 ([9]). *A generalized quadrangle $S$ is isomorphic to $W(q)$ if and only if all the points of $X$ are regular.*

THEOREM 5.1.3 ([82]). *$Q(4,q)$ is isomorphic to $W(q)$ if and only if $q$ is even.*

THEOREM 5.1.4 ([82]). *The dual of $Q(4,q)$ is isomorphic to $W(q)$.*
*The dual of $Q^-(5,q)$ is isomorphic to $H(3,q^2)$.*

The proof is taken from the book **Finite Generalized Quadrangles** by Payne and Thas [82].

PROOF. Let $Q^-(5,q)$ be an elliptic quadric in $PG(5,q)$. Then extend $PG(5,q)$ to $PG(5,q^2)$. The extension of $Q^-(5,q)$ is a hyperbolic quadric, $Q^+(3,q)$, in $PG(5,q^2)$. Thus, by definition, we get that $Q^+$ is the Klein quadric of the lines of $PG(3,q^2)$. So for $Q^-$ in $Q^+$ there corresponds a set $V$ of lines in $PG(3,q^2)$. Thus, for any line $L$ of $Q^-$, there corresponds $q+1$ lines of $PG(3,q^2)$ that all lie in a plane and pass through a point $x \in PG(3,q^2)$. Let $P$ be the set of points on the lines contained in $V$. Then for each point of $Q^-$, there corresponds a line in $V$ and with each line $L$ in $Q^-$, there corresponds a point $x$ in $P$. So for distinct lines $L$ and $M$ in $Q^-$ there correspond distinct points $x$ and $y$ in $P$, as a plane of $Q^+$ contains at most one line of $Q^-$. Now, since a point $y$ of $Q^-$ is on $q^2+1$ lines of $Q^-$, these $q^2+1$ lines are mapped onto the $q^2+1$ points of the image of $y$. Thus, there exists an anti-isomorphism from $Q^-$ onto $(P, V, I)$ where $I$ is the natural incidence relation. Therefore, $(P, V, I)$ is a $GQ$ of order $(q^2, q)$ embedded in $PG(3, q^2)$. But, this says that the $GQ$ $(P, V, I)$ is $H(3, q^2)$ due to a result by F. Buekenhout and C. Lefevre [21]. $\square$

Let $S = (\mathcal{P}, \mathcal{B}, I)$ be a $GQ$ of order $(s, t)$. Given a pair of distinct points $x, y \in \mathcal{P}$, the trace of $(x, y)$ is defined to be the set $x^\perp \cap y^\perp$, denoted by $tr(x, y)$ or $\{x, y\}^\perp$. Define $\mathcal{B}^*$ to be the set of all spans (the set of all hyperbolic lines). Then $S^* = (\mathcal{P}, \mathcal{B}^*, \mathsf{I})$ is a linear space. A *linear variety* of $S^*$ is a subset $\mathcal{P}' \subset \mathcal{P}$ such that $x, y \in \mathcal{P}'$, $x \neq y$, implies that $\{x, y\}^{\perp\perp} \subset \mathcal{P}'$. $\mathcal{P}'$ is a *proper* linear variety if $\mathcal{P}' \neq \mathcal{P}$ and $|\mathcal{P}'| > 1$. With these definitions in place, we can now state Tallini's theorem [95].

THEOREM 5.1.5. *Let $S = (\mathcal{P}, \mathcal{B}, I)$ be a GQ of order $(s, t)$ with $s \neq t$, $s > 1$, and $t > 1$. Then $S$ is isomorphic to $H(3, s)$ if and only if*

*(i) all points of $S$ are regular, and*

*(ii) if the lines $L$ and $L'$ of $\mathcal{B}^*$ are contained in a proper linear variety of $S^*$, then the lines $L^\perp$ and $L'^\perp$ of $\mathcal{B}^*$ are contained in a proper linear variety of $S^*$.*

The proof is several pages long, and will, thus, not be presented here. The proof can be found in [82, 95]. This theorem is also known as Tallini's characterization of $H(3, s)$. While there are further relationships between non-classical generalized quadrangles, they do not play a prominant role (except in the last chapter), so they will not be addressed at this time.

5.1.2. The Group of $H(3,q)$. Not only does the hermitian variety $H(3,q^2)$ play a critical role in constructing the family of strongly regular graphs we are after, but so does its group, $P\Gamma U(4,q)$. Before discussing the characteristics of $P\Gamma U(4,q)$, we will first define some general group properties, coming from Taylor's book **The Geometry of Classical Groups** [99].

Let $G$ be a group and $\Omega$ a non-empty set, then we say that $G$ acts on $\Omega$ if for all $g \in G$, there exists a bijection $\alpha \mapsto g(\alpha)$ of $\Omega$ such that

$$1(\alpha) = \alpha \ \forall \alpha \in \Omega \text{ and}$$

$$(gh)(\alpha) = g(h(\alpha)) \ \forall \alpha \in \Omega \text{ and } \forall g, h \in G.$$

Now for $\Delta \subset \Omega$ and $g \in G$, let $g(\Delta) := \{g(\alpha) | \alpha \in \Delta\}$. Then we call the subgroup

$$G_\Delta := \{g \in G | g(\Delta) = \Delta\}$$

the **stabilizer** of $\Delta$ in $G$. The subgroup

$$G(\Delta) := \{g \in G | g(\alpha) = \alpha$$

$\forall \alpha \in \Delta\}$ is the **pointwise stabilizer** of $\Delta$. The assignment $\Delta \mapsto g(\Delta)$ is defined to be the action of $G$ on the set of all subsets of $\Omega$. The subgroup $G(\Omega)$ of $G$ is defined to be the kernel of the action of $G$ on $\Omega$. If $G(\Omega) = 1$, then we say that $G$ acts **faithfully** on $\Omega$ or we say that $G$ is a **group of permutations** of $\Omega$. If $G_\Delta = G$, then $\Delta \subset \Omega$ is $G$**-invariant**. The smallest, nonempty subset of $\Omega$ such that it is $G$-invariant is called an **orbit**. We then get that $\Omega$ is the disjoint union of all its orbits. If $\alpha \in \Omega$, then the orbit which contains $\alpha$ is defined to be $orb_G(\alpha) := \{g(\alpha) | g \in G\}$. If $|orb_G(\alpha)| = n$ for some $n \in \mathbb{N}$, then the Orbit-Stabilizer Theorem says that $|orb_G(\alpha)| = |G||G_\alpha|$. If there is only one orbit in $G$, then $G$ is **transitive**. Furthermore, if $G$ is transitive and $G_\alpha = 1 \ \forall \alpha \in \Omega$, then $G$ is said to act **regularly** on $\Omega$.

With these definitions in place, we can now construct $P\Gamma U(4,q)$. Let $\pi$ be a unitary polarity of the projective geometry $P(V)$, where $V$ is a vector space of dimension $n$ over the field $\mathbb{F}$. A **polarity** is a correlation $\pi$ of order 2, where by $\pi$ is a bijection from $P(V)$ to $P(V)$ which reverses inclusion. The fact that makes the polarity a unitary polarity is that $\sigma^2 = 1$, $\sigma \neq 1$, and $\beta(u,v) = \sigma\beta(v,u)\ \forall u, v \in V$, whenever $\pi$ arises from a non-degenerate reflexive $\sigma$-sesquilinear form $\beta$. More simply speaking, in $PG(2,q^2)$, a unitary polarity is a polarity $\pi$ having $q^3+1$ absolute points and for any non-absolute line, that line contains exactly $q+1$ absolute points. Let $\pi$ be induced by a $\sigma$-hermintian form $\beta$, where $\sigma$ is an automorphism of $\mathbb{F}$ of order 2. Then the elements $f \in GL(V)$ such that $\beta(f(u), f(v)) = \beta(u,v)\ \forall u, v \in V$ forms the **unitary group** $U(V)$. That is to say, $U(V)$ is the group of square unitary matrices. The full unitary group $\Gamma U(V)$ consists of $\tau$-semilinear transformations $f$ that induce a collineation of $P(V)$ that commutes with $\pi$. In other words, for some $a \in \mathbb{F}$ such that $a = \sigma(a)$,

$$\beta(f(u), f(v)) = a\tau\beta(u,v)\ \forall u, v \in V.$$

From $\Gamma U(V)$ we can then construct the general unitary group, $GU(V)$.

$$GU(V) := \Gamma U(V) \cap GL(V).$$

$GU(V)$ consists of all $n \times n$ matrices $A$ such that $A^*A = bI$ for some $b \neq 0$.

A property that distinguishes $U(V)$ from other classical groups is the fact that $U(V)$ need not contain transvections. Transvections in $U(V)$ correspond to an important set of points in $P(V)$, which we will need later on. Therefore, we will define transvections. Let $t \in GL(V)$ be a linear transformation that fixes every element of a hyperplane $H$ of $V$. Now let $v \in V$ such that $v \notin H$ and let $\phi$ be a linear functional defined by

$$\phi(av+h) := a\ \forall a \in \mathbb{F}, h \in H.$$

Then $ker\phi = H$ and if $w \in V$, then $w - \phi(w)v \in H$. Furthermore, we get that

$$t(w) := w + \phi(w)u, \text{ where } u := t(v) - v.$$

But, if $\phi \in V^*$ and if $u$ is a vector such that $\phi(u) \neq -1$, then

$$t_{\phi,u}(w) := w + \phi(w)u$$

is an element of $GL(V)$ which fixes every vector in $ker\phi$. If $\phi(u) = 0$, then $t_{\phi,u}$ is a **transvection**.

Transvections in $U(V)$ are equivalent to the existence of isotropic points in $P(V)$. When constructing a new family of strongly regular graphs, isotropic points will come into play. An *isotropic point* in $P(V)$ is a nonzero vector in $V$ which is orthogonal to itself. We say that a subspace $W$ is *totally isotropic* if $W \subset W^{\perp}$.

There are two nice consequences based off of the field with are useful for determining if there are isotropic points. But before stating them, we will need a couple definitions. Let $\mathbb{F}_0$ denote the fixed field of $\sigma$, where $\sigma$ is an automorphism of $\mathbb{F}$ of order 2. That is, $\mathbb{F}_0 := \{a \in \mathbb{F} | a = \sigma(a)\}$. The function

$$N : \mathbb{F}^{\times} \to \mathbb{F}_0^{\times}$$

$$: a \mapsto a\sigma(a)$$

is called the *norm*.

LEMMA 5.1.6. *If dim $V \geq 2$ and the norm map $N$ is onto, then $V$ contains isotropic vectors.*

PROOF. Suppose that $v \neq 0$ is not isotropic. Since $v$ is not isotropic, there exists a $b \neq 0$ such that $b = \beta(v, v)$. Then $b = \sigma(b)$ and for $a \in \mathbb{F}$, $u \in <v>^{\perp}$, we get that

$$\beta(u + av, u + av) = \beta(u, u) + a\sigma(a)b.$$

Since $-b\beta(u,u) \in \mathbb{F}_0$ and $N(a) = 1$ if and only if $a = \frac{b}{\sigma(b)}$ for some $b \in \mathbb{F}^{\times}$, we have that there exists an $a \in \mathbb{F}$ such that $u + av$ is isotropic. Therefore, $V$ contains isotropic vectors, provided that the dimension of $V$ is greater than or equal to 2. $\square$

From this Lemma, we obtain the following Corollary.

Corollary 5.1.7. *If dim $V \geq 2$ and $\mathbb{F}$ is finite, then $V$ contains isotropic vectors.*

Now, we have been denoting the unitary group as $U(V)$, but if $\mathbb{F}_0$ is the finite field $\mathbb{F}_q$ and $n := \dim V$, then we denote the unitary group as $U(n,q)$. The same applies to $\Gamma U(V)$, giving us $\Gamma U(n,q)$. By switching to this notation, we can compute the orders of these groups, as well as count the number of isotropic vectors in $V$.

Lemma 5.1.8. *$V$ contains $(q^{n-1} - (-1)^{n-1})(q^n - (-1)^n)$ isotropic vectors.*

The order of $U(n,q)$ is

$$q^{\frac{n(n-1)}{2}} \prod_{i=1}^{n} (q^i - (-1)^i).$$

Another group of particular interest is $PGU(4,q)$, the projective general unitary group. $PGU(4,q)$ is obtained from $GU(4,q)$ by factoring out the scalar matrices contained in that group. The order of $PGU(4,q)$ is

$$q^6(q^4 - 1)(q^3 + 1)(q^2 - 1).$$

Similarly, $P\Gamma U(4,q)$ is obtained from $\Gamma U(4,q)$ by factoring out the scalar matrices contained in $\Gamma U(4,q)$. The order of $P\Gamma U(4,q)$ is

$$h|PGU(4,q)|,$$

where $q^2 = p^h$ and $p$ is prime.

5.1.3. Relationships. As previously mentioned, generalized quadrangles are related to other combinatorial objects. For example, a generalized quadrangle can be defined in terms of a partial geometry. A **partial geometry** is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ in which $\mathcal{P}$ and $\mathcal{B}$ are disjoint, nonempty sets of objects. $\mathcal{P}$ is the set of points, while $\mathcal{B}$ is the set of lines and $I$ is a symmetric point-line incidence relation that satisfies the following axioms:

(i) Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.

(ii) Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.

(iii) If $x$ is a point and $L$ is a line not incident with $x$, then there are $\alpha$ points ($\alpha \geq 1$) $x_1, ..., x_\alpha$ and $\alpha$ lines $L_1, ..., L_\alpha$ such that $xIL_iIx_iIL$, $i = 1, 2, ..., \alpha$.

By definition, if $\alpha = 1$, then a partial geometry is also a generalized quadrangle. Partial geometries were introduced by Bose in 1963 [14].

Another object related to a generalized quadrangle is a partial quadrangle. A **partial quadrangle** is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ of points and lines satisfying conditions (i) and (ii) of a partial geometry, as well as

(iii)' If $x$ is a point and $L$ is a line not incident with $x$, then there exists at most one pair $(y, M) \in \mathcal{P} \times \mathcal{B}$ for which $xIMIyIL$.

(iv)' If two points are not collinear, then there are $\mu$ points ($\mu > 0$) collinear with both.

These objects were introduced by Cameron in 1974 [23]. We get that a partial geometry is a generalized quadrangle if and only if $\mu = t + 1$.

A **semi-partial geometry** was introduced by Debrocy and Thas in 1978 [36] as being an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ of points and lines satisfying (i) and (ii) as given above, as well as

(iii)" If $x$ is a point and $L$ is a line not incident with $x$, then there are 0 or $\alpha$ points ($\alpha \geq 1$) which are collinear with $x$ and incident with $L$.

(iv)" If two points are not collinear, then there are $\mu$ points ($\mu > 0$) collinear with both.

We then get that a semi-partial geometry is a partial geometry if and only if $\mu = (t+1)\alpha$. Thus, it is a generalized quadrangle if and only if $\alpha = 1$ and $\mu = t + 1$.

We also get that generalized quadrangles are related to designs. A **partial geometric design** is an incidence structure $S = (\mathcal{P}, \mathcal{B}, I)$ of points and blocks satisfying the following properties:

(i) Each point is incident with $t + 1$ blocks ($t \geq 1$), and each block is incident with $s + 1$ points ($s \geq 1$).

(ii) For a given point-block pair $(x, L)$ such that

   (a) if $x$ is not on $L$, we have $\sum_{yIL}[x, y] = \alpha$, where $[x, y]$ denotes the number of blocks incident with $x$ and $y$ and

   (b) if $xIL$, we have $\sum_{yIL}[x, y] = \beta$.

The notation used to denote such a partial design is $D(s, t, \alpha, \beta)$. A generalized quadrangle is a partial design in which $\alpha = 1$ and $\beta = s + t + 1$. Such structures are due to Bose, Shrikhande, and Singhi [17].

The underlying connection (which we are primarily concerned with) between all of these objects is strongly regular graphs. Let $S$ be a partial geometry with parameters $s$, $t$, and $\alpha$. Let the points of $S$ be the vertices of a graph. Then two vertices are adjacent if and only if they are collinear in $S$. This graph is known as the **point** or **collinearity graph** of the partial geometry $S$. Whenever $\alpha \neq s+1$, the point graph is a strongly regular graph with parameters

$$((s+1)(st+\alpha)/\alpha, s(t+1), s-1+(\alpha-1)t, \alpha(t+1)).$$

But, as previously stated, if $\alpha = 1$, the partial geometry, $S$, is a generalized quadrangle. Thus, the point graph of a generalized quadrangle is a strongly regular graph with parameters

$$((s+1)(st+1), s(t+1), s-1, t+1).$$

The classicial generalized quadrangle $H(3,q^2)$, for $q$ even, which will be denoted as $H$, is of primary interest. The point graph of $H$ is a strongly regular graph with parameters $((q^2+1)(q^3+1), q^3+q^2, q^2-1, q+1)$.

## 5.2. New Family of Graphs

The new family of strongly regular graphs which we have constructed relies on the point graph of $H(3,q^2)$, for $q$ even.

Theorem 5.2.1. *Let $\Gamma$ be the point graph of $\mathbf{H} = \mathrm{H}(3,q^2)$, $q$ even. Fix a point $P$ of $\mathbf{H}$ and two orthogonal tangent lines $t_1$ and $t_2$ on $P$, and define $\mathcal{A} := \{A \in \mathbf{H} : A \sim P\}$ and $\mathcal{B} := \{Q \in \mathbf{H} : Q \not\sim P\}$. Define a graph $\Gamma'$ as follows:*

- *Remove all edges $\{Q,R\}$ in $\Gamma$ if and only if $Q, R \in \mathcal{B}$.*
- *Add edge $\{Q,R\}$ if and only if there exists a totally isotropic point $Y \in P^\perp \setminus Q^\perp$ such that $Y \neq P$ with $V := t_2 \cap (PQ)^\perp$, $S := VY \cap t_1$, and $R := S^\perp \cap YQ$.*

*Then $\Gamma'$ is a strongly regular graph with parameters $((q^2+1)(q^3+1), q^3+q^2, q^2-1, q+1)$.*

PROOF. By definition, the number of vertices of $\Gamma$ is $(q^2+1)(q^3+1)$, so clearly the number of vertices of $\Gamma'$ is the same. Show that the valency of $\Gamma'$ is $q^3+q^2$. By definition $P$ is adjacent to the $q^2+q^3$ points in $\mathcal{A}$. For $A \in \mathcal{A}$, $A$ is adjacent to $P$, $q^2-1$ points in $\mathcal{A}$, and $q^3$ points in $\mathcal{B}$. Since adjacency between points in $\mathcal{A}$ and $\mathcal{B}$ is left unchanged, the valency of $A \in \mathcal{A}$ is unchanged as well. Let $Q \in \mathcal{B}$, then in $\Gamma$ $Q$ is adjacent to $q+1$ points in $\mathcal{A}$ and $(q^2-1)(q+1)$ points in $\mathcal{B}$. In $\Gamma'$, those $(q^2-1)(q+1)$ edges are removed. It must be shown that the number of edges added equals $(q^2-1)(q+1)$. This is obvious provided that $R$, as defined, is not adjacent to $P$ and is totally isotropic (because the number of $Y \in P^\perp \setminus Q^\perp$ such that $Y \neq P$ and $Y$ is totally isotropic is $(q^2-1)(q+1)$). $R$ is not adjacent to $P$ because $R \neq Y$, as $SY$ is hyperbolic and $P^\perp \cap YQ = Y$. WLOG, let $P =< p >$, $Q =< q >$, $U =< u >$, and $V =< v >$. Then the Gram matrix with respect to $p, u, v, q$ is

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

If we define $R =< r >$, then $r = y+q$ because $< y >=< p+u+v >$, which implies that $S =< p+u >$. So $S^\perp$ consists of points of the form $(x_1, x_2, x_3, x_2)$, which implies that $r = y+q$. Now $< r, r >=< y+q, y+q >=< y, y > + < y, q > + < q, y > + < q, q >=< y, q > + < q, y >$ because $Q$ and $Y$ are totally isotropic. Now $< y, q >=< p+u+v, q >=< p, q >= 1$, so $< y, q > + < q, y >= 1+1 = 0$ in characteristic 2. Therefore, $< r, r >= 0$, and $R$ is totally isotropic. Furthermore, it must be shown that $\Gamma'$ is symmetric. That is, it must be shown that the edge $\{Q, Q'\}$ is undirected for $Q, Q' \in \mathcal{B}$. Suppose $\{Q, Q'\}$ is an edge in $\Gamma'$, then there must exist a $Y \in P^\perp \setminus Q^\perp$ such that $Y$ is totally isotropic and $Y \neq P$. We can then define $V := (PQ)^{\perp} \cap t_2$ and $S := VY \cap t_1$, so that $Q' := S^{\perp} \cap YQ$. But $Q' = S^{\perp} \cap YQ$ implies that $Q' \in YQ$. Thus, $Y \in P^{\perp} \setminus (Q')^{\perp}$. Let $V' := (PQ')^{\perp} \cap t_2$ and $S' := V'Y \cap t_1$. Show that $Q := (S')^{\perp} \cap YQ'$. In order to do so, we must first show that $\{P, S, V', Y\}$ is a quadrangle. $P$ and $V'$ are on $t_2$, for which $S$ and $Y$ are not. $P$ and $S$ are on $t_1$, for which $V'$ and $Y$ are not. $PY$ is a totally isotropic line not containing $S$ or $V'$, as neither of them are isotropic. $S$ cannot lie on the line $YV'$, for if it did, $Q = Q'$. Similarly, $V'$ cannot lie on the line $YS$, for if it did, $Q = Q'$. Therefore, $\{P, S, V', Y\}$ is a quadrangle in characteristic two, which implies that the diagonal points are collinear. The diagonal points are $SY \cap V'P = V$, $SV' \cap YP = A$, where $A \in \mathcal{A}$ is a common neighbor of $Q$ and $Q'$, and $SP \cap YV' = S'$. Thus, $V$, $A$, and $S'$ are collinear. Now $Q' \perp V'$ and $Q' \perp A$ implies that $Q' \perp V'A$, which implies that $Q' \perp S'$. But $Q' = S^{\perp} \cap YQ$, so $Q = (S')^{\perp} \cap YQ'$. Thus, $\Gamma'$ is symmetric.

Show that the number of common neighbors given two adjacent vertices is $q^2 - 1$.

**Case 1:** Suppose $\{P, A\}$ is an edge in $\Gamma'$ for $A \in \mathcal{A}$. $P$ is adjacent to $q^2 + q^3$ points in $\mathcal{A}$, one of which is $A$. $A$ is adjacent to $q^2 - 1$ points in $\mathcal{A}$ and $q^3$ points in $\mathcal{B}$ (as well as $P$). Since $P$ is not adjacent to any points in $\mathcal{B}$, the number of common neighbors between $P$ and $A$ is $q^2 - 1$.

**Case 2:** Suppose $\{A, B\}$ is an edge in $\Gamma'$ for $A, B \in \mathcal{A}$. By definition, both points are adjacent to $P$. In $\Gamma$, $A$ and $B$ have $q^2 - 2$ common neighbors in $\mathcal{A}$ and no common neighbors in $\mathcal{B}$. Since adjacency between points in $\mathcal{A}$ is unchanged in $\Gamma'$, the number of common neighbors is $q^2 - 1$.

**Case 3:** Suppose that $\{A, Q\}$ is an edge in $\Gamma'$ for $A \in \mathcal{A}$ and $Q \in \mathcal{B}$. In $\Gamma$, $A$ and $Q$ have no common neighbors from $\mathcal{A}$. So the same is true in $\Gamma'$. Show that $A$ and $Q$ have $q^2 - 1$ common neighbors from $\mathcal{B}$ in $\Gamma'$. Let $R \in \mathcal{B}$ be a common neighbor. Then $R \sim Q$

implies that there exists a $Y \in P^{\perp} \setminus Q^{\perp}$ such that $Y$ is totally isotropic and $Y \neq P$. Then $V := (PQ)^{\perp} \cap t_2$ and $S := VY \cap t_1$, so that $R := S^{\perp} \cap YQ$. Since $R := S^{\perp} \cap YQ$, $Y \in RQ$. Suppose that $Y \notin PA \setminus \{P, A\}$, then $A \notin (RQ)^{\perp}$. But this contradicts the fact that $Q \sim A$ and $R \sim A$. Thus, $Y \in PQ \setminus \{P, A\}$, and since the number of $Y \in PA \setminus \{P, A\} \subset P^{\perp} \setminus Q^{\perp}$ is $q^2 - 1$, $Q$ and $A$ have $q^2 - 1$ common neighbors from $\mathcal{B}$ in $\Gamma'$.

**Case 4:** Suppose $\{Q, Q'\}$ is an edge in $\Gamma'$ for $Q, Q' \in \mathcal{B}$. In $\Gamma$, $Q$ and $Q'$ have one common neighbor from $\mathcal{A}$. Thus, in $\Gamma'$, they still have one common neighbor from $\mathcal{A}$. Show that $Q$ and $Q'$ have $q^2 - 2$ common neighbors from $\mathcal{B}$. Suppose that $R \in \mathcal{B}$ is a common neighbor. Then $Q \sim R$ implies that there exists a $Y \in P^{\perp} \setminus Q^{\perp}$ such that $Y \neq P$, $Y$ is totally isotropic, $V = t_2 \cap (PQ)^{\perp}$, and $S = VY \cap t_1$ such that $R = S^{\perp} \cap YQ$. Similarly, $Q' \sim R$ implies that there exists a $Y' \in P^{\perp} \setminus (Q')^{\perp}$ such that $Y' \neq P$, $Y'$ is totally isotropic, $V' = t_2 \cap (PQ')^{\perp}$, and $S' = V'Y' \cap t_1$ such that $R = (S')^{\perp} \cap Y'Q'$. If $A \in \mathcal{A}$ is the common neighbor, then $< P, Q, Q' >^{\perp} = A$. By definition, we know that $S, S' \in t_1$ with $R \in S^{\perp}$ and $R \in (S')^{\perp}$. Thus, $S, S' \in (PR)^{\perp}$, which is a line, call it $l_R$, on $A$. So $S = l_R \cap t_1 = S'$. Hence, $S = S'$. Choose $S$ on $t_1$ such that $S \neq P$, $S \notin Q^{\perp}$ and $S \notin (Q')^{\perp}$. Then there are $q^2 - 2$ choices for $S$. Consider the planes $< S, V, Q >, < S, V', Q' >, S^{\perp}$. We get that $< S, V, Q > \cap < S, V', Q' >$ is a tangent line on $S$, call it $t_s$. Then $S^{\perp} \cap t_s = R(S)$ for some $R(S)$, and $R(S)$ is totally isotropic as $R(S) \in S^{\perp}$ and $S$ is not totally isotropic. Thus, the number of common neighbors is $q^2 - 2$, as there are $q^2 - 2$ choices for $S$. Therefore, the number of common neighbors given a pair of adjacent vertices is $q^2 - 1$. Now we must show that the number of common neighbors given two non-adjacent vertices is $q + 1$. This will be done by considering various cases, the first of which deal with the unique point $P$ and a point in $\mathcal{B}$. The final case considers two non-adjacent vertices in $\mathcal{B}$.

**Case i:** Let $Q \in \mathcal{B}$, then $P$ and $Q$ are not adjacent. $P$ is adjacent to all $q^2 + q^3$ points from $\mathcal{A}$. $Q$ is adjacent to $q+1$ points from $\mathcal{A}$. Thus, $P$ and $Q$ have $q+1$ common neighbors in $\mathcal{A}$.

**Case ii:** Suppose that $A$ and $B$ are non-adjacent vertices in $\mathcal{A}$. Then both $A$ and $B$ are adjacent to $P$. Furthermore, in $\Gamma$, $A$ and $B$ have $q$ common neighbors from $\mathcal{B}$ and no common neighbors from $\mathcal{A}$. Since adjacency between points in $\mathcal{A}$ and $\mathcal{B}$ is left unchanged in $\Gamma'$, $A$ and $B$ still have $q$ common neighbors from $\mathcal{B}$. Thus, there are $q + 1$ common neighbors.

**Case iii:** Suppose that $A \in \mathcal{A}$ and $Q \in \mathcal{B}$ are not adjacent. In $\Gamma$, $A$ and $Q$ have 1 common neighbor from $\mathcal{A}$. Since adjacency is unchanged between points from $\mathcal{A}$ and $\mathcal{B}$, $A$ and $Q$ still have one common neighbor from $\mathcal{A}$, call it $B$. Show that $A$ and $Q$ have $q$ common neighbors from $\mathcal{B}$. For all $R \in \mathcal{B}$, if $R \sim Q$, then there exists a $C \in \mathcal{A}$ such that $Q \sim C$ and $R \sim C$ and $B \neq C$. $Q \sim C$ implies that $C \in (PQ)^{\perp}$, and $Q \sim R$ implies that there exists a $Y \in P^{\perp} \setminus Q^{\perp}$ such that $Y$ is totally isotropic and $Y \neq P$. Then $V := t_2 \cap (PQ)^{\perp}$ and $S := VY \cap t_1$, so that $R := S^{\perp} \cap YQ$. Furthermore, suppose that $R \sim A$ (and $Q \not\sim A$), then $C = AS \cap (PQ)^{\perp}$ (because $AC$ is a hyperbolic line in $P^{\perp}$ meeting $t_1$ and contains all points from $\mathcal{A}$ adjacent to $R$) and, hence, $Y = PC \cap VS$. There are $q$ choices for $C$ on $(PQ)^{\perp}$ such that $C \in \mathcal{A}$ and $C \neq B$. Thus, there are $q$ common neighbors from $\mathcal{B}$.

**Case iv:** Suppose that $Q, Q' \in \mathcal{B}$ are not adjacent in $\Gamma'$.

a. If $Q \sim Q'$ in $\Gamma$, then there exists an $A \in \mathcal{A}$ such that $Q \sim A$ and $Q' \sim A$ in both $\Gamma$ and $\Gamma'$. Show that $Q$ and $Q'$ have $q$ common neighbors from $\mathcal{B}$. Suppose that $R \in \mathcal{B}$ is a common neighbor. Then $Q \sim R$ implies that there exists a totally isotropic point $Y \in P^{\perp} \setminus Q^{\perp}$, for $Y \neq P$, $V = t_2 \cap (PQ)^{\perp}$, and $S = VY \cap t_1$ such that $R = S^{\perp} \cap YQ$. Similarly, $Q' \sim R$ implies that there exists a totally isotropic point $Y' \in P^{\perp} \setminus Q'^{\perp}$, for $Y' \neq P$, $V' = t_2 \cap (PQ')^{\perp}$, and $S' = V'Y' \cap t_1$ such that $R = (S')^{\perp} \cap Y'Q'$. By the same argument as in Case 4, $S = S'$. Now since $A \sim Q$, $A \sim Q'$, and $Q \sim Q'$ in $\Gamma$, there exists a totally isotropic line $l$ through $A, Q, Q'$ which does not contain $P$, but is contained in $< P, Q, Q' >$. Furthermore, $P, Y, Y'$ are collinear as $Y, Y' \in P^{\perp}$, which implies that $P \subset (YY')^{\perp}$. This line is contained in $< P, Q, Q' >$. We also know that $S^{\perp} \cap < P, Q, Q' >$ is a hyperbolic line on $P$, containing $q$ totally isotropic points not equal to $P$, and since $S^{\perp} \cap Y'Q' = S^{\perp} \cap YQ = R$, $Q$ and $Q'$ have $q$ common neighbors from $\mathcal{B}$.

b. Suppose that $Q \not\sim Q'$ in $\Gamma$.

1. Suppose that $QQ'$ is a hyperbolic line not containing $P$. Then in $\Gamma$, there does not exists an $A \in \mathcal{A}$ such that $Q \sim A$ and $Q' \sim A$. Thus, $Q$ and $Q'$ have no common neighbors from $\mathcal{A}$ in $\Gamma'$. Furthermore, $< P, Q, Q' >$ is a non-degenerate plane containing either $t_1$ or $t_2$. WLOG, suppose that $t_1 \subset < P, Q, Q' >$. Let $R \in \mathcal{B}$ be a common neighbor of $Q$ and $Q'$ in $\Gamma'$. This implies that there exists a totally isotropic point $Y \in P^{\perp} \setminus Q^{\perp}$ such that $Y \neq P$, $V = t_2 \cap (PQ)^{\perp}$, and $S = VY \cap t_1$ such that $R = S^{\perp} \cap YQ$, and that there exists a totally isotropic point $Y' \in P^{\perp} \setminus (Q')^{\perp}$ such that $Y' \neq P$, $V' = t_2 \cap (PQ')^{\perp}$, and $S' = V'Y' \cap t_1$ such that $R = (S')^{\perp} \cap Y'Q'$. But $S = S'$, because if not, $R \in t_1$, which cannot happen. Thus, $R = S^{\perp} \cap Y'Q' = S^{\perp} \cap YQ$. Now let $W = QR \cap P^{\perp}$ and $m = VW$, then $S = t_1 \cap m$, which implies that $R \in S^{\perp} \cap < Q, Q', V >$ $(R \neq W)$. Since $Q \not\sim Q'$ in $\Gamma'$ and there does not exist a common neighbor from $\mathcal{A}$, $Q \not\sim R$ in $\Gamma$. So $QQ'$ is a hyperbolic line with $q + 1$ totally isotropic points from $\mathcal{B}$. Thus, $W$ is not totally isotropic (because $W \in P^{\perp}$). Hence, the number of common neighbors from $\mathcal{B}$ in $\Gamma'$ is $q + 1$.

2. Suppose that $QQ'$ is a hyperbolic line containing $P$. Then in $\Gamma$, $Q$ and $Q'$ have $q+1$ common neighbors in $\mathcal{A}$, which translates to $Q$ and $Q'$ having $q+1$ common neighbors from $\mathcal{A}$ in $\Gamma'$. There does not exist an $R \in \mathcal{B}$ such that $R$ is a common neighbor, because if there did exist one, then $S = S'$. But if $S = S'$, then $Y = Y'$ (as $(PQ)^{\perp} = (PQ')^{\perp}$ and $V = V'$), which says that $Q = Q'$, a contradiction.

Thus, the number of common neighbors for any two non-adjacent vertices is $q+1$. Therefore, $\Gamma'$ is a strongly regular graph with parameters $((q^2+1)(q^3+1), q^3+q^2, q^2-1, q+1)$. $\square$

It remains to be shown that these graphs are in fact new. When $q = 2$, this graph had been found using backtracking algorithms [34]. A geometrical description of the graph was not given. As for the case when $q \geq 4$, the only known strongly regular graphs with these parameters arise from generalized quadrangles. Thus, in order to show that these graphs are new, it must be shown that there exists a complete graph on 4 vertices minus an edge $e$ $(K_4 \setminus \{e\})$.

THEOREM 5.2.2. *The family of graphs given in Theorem 5.2.1 for* $q \geq 4$ *are new strongly regular graphs.*

PROOF. Since the only known graphs with these parameters are point graphs of generalized quadrangles, we can apply Corollary 3.2.7. That is, it must be shown that there exists a $K_4 \setminus \{e\}$, for some edge $e$, as a subgraph of $\Gamma'$. Pick a point $Q \in \mathcal{B}$. $Q$ is adjacent to $q^3+q^2-q-1$ points from $\mathcal{B}$. And two adjacent points from $\mathcal{B}$ have $q^2-2$ common neighbors in $\mathcal{B}$. Thus, there are $(q^3+q^2-q-1)(q^2-2)/2$ triangles on $Q$. Now $Q$ is adjacent to $q+1$ points from $\mathcal{A}$, and a point from $\mathcal{A}$ that is adjacent to a point from $\mathcal{B}$ has $q^2-1$ common neighbors from $\mathcal{B}$. Thus, there are at most $(q+1)(q^2-1)$ points in $\mathcal{A}$ that can be used to get a complete graph on 4 vertices. But $(q^3+q^2-q-1)(q^2-2)/2 > (q+1)(q^2-1)$ for $q \geq 4$. Therefore, $\Gamma'$ contains at least one subgraph of the form $K_4 \setminus \{e\}$. $\square$

Computer evidence suggests that the automorphism group of $\Gamma'$ is the stabilizer of two orthogonal tangent lines in $P\Gamma U(4, q^2)$. This has been confirmed for $q = 2, 4, 8$.

## 5.3. Miscellaneous Graphs

Using the same search process as the one used to find and construct $\Gamma'$ from Theorem 5.2.1, we were able to find numerous new strongly regular graphs. Some of these graphs provide promising leads into constructing infinite families, but this requires further investigation.

Consider strongly regular graphs with parameters $((s+1)(st+1), (t+1)s, s-1, t+1)$ (the parameters of a point graph from a GQ of order $(s,t)$) for $(s,t) = (q,q)$, $(q^2, q^3)$, and $(q^2, q^2)$, where $q$ is a prime power greater than 3. The canonical examples have groups $\mathrm{PSp}(4,q)$, $\mathrm{PGU}(5,q)$, and $\mathrm{PGO}(5,q)$.

When examing subgroups of $\mathrm{PGO}(5,q)$, the following result was obtained.

Theorem 5.3.1. *Fix a point $P$ and a line $l$ of $Q(4,4^h)$, for $h = 1, 2$. Let $Q$ be a point on the line joining $P$ and the nucleus $N$ of $Q(4,4^h)$. Let $\Gamma$ be the point graph of $Q(4,4^h)$, and let $\Gamma'$ be the graph obtained by deleting all edges $\{R,S\}$ of $\Gamma$ where $RS$ is skew to $l$ and $P$ is not collinear with $R$ and $S$, and adding edges $\{R', S'\}$ whenever $R'$ is not collinear with $P$, $S'$ is not collinear with $P$ if and only if there exists a $Q(4,2^h)$ subquadrangle containing $R', S', P, Q$ and meeting $l$ in $2^h+1$ points. Then $\Gamma'$ is a strongly regular graph with parameters $(85, 20, 3, 5)$ and $(4369, 272, 15, 17)$ whose automorphism groups have order 1536 and 3000, respectively.*

Proof. This is a straightforward computation in Magma. $\square$

Using Magma, we have found new strongly regular graphs whose automorphism groups have orders 1280, 1440, 1536 as subgroups of $\mathrm{PSp}(4,4)$, 4320 in $\mathrm{PGU}(5,2)$, 768 and 1440 in $\mathrm{PGO}(5,4)$, and 2400 in $\mathrm{PGO}(5,5)$.

CHAPTER 6

# Strongly Regular Graphs From Large Arcs in Affine Planes

Continuing with the theme of generalized quadrangles, one of particular interest is the Ahrens-Szekeres generalized quadrangle. It is of order $(q-1, q+1)$ and was constructed using a $q$-arc of $\mathrm{PG}(2,q)$. Before discussing the construction, we will first develop the necessary background information.

## 6.1. Arcs

A **$k$-arc** is a set of $k$ ($k \geq 3$) points in $\mathrm{PG}(2,q)$ such that no three points are collinear. It is possible to have $k$-arcs exist in both Desarguesian and non-Desarguesian planes. If the plane has order $q$, then $k \leq q+2$. However $k$ cannot equal $q+2$ unless $q$ is even. If $k = q+1$, then the arc is called an **oval**, and if $k = q+2$ (and $q$ is even), then the arc is called a **hyperoval**. Hyperovals are **maximal arcs**, as $k$ has reached its upper bound. More generally, in a finite projective plane of order $q$, a **$\{k; n\}$-arc** $K$ is a nonempty proper subset of $k$ points of the plane such that no $n+1$ points are collinear, where $2 \leq n \leq q-1$. If $k = nq + n - q$, then $K$ is a **maximal arc**.

6.1.1. Families of Arcs. Ovals in Desarguesian projective planes $\mathrm{PG}(2,q)$, $q$ odd, are nonsingular conics [92]. This implies that every oval of $\mathrm{PG}(2,q)$ has the parameterization $\{(t, t^2, 1) | t \in \mathrm{GF}(q)\} \cup \{(0,1,0)\}$. For the case when $q$ is even, no such classification has been made. As noted before, ovals need not be constructed in a Desarguesian plane. Furthermore, they can be defined in such a way so as they cannot be embedded in a projective plane.

When $q$ is even and an oval exists, it is possible to construct a hyperoval. Let $O$ be an oval. There exists a unique tangent line through each point in $O$. These tangent lines meet at a point $P \in \mathrm{PG}(2,q) \setminus O$. $P$ is known as the **nucleus** of the oval. Then $O \cup \{P\}$ is a hyperoval. Thus, all hyperovals in $\mathrm{PG}(2,2^h)$ can be parameterized as $\{(t,t^2,1)|t \in \mathrm{GF}(2^h)\} \cup \{(0,1,0)\} \cup \{(1,0,0)\}$. These, however, are not the only hyperovals in $\mathrm{PG}(2,2^h)$. We do know that by the Fundamental Theorem of Projective Geometry we can assume that the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1,1,1)$ are contained in any hyperoval. As for the other points, when $h > 1$, they will be of the form $(t, f(t), 1)$, where $t \in \mathrm{GF}(2^h)$ and $f$ is a permutation polynomial of degree less than or equal to $2^h - 2$.

A plethora of hyperovals have been constructed [28, 29, 32, 33, 54, 77, 80, 91, 93]. All of these were constructed in the Desarguesian setting. On the other hand, constructing ovals in non-Desarguesian planes is an unsolved problem, considered to be of immense difficulty.

6.1.2. Non-Desarguesian Arcs. Various methods have been used to construct ovals in non-Desarguesian planes. Some methods include the use of polarities and collineations.

In a particular setting, an irreducible conic $C$ of the affine plane $\mathrm{AG}(2,q)$, for $q = p^h$, $p$ prime, is an **inherited arc** of $\sigma$ (a modified affine plane) if no three points are collinear in $\sigma$. If $C$ is an ellipse in $\mathrm{AG}(2,q)$, then $C$ is an oval in the projective closure of $\sigma$. Ovals obtained in such a way are called **inherited ovals**. Inherited ovals make up a fair bit of the ovals constructed in the non-Desarguesian setting. They consist of hyperbolic, parabolic, and elliptic inherited ovals. Most of these have been constructed in non-associative Moulton planes [7, 69].

In addition to inherited ovals, Room [87] showed that in the Hughes plane there exists a ploarity such that the set of absolute points contains an oval. Constructions involving the Suzuke group and collineation groups have also yielded new ovals [66]. Nevertheless, it is believed these techniques will not work in other settings. Various other ovals have been constructed using ad hoc techniques [28, 66] and computer searches [29, 31, 45, 46, 49].

Those of most interest are as follows. In 1992, O'Keefe, Penttila, and Pascasio constructed an infinite family of ovals in Hall planes [76], while Cherowitzo (1990) [30] and de Resmini and Hamilton (1998) [47] constructed infinite families in Figueroa planes. In 1972, Ganley constructed an infinite family of ovals in commutative semifield planes [52].

## 6.2. Generalized Quadrangles and Arcs

Recall that a generalized quadrangle of order $(s,t)$ is a point-line incidence structure $\Pi$ such that every pair of points is incident with at most one line, given a point $P$ and a line $l$ not incident with it, there is a unique point incident with $l$ and collinear with $P$, and every line is incident with $s+1$ points, while every point is incident with $t+1$ lines. By definition, there are $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines.

Theorem 6.2.1. *([1, 57]) Every hyperoval of* $\mathrm{PG}(2,q)$ *gives rise to a generalized quadrangle of order* $(q-1,q+1)$.

Theorem 6.2.2. *([105]) Every oval of* $\mathrm{PG}(2,q)$ *gives rise to a generalized quadrangle of order* $(q,q)$.

Theorem 6.2.3. *([79]) Every* $q$*-arc of* $\mathrm{PG}(2,q)$ *gives rise to a generlized quadrangle of order* $(q+1,q-1)$.

As previously mentioned, the point graph of a generalized quadrangle of order $(s,t)$ is a strongly regular graph with parameters $((s+1)(st+1), s(t+1), s-1, t+1)$. Similarly, the concurrency graph is also a strongly regular graph with parameters $((t+1)(st+1), t(s+1), t-1, s+1)$. Therefore, two families of strongly regular graphs arise for each of the three before mentioned theorems. But, it should be noted that each of these graphs come about in a Desarguesian plane. So the question that stems from this knowledge should be, can strongly regular graphs be constructed from large arcs of non-Desarguesian planes? The answer is yes. The following results are due to joint work with Stan Payne.

THEOREM 6.2.4. *Let $\mathcal{A}$ be an affine plane of order $q$ (with point set $\mathcal{P}$) containing a $q$-arc $K$. Define a graph $\Gamma(\mathcal{A}, K)$ as follows: the vertex set is $\mathcal{P} \times K$ and $(P, A)$ is adjacent to $(Q, B)$ if and only if $A \neq B$ and either $P = Q$ or $PQ||AB$ or $A = B$ and $P \neq Q$ and $PQ$ is parallel to some tangent line of $K$ at $A$. Then $\Gamma(\mathcal{A}, K)$ is a strongly regular graph with parameters $(q^3, q^2+q-2, q-2, q+2)$.*

PROOF. The number of vertices is clearly $q^3$, as $|\mathcal{P}| = q^2$ and $|K| = q$. In order to show that the graph is regular, consider the vertex $(P, A)$. There are $q-1$ points $B$ in $K \setminus \{A\}$, so there are $q-1$ vertices of the form $(P, B)$. Furthermore, if $B \neq A$, then let $l$ be the line of $\mathcal{A}$ through $P$ parallel to $AB$. There are $q-1$ points of $l \setminus \{P\}$, so there are $q-1$ vertices of the form $(Q, B)$ which are adjacent to $(P, A)$. Thus, for the first case of adjacency, $(P, A)$ is adjacent to $q-1+(q-1)(q-1) = q(q-1)$ vertices. For condition (ii), it should be noted that there are two lines of $\mathcal{A}$ through $A$ tangent to $K$ and $q-1$ points $Q$ different than $P$ on each of the two lines through $P$ parallel to a tangent line of $A$. Thus, there are $2(q-1)$ vertices adjacent to $(P, A)$. Hence the valency is $q(q-1) + 2(q-1) = q^2 + q - 2$.

Next, we show that the number of common neighbors given two adjacent vertices is $q-2$.

**Case 1:** Let $(P, A)$ be adjacent to $(P, B)$ $(A \neq B)$. There are $q-2$ common neighbors of the form $(P, C)$ for $C \in K \setminus \{A, B\}$. Suppose $(X, C)$ is a common neighbor with $X \neq P$. Let $l$ be the line through $A$ parallel to $PX$ and let $k$ be the line through $B$ parallel to $PX$. Then $(X, C)$ is adjacent to $(P, A)$ if and only if $A \neq C$ and $l = AC$ or $A = C$ and $PX$ is parallel to a tangent line at $A$. $(X, C)$ is adjacent to $(P, B)$ if and only if $B \neq C$ and $k = BC$ or $B = C$ and $PX$ is parallel to a tangent line at $B$. But, $A \neq B \neq C$, because that would imply that $l = AC = BC$, which contradicts the definition of an arc. Thus, $A = C$ or $B = C$. Without loss of generality, suppose that $A = C$, then $k = BC$, $PX$ is parallel to a tangent line at $A$ and also parallel to $AB$, which is not tangent to $A$. Thus, $(P, A)$ and $(P, B)$ cannot have a common neighbor of the form $(X, C)$. Hence, if $A \neq B$, then $(P, A)$ and $(P, B)$ have exactly $q-2$ common neighbors.

**Case 2:** Suppose that $A \neq B$, $P \neq Q$, and $PQ||AB$. That is $(P, A)$ and $(Q, B)$ are adjacent. Let $(X, C)$ be a common neighbor. Suppose that $X = P$. Then $(P, C)$ being adjacent to $(P, A)$ implies that $C \neq A$, and $(P, C)$ being adjacent to $(Q, B)$ with $P \neq Q$ implies that $PQ||BC$. Thus, $BA = BC$, which implies that $A = C$ since $K$ is an arc. Therefore, $X \neq P$. Similarly, $X \neq Q$. Thus, we have that $P \neq X \neq Q$ with $PQ||AB$. Let $l$ be the line through $A$ parallel to $PX$, and let $k$ be the line through $B$ parallel to $QX$. Then $(X, C)$ is adjacent to $(P, A)$ if and only if $A \neq C$ and $l = AC$ or $A = C$ and $PX$ is parallel to a tangent line at $A$. Similarly, $(X, C)$ is adjacent to $(Q, B)$ if and only if $B \neq C$ and $k = BC$ or $B = C$ and $QX$ is parallel to a tangent line at $B$. Since $C$ cannot equal $A$ and $B$ simultaneously, without loss of generality, assume that $A \neq C$ and $AC$ is the line through $A$ parallel to $PX$. Suppose that $C = B$ and $QX$ is parallel to a tangent line at $B$. Then $k = AC = AB$ is the line through $A$ parallel to $PX$. But that means that $AB$is parallel to both $PQ$ and

$PX$, implying that $X$ is on $PQ$. Then $QX = PQ$ is parallel to $AB$, which is not a tangent through $B$. This implies that $A$, $B$, and $C$ are distinct. Furthermore, $AC$ is the line through $A$ parallel to $PX$; $BC$ is the line through $B$ parallel to $QX$, and $AB||PQ$. Thus, the number of $C \in K \setminus \{A, B\}$ is $q-2$. Let $l$ be the line through $P$ parallel to $AC$, and let $k$ be the line through $Q$ parallel to $BC$. Since $AC$ and $BC$ cannot be parallel (because $K$ is an arc), $l$ and $k$ are not parallel either and meet at a unique point $X$. Thus, there are exactly $q-2$ vertices $(X, C)$ which are common neighbors of $(P, A)$ and $(Q, B)$.

**Case 3:** Let $P \neq Q$ and $PQ$ be parallel to some line tangent to a point $A$ in $K$. That is, $(P, A)$ is adjacent to $(Q, A)$. Suppose that $(X, C)$ is a common neighbor. If $X = P$, then $(P, C)$ being adjacent to $(P, A)$ implies that $C \neq A$. Furthermore, $(P, C)$ being adjacent to $(Q, A)$ imples that $PQ||CA$, which is impossible because $PQ$ is parallel to some line tangent to $A$ in $K$. Thus, $X \neq P$. Similarly, $X \neq Q$. Now suppose that $C \neq A$ (and $P \neq Q \neq X$). Then $(X, C)$ being adjacent to $(P, A)$ imples that $PX||AC$. Furthermore, $QX||AC$ implies that $X \in PQ$, which forces $PQ||AC$. But this means that $A = C$, since $PQ$ is parallel to a tangent line at $A$. Thus, the common neighbors are vertices of the form $(X, A)$ where $X \in PQ \setminus \{P, Q\}$. Thus, there are $q-2$ choices for $(X, A)$.

Therefore, given any two adjacent vertices, there are $q-2$ vertices adjacent to both.

Next we will show that given any two non-adjacent vertices, there are $q+2$ vertices adjacent to both.

**Case i:** Suppose that $A \neq B$ and that $(P, A)$ is not adjacent to $(Q, B)$, which implies that $P \neq Q$ and $PQ \not|| AB$. For each $C \in K \setminus \{A, B\}$, let $l$ be the line on $P$ parallel to $AC$ and $m$ the line on $Q$ parallel to $BC$. Since $AC \not|| BC$, $l$ must meet $m$ in a point $X$. Then $(X, C)$ is a common neighbor of $(P, A)$ and $(Q, B)$. (Note that $PQ||BC$ if and only if $(X, C) = (P, C)$.

Similarly, $PQ||AC$ if and only if $(X,C) = (Q,C)$.) Thus, $(P,A)$ and $(Q,B)$ have $q-2$ common neighbors so far. Next, we will consider neighbors of the form $(X,A)$ or $(X,B)$, and if they can be common neighbors. If $(X,A)$ is adjacent to $(P,A)$ then $PX$ is parallel to a tangent line on $A$ in $K$. Since $PQ \not|| AB$, the line $l$ on $Q$ parallel to $AB$ does not contain $P$. If $m$ is eigher one of the two lines on $P$ parallel to a tangent line on $A$ in $K$, then $X = l \cap m$ yields a vertex $(X,A)$, which is a neighbor of both $(P,A)$ and $(Q,B)$. Similarly, there are two common neighbors of the form $(X,B)$. Therefore, there are $q+2$ common neighbors.

**Case ii.** Suppose that $A = B$ and $(P,A)$ is not adjacent to $(Q,A)$ for $P \neq Q$. Consider vertices of the form $(X,A)$. We must have that $P \neq X \neq Q$, $PX$ is parallel to a tangent line at $A$ in $K$, and $QX$ is parallel to a tangent line at $A$ in $K$. Let $l$ be one tangent line through $A$ and let $k$ be the other. Now let $l_P$ be the line on $P$ parallel to $l$ and $k_P$ the line on $P$ parallel to $k$. Similarly, $l_Q$ and $k_Q$ are the lines on $Q$ parallel to $l$ and $k$, respectively. Since $PQ$ is not parallel to either $l$ or $k$ (by the hypothesis), it must be that $l_P$ meets $m_Q$ at a point $X_1$ while $m_P$ meets $l_Q$ at a point $X_2$, giving two common neighbors, $(X_1,A)$ and $(X_2,A)$. Finally, consider neighbors of the form $(X,C)$ for $C \neq A$. Then we get that $P = X$ or $PX||AC$ and $Q = X$ or $QX||AC$. In either case, we end up with $PQ||AC$. Since we are given that $PQ$ is not parallel to the tangent lines on $A$ in $K$, it must be that $PQ$ is parallel to a unique secant line, namely $AC$, $C \in K$. So, if $P \neq X \neq Q$, $PX||AC||PQ$ implies that $X$ lies on the line $PQ$. Conversely, for any point $X$ on $PQ$, $(X,C)$ is a common neighbor. Thus, there are $q+2$ common neighbors.

Thus, given any two non-adjacent vertices, there are $q+2$ vertices adjacent to both.

Therefore, $\Gamma(\mathcal{A},K)$ is a strongly regular graph with parameters $(q^3, q^2+q-2, q-2, q+2)$. $\square$

We can also construct strongly regular graphs from maximal $\{k;n\}$-arcs in non-Desarguesian planes, as is witnessed in the following theorem. (Thank you to Stefaan De Winter for the encouragement to consider this case.)

THEOREM 6.2.5. *Let $\mathcal{A}$ be an affine plane of order $q$ (with point set $\mathcal{P}$) containing a maximal $\{k;n\}$-arc $M$, $2 \leq n \leq q-1$. Define a graph $\Gamma'(\mathcal{A}, M)$ as follows: the vertex set is $\mathcal{P} \times M$ and $(P, A)$ is adjacent to $(Q, B)$ if and only if $A \neq B$ and $P = Q$ or $PQ||AB$. Then $\Gamma'(\mathcal{A}, M)$ is a strongly regular graph with parameters $(nq^3 - q^3 + nq^2, nq^2 - q^2 + nq - q, 2qn - 3q, qn - q)$.*

PROOF. The number of vertices is clearly $nq^3 - q^3 + nq^2$, as the number of points in $\mathcal{P}$ is $q^2$ and $nq + n - q$ in $M$. The fact that $\Gamma'$ is regular follows from the observation that the neighbors of $(P, A)$ with second coordinate $B$ must have $B \neq A$, for which there are $nq + n - q - 1$ such choices, and first coordiante $Q$ must lie on the line parallel to $AB$ on $P$, on which there are $q$ choices. Therefore the valency is $q(nq + n - q - 1) = nq^2 - q^2 + nq - q$. We now consider the number of common neighbors given two adjacent vertices.

**Case 1:** Suppose that $(P, A) \sim (P, B)$ for $A \neq B$. There are $nq+n-q-2$ common neighbors of the form $(P, C)$ with $C \in M \setminus \{A, B\}$. Suppose that $(X, C)$ is a common neighbor for $X \neq P$. If $(X, C) \sim (P, A)$ then $PX||AC$ and if $(X, C) \sim (P, B)$ then $PX||BC$. Since $A \neq B \neq C$, $BC = AC$. Thus, there are $(q - 1)(n - 2)$ common neighbors of the form $(X, C)$. Hence, if $A \neq B$, $(P, A)$ and $(P, B)$ have $2qn - 3q$ common neighbors.

**Case 2:** Suppose that $(P, A) \sim (Q, B)$ for $A \neq B$ and $PQ||AB$. Suppose $(X, C)$ is a common neighbor. Then $PX||CA$ and $QX||CB$ with $A \neq B \neq C$. If $A, B, C$ are collinear, then $CA = CB$ and there are $q(n - 2)$ such neighbors. If $A, B, C$ are not collinear, then $X$ is uniquely determined by $C$; in which case, there are $qn - q$ such choices for $C$. Thus, the number of common neighbors is $2qn-3q$. Thus, given any two adjacent vertices, there exist $2qn-3q$ vertices adjacent to both. We now consider the number of common neighbors given two non-adjacent vertices.

**Case i:** Let $(P, A)$ and $(Q, A)$ be two distinct (non-adjacent) vertices. Let $l_A$ be the line parallel to $PQ$ on $A$ and let $B$ be one of the $n-1$ points in $M$ on $l_A$. Then there are $q$ choices for $X$ such that $X$ is on $PQ$ which is parallel to $l_A$. Thus there are $q(n-1) = qn-q$ common neighbors.

**Case ii:** Let $(P, A) \not\sim (Q, B)$ with $A \neq B$. Then $P \neq Q$ and $PQ \not\parallel AB$. Let $C \in M \setminus \{A, B\}$ and let $l$ be the line on $P$ parallel to $AC$ and $m$ be the line on $Q$ parallel to $BC$. If $A, B, C$ are not collinear, then $l \cap m = X$. So $X$ is uniquely determined by $C$, of which there are $qn-q$ to choose from. If $A, B, C$ are collinear then $l||AC$ and $m||BC$, which implies that $l||AB$ and $m||AB$. But $l \neq m$, as $(P, A) \not\sim (Q, B)$. Thus, there does not exist a common neighbor $(X, C)$ with $C$ being collinear to $A$ and $B$.

Therefore, the number of common neighbors given two non-adjacent vertices is $qn-q$.

$\square$

When $\mathcal{A}$ is Desarguesian, $\Gamma'(\mathcal{A}, M)$ is the point graph of the partial geometry $T_2^*(M)$ of Thas [101].

When $n = 2$, $q$ is even and $M$ is a hyperoval.

Corollary 6.2.6. *Let $\mathcal{A}$ be an affine plane of order $q$ (with point set $\mathcal{P}$) containing a hyperoval $H$. Define a graph $\Gamma'(\mathcal{A}, H)$ as follows: the vertex set is $\mathcal{P} \times H$ and $(P, A)$ is adjacent to $(Q, B)$ if and only if $A \neq B$ and either $P = Q$ or $PQ||AB$. Then $\Gamma'(\mathcal{A}, H)$ is a strongly regular graph with parameters $(q^3+2q^2, q^2+q, q, q)$.*

This family of strongly regular graphs yields a symmetric $(q^3 + 2q^2, q^2 + q, q)$-design whose polarity has no absolute points. Additionally, the sets $\{(P, A) : P \in \mathcal{P}, A \in H\}$ are cocliques of $\Gamma'(\mathcal{A}, H)$, which partitions $\Gamma'(\mathcal{A}, H)$ into cocliques. So if $\Gamma'(\mathcal{A}, H)$ is geometric, the corresponding $GQ(q+1, q-1)$ admits a partition into ovoids. Moreover, for each ovoid in the partition, every pair of distinct points of the ovoid is regular. It remains to be shown whether $\Gamma'(\mathcal{A}, H)$ is geometric or not.

THEOREM 6.2.7. *Let $\mathcal{A}$ be an affine plane of order $q$ (with point set $\mathcal{P}$) containing an oval $O$. Define a graph $\Gamma''(\mathcal{A}, O)$ as follows: the vertex set is $(\mathcal{P} \times O) \cup O$ and $(P, A)$ is adjacent to $(Q, B)$ if and only if either $A \neq B$ and either $P = Q$ or $PQ||AB$ or $A = B$ and $P \neq Q$ and $PQ$ is parallel to the tangent line of $O$ at $A$; $A$ is adjacent to $B$ for all $A \neq B$ in $O$; and $(P, A)$ is adjacent to $B$ if and only if $A = B$. Then $\Gamma''(\mathcal{A}, O)$ is a strongly regular graph with parameters $(q^3 + q^2 + q + 1, q^2 + q, q - 1, q + 1)$.*

PROOF. The number of vertices is clearly $q^3 + q^2 + q + 1$, as $|\mathcal{A}| = q^2$ and $|O| = q + 1$. A vertex $(P, A)$ having neighbors with the second coordinate $B$ must have that $B \neq A$ and consist of pairs $(X, B)$ with $X$ on the line parallel to $AB$ on $P$ ($q^2$ of these) or have $B = A$ and consist of pairs $(X, A)$ with $X$ a point, other than $P$, on the line through $P$ parallel to the tangent line on $A$ in $O$ ($q - 1$ of these). The other neighbor of $(P, A)$ is $A$. Therefore, the valency of $(P, A)$ is $q^2 + q$.

**Case 1:** Let $(Q, C)$ be a vertex distinct from $(P, A)$. We will show that the number of common neighbors is $q-1$. Suppose that $(X, B)$ is a common neighbor such that $A \neq B \neq C$. Suppose that $A \neq C$, then a vertex $(X, D)$ is a common neighbor of $(P, A)$ and $(Q, C)$ if and only if $A \neq D \neq C$ and $P = X$ or $PX||AD$ and $Q = X$ or $QX||CD$. So $X$ lies on the line through $P$ parallel to $AD$ and on the line through $Q$ parallel to $CD$. Since $A$, $C$, and

$D$ are on $O$, they cannot be collinear. Thus, $AD$ and $CD$ are not parallel, and, for each $D$, $X$ is uniquely determined. Hence, in either case, there are exactly $q-1$ common neighbors of this form.

Suppose that $(Q, C)$ and $(P, A)$ are common neighbors. We will show there are not any common neighbors of another form. Since $(Q, C)$ and $(P, A)$ are adjacent, $P = Q$ or $PQ||AC$. Since the tangent to $O$ at $A$ and the tangent to $O$ at $C$ are distinct, if $P = Q$, there are no further common neighbors. If $P \neq Q$, then $PQ||AC$, so $PQ$ cannot be parallel to the tangents of $O$ and either $A$ or $C$. Therefore, there are no further common neighbors.

**Case 2:** Suppose that $(P, A)$ and $(Q, A)$ are adjacent. The vertex $(X, A)$ with $PX$ parallel to the tangent to $O$ at $A$ parallel to $QX$ is a common neighbor. Such vertices only exist if $PQ$ is parallel to the tangent to $O$ at $A$. There are $q-2$ of them. Now consider $(X, B)$ for $A \neq B$. If $(X, B)$ is a common neighbor, then $PX||AB||QX$. There are $q$ such neighbors; those $(X, B)$ with $X$ on the line $PQ$ and $B$ the other point of $O$ on the line parallel to $PQ$. But this means that $(P, A)$ and $(Q, A)$ are not adjacent, because $PQ$ is not parallel to the tangent to $O$ at $A$. Therefore, there are no neighbors of the form $(X, A)$ whenever $(P, A)$ and $(Q, A)$ are adjacent. Lastly, $A$ is a common neighbor of $(P, A)$ and $(Q, A)$. Thus, the number of common neighbors is $q-1$.

Note that $(P, A)$ and $B$ are not adjacent whenever $A \neq B$. So we will consider this case later on.

**Case 3:** Vertices $A$ and $B$ are adjacent, and their common neighbors are $C \in O \setminus \{A, B\}$. Therefore, given two adjacent vertices, the number of common neighbors is $q-1$. Now we will consider two non-adjacent vertices.

**Case i:** Suppose that $(Q, C)$ and $(P, A)$ are not adjacent. As shown in the adjacency case, there are $q - 1$ common neighbors, whether or not $(Q, C)$ and $(P, A)$ are adjacent. Furthermore, if $(Q, C)$ and $(P, A)$ are not adjacent, then $Q \neq P$ and there are two addition common neighbors, namely, $(X, A)$ where $X$ is the intersection of the line through $P$ parallel to the tangent line to $O$ at $A$ with the line through $Q$ parallel to $AC$, and $(Y, C)$ where $Y$ is the intersection of the line through $Q$ parallel to the tangent line to $O$ at $C$ with the line through $P$ parallel to $AC$. Therefore, the number of common neighbors given to non-adjacent vertices of the form $(Q, C)$ and $(P, A)$ is $q + 1$.

**Case ii:** As mentioned before, given the vertices $(P, A)$ and $(Q, A)$, they are not adjacent whenever they have a common neighbor of the form $(X, B)$ for $B \neq A$. There are $q$ points of this form. Additionally, they are both adjacent to the point $A$. Thus, there are $q + 1$ common neighbors.

**Case iii:** If $A \neq B$, then $(P, A)$ and $B$ are not adjacent. Their common neighbors are the vertices $(X, B)$ with $X$ on the unique line through $P$ parallel to $AB$ ($q$ of these), together with the vertex $A$.

Therefore, given two non-adjacent vertices, there are $q + 1$ vertices adjacent to both.

Therefore, $\Gamma''(\mathcal{A}, O)$ is a strongly regular graph with parameters $(q^3 + q^2 + q + 1, q^2 + q, q - 1, q + 1)$. □

Each of these strongly regular graphs gives rise to a symmetric $(q^3 + q^2 + q + 1, q^2 + q + 1, q + 1)$-design with a polarity with all points absolute. It does remain to be answered whether or not these graphs are geometric. As it stands, we only know that these graphs are pseudo-geometric.

# CHAPTER 7

# Godsil-Hensel Strongly Regular Graph Construction

We can extend the definition of an oval in PG$(2, q)$ to an ovoid in PG$(3, q)$. This extension will be necessary, as we will examine results due to Godsil and Hensel, who constructed a number of strongly regular graphs with the parameters of the point graph of a generalized quadrangle from an ovoid.

## 7.1. Motivation

In PG$(3, q)$, $q$ a prime power greater than 2, an **ovoid** is a set of $q^2+1$ points in PG$(3, q)$ such that no three are collinear. An additional property of ovoids is that there are exactly $q+1$ tangent lines for each point on the ovoid. This can be restated, yielding the following interpretation. Through every point $P$ on the ovoid, there is a unique plane intersecting the ovoid at that point. The $q+1$ tangents lie on that unique plane. The $q^2+1$ points of an elliptic quadric is an example of an ovoid in any projective three-space. Interestingly enough, when $q$ is odd or $q = 4, 16$, no ovoids exist other than the elliptic quadrics. Furthermore, every plane which is not a tangent plane meets the ovoid in a oval;for the elliptic quadric, this is a conic. For the case when $q = 2^{2h+1}$, it has been conjectured that the only ovoid, other than an elliptic quadric, in PG$(3, q)$ is the Tits (or Suzuki) ovoid, and this has been proved for $q = 8$.

More specifically, an ovoid of a generalized quadrangle is a set of points such that every line of the generalized quadrangle meets the set in exactly one point. (Theorems of Segre and Thas [103] connect the two concepts by showing that every ovoid of PG$(3, q)$, $q$ even, is an ovoid of the GQ $W(q)$, and conversely.) In 1992, Godsil and Hensel (and Godsil, [55, 56]) constructed a number of strongly regular graphs with the parameters of the point graph of a generalized quadrangle of order $(s-1, s+1)$, $s$ odd, from an ovoid of a generalized quadrangle of order $(s^2, s)$ that has an automorphism $t$ of order 2 fixing every point of the ovoid. They note that the $\mathrm{U}(4, q)$ generalized quadrangle $\mathrm{H}(3, q^2)$ has such an ovoid. The ovoid is the intersection of the perp of a point not on the Hermitian variety with the Hermitian variety. However, they did not state the isomorphism type of this strongly regular graph. In fact, it is isomorphic to the point graph of the Ahrens-Szekeres generalized quadrangle. Therefore, their construction gives nothing new. Nevertheless, their construction procedures can be used in another setting to obtain new strongly regular graphs. More generally, they constructed other strongly regular graphs from an ovoid of a generalized quadrangle of order $(s^2, s)$ that has an automorphism $t$ of order $r \neq 1$ fixing every point of the ovoid, and this does yield something new when applied to the above ovoid, which admits automoprihsms of all orders dividing $q + 1$ fixing every point of the ovoid.

7.1.1. Covers. One such object necessary to apply the Godsil-Hensel construction is a cover. A **covering graph** of a graph $H$ is a graph $G$ together with an equitable partition $\pi$ of $G$ such that $G/\pi$ is isomorphic to $H$. An **equitable partition** of a graph $G$ is a partition $\pi$ of the vertices of $G$ such that if $C_i, C_j \in \pi$ then $c_{ij} = |\{v \in C_j : v \text{ adjacent to } w_i\}|$ is constant, for all $w_i \in C_i$. The quotient $G/\pi$ is a directed multigraph with loops where the vertices are the cells $C_i \in \pi$ and the directed edges are $c_{ij}$ going from $C_i$ to $C_j$ if and only if there is an edge from an element of $C_i$ to an element of $C - j$.

An equitable partition of a graph are the orbits of the graphs automorphism group. However, there are usually many other partitions. For example, if we let $G$ be the incidence graph of a finite projective plane, then each equitable partition with four cells corresponds to a complementary pair of two intersection sets and their dual sets. Nevertheless, such an equitable partition will arise as the orbits of a group if and only if the corresponding strongly regular graph is a rank three graph.

As we have only been considering simple graphs, $H$ must be loopless. Thus, $c_{ii} = 0$ for all $i$. This implies that the induced subgraph of $G$ on a cell $C_i$ of $\pi$ is null (a coclique-a subgraph whose complement is a clique, i.e. an independent set). Furthermore, $H$ has no multiple edges, so there is exactly one edge from each $w_i \in C_i$ to a vertex in $C_j$, for $i \neq j$, provided the corresponding vertices of $H$ are adjacent. And since $H$ is undirected, the same is true with $i$ and $j$ interchanged. Thus, the preimage under $p$ of an edge is a perfect matching. We then get that the cells have the same size if $H$ is connected. The common size $r$ is the index of the cover, and we refer to the cover as an **r-cover**. Therefore, these (special) types of equitable partitions give rise to covers of connected graphs where each cell is a coclique of the same cardinality and there are either no edges between a pair of cells or there is a perfect matching between a pair of cells. Such partitions are automatically equitable.

Interestingly enough, a cover of a trivial graph may be non-trivial [10]. It is known that for a distance-regular graph that is not bipartite, if it is a cover, then it is an antipodal cover; that is, the relation of being antipodal (at maximum distance) is an equivalence relation and the cells (fibers) consist of the equivalence classes of this relation. This is what happens for the distance-regular graph of diameter three constructed from the collinearity graph of a generalized quadrangle by deleting a spread (the dual to an ovoid). It is also known that every distance-regular graph of diameter three that is a cover is an antipodal cover of a complete graph. In addition to that, an r-cover of the complete graph on $n$ vertices is distance-regular if and only if non-adjacent vertices from distinct fibers have a constant number of common neighbors.

7.1.2. Ahrens-Szekeres GQ. The Ahrens-Szekeres generalized quadrangle is an example of a non-classical generalized quadrangle. For each prime power of $q$, the Ahrens-Szekeres generalized quadrangle has order $(q-1, q+1)$ [1]. Hall found these examples independently for the case when $q$ is even [57]. Later, Payne [79] constructed generalized quadrangles which included the Ahrens-Szekeres generalized quadrangle.

Let $q$ be odd, and let $\mathcal{P}$ be the points of the affine 3-space $\mathrm{AG}(3, q)$. Define $\mathcal{L}$ to be the curves of $\mathrm{AG}(3, q)$ such that

(i) $x = \sigma$, $y = a$, $z = b$,

(ii) $x = a$, $y = \sigma$, $z = b$,

(iii) $x = c\sigma^2 - b\sigma + a$, $y = -2c\sigma + b$, $z = \sigma$

for $a, b, c \in \mathrm{GF}(q)$ and for $\sigma$ ranging over $\mathrm{GF}(q)$. If incidence $I$ is defined as the natural one, then $(\mathcal{P}, \mathcal{L}, I)$ is a generalized quadrangle of over $(q-1, q+1)$. The parameters of the strongly regular point graph of this generalized quadrangle are $(q^3, q^2+q-2, q-2, q+2)$. (This is isomorphic to the graph $\Gamma''$ constructed from a $q$-arc of $\mathrm{PG}(2, q)$; see [85].)

## 7.2. Construction Extension

The Godsil-Hensel construction gives a graph $\Gamma$, which is a distance regular graph of diameter three, with vertices the orbits of length two of $< t >$ (where $t$ is an order 2 automorphism of the ovoid of a generalized quadrangle of order $(q^2, q)$), such that the neighborhood graph $\Gamma'(v)$ of $\Gamma$ at a vertex $v$ is strongly regular with the parameters of a point graph of a generalized quadrangle of order $(q-1, q+1)$. More generally, the Godsil-Hensel construction gives a graph $\Gamma$ with vertices the orbits of length two of $< t >$ (where $t$ is an order $r$ automorphism of the ovoid of a generalized quadrangle of order $(q^2, q)$), such that the neighborhood graph $\Gamma'(v)$ of $\Gamma$ at a vertex $v$ is strongly regular with the parameters

$$(q^3, (q-1)(\frac{(q+1)^2}{r} - q), r(\frac{q+1}{r} - 1)^3 + r - 3, (\frac{q+1}{r} - 1)(\frac{(q+1)^2}{r} - q)).$$

It is known that $r$ must be a divisor of $q+1$.

7.2.1. Flocks and GQs. A **quadratic cone** is a set of points in $\mathrm{PG}(3, q)$ that arise when a point $V$ not on a plane is joined to every point of a non-degenerate conic in that plane. A **flock** of the quadratic cone is a partition of the points of the cone, other than the vertex $V$, into conics. In 1979, in a joint paper with Thas, Fisher constructed a flock of the quadratic cone which now bears his name over fields of odd order greater than 3 [51]. In 1987, by reinterpreting a number of papers by Kantor and by Payne, Thas constructed a generalized quadrangle of order $(q^2, q)$ from any flock of the quadratic cone [102]. In 1994, Payne and Thas constructed an ovoid of these flock quadrangles [104]. The Payne-Thas ovoid will be discussed later. Interestingly enough, the Payne-Thas ovoid of the Fisher flock GQ has an automorphism of order 2 fixing every point of the ovoid, provided the field order is 3 modulo 4. Thus, it appears that if we apply the same process of the Godsil-Hensel construction to the Payne-Thas ovoid of the Fisher flock generalized quadrangle, we get two orbits on vertices, one short and one long, provided the ovoid is stabilized. A vertex from the short orbit gives the point graph of the Ahrens-Szekers GQ yet again. But we believe that a vertex from the long orbit gives a new graph. This remains to be proven.

7.2.2. Knarr Construction. As the Knarr construction is necessary to extending the Godsil-Hensel result, it will be discussed in depth here. We begin by introducing much of the necessary notation. Let $F = \{C_1, C_2, ..., C_q\}$ be a flock of the quadratic cone $K$, whose vertex is $x$ in PG$(3, q)$, $q$ odd. Now let $\pi_i$ represent the plane of $C_i$ for $i \in \{1, ..., q\}$. Assume that $K$ is embedded in the nonsingular quadric $Q$ of PG$(4, q)$, and let $L_i$ respresent the polar line of $\pi_i$ with respect to $Q$. Then $L_i \cap Q = \{x, x_i\}$ for $i \in \{1, ..., q\}$. This says that no point of $Q$ is collinear with all three points $x, x_i, x_j$, $1 \leq i < j \leq q$. Furthermore, in 1990, Bader, Lunardon, and Thas extended this concept by proving that no point of $Q$ is collinear with all three points $x_i, x_j, x_k$, $1 \leq i < j < k \leq q$ [4]. Hence, this set is known as a BLT-set. BLT-sets can be extended to refer to sets of lines (or planes). This is due to the fact that the generalized quadrangle $Q(4, q)$ arising from $Q$ is isomorphic to the dual of the generalized quadrangle $W(q)$. Hence, a BLT-set (of points) in $Q$ corresponds to a set of $q + 1$ lines in $W(q)$ such that no line of $W(q)$ is concurrent with three distinct lines of the BLT-set.

As mentioned in the last section, given a flock $F$, we can obtain a generalized quadrangle of order $(q^2, q)$. Let $G$ be such a generalized quadrangle arising from $F$. Then in 1992, Knarr proved that $G$ is isomorphic to the following structure: Let $\xi$ be a symplectic polarity of PG$(5, q)$, with $p \in$ PG$(5, q)$. Let PG$(3, q)$ be a 3-dimensional subspace of PG$(5, q)$ not containing $p$ such that PG$(3, q) \subset p^{\xi}$. Then $\xi$ induces a symplectic polarity $\xi'$ in PG$(3, q)$, which yields $W(q)$. Let $V$ be a BLT-set of $W(q)$ arising from $F$ (see Bader, Lunardon and Thas [4]) and construct the incidence structure as follows:

- Points:
  - $p$;
  - lines of PG$(5, q)$ not containing $p$ but contained in one of the planes $\pi_i$, where $L_i$ is a line of V;
  - points of PG$(5, q)$ not in $p^\xi$.
- Lines:
  - planes $\pi_i$, with $L_i \in V$;
  - totally isotropic planes of $\xi$ not contained in $p^\xi$ and meeting $\pi_i$ in a line (not through $p$).

Incidence is defined by the natural incidence relation inherited from PG$(5, q)$. Once again, this is isomorphic to $G$ and is, thus, a generalized quadrangle of order $(q^2, q)$ [68].

The Knarr GQ is classical (isomorphic to $H(3, q^2)$) if and only if the flock $F$ is linear.

7.2.3. Automorphism group of Knarr GQs. In [81], Payne and Thas showed that the automorphism group of a non-classical Knarr GQ is the stabilizer of $B = \{\pi_i : i\}$ in $P\Gamma Sp(6, q)$.

7.2.4. Payne-Thas ovoid. In the Knarr construction, if $\Sigma$ is a 3-space meeting $p^\xi$ in a plane on $p$, disjoint from the planes $\pi_i$, then the set of points of $\Sigma$ not in $p^\xi$ together with $p$ forms an ovoid of $G$.

Thus, if there is an automorphism of order $r \neq 1$ of a Knarr GQ fixing a Payne-Thas ovoid pointwise, then this is induced by an element of $P\Gamma Sp(6, q)$ stabilizing $B$ (which implies that it is an element of $PSp(6, q)$), and thus there is an element of order $r \neq 1$ of $PSp(4, q)$ stabilizing the BLT-set of $W(q)$ arising from the flock $F$ and also fixing a line of PG$(3, q)$, disjointwise from the lines of the BLT-set, pointwise. It turns out that the line involved is hyperbolic. Dualizing, there is an element of order $r \neq 1$ of $PGO(5,q)$ stabilizing the BLT-set of $Q(4,q)$ arising from the flock $F$ and also fixing all the lines on a regulus in a hyperbolic quadric hyperplane section disjoint from the BLT-set. Which known infinite families of BLT-sets of $Q(4,q)$ have this property? The groups of the known BLT-sets have been calculated, with the last family worked out in Eric Nelson's 2012 Ph.D. thesis at CSU [74]. Only the family of Fisher BLT-sets have this property. (The BLT-sets of $Q(4,q)$ of Bader-Durante-Law-Lunardon-Penttila [3] for $q = 23, 47$ also have this property.)

7.2.5. The Fisher BLT-sets. This section follows Penttila [83]. Let $q$ be an odd prime power, and let $Q$ denote the function with domain $V = GF(q^2) \times GF(q^2) \times GF(q)$ and $Q(x,y,a) = x^{(q+1)} + y^{(q+1)} + a^2$. Then $Q$ is an non-degenerate quadratic form on $V$. Let $\eta$ be a primitive $q+1$-st root of unity in $GF(q^2)$, and $\gamma$ be a primitive $2(q+1)$-st root of unity in $GF(q^2)$. Then $B = \{(\gamma\eta^2 i, 0, 1) : i \in \{0, \dots, \frac{q-1}{2}\}\} \cup \{(0, \gamma\eta^2 i, 1) : i \in \{0, \dots, \frac{q-1}{2}\}\}$ is a Fisher BLT-set of $Q(4,q)$. Moreover, $\{(x,y,0) : x, y \in GF(q^2)\}$ is a hyperplane section of $Q(4,q)$ meeting it in a hyperbolic quadric, one the reguli of which is

$$\{\{(\delta\eta^j, \eta^j, 0) : j+1, \dots, q\} : \delta a^{(q+1)} = -1\} = R$$

The map $(x,y,a) \mapsto (\eta^2, \eta^2, 1)$ is an isometry of $Q$ of order $\frac{q+1}{2}$, fixing $B$ and every line on $R$. Moreover, the lines of $R$ are disjoint from $B$.

7.2.6. Construction Procedure. Consider the polar space admitting $\mathrm{Sp}(6,q)$, $q$ odd. (Note that this is $\mathrm{W}(5,q)$, $q$ odd, and $\mathrm{Sp}(6,q)$ is necessary for constructing a BLT-set.) Apply the Knarr construction to a Fisher BLT-set. Choose a 3-space $\Sigma$ that meets $p^\xi$ in the plane $\pi$ such that $\pi/p$ is the special hyperbolic line fixed pointwise by the stabilizer of the Fisher

BLT-set in $W(5,q)/p$. Then the Payne-Thas ovoid arising from $\Sigma$ has elements of order $\frac{q+1}{2}$ in its pointwise stabilizer the stabilizer in $PSp(6,q)$ of the BLT-set of planes. Then the Godsil-Hensel construction yields a distance regular graph of diameter three with vertices the orbits of the subgroup generated by an element of order $r$ fixing the Payne-Thas ovoid pointwise on the lines of the Knarr GQ. There are two orbits on vertices of this graph. By then considering the neighborhood graph of a vertex in the long orbit after having stabilized the Payne-Thas ovoid, we believe there exist new strongly regular graphs with parameters $(q^3, q^2+q-2, q-2, q+2)$ for $q \geq 7$, $q$ congruent to 3 modulo 4. (The short orbit seems to give the previously constructed examples.) More generally, there seem to be new Godsil-Hensel strongly regular graphs with the parameters

$$(q^3, (q-1)(\frac{(q+1)^2}{r} - q), r(\frac{q+1}{r} - 1)^3 + r - 3, (\frac{q+1}{r} - 1)(\frac{(q+1)^2}{r} - q)),$$

where $r$ is be a divisor of $\frac{q+1}{2}$. These conjectures are supported by computer evidence for small field orders.

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