A COMPUTER STUDY OF TRANSITION OF WALL BOUNDARY LAYERS

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A direct solution by computer of the governing equations for the mean and fluctuating motions in incompressible, two dimensional fluid flow has been obtained using an invariant modelling technique for the triple and higher order velocity correlation terms. This provides a reasonably satisfactory data-generating procedure for predicting general two dimensional wall boundary layer data (laminar through transition to turbulent flows) for a wide variety of wall and free stream conditions, often impossible to model in wind tunnels.

It has also been shown that differential methods of wall boundary layer prediction, utilizing the eddy viscosity concept, and integral relation methods, can be adapted through an implicit transition or intermittency function to provide a continuous prediction of general two dimensional wall boundary layers from laminar through transition to the turbulent regime. Such a transition function has been evaluated and tested in this work.

From a force-field theory developed in this work, the phenomena of laminar instability and laminar to turbulent transition have been reexamined with satisfactory predictions of incipient laminar instability, transition and non-linear disturbance amplification characteristics. The force-field theory emphasizes a dynamic fluid property which defines the fluid cohesiveness or ability to resist perturbations. This property, it appears, solely determines the flow characteristics. The force-field theory derives, in addition, a simple non-linear flow transfer function which provides a very simple procedure for the

continuous prediction of simple two dimensional wall boundary layers.

Finally, by assuming that conceptual fluid particles in a boundary layer fluid would be arranged in continuous mean energy levels, a statistical collision theory has been initiated to describe and predict turbulence characteristics in wall boundary layers. All preliminary results are satisfactory and justify continued pursuance of the methods demonstrated in this work.

TABLE OF CONTENTS

Chapter		Page
	ABSTRACT	ii
	LIST OF FIGURES	vi
	NOMENCLATURE	vii
1	INTRODUCTION	1
2	A BRIEF SURVEY OF CURRENT WALL BOUNDARY LAYER THEORIES	7
	2.1 Laminar Boundary Layer Research	8
	2.2 Transitional Boundary Layer Research	13
	2.3 Turbulent Boundary Layer Research	16
3	THE NUMERICAL EXPERIMENTAL TECHNIQUE	21
	3.1 The Analytical Model	22
	3.2 Reduction of the Governing Equations of Motion	26
	3.3 The Numerical Method	34
	3.4 The Computer Program	41
4	FLOW INSTABILITY AND TRANSITION	44
	4.1 The Force Field Concept	44
	4.2 An Energy Method	55
	4.3 A Flow Describing Function	75
	4.4 Use of a Describing Function Method for Continuous Solution of the Wall Boundary Layer	84
5.	A TRANSITION FUNCTION METHOD FOR CONTINUOUS SOLUTION OF THE WALL BOUNDARY LAYER	88
	5.1 The Form of the Transition Function	89
	5.2 A Differential Method	95
	5.3 An Integral Relation Method	97
6	A STATISTICAL ENERGY THEORY OF TURBULENCE	101
	6.1 The Conceptual Energy-Surface Structure of Wall Bounded Flows	101
	6.2 A Statistical "Collision" Theory	104
7	A DISCUSSION OF PRESENT RESULTS	115
8	CONCLUDING REMARKS	119
9	REFERENCES	124

TABLE OF CONTENTS (Continued)

Chapter | Page

APPENDICES 129

A. THE METHOD OF INVARIANT MODELLING 130
B. APPENDIX TO CHAPTER III OF TEXT 138
C. DERIVATION AND SOLUTION OF EQUATION (4.39) IN THE TEXT. 147
D. THE COMPUTER PROGRAM FOR TURBULENCE FIELD CLOSURE 154

FIGURES 172

LIST OF FIGURES

Figure		Page
1	Schema of the Conventional Wall Boundary Layer Regimes	173
2	Orthogonal Curvilinear Coordinate System	174
3	Schema of Boundary Layer Sections	175
4	Schema of Force Field/Stability Number Relationship	176
5	Functional Relation Between $(U_1\delta^*/\nu_1)_{ci}$ and the Shape Factor, H	177
6	Herring-Mellor Coordinate System for Difference-Differential Method	178
7	Sketch of Force Field Coordinate System	45
8	Schema of Conceptual Energy-Volumes on a Flat Plate Boundary Layer	179
9	Effect of Initial Turbulence Intensity on $(SN_{ct} - SN_{ci})$	180
10	Continuous Predictions of Boundary Layer Thicknesses	181
11	Continuous Predictions of Mean Velocity Profiles . .	182
12	Continuous Predictions of Skin Friction	183
13	Statistical Theory Prediction of Turbulence Distribution	184
14	Turbulence Energy Production and Dissipation Rates in Wall Region	185
15	Turbulence Energy Production and Dissipation Away from Wall	186
16	Typical Streamwise Variation of Maximum Amplification Factor	187
17	The Describing Function Coefficients	188

NOMENCLATURE

Symbol	Definition
(A,B,...Z) (a,b,...z)	Constants or function, as defined in text
A_t, A_{tx}	Transpiration Number, $(\frac{U_1 L}{\nu_1} \cdot (\frac{v_o}{U_1}))$; $(\frac{v_o L}{\nu_1})^2$
a^*	Amplification factor, or disturbance amplitude.
c,(C)	Wall curvature, $1/r_o$, $(c\delta^*)$
$C_f/2$	Skin friction
c_p	Specific heat at constant pressure
c_v	Specific heat at constant volume
d	Δx, x-spacing
e,(E)	Kinetic energy (or Eckert Number)
f'	Mean velocity defect, $(1 - u/U_1)$
F	Pressure gradient parameter, $\frac{\nu}{U_1^2} \frac{dU_1}{dx}$
f_p	Disturbance frequency
Ga	Görtler Number
$G_{L(.)}$	Grashof Number
g	Gravitational constant
H	Velocity profile shape factor, δ^*/θ
I	Local total turbulence intensity, $\frac{1}{U_1}\sqrt{\frac{1}{3}(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}$
I_1	Initial free stream total turbulence intensity
i,j,k	Indices
K	Total disturbance kinetic energy $(\overline{u'^2} + \overline{v'^2} + \overline{w'^2})$
k_i(i=1,2,3)	Double velocity correlations, $(\overline{u'u'}, \overline{v'v'}, \overline{w'w'})$

NOMENCLATURE (Continued)

Symbol	Definition
k	Coefficient of thermal conductivity
k_s	Wall roughness height
L	Characteristic length
Mini (.)	Minimum value of (.)
p	Local mean pressure
Pr	Prandtl Number, $\mu C_p/k$
q	Local total energy, (kinetic or thermal)
q'	Mean temperature defect, $(T-T_1)/(T_o-T_1)$
r	Local radius of curvature
Ra	Rayleigh Number, $[C_p(T_o-T_1)gL^3/\nu_1 T_1 k]$
Re_L	Section Reynolds Number, $U_1 L/\nu_1$
R_L	Local Reynolds Number, $U_1 L/\nu$
S_c	Sutherland's constant, (= 110^{o}K, for air)
SN	Section stability number
SN_D	$[SN(x+\Delta x) - SN(x)]/SN(x+\Delta x)$
T	Local temperature, (^{o}K)
Ta	Taylor Number, $(2C\ Re_L^2)$
U_1	Free stream velocity
u	Local streamwise velocity
v	Local transverse velocity
w	Local lateral velocity
x	Streamwise coordinate
y	Transverse coordinate
z	Lateral coordinate

NOMENCLATURE (Continued)

Symbol	Definition
α,β,γ	Constants or functions, as defined in text
$\Gamma(.)$	Gamma function, or scalar variable
Δ	Modified boundary layer thickness, $[= \delta^* \, 10^{-2.34\tanh(\frac{10}{H} - 4)}]$
δ	Boundary layer physical thickness $(y_{u= 0.995U_1})$
δ^*	Boundary layer displacement thickness, $\int_o^\infty (1-u/U_1)/(1-cy)\, dy$
ε	Flow transport coefficient (e.g., viscosity)
θ	Boundary layer momentum thickness, $\int_o^\infty \frac{u}{U_1} (1 - \frac{u}{U_1})/(1 - cy)dy$
$\lambda,(\Lambda)$	Local disturbance characteristic length
κ	Von Kármán's constant
ρ	Local fluid density
η,η^*,η_δ	Dimensionless height from wall, y/δ^*, y/Δ, y/δ
σ	Double velocity cross correlation, $(-\overline{u'v'})$
ξ	Local vorticity, or function, as defined
μ	Molecular viscosity
ν, (ν_t)	Kinematic viscosity, (eddy viscosity)
τ	Local shear stress, or characteristic period of disturbance
Ω	Ratio of maximum total disturbance kinetic energy to same at point of neutral stability K/K_{ci}, (also, force field phase angle)
ω	Ratio of stability number at point of neutral stability to local value, SN_{ci}/SN

Symbol	Definition
ϕ, Φ ; ψ, Ψ	Functions as defined in text
χ	Pressure Gradient function, $(\frac{10}{H} - 4)$
$(\ )_{ci}$	Value at point of neutral stability
$(\ )_{ct}$	Value at point of incipient transition
$(\ )_1$	Value at free stream
$(\ )_{max}$	Maximum value
$(\ )_o$	Value at the wall
$(\ )'$	Differentiation with respect to appropriate independent variable, or fluctuating quantity
$(\ )_{et}$	Value at end of transition region
$(\ )_e$	Equilibrium value

Chapter I

1. INTRODUCTION

In the study of wall boundary layers one is dealing with those shear layers whose structure is directly influenced by the presence of a solid boundary. A variety of such boundary layers exists depending on the nature and configuration of the boundary. The present study is restricted to consideration of the incompressible flow over a rigid curved wall with heat and mass transfer across the wall. Attention is focussed on the structure and characteristics of the relatively thin shear layer along the surface of the wall outside of which is the undisturbed free stream. The main flow is considered two dimensional but all disturbances in the boundary layer are assumed to be three-dimensional. An arbitrariness in the pressure distribution and mass transfer over the boundary is allowed but only cases involving the transfer of the same material as in the main flow, across the boundary, have been studied. Extension of the results to more arbitrary transfer can be made by suitable choice of transfer parameters.

The wall boundary layer as defined above is encountered, in one form or the other, in many practical flows and has therefore received a great deal of attention both experimentally and theoretically. So far, however, the boundary layer has been studied only in such obvious regimes as the laminar, the transitional, the turbulent, and the separated wall boundary layer. The laminar regime is almost completely understood both analytically and experimentally. Such however, is not the case with the transitional, the turbulent or the separated regime. By analogy with the laminar boundary layer, however, it has been possible to formulate plausible mathematical models (which give reasonable practical results) for the isolated fully turbulent boundary layer. It is very disconcerting to note, however, that quite often, because of the unsound understanding of turbulence, very wrong physical reasoning can lead to models of the turbulent regime which give answers in fair agreement with some experimental results. The observation is often made that turbulence is so turbulent that it draws from every source, sound and unsound. That is, almost any reasoning which satisfies the boundary conditions of the flow yields answers close to some experimental results. Much less success, however, is evident for the transitional regime. It has been common practice to assume that the turbulent regime starts where the laminar regime ends. Experimental studies, notably those of Dryden, Schubauer, Skramstad, and Klebanoff, (1947; 1948, 1955), clearly show that oscillations in the laminar regime may cause transition to the turbulent regime and that there exists a usually well-defined finite zone between the laminar and the turbulent regimes where the boundary layer is intermittently laminar and turbulent. The importance of the knowledge of the correct position of the transition point may be appreciated when it is noted that in airfoil design, for example, a small inaccuracy in the choice of the transition point can cause an error of as great as 30% to 40% in the theoretical calculation of the drag of an airfoil. In fact, in turbines and compressors, the transitional boundary layer often extends over 80% or more of the blade surfaces. Mathematical analysis of the transitional boundary layer has been difficult because of the near complete ignorance of the nature of boundary layer oscillations.

The simplification of boundary layer study by dividing it into conceptually obvious regimes, the conventional approach, has indeed been most helpful especially in gross experimental understanding of the general boundary layer. It has however apparently led to a difficult direction for the mathematical description of the general boundary layer. The conventional approach to boundary layer study implies that flows in nature should be laminar but that certain events have taken place in these laminar flows which have made them turbulent. The emphasis is then on the origin, growth and consequent effects of these events on the laminar flow. But nature around us is beset with turbulence, laminarity or order among diversity being particularly very rare. "Indeed, I can excuse turbulence in flows because that seems to be what has been around us for time immemorial. What I cannot understand is how laminar flows can exist, with all those 'particles' so nicely and orderly arraigned." By the conventional approach to boundary layer study one may not hope to achieve much more than have already been achieved. Universal theories and laws, it seems, must continue to be inexact and incompletely universal. A new and physically more sound approach is needed if further advances are to be made in boundary layer studies, because, as it stands now, the boundary layer has become one very difficult mathematical problem, which must wait for the advancement of the proper mathematical tools, for its solution.

The Navier-Stokes equations, which have been accepted as the governing equations for fluid flow were derived from the basic concept that the product of a mass in motion and its acceleration must balance the net force on the mass. This same basic theory contains the force-field concept, namely, that the dynamic characteristics of any body in motion are determined by and only by the forces acting on the body. The force field concept, however, has the advantage over the Navier-Stokes equations of allowing more room for a philosophical pursuance of the flow problem using physical analogies. Such an approach should lead to a greater insight for the physical explanation of some previously baffling flow phenomena such as flow instability and transition, and turbulence characteristics. Further, the force-field concept treats the flow problem as the continuous problem it is.

The approach suggested herein is based on the following fundamental hypotheses:

i) All particles in nature are essentially similar in their reactions to the force field (external and internal) in which they find themselves. Each will tend to go its own way. It requires a finite and continuous directional force above a certain minimum magnitude to produce order among any group of particles. The magnitude of this directional force depends essentially on how much progress the individual particles have made in executing their separate motions, that is, on the degree of the existing chaos.

ii) The basic particles of any system in nature tend to be arranged in continuous (or discrete) energy levels according to their statistical energy content. Transfer from one energy level to another is possible in any direction, although, under certain circumstances, one direction may be preferred over another. Each transfer involves a finite work (positive or negative) which is due to the local force field.

The first hypothesis implies that all particles in nature including human beings fall into one of two classes, passive particles which produce no internal force and active particles (the class of human beings and other animals) which produce varying degrees of internal force. The second hypothesis implies among other things that the boundary layer can be considered as a conglomeration of non-intersecting closed energy 'volumes' in which the particles in nature execute their separate motions. Collisions among the particles in any 'volume' are possible and such collisions may result in energy loss or gain and consequently in particle transfer from one energy volume to another. Thus, the structure of a natural system is essentially the same, on a large scale, as the structure of the atom. Turbulence is then a measure of the rate of particle transfer within and among the energy volumes, or more generally the distribution of the rate of collision of particles. The above ideas are by no means novel. They have been proved and used in science for decades before now. Their application to the study of fluid flow and to the consequent explanation of laminar and turbulent flows is what is new, because that has been overlooked before now. It is clear that the implication of this new approach, unlike the conventional approach, is that flows in nature should indeed be turbulent and that laminar flows are the ones which require certain events to effect. Furthermore, energy in a statistical sense, is made the primary quantity of interest. G. I. Taylor apparently sensed this approach when in his statistical theories of turbulence he introduced the concepts of velocity correlations and spectral densities. In the new approach, these quantities are explicitly called statistical energies because that is what they really are.

With these ideas, it is hoped to initiate a continuous model for two dimensional wall boundary layers. The problem of the structure and characteristics of the fully laminar regime will be assumed to be completely known, but the stability of the laminar regime to external and internal disturbances will be studied in detail by means of the statistical energy method described later in this work. A method of describing function which assumes the boundary layer to be a non-linear system with a specific transfer function will then be developed as a method of continuous solution of at least the simple zero pressure gradient boundary layer. Modifications to existing methods of boundary layer computation for the laminar and turbulent boundary layers will be presented and coupled through a transition function to provide a continuous solution of the general boundary layer.

The methods presented in this work are valid up to the point of separation of the boundary layer from the wall. Beyond this point, further assumptions have to be made for continuity, than are considered in this work.

Chapter II

2. A BRIEF SURVEY OF CURRENT WALL BOUNDARY LAYER THEORIES

The history of the developments in boundary layer theory since its conception by Prandtl in 1904, is a long one full of many frustrations and sparks of genius. Comprehensive and sometimes detailed accounts of these developments are available in so much of the common literature on boundary layer theory that it seems unwise to repeat that long history in this work. Rather, this section of this work will discuss briefly some of the more recent theories and analytical methods in boundary layer research.

Earlier, in the introduction to this work, it was pointed out that boundary layer research, experimental and analytical, has hitherto been carried out separately for either the laminar, transitional or turbulent boundary layer. New theories for each of these boundary layer regimes appear in print with such regularity that it would seem a difficult task to try to document them. Fortunately, however, all these theories, past and present, seem to fall into one of a few basic theories. That is, they differ mostly in their method of practical application rather than in their fundamental. Some of these basic theories are discussed in this chapter. Physical experimental techniques are not considered although their results are invaluable in judging any analytical theories. The conventional boundary layer is schematically presented in Figure (1) where the various sections that have been isolated and studied are shown. The research efforts in these isolated boundary layer regimes will now be discussed.

2.1 Laminar Boundary Layer Research

As initiated by Prandtl the laminar boundary layer theory assumed that in the case of small coefficients of viscosity the action of viscosity on the flow past a solid body is confined to a thin layer of fluid close to the boundary; the motion outside the boundary layer being of inviscid flow type. This assumption is used to simplify the dynamic equations (the Navier-Stokes equations) of fluid motion to yield the so-called boundary layer equations. For steady two-dimensional incompressible main flow, these equations are:

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\nu\frac{\partial u}{\partial y}\right) . \end{aligned}\right\} \qquad (2.1)$$

The mathematical problem is thus to find the motion in the boundary layer, given the appropriate boundary conditions. This problem redounds to finding closed solutions for the above equations (2.1). Such analytical solutions have been found for most laminar boundary layer flows in an exact or approximate form, and have been discussed at length by Schlichting (1968), Walz (1969), Meksyn (1961) and others. The exact solutions reduce to essentially two methods, the method of power series expansion and the method of asymptotic expansions. The relative merits and demerits of these methods are extensively discussed by Meksyn (1961). The approximate methods involve the solution of either a difference-differential or an integral equation obtained from equation (2.1) using plausible assumptions. One of the best known approximate methods is that due originally to Pohlhausen (1921) in which the Karman integral equation was solved with an assumed polynomial velocity profile. This method was later modified and made more accurate by Thwaites (1949). The problem of singularity at the leading edge or entrance region of the laminar boundary layer has been partially resolved notably by Iglisch (1944) and Carrier and C. C. Lin (1948). Extensions of the two-dimensional laminar boundary layer theories to treat three-dimensional and axi-symmetric flows and also to the study of heat and mass transfer in the laminar boundary layer have been very successful. Such results have been documented, for example, by Moore (1964) and Rosenhead (1963).

The basic boundary layer theory was developed for the laminar boundary layer. Observations in nature however showed that most boundary layers are turbulent. For continuity in the boundary layer research, it has been assumed that disturbances, from within or without, enter the laminar boundary layer and may cause it to become unstable and go turbulent. The problem of the stability of the laminar boundary layer has therefore been extensively studied. Historically, the study of the stability of laminar flows started with the classical paper of Rayleigh (1892) in which he introduced the method of small perturbations. Heisenberg (1924), Tollmien (1929), Schlichting (1933) and C. C. Lin (1945) advanced Rayleigh's work to the current classical stability theory for infinitessimal disturbances (the so-called Tollmien-Schlichting waves). Taylor (1923) and Görtler (1940), on the other hand, studied the stability of laminar flows in the presence of longitudinal (so-called Taylor-Görtler) vortices. These two modes of instability, Tollmien-Schlichting waves and Taylor-Görtler vortices are, for mathematical simplicity, usually considered the principal modes for stability investigations. Any one mode plays the more dominant role if the body forces on the flow are such as to excite disturbances of that particular type. Classical stability theory explains rationally, at least, the nonoccurrence of laminar flow under certain combinations of the flow parameters. At present, the achievement of the stability theory are mainly limited to the description of the initial breakdown of the laminar flow due to the disturbances mentioned above. Fundamental aspects of the stability theory have been summarized in a monograph by C. C. Lin (1955). Some of the applications of this theory to problems of technical interest have been considered by Shen (1964). However limited, the results of stability theory are of great practical significance especially when nothing better appears available.

A second theory on the possible mechanism of transition which offered an opposing school of thought to classical stability theory was that due to Taylor (1936). Taylor assumed that transition to turbulence originated from momentary separation produced by fluctuating pressure gradients accompanying the free stream turbulence. According to this theory, the Reynolds number at transition is a function of $(u'/U_1)(\ell/L)^{1/5}$ where ℓ is the reference length used in defining the Reynolds number and L is the scale length of the turbulence. The conflict between stability theory and the Taylor theory was resolved principally by the extensive experiments of Dryden and his colleagues Schubauer, Skramstad, and Klebanoff (1947, 1948, 1955). The current understanding is that the Taylor theory explains the origin of turbulence when the free stream turbulence intensity is greater than 0.2 percent, while turbulence originates in the manner suggested by stability theory, from amplification of small disturbances which takes place when the Reynolds number exceeds a certain critical value, when the free stream turbulence is less than 0.2 percent. A very comprehensive survey of up-to-date theories on transition mechanisms (particularly for compressible flows) with very extensive references was published by Morkovin (1968).

The growth or decay of the energy of the disturbances certainly should be a more significant parameter for determining the stability of laminar flows. Such energy methods, unfortunately, have not been exploited very much since Rayleigh's early failure with an energy method. It can be shown that the time rate of change of the kinetic energy of disturbances in a flow is given by:

$$\frac{\partial E}{\partial t} = \rho M - \mu N \tag{2.2}$$

where

$$\left.\begin{aligned} E &= \int\int \frac{1}{2}\rho(\overline{u'^2} + \overline{v'^2})\, dxdy \\ \rho M &= -\int\int \rho\overline{u'v'}\,\frac{du}{dy}\, dxdy \\ \mu N &= \int\int \mu\xi'^2\, dx\, dy \end{aligned}\right\} \tag{2.3}$$

and ξ' is the fluctuating vorticity. Equation (2.2) suggests a way to estimate the minimum critical Reynolds number, since neutral stability may be defined to occur when $\partial E/\partial t = 0$. In dimensionless form, at neutral stability

$$Re_{crit} = \frac{N}{M} \tag{2.4}$$

The minimum critical Reynolds number is then given by minimizing the quotient N/M with respect to all possible disturbance modes u', v'. The principal objection to the energy method is that it gave very low predictions of the minimum critical Reynolds number, in practical applications. The explanation for the low predictions was that the energy method usually included all possible modes of disturbances whether or not they are present in the case being investigated. Stuart's (1958) partial success with the energy method appears to have dispelled this criticism. Apparently what has been the problem with the energy method is that without sufficient knowledge of the proper distributions of u' and v' etc., the trial disturbance functions assumed in the solution of the integral equations (2.3) were too inaccurate. Stuart obtained the disturbance modes from classical stability theory and as these would certainly be very good approximations of the actual modes present, his results were very impressive. With caution, Stuart's success with the energy method could be extended to a wide class of flow stability problems. Shen (1964; pages 839-843) discusses the energy method in some detail. The advantage of the energy method is that it is not restricted to just the region of laminar instability since disturbuances of any magnitude may be considered. The energy method predicts the linear and non-linear amplification rates of the disturbances and the final equilibrium amplitude.

These theories on the stability of laminar boundary layers have had quite some success with the two-dimensional boundary layers. Their extension to more general types of boundary layer flows with appreciable boundary influences have not been very impressive. Laminar boundary layer stability theories, in short, still remain far from being universal.

2.2 Transitional Boundary Layer Research

Theories on the stability of laminar boundary layers predict the qualitative effectiveness of the various parameters in promoting or supressing stability. These predictions appear to have a direct correlation with the role of the same parameters in promoting or suppressing transition, as evidenced by comparison with available experimental investigations. Research in the transitional boundary layer has been very much experimental; mathematical theories in this area being rather very difficult to formulate. These experiments, notably those of Schubauer and Skramstad (1948), Schubauer and Klebanoff (1955), and Elder (1960), and Emmons (1951), for the incompressible boundary layer, have nevertheless remarkably defined certain very important general characteristics of the transitional boundary layer, which then make it possible to formulate some plausible analytical theories or models for the transitional boundary layer. The following characteristics of the transitional boundary layer may be stated:

i) Downstream of the laminar instability is a point of initial breakdown in the laminar flow, at which turbulence spots begin to form (Emmons (1951)). This transition point of initial breakdown of the laminar flow is rarely, if ever, seen in a transition experiment, which implies (Schubauer and Skramstad (1948)) that the probability of having a probe at the point where the event occurred is very small.

This is evidence that laminar breakdown is pointlike as opposed to a simultaneous breakdown along a line or over a considerable area. Furthermore, a pointlike breakdown would support the existence of an opportunity for lateral growth of the turbulence spots as has been amply evidenced by experiment. The possibility that conditions for breakdown may be met simultaneously over a region of some extent cannot be ruled out; cases may differ in this respect.

ii) Throughout the transitional boundary layer, the mean characteristics of the boundary layer change gradually from those characterising fully laminar flow to those characterising fully turbulent flow, such that the transitional boundary layer appears to be intermittently laminar and turbulent. An intermittency factor (proportion of flow that is fully turbulent) can be defined and varies from a value of zero for fully laminar flow to a value of unity for the fully turbulent regime.

iii) The transitional boundary layer is universally statistically similar, whether long or short, and whether the disturbances are strong or weak and irrespective of whether they are introduced from the free stream or from the solid boundary (Schubauer and Klebanoff (1955)).

iv) The quantitative distribution of the intermittency factor in the transitional boundary layer is a function of the flow environment characterised chiefly by the pressure gradients, wall conditions and free stream turbulence.

v) The qualitative distribution of the intermittency factor in the transitional boundary layer is independent of the flow. Its close resemblance to a Gaussian integral curve seems to confirm the earlier notion that transition in a boundary layer depends on random perturbations superimposed on nearly regular pattern of amplified oscillations present in the boundary layer.

These experimental conclusions on the transitional boundary layer point to the model of the wall boundary layer sketched in Figure (1). Perturbations in the boundary layer, basically three dimensional amplify to a threshold magnitude necessary for the formation of turbulence spots. Elder (1960) showed in his experimental work that the condition for the formation of turbulence spots is $\sqrt{\overline{u'^2}}/U_1 \geq 0.18$. This breakdown in the laminar flow moves downstream and spreads like a wedge as it occurs. Turbulence spots act basically as arrestors to infinite amplification. By virtue of turbulence spot formation the boundary layer is able to adjust the fluctuations in it to the stable state found in fully turbulent boundary layer. The transitional boundary layer represents therefore the region of readjustment of the fluctuating motion, to a stable configuration, that is, the region of the developing turbulent boundary layer. It seems clear from all the above discussion on the transitional boundary layer that a realistic analytical model of the transitional boundary layer must necessarily be statistical. The fractional time that any portion of the transitional boundary layer is fully turbulent can be discussed meaningfully only in terms of probabilities that turbulence spots formed upstream pass over the point. Emmons (1951) initiated such a model and has successfully applied it to quite a wide class of flows. McCormick (1968) also has formulated a slightly different model from Emmons'. It seems that these statistical models are indeed the correct representation of the actual transition of wall boundary layer. The major problem in closing the conventional approach from laminar through transitional to turbulent flows seems to lie just in the proper understanding of laminar boundary layer perturbations, their amplification and the conditions for the formation of turbulence spots. If these are properly understood the conventional approach to boundary layer study can give a continuous closed solution at least for simple cases, of the entire boundary layer. The present author is of the opinion that these unresolved difficulties will be more easily and finally resolved if flows are considered as basically turbulent; laminar flows being only special and unstable cases. Hence the primary justification for the present attempt to follow the natural process of moving from stable to unstable that is, from turbulent flows to laminar flows.

2.3 Turbulent Boundary Layer Research

Physical experimental observations indicate that fluid motion may be considered to be made up of a mean part and a fluctuating part such that any physical quantity g, say, may be written as

$$g = \bar{g} + g' ,$$

where $\bar{g}$ is the average value of g and g' is the instantaneous fluctuation whose average is zero. If the physical quantities in the boundary layer equation (2.1) are replaced by their mean and fluctuating components and Reynolds time averages taken, one obtains the classical equations for the turbulent boundary layer. For the steady incompressible two-dimensional main flow over a smooth flat plate these equations are

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\bar{u}\,\frac{\partial \bar{u}}{\partial x} + \bar{v}\,\frac{\partial \bar{u}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2\,\bar{u} - \left(\frac{\partial \overline{u'^2}}{\partial x} + \frac{\partial \overline{u'v'}}{\partial y} + \frac{\partial \overline{u'w'}}{\partial z}\right)$$

$$\bar{u}\,\frac{\partial \bar{v}}{\partial x} + \bar{v}\,\frac{\partial \bar{v}}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\nabla^2\bar{v} - \left(\frac{\partial \overline{u'v'}}{\partial x} + \frac{\partial \overline{v'^2}}{\partial y} + \frac{\partial \overline{v'w'}}{\partial z}\right) \qquad (2.5)$$

Equations (2.5) are identical with similar equations for the laminar boundary layer except for the presence of correlation terms, $\overline{u'v'}$ etc. of the fluctuating quantities. Once again, the problem redounds to that of obtaining closed solutions for the physical quantities ($u(x,y)$, for example) in the boundary layer, given the free-stream velocity, $U_1(x)$. The difficulty in obtaining closed solutions to these equations results from the scarcely understood turbulence correlation terms present in the equations. Further assumptions and theories must be made in order to close the turbulent boundary layer equations. Until recently, the basic approach has been to make explicit or implicit assumptions concerning the local turbulence correlation. In the explicit method the equation of motion is reduced to the same form as for laminar boundary layer with the molecular viscosity replaced by an effective viscosity which incorporates the turbulence correlation term as an apparent or turbulent shear stress. That is,

$$\underset{\text{(Effective Viscosity)}}{\nu_e} = \underset{\text{(Kinematic Viscosity)}}{\nu} + \underset{\text{(Eddy Viscosity)}}{\varepsilon} \qquad (2.6)$$

where ν_e is the effective viscosity whose form can explicitly be deduced for example in the manner suggested in reference (3). Although the theory behind this effective viscosity (or mixing length) method appears to be physically unsound, the method nevertheless gives satisfactory answers in some cases.

The implicit method attributable to Dorodnitsyn was originally developed as a tool for systematically reducing partial differential equations to a few coupled ordinary differential equations of the initial value type. Again the turbulence velocity correlation terms are treated as apparent shear stresses but in this case, rather than make explicit statements about the effective viscosity, an integral equation is obtained from the resulting equation in the same manner as the Karman integral equations were obtained for the laminar boundary layer. The chief virtue of the integral relation method lies thus in the implicit and global manner in which the effect of turbulence is incorporated. It must however be noted that it is not easy to extend the integral methods to a wide class of flows without making explicit assumptions about the local turbulence. Most integral methods require some assumption about the velocity profile to permit the effects of non-uniformity in mass and momentum flux across the boundary layer to be considered in the governing integral equations. Apparently the velocity profile need not be very precise to allow adequate consideration of these non-uniformities.

It seems, in fact, that a simple profile which adequately fits the outer region and scales on the outer region parameters (U_1, δ) suffices for this purpose. However, a very accurate velocity profile which must be scaled on the wall region parameters (U_τ, ν) is essential for accurate calculation of wall shear stress from the Newtonian constitutive equation, $\tau = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0}$. The Stanford Conference (1968), on turbulent boundary layer prediction produced a remarkable source of both experimental data and the most recent analytical methods in turbulent boundary layer research. Its forecast for future lines of attack appear to have been fruitfully heeded.

The more recent approach to the solution of the turbulent boundary layer equations is to utilize the raw power of modern automatic computing machines in solving directly the partial differential equations. Equations can be derived for the second order correlations found in the basic equations of turbulent motion. These latter equations will not be closed, however, for their non-linear nature causes correlations of order $(n + 1)$ to appear in the equations for the nth. order correlations. An infinite system of such equations could however be written but no method is known yet for solving such infinite systems. Alternatively, the system of equations is terminated after the third or fourth order correlations by modelling these higher order correlations in terms of lower order correlations. It is in this latter approach that the turbulence theories of Taylor (1936), Karman and Howarth (1938), Kolmogoroff (1941) and Heisenberg (1948) to name but a few, are very useful in providing closure theories. The efforts of Glushko (1965), Bradshaw et al. (1967) and most recently Donaldson (1971) are remarkable examples of the power of this latter method of turbulence closure.

Other approaches to the turbulence problem based on the stochastic nature of turbulence have resulted in promisingly outstanding turbulence theories such as those of Kraichnan (1968), Burgers (1964), Hopf (1962), etc. These theories yield reasonable answers in most of their applications but there is not a single one of them which has overall acceptability or compatibility with the physical world.

The extensions of the turbulence theories and analytical methods to the study of heat and mass transfer in turbulent boundary layers have been about as successfully approximate as for the momentum transfer studies.

In conclusion, it can safely be said that turbulence is yet poorly understood.

Chapter III

3. THE NUMERICAL EXPERIMENTAL TECHNIQUE

A number of boundary layer Researchers, for example, Morkovin (1968), and Donaldson (1970), have pointed out that the influence of the particular physical environment upon the phenomenon of boundary layer laminar to turbulent transition, is so important that every observed case of such a transition should in itself be a unique case. In this respect, an exhaustive experimental study of boundary layer transitions would be practically very tasking since it would involve the repetition of a large number of identical experiments but over a well controlled variation of the experimental environment. This idealization is certainly an impossible task for any one laboratory and the delicacy of such experimentation would restrict the amount of useful generalizing conclusions that can be drawn from studies made in different laboratories. An alternative way to physical experimentation must therefore be sought. The next best method that suggests itself is numerical experimentation utilizing the power of modern automatic computing machines. The major demerit with numerical experimentation is the great difficulty of establishing satisfactory mathematical models for the physical phenomena. In any situation, however, where such models can be satisfactorily formulated, numerical experimentation shows great superiority and desirability, over physical experimentation. Not only can numerical experimentation be faster and often more accurate than physical experimentation, it is also usually much less expensive and is much more easily reproducible permitting similar studies to be made in a wide variety of conditions without any doubt as to their similarity.

The experiment discussed in this chapter is a numerical experimental study of the wall boundary layer for general types of flows, involving the mathematical modelling of wall boundary layers and their environment and the numerical solution of the resulting mathematical equations with an automatic computing machine.

3.1 The Analytical Model

In tensor notation, the basic equations governing the flow of fluid are as follows:

Conservation of mass:

$$\frac{\partial \rho}{\partial t} + (\rho u^j),j = 0 \tag{3.1}$$

Conservation of momentum:

$$\rho \frac{\partial u_i}{\partial t} + \rho u^j \, u_{i,j} = - \frac{\partial p}{\partial x^i} + (\tau_i^j),j \tag{3.2}$$

Conservation of energy:

$$\rho c_v(\frac{\partial T}{\partial t} + u^j T,j) = p \, u^j,j + g^{ij}(kT,i),j + \tau_i^j \, u^i,j \tag{3.3}$$

where

$$\tau_{ij} = \mu(u_{i,j} + u_{j,i}) + g_{ij} \, \lambda u^k,k \tag{3.4}$$

and

$$i,j,k = 1,2,3 \; .$$

If the fluid motion is considered to be made up of a mean part and a fluctuating part such that any physical quantity, q , may be written as $q = \bar{q} + q'$ where $\bar{q}$ is the average value of q and q' is the instantaneous fluctuation whose average is zero, then equations may be written for both the mean and fluctuating portions of the fluid motion. For the case of steady state flow, such equations have been derived, and it will suffice to discuss briefly only the basic approach.

Physical quantities in the basic equations of motion are replaced by their mean and fluctuating components to obtain generalized flow equations. Taking the time average of these generalized equations yields the equations for the mean motion. Subtracting the equations for the mean motion from the appropriate generalized equations yield the equations for the fluctuations.

In the equations of motion thus obtained one always finds terms representing the correlations of the fluctuating quantities (usually the second and third order correlations in this case). Equations may be obtained for the second order correlations by multiplying the equations of motion for the fluctuations by appropriate fluctuation quantities and averaging in time. These equations, however, will not be closed, for their non-linear nature causes correlations of order $(n + 1)$ to appear in equations for the nth. order correlations.

Donaldson et al. (1970) developed a "method of invariant modelling" which enables one to obtain a closed system of equations. In this method, a closure of the set of equations for the mean flow quantities and the second-order velocity correlations was achieved through a modelling of the higher order and pressure correlation terms in these equations using the second-order correlations themselves and two scalar lengths, λ (microscale) and Λ (macroscale) which are related to the microscale and integral scale of the velocity fluctuations. A brief discussion of the method of invariant modelling is given in Appendix A together with the relevant equations of motion for the general incompressible boundary layer.

The equations obtained by the invariant modelling technique discussed above contain two unknown constants a and b in addition to the length scales λ and Λ. If one examines the equations of motion given in Appendix A, it becomes clear that Λ is of importance only in the outer boundary region while λ is important in the wall region. This is not surprising since λ and Λ are essentially the length scales of the disturbances, or more practically, represent the sizes of the eddying motion. The length scales λ and Λ will be assumed to be linearly proportional to y, the vertical distance from the wall. This assumption is not a good one, especially for the microscale, λ, and should not be misconstrued as implying that $\lambda = \Lambda$. The relation between λ and Λ has been discussed by Donaldson et al. (1968) and Glushko (1965). The assumption of $\lambda = y$ and $\Lambda = y$ is used only to simplify the disturbance kinetic energy equations.

The two parameters a , and b occurring in the disturbance equations have been obtained by Donaldson et al. based on the following reasoning.

i) a , and b must be such that results obtained by using the disturbance equations satisfy the classical stability critical Reynolds numbers ($R_{\delta *} \approx 420$ or $R_\delta \approx 1200$) for incompressible flows.

ii) The modal shape of the profile for $\overline{K}(=\overline{k}_1 + \overline{k}_2 + \overline{k}_3)$ and the decay rate of disturbances should have the general characteristics of the same profile as calculated by classical stability theory. (Here the overbars indicate dimensionless quantities.)

Donaldson et al. (1970) obtained after many calculations that a satisfactory choice of parameters would be a = 50 and b = 0.01. The small value of b was largely due to the requirement that the disturbance profile shape, $\overline{K}(y)$, should be peaked in the neighborhood of $y/\delta = 0.5$, which seems to be the case in actual boundary layers.

Further, Donaldson et al. found that for such small values of the parameter, b , the terms in the disturbance equations, containing b become insignificant, that is, $b \approx 0$. These results of Donaldson et al. for the values of a and b will not be checked in the present work. Nevertheless, their use in preliminary computations yielded answers reasonably in accord with experimental results. In the present work, therefore, the parameters will be adopted as

$$a = 50 , \quad b = 0.$$

3.2 Reduction of the Governing Equations of Motion

With the assumptions discussed in (3.1) and the values of the disturbance parameters and length scales thus obtained, the governing equations of motion for an incompressible, two dimensional boundary layer, with three dimensional velocity perturbations, are as given in Appendix A, in the orthogonal coordinate system of parallel curves (Figure 2).

The present analyses are desired to cover the following range of principal parameters:

$C = \frac{\delta^*}{r_o}$	from 0 to 0.01, that is, to about 100 times the boundary layer displacement thickness.
$Re_{\delta^*} = \frac{U_1 \delta^*}{\nu_1}$	from about 50 to ∞ .
$\frac{L}{r_o} =$	Characteristic Length ÷ radius of curvature from 0 to 10.
$\frac{dc}{dx} =$	Streamwise curvature gradient from 0 to 200.

If the system of equations of Appendix A are made dimensionless and the relative magnitudes of their terms examined, in the manner suggested by A.M.O. Smith (1953), it is easy to see that some terms are very small, in order of magnitude, compared to other terms. These smallest terms will be neglected in comparison with other terms because at no time will they ever approach dominance in the boundary layer.

Terms in $(1/r_o)^2$ and higher order will be neglected too.

The following simplification will also be made:

$$\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right) \doteq 0 \qquad \text{since}$$

$$\frac{1}{(1-cy)} = 1 + cy - O(c^2) \tag{3.5}$$

and $\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right) \doteq c(= 1/r_o)$ which for the range of curvatures of interest is of magnitude from zero to $(0.01/\delta^*)$. That is, only for extremely accelerated flows for which δ^* becomes very small, will $\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right)$ be appreciable. In such extreme cases, however, the inertial terms are usually very dominant so that terms in $\frac{\partial}{\partial y}\left(\frac{1}{1-cy}\right)$ will still be small compared to most other terms in the equation.

The simplified equations of motion for the boundary layer, for the case where the molecular viscosity is temperature dependent now become in dimensional form:

Continuity:

$$\frac{1}{(1-cy)}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{cv}{(1-cy)} = 0 . \tag{3.6}$$

Combined momentum (or vorticity transport):

$$\frac{u}{(1-cy)}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] + v\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right] - \frac{cu}{(1-cy)}\frac{\partial u}{\partial x} - \frac{cv}{(1-cy)}\frac{\partial u}{\partial y}$$

$$= \frac{\partial^2\sigma}{\partial y^2} - \frac{\partial^2\sigma}{\partial x^2} + \frac{\partial^2}{\partial y^2}\left(\nu\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\nu c}{(1-cy)}\frac{\partial u}{\partial y}\right) - (1-cy)\frac{\partial\sigma}{\partial y} \tag{3.7}$$

Energy:

$$c_p \left[\frac{u}{(1-cy)} \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right] = c_p \alpha \frac{\partial^2 T}{\partial y^2} + \nu \left(\frac{\partial u}{\partial y}\right)^2 - \frac{\partial \overline{v'T'}}{\partial y} \tag{3.18}$$

neglected in the early transitional regime.

Streamwise velocity correlation: $\overline{u'u'} = k_1$

$$\frac{u}{(1-cy)} \frac{\partial k_1}{\partial x} + v \frac{\partial k_1}{\partial y} = 2\sigma \frac{\partial u}{\partial y} - \frac{2k_1}{(1-cy)} \frac{\partial u}{\partial x} + \frac{2k_1 cv}{(1-cy)} + \frac{\partial \nu}{\partial y} \frac{\partial}{\partial y} [2k_1 - \frac{K}{3}]$$

$$+ \frac{2}{(1-cy)} \frac{\partial \nu}{\partial y} \frac{\partial \sigma}{\partial x} + 50 \frac{\partial u}{\partial y} [\frac{K}{3} - k_1] + \nu \frac{\partial^2 k_1}{\partial y^2} - \frac{2\nu k_1}{y^2} \tag{3.9}$$

Transverse velocity correlation: $\overline{v'v'} = k_2$

$$\frac{u}{(1-cy)} \frac{\partial k_2}{\partial x} + v \frac{\partial k_2}{\partial y} = - 2k_2 \frac{\partial v}{\partial y} + \frac{\partial k_2}{\partial y^2} - \frac{2\nu k_2}{y^2}$$

$$+ 50(\frac{K}{3} - k_2) \frac{\partial u}{\partial y} + \frac{\partial \nu}{\partial y} \frac{\partial}{\partial y} [4k_2 - \frac{K}{3}] . \tag{3.10}$$

Lateral velocity correlations: $\overline{w'w'} = k_3$

$$\frac{u}{(1-cy)} \frac{\partial k_3}{\partial x} + v \frac{\partial k_3}{\partial y} = \frac{\partial \nu}{\partial y} \frac{\partial}{\partial y} [2k_3 - \frac{K}{3}] + 50(\frac{K}{3} - k_3) \frac{\partial u}{\partial y} + \nu \frac{\partial^2 k_3}{\partial y^2} - \frac{2\nu k_3}{y^2} \tag{3.11}$$

Cross-correlation: $\overline{u'v'} = -\sigma$

$$\frac{u}{(1-cy)} \frac{\partial\sigma}{\partial x} + v\frac{\partial\sigma}{\partial y} = 3\frac{\partial\nu}{\partial y}\frac{\partial\sigma}{\partial y} + k_2\frac{\partial u}{\partial y} + \nu\frac{\partial^2\sigma}{\partial y^2} - 50\sigma\frac{\partial u}{\partial y} - \frac{2\nu\sigma}{y^2} \qquad (3.12)$$

To facilitate further analysis of the system of equations (3.6 - 3.12), it is convenient to define a new set of variables.

The following transformations are therefore used: (Overbars, here denote dimensionless quantities).

$$u = U_1(1 - f') \qquad \text{i.e.} \quad f'(x,\eta) = 1 - u/U_1$$

$$y = \eta\delta^*$$

$$k_i = U_1{}^2\overline{k}_i \qquad i = 1,2,3$$

$$\sigma = U_1^2\,\overline{\sigma}$$

$$K = U_1^2\,\overline{K}$$

$$c\delta^* = C$$

$$v_o = U_1\,\overline{v}_o$$

L = any suitable characteristic length in the x-direction

$$x = \overline{x}L$$

$$\delta^* = \overline{\delta}^*L$$

$$\frac{dU_1}{dx} = \frac{FU_1^2}{\nu_1} \quad \text{where} \quad F = \frac{\nu_1}{U_1^2}\frac{dU_1}{dx}, \text{ the pressure gradient parameter.}$$

$$T = T_o[q'(1 - T_1/T_o) + T_1/T_o] \quad \text{where} \quad q' = \frac{T-T_1}{T_o-T_1}.$$

It follows from the above transformation that:

$$\frac{\partial k_i}{\partial y} = \frac{U_1^2}{L\overline{\delta}^*}\frac{\partial\overline{k}_i}{\partial\eta} \qquad i = 1,2,3$$

$$\frac{\partial\sigma}{\partial y} = \frac{U_1^2}{L\,\overline{\delta}^*}\frac{\partial\overline{\sigma}}{\partial\eta}$$

$$\frac{\partial k_1}{\partial x} = \frac{U_1^2}{L} \frac{\partial \bar{k}_i}{\partial \bar{x}} + 2 \frac{FU_1^3}{\nu_1} \bar{k}_i$$

$$\frac{\partial u}{\partial y} = - \frac{U_1}{L\bar{\delta}^*} f''$$

$$\frac{\partial u}{\partial x} = (1 - f') \frac{FU_1^2}{\nu_1} - \frac{U_1}{L} \frac{df'}{d\bar{x}}$$

$$\frac{\partial T}{\partial y} = \frac{T_o}{L} \frac{q''}{\bar{\delta}^*} (1 - T_1/T_o)$$

$$\frac{\partial T}{\partial x} = \frac{q'}{L} \frac{\partial T_o}{\partial x} + \frac{T_o}{L} (1 - T_1/T_o) \frac{dq'}{d\bar{x}}$$ where it is assumed

that $dT_1/d\bar{x} = 0$ but $dT_o/d\bar{x}$ may be different from zero.

$$\nu = \nu_1 (\frac{T}{T_1})^{3/2} (\frac{T_1 + S_c}{T + S_c})$$ where $S_c = 110^oK$ for air.

$$= \nu_1 [q'(T_o/T_1 - 1) + 1]^{3/2} \{ \frac{1 + S_c/T_1}{1 + q'(T_o/T_1 - 1) + S_c/T_1} \}.$$

Further, it will be assumed that from the continuity equation

$$\frac{1}{(1-cy)} \frac{\partial u}{\partial x} \doteq - \frac{\partial v}{\partial y} . \qquad (3.13)$$

That is,

$$v \doteq U_1 \bar{v}_o + \frac{(f - \eta)}{(1-C\eta)} [U_1 \frac{d\bar{\delta}^*}{d\bar{x}} + FU_1 R_{\delta^*}] + \frac{U_1 \bar{\delta}^*}{(1-C\eta)} \frac{df}{d\bar{x}} . \qquad (3.14)$$

and

$$\frac{dv}{dx} = \frac{U_1(1+C\eta)}{L} [(2\bar{\delta}^*_x + F\ Re_{\delta^*}) \frac{df}{d\bar{x}} + F\ Re_{\delta^*}\{ \frac{\bar{v}_o}{(1+C\eta)} + (f-\eta)(\bar{\delta}^*_x + F\ Re_{\delta^*})$$

$$+ \bar{\delta}^* \frac{df}{dx} \}]$$

The equations for the fluid motion can now be written in a dimensionless form as follows:

Combined momentum (or Vorticity Transport):

$$[\frac{1}{R_{\delta^*}(1+C\eta)}]f^{iv} + [\frac{2}{(1+C\eta)} \frac{\partial}{\partial\eta} (\frac{1}{R_{\delta^*}}) - \frac{\bar{v}_o}{(1+C\eta)} - (F-\eta)(\frac{d\bar{\delta}^*}{d\bar{x}} + F\ Re_{\delta^*})]f'''$$

$$+ [\frac{C\bar{v}_o}{(1+C\eta)} + C(f-\eta) \frac{d\bar{\delta}^*}{d\bar{x}} + (1-f') \frac{d\bar{\delta}^*}{d\bar{x}} + (C(f-\eta)-1)F\ Re_{\delta^*} + \frac{1}{(1+C\eta)} \frac{\partial^2}{\partial\eta^2} (\frac{1}{R_{\delta^*}})]f''$$

$$+ [(2-f')C\ F\ Re_{\delta^*} + F''\ F\ Re_{\delta^*}]\ f' = (f'''\bar{\delta}^* - Cf''\ \bar{\delta}^*) \frac{df}{d\bar{x}} - C(1-f')\ \bar{\delta}^* \frac{df'}{d\bar{x}}$$

$$+ (1-f')\ \bar{\delta}^* \frac{df''}{d\bar{x}} + F\ Re_{\delta^*}[C - \frac{4\bar{\delta}^*}{(1+C\eta)} \frac{\partial\sigma}{\partial\bar{x}} - \frac{6F\ Re_{\delta^*}}{(1+C\eta)}\ \bar{\sigma}\ ]$$

$$+ \frac{1}{(1+C\eta)} [\frac{\partial^2\bar{\sigma}}{\partial\eta^2} - \frac{C\partial\bar{\sigma}}{\partial\eta} - 2\bar{\sigma}\ Re_{\delta^*}\ \bar{\delta}^* \frac{dF}{d\bar{x}}] \qquad (3.15)$$

Energy:

$$[\frac{1}{Pr\ Re_{\delta*}(1+C\eta)}]\ q''' - [\frac{\overline{v}_o}{(1+C\eta)} + (f-\eta)\ \frac{d\overline{\delta}*}{d\overline{x}} + (f-\eta)\ F\ Re_{\delta*}]q''$$

$$- [(1-f')(\frac{T_1}{T_o-T_1})\ \overline{\delta}*\ \frac{d}{d\overline{x}}\ (T_o/T_1)]q'$$

$$= - \frac{E}{Re_{\delta*}}\ \frac{(f'')^2}{(1+C\eta)} + q''\overline{\delta}*\ \frac{df}{d\overline{x}} + (1-f')\overline{\delta}*\ \frac{dq'}{d\overline{x}}$$

(3.16)

Streamwise correlation: $\overline{u'u'} = k_1$

$$[\frac{1}{R_{\delta*}(1+C\eta)}]\overline{k}_1'' + [\frac{1}{(1+C\eta)}\ \{2\ \frac{\partial}{\partial\eta}\ (\frac{1}{R_{\delta*}}) - \overline{v}_o\} - (f-\eta)(\overline{\delta}_x^* + F\ Re_{\delta*})]\overline{k}_1'$$

$$+ [2(f'-1)\ \{2 + C\eta\}\ F\ Re_{\delta*} + 50f'' - \frac{2}{R_{\delta*}\eta^2}\ ]\ \frac{\overline{k}}{(1+C\eta)}$$

$$= \overline{\delta}*(1-f')\ \frac{\partial\overline{k}_1}{\partial\overline{x}} + \frac{2\overline{\sigma}}{(1+C\eta)}\ f'' + \overline{\delta}*\ k_1'\ \frac{df}{dx}$$

$$- 2\ k_1\ \overline{\delta}*\ \frac{df'}{dx} + \frac{1}{3(1+C\eta)}\ [\frac{\partial}{\partial\eta}\ (\frac{1}{R_{\delta*}}\ \frac{\partial K}{\partial\eta} + 50\ f''K]\ . \qquad (3.17)$$

Transverse correlation: $\overline{v'v'} = k_2$

$$[\frac{1}{R_{\delta*}(1+C\eta)}]\overline{k}_2'' + [\frac{1}{(1+C\eta)}\ \{4\ \frac{\partial}{\partial\eta}\ (\frac{1}{R_{\delta*}}) - \overline{v}_o\ \} - (f-\eta)(\overline{\delta}_x^* + F\ Re_{\delta*})]\overline{k}_2'$$

$$+ [2(f'-1)(1+C\eta)\ F\ Re_{\delta*} + 50f'' - \frac{2}{R_{\delta*}\eta^2}\ ]\ \frac{\overline{k}_2}{(1+C\eta)}$$

$$= \overline{\delta}*(1-f')\ \frac{\partial\overline{k}_2}{\partial\overline{x}} + \overline{\delta}*\ k_2'\ \frac{df}{dx}$$

$$+ \frac{1}{(3(1+C\eta)} \left[50f'' K + \frac{\partial}{\partial\eta}\left(\frac{1}{R_{\delta*}}\right)\frac{\partial K}{\partial\eta}\right] \tag{3.18}$$

Lateral Correlations: $\overline{w'w'} = k_3$

$$\left[\frac{1}{R_{\delta*}(1+C\eta)}\right]\bar{k}_3'' + \left[\frac{1}{(1+C\eta)}\left\{2\frac{\partial}{\partial\eta}\left(\frac{1}{R_{\delta*}}\right) - \bar{v}_o\right\} - (f-\eta)(\bar{\delta}^*_x + F\,Re_{\delta*})\right]\bar{k}_3'$$

$$+ \left[2(f'-1)(1+C\eta)F\,Re_{\delta*} + 50f'' - \frac{2}{R_{\delta*}\eta^2}\right]\frac{\bar{k}_3}{(1+C\eta)}$$

$$= \bar{\delta}^*(1-f')\frac{\partial\bar{k}_3}{\partial\bar{x}} + \bar{\delta}^*\,\bar{k}_3'\,\frac{df}{dx}$$

$$+ \frac{1}{3(1+C\eta)}\left[50f'' K + \frac{\partial}{\partial\eta}\left(\frac{1}{R_{\delta*}}\right)\frac{\partial K}{\partial\eta}\right] \tag{3.19}$$

Cross correlation: $\overline{u'v'} = -\sigma$

$$\left[\frac{1}{R_{\delta*}(1+C\eta)}\right]\bar{\sigma}'' + \left[\frac{1}{(1+C\eta)}\left\{3\frac{\partial}{\partial\eta}\left(\frac{1}{R_{\delta*}}\right) - \bar{v}_o\right\} - (f-\eta)(\bar{\delta}^*_x + F\,Re_{\delta*})\right]\bar{\sigma}'$$

$$+ \left[2(f'-1)(1+C\eta)F\,Re_{\delta*} + 50f'' - \frac{2}{R_{\delta*}\eta^2}\right]\frac{\bar{\sigma}}{(1+C\eta)}$$

$$= \bar{\delta}^*(1-f')\frac{d\bar{\sigma}}{d\bar{x}} + \bar{\delta}^*\,\bar{\sigma}'\,\frac{df}{dx} + \frac{\bar{k}_2}{(1+C\eta)}\,f''. \tag{3.20}$$

The boundary conditions are:

$f(\bar{x},0) = 0 \qquad\qquad \lim_{\eta\to\infty} f(\bar{x},\eta) = 1$

$f'(\bar{x},0) = 1 \qquad\qquad \lim_{\eta\to\infty} f'(\bar{x},\eta) = 0$

$\bar{k}_i(\bar{x},0) = 0$

$\bar{\sigma}(\bar{x},0) = 0 \qquad\qquad \lim_{\eta\to\infty} f''(\bar{x},\eta) = 0$

$$q'(\bar{x},0) = 1 \qquad \lim_{\eta\to\infty} q'(\bar{x},\eta) = 0$$

$$\lim_{\eta\to\infty} \bar{k}_i(\bar{x},\eta) = 0$$

$$\lim_{\eta\to\infty} \overline{\sigma}(x,\eta) = 0$$

$$\Delta^*(x) = \int_0^\infty [(T-T_1)/(T_o-T_1)(1-y/r_o)]dy$$

$$\delta^*(x) = \int_0^\infty [(1-\frac{u}{U_1})/(1-y/r_o)]dy; \quad \theta(x) = \int_0^\infty [\frac{u}{U_1}(1-u/U_1)/(1-y/r_o)]dy$$

In these analyses, primes denote differentiation with respect to η and subscript x denotes differentiation with respect to x. Over bars denote, up to this point, non-dimensional variables.

3.3 The Numerical Method

The dimensionless equations for the mean and fluctuating motion in the boundary layer (equations 3.15 through 3.20) constitute a system of six non-linear partial differential equations in the six variables f, q', $\bar{k}_1$, $\bar{k}_2$, $\bar{k}_3$, and $\bar{\sigma}$, and hence can be solved. In this work, a numerical method is used which computes solutions for the boundary layer equations in a downstream march from prescribed conditions and profiles at an initial x-position. At this point, it would be useful to convert the system of equations to a set of ordinary differential equations by using finite differences in the x-derivatives. An implicit type method which is essentially an adaptation of the Crank-Nicholson (1947) scheme will be used. The method is always stable and the error is of second order in the x-step size.

First the system of equations (3.15-3.20) are rewritten in terms of average functions at a point half way between the x-position of the known profiles x_{i-1} and that of the profiles to be calculated x_i . Then using the relation:

$$\bar{g} = \frac{1}{2}[g_i + g_{i-1}] \qquad (3.21)$$

where the bar over g denotes averaging, the system of equations can be written in terms of functions at position x_i . From this point, bars over a quantity denote averaging in the above sense. All variables are now dimensionless and overbars previously denoting that are considered superfluous. The combined momentum equation (3.15), for example, becomes:

$$a_1(f_i^{iv} + f_{i-1}^{iv}) + a_2(f_i''' + f_{i-1}''') + a_3(f_i'' + f_{i-1}'') + a_4(f_i' + f_{i-1}')$$

$$= a_5(f_i - f_{i-1}) - a_6(f_i' - f_{i-1}') + a_7(f_i'' - f_{i-1}'') + 2a_8$$

$$- a_9(\bar{\sigma}_i - \bar{\sigma}_{i-1}) - a_{10}(\bar{\sigma}_i + \bar{\sigma}_{i-1}) - a_{11}(F_i - F_{i-1}) \qquad (3.22)$$

The variable coefficients, $a_1 - a_{34}$, are as defined in Appendix B.

The final form in which the equations will be solved is as follows:

Momentum:

$$[a_1 f^{iv} + a_2 f''' + (a_3 - a_7)f'' + (a_4 + a_6)f' - a_5 f]_i = D(1)_{i-1}$$

where

$$D(1)_{i-1} = [-a_1 f^{iv} - a_2 f''' - (a_3 + a_7)f'' + (a_6 - a_4)f' - a_5 f]_{i-1}$$

$$- a_{11}(F_i - F_{i-1}) - a_{10}(\sigma_i + \sigma_{i-1}) - a_9(\sigma_i - \sigma_{i-1}) + 2a_8 . \qquad (3.23)$$

Energy:

$$\left[\frac{a_1}{Pr} q''' + \left(-a_{14} - \frac{\overline{\delta}^*}{\Delta x}(f_i - f_{i-1})\right) q'' + (-a_{15} - a_7)\, q'\right]_i = D(2)_{i-1}$$

where

$$D(2)_{i-1} = \left[-\frac{a_1}{Pr} q''' + \left(a_{12} + \frac{\overline{\delta}^*}{\Delta x}(f_i - f_{i-1})\right) q'' + (a_{15} - a_7) q'\right]_{i-1}$$

$$- a_{16}(f_i''^2 + f_{i-1}''^2) \; . \qquad (3.24)$$

Streamwise velocity correlation:

$$[a_{17}k_1'' + a_{18}k_1' + (a_{19} - a_7)\, k_1]_i = D(3)_{i-1}$$

where

$$D(3)_{i-1} = [-a_{17}k_1'' - a_{18}k_1' - (a_7 + a_{19})k_1]_{i-1} - a_{20}(f_i - f_{i-1})$$

$$- a_{21}(f_i' - f_{i-1}') - a_{22}(\sigma_i - \sigma_{i-1}) - a_{23}(\sigma_i + \sigma_{i-1}) + 2a_{24}$$

(3.25)

Transverse velocity correlations:

$$[a_{17}k_2'' + a_{25}\, k_2' + (a_{26} - a_7)k_2]_i = D(4)_{i-1}$$

where

$$D(4)_{i-1} = [-a_{17}k_2'' - a_{25}k_2' - (a_{26} + a_7)k_2]_{i-1} - a_{27}(f_i - f_{i-1}) + 2a_{24} \; . \qquad (3.26)$$

Lateral velocity correlation:

$$[a_{17}k_3'' + a_{18}k_3' + (a_{29} - a_7)k_3]_i = D(5)_{i-1}$$

where

$$D(5)_{i-1} = [-a_{17}k_3'' - a_{18}k_3' - (a_{29} + a_7)k_3]_{i-1} + a_{30}(f_i - f_{i-1}) + 2a_{24} \; .$$

(3.27)

Cross correlation:

$$[a_1\sigma'' + a_{31}\sigma' + (a_{32} - a_7)\;\sigma]_i = D(6)_{i-1}$$

where

$$D(6)_{i-1} = [-a_1\sigma'' - a_{31}\sigma' - (a_{32} + a_7)\sigma]_{i-1} + a_{33}(f_i - f_{i-1}) + 2a_{34} \; .$$

(3.28)

All the equations in their final form, except the combined momentum equation, can be expressed in the following finite difference f orm:

$$[-A_j\; u_{j+1} + B_j u_j - C_j\; u_{j-1}]_i = [D_j]_{i-1} \qquad (3.29a)$$

The system of equation (3.29a) from $j = 1$ to J has the tridiagonal matrix form, and so can be solved very easily by the simplified direct inversion method as explained, for example, by Stiefel (1956). The solutions have the form

$$u_j = E_j\, u_{j+1} + F_j \;. \qquad (3.29b)$$

The boundary conditions at the wall determine E_1 and F_1, and E_j and F_j $(j > 1)$ can be determined in a general form so that the family of solutions given by equations (3.29a) and (3.29b) are the same. To achieve this, u_{j-1} is replaced by $(E_{j-1}\, u_j + F_{j-1})$ in equation (3.29a). The result of this operation is a relation between u_j and u_{j+1} which can be written as

$$u_j = \frac{A_j}{(B_j - C_j E_{j-1})}\, u_{j+1} + \frac{D_j + C_j F_{j-1}}{(B_j - C_j E_{j-1})} \qquad (3.30)$$

If the right hand sides of equations (3.29b) and (3.30) are equated, and if it is recalled that the result must hold for a one-parameter set of values of u_{j+1} , then E_j and F_j can be identified as follows:

$$E_j = \frac{A_j}{(B_j - C_j E_{j-1})}$$

$$\text{for } j \geq 2 \qquad (3.31)$$

$$F_j = \frac{D_j + C_j F_{j-1}}{B_j - C_j E_{j-1}}$$

From these equations, together with the boundary conditions at the wall and free stream, E_j and F_j can be calculated inductively in order of increasing $j(j = 2,3,\ldots,J-1)$. Now, u_{j+1} is given for $(J-1)$ by the outer region boundary condition. Therefore all the u_j can now be calculated inductively from equation (3.29b) in order of decreasing $j(j = J-1, J-2, \ldots,2)$. This completes the calculation. The direct inversion method definitely is simpler and faster than any iterative techniques. Moreover, for the tridiagonal case, Wilkinson (1961) has shown that the direct method is extremely stable with respect to the growth of rounding errors.

In finite difference form, the system of equations corresponding to the combined momentum equation (3.23) has the five-diagonal matrix form. Direct inversion methods for solving such matrix equations suffer from serious round-off errors and may not always converge to the true solutions. Standard iterative techniques have been ruled out for the solution of equation (3.23) in view of the very large number of computations that must be made, in this study, in order to draw general conclusions on wall boundary layers. The alternative that suggests itself and the one adopted for the present study is to solve equations (3.23) for f''. By a simple integration f' is computed from f'' and similarly f from f' Apart from increasing slightly the number of iterations required for simultaneous convergence on f, f' and f'', this method has shown no serious inferiority to alternative methods.

A_j , B_j , C_j and D_j have been computed for the six sets of equations (3.23 through 3.28) and are as tabulated in Appendix B. At $j = 1$, the boundary conditions for equations (3.25 through 3.28) yield that $E_1 = F_1 = 0$. For equations (3.23) and (3.24), E_1 and F_1 are a bit more difficult to determine. In the present analyses, they are determined as follows:

$$u_1 = E_1 u_2 + F_1 \ . \tag{3.32}$$

Since u_2 is not much different from u_1 , it is assumed that $E_1 = 1$. Then $F_1 = u_1 - u_2$, i.e.,

$$E_1 = 1$$
$$F_1 = - u_1'(\eta_2 - \eta_1) \ . \tag{3.33}$$

Hence, for equations (3.23) and (3.24), q_1'' and f_1''' must be found, in some way, before the beginning of each iteration. For the combined momentum equation, f_1''' , is roughly estimated as follows:

$$f_1''' = F \, \overline{Re}^2_{\delta*} + f_1'' \, [\overline{C} + \overline{Re}_{\delta*}(v_o - \frac{\partial}{\partial\eta} (\frac{1}{R_{\delta*}})_1)] \ .$$

The quantity f_1'' is taken as the same from the previous iteration while q_1'' is estimated as

$$q_1'' = -0.332 \ \sqrt[3]{Pr} \cdot \sqrt{\overline{Re}_{\delta*} \ \overline{\delta}*/x}$$

Then:

$$\left.\begin{aligned} E_1 &= 1 && \text{Energy and momentum} \\ F_1 &= -q_1''(\eta_2 - \eta_1) && \text{Energy} \\ F_1 &= - f_1'''(\eta_2 - \eta_1) && \text{Momentum .} \end{aligned}\right\} \tag{3.34}$$

3.4 The Computer Program

The computer program listing is presented in the Appendix D. It indicates two main parts for the main program, and four subroutines. The first part of the main program accepts the input data and prepares the appropriate mean flow and perturbation profiles and associated parameters for the initial x-station. The first few instructions read in all the input data required, according to the formats as listed below. Then all flow parameters for the initial x-station profiles are computed. This is the end of the initialization portion of the program.

The forward motion part of the main program consists of a loop which cycles for each x-station calculation. The loop begins by moving the known profiles into storage for the profile at the x-station before the one to be calculated. This is followed by the iterative loop to calculate a new set of profiles at the new x-station. Within this loop, there is an inner loop to iterate for the mean flow profiles When these calculations have simultaneously converged, the perturbation profiles and integral parameters for that station are calculated, and the integral test for accuracy is printed out. This process continues until profiles have been calculated at all x-stations. Finally, for convenience, a summary of the parameters of the flow is printed out. The input to the initialization section is, in order, as follows:

i) Fixed input data in FPS units.

a) JD, ID, EPS

b) CP (BTU/lbm^{o}k), PR, SC, VKC

ii) Label describing the Program Title (first 72 columns of first data card)

iii) Total actual numbers of profile data points

Y, Fl, Ql, AKl, AK2, AK3, SIG

(Format 7I2)

iv) Initialization parameters

TIR, DT(1), CHL

(Format F8.8, 5x, F8.4, 5x, F10.3, 5x)

v) Fixed flow parameters

RHO, GC, TE, VEE, L

(Format F10.9, F10.3, F10.8, F10.9, 10x, I1)

vi) Initial Profiles data points

Y, FI, Ql, AKl, AK2, AK3, SIG

(Format 10F 8.8 (4)) i.e., 10 per data card (in the order listed above).

vii) x-Stations and primary descriptive parameters

X(I); UE(I); TW(I); VW(I); CW(I); RUF(I)

(ft) (fps) (^{o}K) (fps) (ft) (ft)

(Format 6F10.3)

10000. (serves as end card for the input).

Subroutine PRØFYL: computes all thy profiles for the mean flow and perturbations by a direct inversion method as discussed in section (3.3).

Subroutine INTEG: performs a simple trapezoidal quadrature.

Subroutine DIVIDE: subdivides the interval between the values which define a function using a linear interpolation. L defines the number of subdivisions required. It is suggested that for very small x-step size, smaller spacing should be used throughout the layer. x-spacings of about ten times the displacement thickness are usually adequate for most computations.

Since, most often, the initial input profiles are inaccurate, it has been found necessary, for reduction in the number of iterations required for the convergence of subsequent solutions, to smooth these initial curves before they are subjected to any operations.

Subroutine SMØØTH: performs this smoothing by fitting the desired degree polynomial through the data points.

Subscripts are, as far as possible, the same thoughout the program. Additional subroutines may readily be incorporated to facilitate any specific detailed studies.

Chapter IV

4. FLOW INSTABILITY AND TRANSITION

The philosophy promulgated in this work about flow instability is that flows are generally turbulent except when the turbulence is "locked-in" by a strong force field. Such constrained flows are the so-called laminar flows. Reversion to the more normal turbulent situation, through the so-called laminar instability and transition, occurs when the constraining force becomes overwhelmed by other forces in the flow field.

The force field concept will be introduced in greater detail now, and the instability characteristics of laminar flows deduced by an energy method very similar, in some ways, to the method used by Stuart (1958). An implication of the force field concept namely that the fluid flow may be considered as a non-linear system with an explicit response function from which the stability or instability characteristics of the flow may be deduced, is discussed. In fact, an attempt is made to deduce the functional form of this flow describing function, for the simple flow with zero pressure gradient and no heat or mass transfer at the wall.

4.1 The Force Field Concept

Essentially, the force field philosophy implies that, given an elemental volume as in Figure (3a), what happens to a flow quantity between the locations A to B (the shaded area) depends only on the force field within that volume. This implies the conservation of such invariant flow quantities as mass, momentum and energy.

i.e. $q_B = \psi[q_A \text{ , Local force field}]$ (4.1) where q is any invariant flow variable. Let the force (per unit volume) in an elemental fluid volume be identified by the notation of Figure (7), below

Flow direction

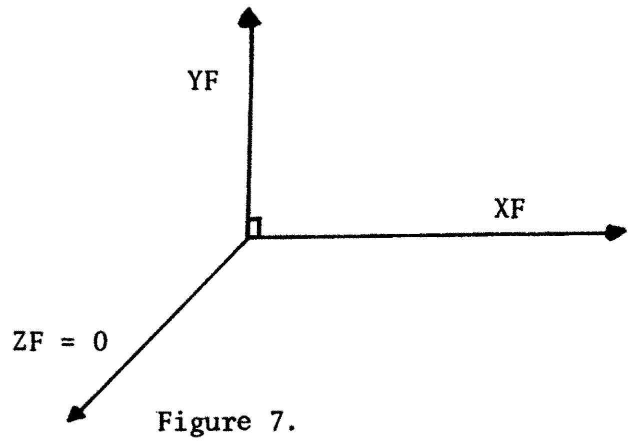

Figure 7.

The streamwise driving force is essentially the algebraic sum of

(a) An x-pressure force.

(b) A viscous force of scale $U_r \mu / L^2$

(c) A force proportional to the momentum inbalance due to mass transfer.

The vertical driving force is essentially the algebraic sum of

(a) A bouyancy force.

(b) A centrifugal force.

(c) A shear force of scale $\mu^2/\rho L^3$ or, for a bouyant system, $\frac{\mu k}{C_p L^3}$.

(d) A force proportional to the momentum imbalance due to mass transfer.

If it is desired to bring the fluid to rest, within a distance commensurate with the flow velocity, it will be necessary to apply a force, $\vec{FP}$, proportional to the vector sum of $\vec{XF}$, $\vec{YF}$ and $\vec{ZF}$

i.e. $\vec{FP} \propto \vec{XF} + \vec{YF} + \vec{ZF}$ (4.2)

The direction, Ω, and gradient, ΔFP, of this force are given by:

$$\Omega \propto \mathrm{Tan}^{-1}(YF/XF) \qquad (4.3)$$

$$\Delta FP \propto \nabla . \vec{FP} \qquad (4.4)$$

Thus, $|FP|$, Ω , and ΔFP describe the force field of the flow problem, and equation (4.1) implies that:

$$q(x + \Delta x) = \Phi[q(x), |FP| , \Omega , \Delta FP] \qquad (4.5)$$

Obviously, the direct use of these force quantities is not feasible. One would want to represent the force field with some characteristic dimensionless force quantity. Such dimensionless quantities will now be sought. Consider the two simple one-dimensional boundary layer problems sketched in Figures (3b, 3c). In Figure (3b), the force field is quantitatively determined by the inertial force and the resistive force (which in this case is essentially viscous force). Hence the descriptive force quantity is the function:

$$\Psi_1 \text{ [Inertial force, Viscous force].}$$

The most appropriate dimensionless quantity derivable from the above function is the Reynolds number, R_L , based on a characteristic vertical length, L, in the boundary layer and on a characteristic local velocity, (i.e., $R_L = U_r L/\nu$). If the influence of mass transfer across the solid boundary is considered, the appropriate dimensionless force quantity in the x-direction is given by the sum of the Reynolds number and a transpiration (or aspiration) parameter $A_{tx} = (\frac{U_r L}{\nu_1}) \frac{v_o}{U_r}$. Thus, for such a general one dimensional flow, the appropriate dimensionless force quantity is given by:

$$\text{Force Quantity} = R_L[1 + \frac{v_o}{U_r}] = RX .$$

In the second case, Figure (3c), consider first the case where there are no centrifugal forces. The force field is quantitatively described by the bouyancy and viscous forces. Thus, the descriptive force quantity is the function:

$$\Psi_2 \text{ [Bouyancy force, Viscous force] .}$$

The appropriate dimensionless quantity is the Rayleigh number, Ra (= $C_p g(T_o - T_1)L^3/\nu T_1 k$ ≡ Grashof number x Prandtl number). If there is no bouyancy force but rather centrifugal force, the descriptive force quantity is the function:

$$\Psi_3 \text{ [Centrifugal force, Viscous force].}$$

The appropriate dimensionless quantity is then the Taylor number, Ta (= 2 $C^*R_L^2$). The Görtler number, Ga, which is essentially the square root of the Taylor number is quite commonly used.

If, in Figure (3c), centrifugal and bouyancy forces are simultaneously present, the appropriate dimensionless quantity would be, simply, a function of the Rayleigh and Taylor numbers. If the bouyancy and centrifugal forces are locally in line, then the appropriate dimensionless force quantity is the algebraic sum, (Ra + Ta). In some cases, however, these two forces are not locally aligned, as the bouyancy force must act only in the natural or geographic vertical direction and the centrifugal force is perpendicular to the curved boundary. This latter situation occurs usually in flows with very severely curved boundaries, which are quite rare. In most such cases, however, the influence of one of the forces may usually be neglected in comparison with that of the other. If one includes the effect of the force due to mass transfer across the boundary, then one may state that in the case of no mean flow parallel to the wall, the appropriate dimensionless force quantity is given by:

$$\text{Force Quantity} = (Ra + Ta + A_t) = RY,$$

where A_t is a dimensionless transpiration (or aspiration) parameter for the vertical direction and is defined as

$$A_t = (v_o L/\nu_1)^2 \ . \tag{4.6}$$

In the more general two-dimensional boundary layer problem, the force field is more complicated. For the case of the present study, the viscous and roughness forces are the principal resistive forces. It will be assumed that these forces are very similar in their influence on the boundary layer so that their effective sum may simply be called a viscous force, without loss of meaning. The other forces acting are the x-inertial force, the bouyancy force, some centrifugal force and the force due to transpiration or aspiration. The descriptive dimensionless force quantity is the ratio of the net driving force to the net resistive force.

At first glance, one may be tempted to define this force quantity as follows:

$$\text{Force Quantity} = \left[\frac{XF^2}{F_x^2 + F_y^2} + \frac{YF^2}{F_x^2 + F_y^2}\right]^{1/2} \tag{4.7a}$$

which is the ratio of the vector sums of the appropriate forces, with F_x and F_y as the resistive forces in the x and y-directions respectively and XF and YF as the corresponding driving forces. Examination of the equation (4.7a) shows immediately that such a force quantity does not differentiate between positively and negatively directed YF. It requires a statement of the direction of the net force for completeness. If however one redefines a vertical force term, $\overline{YF}$, which is taken to act in the positive y-direction but which responds directly to the directional sign of YF, and a similar term $\overline{F}_y$ for F_y, then

$$\frac{\overline{YF}^2}{F_x^2 + \overline{F}_y^2} = \text{Function}\left(\frac{RY}{\left(1 + \frac{F_x}{\overline{F}_y^2}\right)^2}, \text{sgn}\,\Omega\right). \tag{4.7b}$$

where $\text{sgn}\Omega$ indicates the direction of the force YF and is positive when YF is directed in the positive y-direction. This new force ratio, $\overline{YF}^2/(F_x^2 + \overline{F}_y^2)$, is the more appropriate force quantity to use for the vertical direction because in the physical boundary layer, the instability modes are sensitive to the direction of YF, and it does not suffice to define this direction separately from YF. A function of two variables can usually be expressed in powers of the one variable with functional coefficients of the other variable. That is

$$\frac{\overline{Y}F^2}{F_x^2 + \overline{F}_y^2} = a_1 RY + a_2 RY^2 + a_3 RY^3 + \ldots \ldots$$

where the a's are functions of (sgnΩ)and $1/(1 + \frac{F_x^2}{\overline{F}_y^{\,2}})^i$. Hence, the appropriate dimensionless force quantity for the two dimensional boundary layer considered above is:

$$\text{Force Quantity} = [bRX^2 + \sum_{i=1}^{\infty} a_i YA^i]^{1/2} . \tag{4.7c}$$

where $b = (1 + \frac{\overline{F}_y^2}{F_x^2}) \doteq 1$, since $\overline{F}_y^2/F_x^2 = 0(1/R_L^2)$. If one terminates the above series after the first term, then a_1 must be a positive constant in order for the truncated series to express the proper physical and mathematical meaning of the complete series. It is not obvious that such a truncation is valid in this particular case. One may, however, argue that, since $\frac{F_x^2}{\overline{F}_y^2} = 0(R_L^2)$, $a_i \to 0$ as i becomes large. This reasoning is strongly supported by the results of D. Joseph (1966) in which he deduced the form of the appropriate force quantity for a plane Couette flow heated from below, by a variational method on the appropriate energy equations.

Since the total characteristics of the general two dimensional boundary layer are determined exclusively by the above force quantity, (4.7c), one may define that force quantity as the boundary layer stability number, SN. That is,

$$SN = [R_L^2 \ (1 + \frac{v_o}{U_r})^2 + a_1(Ra + Ta + A_t)]^{1/2} \tag{4.7d}$$

where,

U_r = Some characteristic velocity, conventionally U_1 .

C^* = Δ/r_o , the curvature parameter

$$R_L = \int_o^x \int_o^\infty \left(\frac{\text{Local Inertial Force}}{\text{Local Viscous Force}} \right) dy\, dx$$

$$\equiv U_r L/\nu_r \ . \qquad (4.8)$$

For Equation (4.8) to define the proper Reynolds number for a section perpendicular to the flow direction, the characteristic vertical length, L , and the characteristic velocity, U_r , must both be flow-history oriented. Conventionally, the free stream velocity, U_1 , is used for U_r , and the momentum or displacement thickness used for L. In this work, it is necessary to keep as close to conventional practice as possible, for the sake of comparison. The boundary layer displacement thickness will therefore be used as the fundamental characteristic vertical length scale. For any chosen characteristic length, however, the corresponding characteristic velocity by inspection from equation (4.8) should be given by:

$$U_r\ L \equiv \Psi\left(\int_o^{\bar{y}} u\ dy \right) \qquad (4.9)$$

where $\bar{y} = y|_{u\ =\ U_1}$. This implies that if the free stream velocity, U_1 , is used as the characteristic velocity, an appropriate characteristic length should be related to $(\delta - \delta^*)$.

$$\text{i.e. } L \sim (\delta - \delta^*). \qquad (4.10)$$

This characteristic length has been noticed before now as an appropriate length in boundary layer research, for example by Head (1958) in his entrainment theory.

If the conventional Reynolds number, $U_1\delta^*/\nu_1$, is used, boundary layer characteristics are observed to depend not only on the Reynolds number but also on the initial velocity profile. By the dimensional argument presented above, boundary layer characteristics should depend only on the appropriate force quantity. Hence, the desire to redefine a Reynolds number which implicitly contains the influence of the velocity profile. The length scale, $(\delta-\delta^*)$, contains this desirable trait, as it may be written, for instance as,

$$(\delta-\delta^*) \equiv (\delta^*(\frac{\delta}{\theta} \cdot \frac{1}{H} - 1)) \qquad (4.11a)$$

If one examines the data for critical Reynolds numbers based on the displacement thickness, δ^*, as shown in Figure (5), it is seen that a best fit curve may easily be obtained if the shape factor parameter used is $(\frac{10}{H} - 4)$. With this parameter one obtains from Figure (5) that:

$$\log_{10}(\frac{U_1\delta^*}{\nu_1})_{ci} \doteq 3 + 2.34 \tanh (\chi) \qquad (4.11b)$$

i.e. $\frac{U_1\delta^*}{\nu_1} 10^{-2.34 \tanh (\chi)} \doteq 10^3$, at the point of instability

where $\chi = (\frac{10}{H} - 4)$ and $H = \frac{\delta^*}{\theta}$, the velocity profile shape factor. Thus, if a new length scale, $\Delta \equiv \delta^*[10^{-2.34 \tanh (\chi)}]$, is defined, a universal constant value, 10^3, is obtained for the critical Reynolds number, $U_1\Delta/\nu_1$, at the point of incipient laminar instability. This value, 10^3 , should then define the critical value of SN in equation (4.7d). That is, for the more general boundary layer, $SN_{ci} = 1000$, and this value becomes the universal value of the critical stability number for laminar instability, for the new length scale, Δ , The conclusion one draws from the above discussion is that the Reynolds number, U_rL/ν_r , is meaningful for any case under consideration, if and only if the velocity, U_r , length, L , and viscosity, ν_r are correspondingly chosen to describe the appropriate ratio:

$$R_L = (\text{Inertial Force/Viscous Force}).$$

Explicitly, the force field concept emphasizes the fluid "tenacity" or "cohesiveness" or the ability of the fluid to resist perturbations. That is, the degree of the bonding among the fluid particles. A high "cohesiveness" is taken to correspond to a large force field and implies that the fluid particles are strongly bonded so that external perturbations or fluctuational motions of the particles can be completely resisted or damped. A zero force field would then be analogous to a critical "cohesiveness" for which the fluid could resonate to certain types of disturbances present in it. Since a negative "cohesiveness" is physically meaningless, a fluid will readjust to increase its "cohesiveness" if this property tends to disappear. This is precisely what happens in the transition from laminar to turbulent flow. The fluid undergoes a phase change. Thus, the laminar, transitional, turbulent and relaminarized flows represent different phases of the same fluid, at least with respect to the fluid cohesive property. This situation is observed on a more severe scale in the phase change from liquid to gas or vice versa.

The force field of a fluid is a function of only the local forces on the fluid,

i.e. Force Field = Function [Local Resistance, Local Drive].

In dimensionless form:

$$\text{Force Field} = \text{Function}\left[\frac{\text{Section Driving Force}}{\text{Section Resisting Force}}, \eta^*\right]$$

$$= \text{Function (Stability Number, } \eta^*)$$

That is, the force field is completely defined by a function of the section stability number, SN, and the local position. So equation (4.5) can further be written as:

$$q(x + \Delta x) = \Phi[q(x), \eta^*, SN] \tag{4.12}$$

where $\eta^* = y/\Delta$. This author represents the quantitative relation between the force field and the stability number as sketched in Figure (4). It will certainly require the analysis of many experiments and further experimentation to conclusively define the force-field theory. Nevertheless, the hypothetical behaviour shown in Figure (4), portrays some trends that are quite familiar in boundary layer research, for instance, the presence of a weak dependence on Reynolds number of fully turbulent boundary layer characteristics, and the typical non-linear hysteresis loop of relaminarization. The separation characteristics of wall boundary layers cannot be deduced easily from Figure (4). One must necessarily use the phase angle characteristic of the force field, the form of which is not clear at this stage of the theory.

4.2 An Energy Method

An attempt will now be made to deduce the stability (or instability) and transitional characteristics for laminar flows in the conventional sense, by considering the energy of the disturbances. The principal characteristics of interest are:

(i) the point of incipient laminar instability

(ii) the most favored disturbance frequency

(iii) the amplification characteristics (amplification factor and rate)

(iv) the point of incipient transition.

These will now be considered in that order.

(i) Consider a two-dimensional laminar flow in which, somehow, a perturbation has entered such that the velocity components can be represented by a mean portion and a fluctuating portion, whose mean is zero. That is:

$$\left.\begin{aligned} u &= \bar{u} + u' \\ v &= \bar{v} + v' \\ w &= 0 + w' \;. \end{aligned}\right\} \tag{4.13}$$

Consider, further, a control volume whose dimensions are proportional to the dimensions of the local perturbation length scale, λ . If the local amplitude of the perturbation in the control volume are, in the relevant Cartesian coordinate system, u', v', and w' which may be finite or infinitessimal, the stability problem now reduces to that of finding the conditions under which these amplitudes will be sustained and amplified or damped out.

The characteristic local fluctuational velocity scale within the above control volume is related to the local r.m.s. velocity,

$$J \equiv \sqrt{\frac{1}{3}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)}.$$

Also, the characteristic period which specifies the order of magnitude of the time required for the occurrence of the fluctuation, is related to λ/J.

The energy of the fluctuation in the control volume mentioned above is derived from the main flow, through some mechanism, and manifests itself as primarily kinetic energy. When this fluctuation occurs, the energy of the fluctuation, per unit mass, is of the order of magnitude of J^2 . Thus, one may write that when the fluctuation occurs, the amount of energy which goes over from the main flow to the fluctuation in the control volume, per unit time and mass, is equal in order of magnitude to:

$$J^2 \, \frac{J}{\lambda} \; .$$

Ideally, one would like to deduce the above result from some appropriate turbulence energy production terms in the equations of motion. For a very simple boundary layer flow with no influence of heat or mass transfer, such a turbulence energy production term is $\overline{u_i'u_j'} \, \frac{\partial \bar{u}_i}{\partial x_j}$ per unit time and mass. One may then apply the mixing length concept of Prandtl to show that $\frac{\partial \bar{u}_i}{\partial x_j}$ is given in order of magnitude by the ratio J/λ . Thus, that the turbulence energy production per unit mass and time, is given in order of magnitude by J^3/λ , as obtained before. However, for more general flows with heat transfer, for instance, the turbulence energy production is represented, in addition to $u_i'u_j' \, \frac{\partial \bar{u}_i}{\partial x_j}$, by terms due to the heat transfer. It becomes difficult, therefore, to show that, generally, the turbulence energy production is of order of magnitude J^3/λ , without making unacceptable assumptions about the order of magnitude of the influence of the heat transfer. Obtaining the order of magnitude of the production term from the magnitude of the initial fluctuation in the manner demonstrated above, gives a global estimate and circumvents any assumptions concerning the influence of the flow conditions.

The fluctuational energy dissipation in an incompressible flow with velocity fluctuations is given generally, per unit mass and time, in the Cartesian tensor notation by:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i}\right) \frac{\partial u_i'}{\partial x_j}} \quad .$$

If it is assumed that the local velocity gradients of the fluctuations, ($\frac{\partial u_i'}{\partial x_j}$, etc), are given by the ratio, J/λ , of the local characteristic values, J and λ of velocity and length respectively, then the energy loss by fluctuations in the control volume per unit time per unit mass is equal in order of magnitude to:

$$\nu \frac{J^2}{\lambda^2} \; .$$

If all other perturbations on the energy of the main flow in the control volume are small compared to those mentioned above, then the fluctuation in the fluid will be sustained only if the net energy is positive. That is, if the production quantity is greater in magnitude than the dissipation quantity. Growth of the fluctuation implies, in the conventional sense, instability of the flow. Hence the flow in the control volume is unstable if:

$$\frac{J^3}{\lambda} > \nu \frac{J^2}{\lambda^2} . \tag{4.14}$$

The instability criterion, (4.14), will now be reduced to a more conventional form. Let vertical distances be non-dimensionalized with the characteristic length, Δ, and all velocities be non-dimensionalized with the free stream velocity, U_1 . Then expression (4.14) is reduced to the following dimensionless form:

$$\frac{U_1 \Delta}{\nu} > \frac{1}{I \ \lambda/\Delta} , \quad \text{for flow instability} \tag{4.15}$$

where I is the local disturbance intensity defined as

$$I = \sqrt{\frac{1}{3} (\overline{u'^2} + \overline{v'^2} + \overline{w'^2})}/U_1 \equiv J/U_1 .$$

The inequality (4.15) indicates that for the fluctuation in a laminar flow to be sustained, that is, for laminar flow instability, the appropriate dimensionless force quantity for the flow must be greater than some function of the local disturbance intensity, I , and the local dimensionless disturbance length scale, (λ/Δ). If it may be assumed that the characteristic disturbance length scale, λ , is approximately proportional to the distance, y , from the solid boundary in a wall boundary layer, then the inequality (4.15) suggests that in general,

$$SN > \Phi(I,\eta^*), \quad \text{for laminar flow instability} \tag{4.16}$$

where $\eta^* = y/\Delta$.

The local disturbance intensity, I , depends on the initial free stream disturbance intensity and the relative local force field. As the force field is completely defined by the stability number, SN , and the local position, η^*, it is clear that

$$SN > \Phi_3(\eta^*, SN, I_1) \ , \quad \text{for laminar flow instability} \qquad (4.17)$$

The problem now is to define what exactly is Φ_3 . The present author believes that Φ_3 is simply the integrated effect of a single general distribution profile of the total disturbance kinetic energy, across the boundary layer. The determination of such a function is quite a task. Although an attempt will be made to deduce it later in this work, an equation may still be given for the critical Reynolds number for laminar flow instability without an explicit derivation of Φ_3 .

It was obtained earlier that the critical stability number SN_{ci} , for laminar flow instability according to the definition of the length scale Δ , is 10^3 . This derivation was made for the cases where I_1 was of such a magnitude as not to appreciably affect the value of SN_{ci} . If the expression (4.17) may be written as:

$$SN > \Phi_o(\eta^*, SN) \cdot \phi(I_1) \qquad (4.18)$$

then

$$SN_{ci} \equiv \text{Mini} \ \{\Phi_o(\eta^*, SN) \cdot \phi(I_1)\} \doteq 10^3 \ \phi(I_1) \ . \qquad (4.19)$$

In a more conventional manner, it may be stated that for laminar flow instability the critical Reynolds number, $U_1\delta^*/\nu_1$, is given by:

$$Re_{\delta*} \doteq \frac{\phi(I_1)\{1-a_2(Ra + Ta + A_t)\}^{1/2}}{(1 + v_o/U_1)} \cdot 10^{\{3+2.34 \tanh(\frac{10}{H} - 4)\}} \tag{4.20}$$

where a_2 is a numerical constant. The function $\phi(I_1)$ has been empirically estimated by examination of the data of Schubauer and Skramstad (1948) to be of the form

$$\phi(I_1) \doteq 1/(1 + 280\ I_1^2) \tag{4.21}$$

The constant a_2 may be fixed from the results of the critical Reynolds number for suction profiles, for example as calculated by C. C. Lin (1946). For zero suction on a flat plate, C. C. Lin obtained that $(U_1\delta^*/\nu_1)_{ci} = 465$, which is slightly higher than the accepted value of 420. Otherwise, Lin's results agree reasonably with experimental results. With this slight discrepancy in mind, the constant a_2 is obtained as follows: From equation (4.20), for a flat plate flow with suction and no heat transfer at the boundary:

$$Re_{\delta*_{ci}} \doteq (1 - a_2A_t)^{1/2}\ 10^{\{3+2.34 \tanh (\frac{10}{H} - 4)\}} \tag{4.22}$$

if the effect of free stream turbulence is neglected. Applying the result of C. C. Lin (1946) for the case when $\frac{v_o}{U_1}(\frac{U_1x}{\nu_1}) = 0.5$, corresponding to $H \approx 2.25$ and $(v_o\delta^*/\nu_1) \approx 0.6$, one obtains that approximately, $Re_{\delta*_{ci}} = 0.667 \times 10^4$. This yields that $a_2 \doteq 0.012$.

With this value of a_2, one obtains that for small and zero Reynolds number parallel flows with heating at the wall, the critical Rayleigh number based on the depth of the unstable layer is in the neighborhood of 140. No experimental data appear to be available for this type of flow, with which to compare the present result. Some numerical results, nevertheless, have been obtained by other workers, for the related case of penetrative cellular perturbations in a horizontal layer of fluid composed of a lower layer of unstable density gradient above which is a layer of stable density gradient. For the classical rigid-free boundary solution corresponding to the limiting case of infinite stability on top of the unstable layer, Chandrasekhar (1961) gives a critical Rayleigh number based on the depth of the layer of about 1100. In reality, of course, the convective cells do penetrate the unstable layer. Rintel, (1967) and Stix, (1970) have shown that for small stability on top of the unstable layer, the critical Rayleigh number is much smaller than that obtained by not taking penetration into account. For various degrees of such penetrative cellular perturbations close to the limiting case of infinite penetration, Stix (1970) and Rintel (1967) obtained critical Rayleigh numbers based on the depth of the layer of 225 and 172, respectively. These results indicate that the present estimate of 140 is very good.

Further, for the asymptotic suction profile $(\frac{v_o \delta^*}{\nu_1} = 1; H = 2)$, one obtains by the present method, a critical Reynolds number, $(U_1 \delta^*/\nu_1) = 63 \times 10^3$. Estimates for this number, C. C. Lin (1946), are accepted as close to 70×10^3 which is quite close to the present estimate.

For flow over a concavely curved boundary, the present method predicts that if the curvature is appreciable, the influence of the Taylor number dominates over the Reynolds number. For this situation, the Görtler number, $Re_\theta \sqrt{\frac{\theta}{r_o}}$, is given by the constant value, 0.45, for all Re_θ . This estimate compares very well with the values 0.34 and 0.58 given respectively by A. M. O. Smith (1953) and Görtler (1940).

The final form of equation (4.20) is presented as:

$$Re_{\delta^*_{ci}} \doteq \frac{\{1 - 0.012(Ra + Ta + A_t)\}^{1/2}}{(1 + 280\ I_1^2)(1 + v_o/U_1)}\ 10^{\{3+2.34\ \tanh(\frac{10}{H} - 4)\}} \qquad (4.23)$$

As a check, it is noted that for the Blasius profile (H = 2.6), equation (4.23) gives that $Re_{\delta^*_{ci}} \doteq 438$, which is quite close to the generally accepted value of 420.

(ii) The time scale of a fluctuation wave in the elemental volume so far considered is:

$$\tau \equiv \frac{\lambda}{\hat{u}} \qquad (4.24)$$

where $\hat{u}$ is the local characteristic group velocity of the wave.

The period, T_p , of a characteristic oscillation within this control volume should be expected to be related to this time scale. Obviously, the most favored oscillatory motion within the control volume is the one with a period closest to the characteristic local time scale. That is, the frequency of the most favored local perturbation may be written as:

$$f_p \propto \frac{U_1}{\lambda} \text{ , if } \hat{u} \text{ is taken to be proportional to } U_1 \text{ .} \qquad (4.25)$$

Although no conclusive statement can be made at this point concerning the functional form of f_p , the following form is suggested from the observed data of Schubauer and Skramstad (1948), and the assumption of a linear relation,

$$f_p \doteq 0.016 \frac{U_1^2}{\nu\eta^*} \cdot \frac{1}{Re_\Delta} \tag{4.26}$$

(iii) The amplification characteristics of the boundary layer for a fixed frequency of disturbance would be a function of only the relative local force field. Considering the case of the most favored frequency, which corresponds to maximum amplification characteristics, or more generally, for a given η^* layer, the local amplification rate should be a function of a quantity which indicates by how much the local force field has fallen below the critical value,

i.e. Amplification rate = $\Phi_1(1 - SN_{ci}/SN) \equiv \alpha_i^*\delta^*$ (for exponential type disturbances).

The above result, however, is not limited to any particular mode of disturbance. Also, the local amplification factor defined as

$$a^* = \frac{A_x}{A_{ci}} \equiv \left(\frac{\text{Amplitude at x-station}}{\text{Amplitude at point of neutral stability}}\right)$$

should be a function of the local force field relative to the critical minimum force field for stability,

i.e., $a^* = \Phi_2(\alpha_i^*\delta^*\,\Delta\overline{x})$

where $\Delta\overline{x} = (x - x_{ci})/L(x)$ and the subscript (ci) denotes quantity at the point of neutral stability.

If a linear theory is assumed, that is, if the functions Φ_1 and Φ_2 are assumed to be linear functions, one obtains then that:

$$\alpha_i^* \delta^* = d[1 - SN_{ci}/SN]$$
$$a^* = [1 + b(\alpha_i^* \delta^* \, \Delta \overline{x})] \tag{4.27}$$

where b, d are numerical constants and the condition that $a^* = 1$ when $\Delta \overline{x} = 0$, has been applied. For small values of the argument, the relation for the amplification factor could be written as:

$$a^* = \exp[b_1 \Delta \overline{x}(1 - SN_{ci}/SN)] \tag{4.28}$$

where b_1 is a numerical constant.

By assuming that a single harmonic, the fundamental harmonic ($\alpha = 1$), regarded as the dominant term in the disturbance, and an equilibrium perturbation function, $u_1(y)$, may be used throughout the growth period of disturbances in a boundary layer, Stuart (1958) obtained by an energy method that:

(i) Amplification rate, in linear theory, is

$$\alpha \frac{\gamma_2}{\gamma_1} [1 - \frac{Re_{ci}}{Re}] = \alpha_i^* \delta^* \ . \tag{4.29}$$

(ii) Amplification factor according to linear theory is

$$a^* = \exp[\frac{\alpha}{2} \frac{\gamma_2}{\gamma_1} (1 - Re_{ci}/Re) \frac{x}{\delta^*}] \tag{4.30}$$

(iii) Equilibrium amplitude is

$$K_e \doteq \frac{\gamma_2}{\alpha\gamma_3} [(Re - Re_{ci})/Re^2] \ . \tag{4.31}$$

where the critical Reynolds number for instability is given by:

$$Re_{ci} \ (= \frac{U_1\delta^*}{\nu})_{ci} \doteq \frac{\gamma_4}{\alpha\gamma_2}$$

and

$\alpha \approx 1$

$\gamma_1 \approx 2.05146$

$\gamma_2 \approx 0.040192$

$\gamma_3 \approx 0.002308$

$\gamma_4 \approx 247.62$

for Poiseuille flow between parallel planes.

These results agree with the deductions made from the force-field concept if the conventional Reynolds number is replaced by the Stability number.

Applying the same assumptions but without the restriction of linearity, Stuart (1958) showed that for a disturbed basic flow, the square of the amplitude, a^* , of the disturbance could be given by:

$$a^{*2} = \frac{C_1 \exp[(\beta_1 - \beta_2)\,\overline{t}]}{1 + (\frac{\beta_3}{\beta_1-\beta_2})\, C_1 \exp[(\beta_1-\beta_2)\overline{t}]} \tag{4.32}$$

where $\beta_1 = \alpha\gamma_2/\gamma_1$

$\beta_2 = \gamma_4/\gamma_1 Re$

$\beta_3 = \alpha^2 \, Re \, \gamma_3/\gamma_1$

and C_1 is a constant which is probably related to the initial disturbance amplitude. Equation (4.32) gives a qualitatively plausible description of the linear and non-linear effects, depending on the value of the time, t. This equation may be re-written for spatial rather than time amplitudes if the approximation is made that:

$$\bar{t} \doteq (x/\hat{u}) \cdot \frac{U_1}{\delta^*}$$

(dimensionless time)

where $\hat{u}$ is the disturbance group velocity taken to be approximately directly proportional to U_1 . One then obtains the following:

$$a^{*2} \doteq \frac{C_1 \exp (\xi)}{1 + C_1 m \cdot \exp(\xi)} \tag{4.33}$$

where $\xi = \alpha \frac{\gamma_2}{\gamma_1} [1 - Re_{ci}/Re] \frac{x}{\delta^* \Delta}$

$$m = (\alpha Re \frac{\gamma_3}{\gamma_2})/(1 - Re_{ci}/Re).$$

Equation (4.33) implies that for a given location, the amplitude of a disturbance is a function of only the location, the deviation of its dimensionless force quantity (in this case the Reynolds number) from its critical value, and the initial amplitude of the disturbance. This implication is basically the same as that deduced from the force-field theory. Moreover, the force-field theory implies that the disturbance field may be represented by a generalized function, $\Phi_3(I_1, y/\Delta, SN)$ which was a basic assumption of Stuart's. Stuart (1958) in fact, represented his disturbance field as, $K\ Re\ u'(y,t)$, which is fundamentally related to the function, Φ_3 . Relative to the force field theory, the limitations in Stuart's analyses stem only from the incompleteness of his dimensionless force quantity, Re , and his disturbance field. For boundary layers with mean velocity profiles very close to the Blasius profile, these errors will be very small. Hence the slight deviations of Stuart's predictions from experimental measurements. Stuart's analyses proved excellent when he extended them to flows with curved boundaries, using the appropriate dimensionless force quantities.

It should now be possible to obtain the general disturbance characteristics for more general boundary layer flows by a method very similar to that used by Stuart (1958) for plane Poiseuille flow. The method used here involves partial derivation of the generalized disturbance profile discussed earlier in this section.

An equation for the total kinetic energy of three dimensional disturbance in a two-dimensional basic flow may be obtained by summing equations (3.24), (3.25) and (3.26) of section (3.3), where $K = k_1 + k_2 + k_3$. One obtains:

$$a_{17}[K''_i + K''_{i-1}] + a_{18}[K'_i + K'_{i-1}] + (a_{29} - a_7)K_i + (a_{29} + a_7)K_{i-1}$$
$$= (-a_{20} - a_{27} + a_{30})[f_i - f_{i-1}] + (a_{28}-a_{21})(f'_i-f'_{i-1}) \tag{4.34}$$

where the quantities are as defined in Chapter 3. The solution of equation (4.34) requires an input profile of $K(\eta^*, SN)$ at a given upstream position (i-1) for a profile at a downstream postion (i). If the upstream position (i-1) is taken constantly as the point of neutral stability in the laminar boundary layer, one may then calculate the disturbance characteristics downstream of the point of neutral stability in parameters that are relative to the characteristics at the point of neutral stability. It is true that as such a calculation progresses downstream, that is, the x-spacing is increasing, the computational error gets larger. However, the Crank-Nicholson scheme used in obtaining equation (4.34) is always stable so that resulting profiles will be similar, at least in shape, to the exact solutions. In fact, if one leaves constants arbitrarily until the final form of the required solution is obtained, these constants may be evaluated empirically to eliminate most of the quantitative numerical error. Another point of dispute in the method being introduced now, is that a general solution thus obtained should be valid only downstream of the point of neutral stability. This however should not be the case. By making Δx negative in a Crank-Nicholson scheme one may, under certain circumstances, retrace the upstream flows from a given (i-1) position. Such a retrace is unstable if numerical computation methods are used to evaluate the final result. If, however, a pure analytical result is obtained for the $K(\eta^*,SN)$ distribution, say, such a result is certainly not limited to only downstream of the instability point. Its only restrictions may be those imposed by the basic boundary layer equations from which it is derived.

Equation (4.34) may now be written as:

$$a_{17}K_{ci}\ \Omega'' + \Omega'[2a_{17}K'_{ci} + a_{18}K_{ci}] + \Omega[a_{17}K''_{ci} + a_{18}K'_{ci} + (a_{29}-a_7)K_{ci}]$$

$$+ a_{17}K''_{ci} + a_{18}K'_{ci} + (a_{29} + a_7)K_{ci} + (a_{20} + a_{27} - a_{30})(f_i - f_{ci})$$

$$- (a_{28} - a_{21})(f'_i - f'_{ci}) = 0\ . \qquad (4.35)$$

where $\Omega = K/K_{ci}$ (and is always >1).

With this apology, the following assumptions will now be made to facilitate the solution of equation (4.34).

(i) Upstream of the point of neutral stability all disturbance in the basic flow are damped with a damping factor which decreases with increasing distance from the leading edge. The total energy of the disturbance at the point of neutral stability is the minimum value with a distribution across the boundary layer given approximately by the following curve:

$$K_{ci}(\eta^*) = A\ \ SN_{ci}^{1/2}\ \ \eta^{*2}\ \exp(-\ \eta^{*2}) \qquad (4.36)$$

The form of the equation (4.36) is obtained empirically from the necessary boundary conditions and the requirement that $K_{ci}(\eta^*)$ has a maximum in the neighborhood of $\eta^* \simeq 1.0$, (or $y/\delta \simeq 0.3$). A is some function of I_1, the free stream turbulence.

(ii) The mean velocity profile is given approximately by a Blasius profile distorted by the disturbance. In the region of the critical layer of the wall boundary layer, the simplified distorted profile will be represented as follows:

$$\frac{u}{U_1} \doteq b_1 \eta^* + b_2 \, SN \, . \, K \tag{4.37}$$

Stuart (1958) obtained in a similar manner, that for steady plane Poiseuille flow

$$\frac{u}{U_{max}} \doteq 1 - (y/d)^2 + Re \int_0^{y/d} \overline{u'v'} \, d(y/d) \tag{4.38}$$

where d is the distance between the plates, y is measured from the centerline, and Re is a characteristic Reynolds number.

If substitutions are made for the a's and for K_{ci} in equation (4.35) from equations (4.36) and (4.37), one obtains after some algebraic manipulation and simplification that for the region of the boundary layer in the neighborhood of the critical layer equation (4.38) reduces to the following:

$$\Omega'' + \frac{1}{(1+\Omega)} \Omega'^2 - d_1 \, SN(1-\omega)\Omega' - \frac{d_2 A^{1/2} SN^{3/2}}{(1+\Omega)^{1/2}} \Omega^2 - \frac{d_3 A^{1/2} SN^{3/4}}{(1+\Omega)^{1/2}} \omega \cdot \Omega$$

$$- (d_4 A^{1/2} + d_s \, \omega^{1/2}) \cdot \frac{1}{(1+\Omega)} = 0 \, , \tag{4.39}$$

where $\omega = SN_{ci}/SN$.

Equation (4.39) has the following general form

$$Y'' + F_1(Y)Y'^2 + F_2(Y)Y' + F_3(Y) = 0 \, , \tag{4.40}$$

which is the classical polynomial class non-linear differential equation of the second order. Notable examples of this type of equation include the Volterra equation for the problem of the prey and predator, Rayleigh's (1883) equation for sound motion and the Van der Pol equation. These analogies are interesting because the equations mentioned above were derived for systems in which "energy" is taken from a basic supply, a situation which is quite typical in the wall boundary layer. Hence, although equation (4.39) was obtained after a series of assumptions and approximations, its solution should contain most of the qualitative disturbance characteristics observable in the physical wall boundary layer. Rayleigh (1883) noted that equations of the type (4.39) will usually attain some steady state or equilibrium solution after an initial non-linear instability. This behavior is quite in accord with the physical situation in a wall boundary layer.

If one makes the following transformations

$$Y' = P(Y) = 1/U(Y) \tag{4.41}$$

equation (4.40) reduces further to the following form:

$$U' = F_3U^2 + F_2U + F_1 \tag{4.42}$$

Equation (4.42) is the classical Abel equation which has been solved explicitly only in a few special cases. The derivation and approximate solution of equation (4.39) have been presented in some more detail in Appendix C. The following solution is stated:

$$\Omega = \frac{MB_2\omega^{1/2} \exp (\xi_2)}{1 + M(1-\omega) \exp (\xi_2)} \tag{4.43}$$

where

$$M = B_1A$$

$\xi_2 = B.SN(1-\omega)$

B_2 = Function of $SN(\approx B_4/(A.SN_{ci}^{1/2}))$

B_1, B, B_4 are numerical constants.

The above analysis was performed for a laminar flow with some initial disturbance (finite or infinitesimal) in the subcritical region, even up to the point of neutral stability. If a situation arises where the initial disturbance is completely damped prior to the point of neutral stability and no further disturbances arise until downstream of the critical point, that is in the supercritical region, it is obvious that the result (4.43) must be modified. This latter case is identical to the one studied by Stuart (1958) and for which the following differential equation was given:

$$\frac{dy}{d\tau} = \beta_1 y - \beta_2 y - \beta_3 y^2 \; . \tag{4.44}$$

Stuart quoted that Landau (1944) gave a general solution for (4.44) but without stating what approximations were used or how the final solution was obtained. Landau's solution is

$$y = \frac{C_1 \exp\left[(\beta_1 - \beta_2)\tau\right]}{1 + C_1 \left(\frac{\beta_3}{\beta_1 - \beta_2}\right) \exp\left[(\beta_1 - \beta_2)\tau\right]} \tag{4.45}$$

where y represents the square of the amplitude of the velocity fluctuation in a disturbed plane Porseuille flow. The similarity between Stuart's (1958) solution for supercritical disturbances (4.44 and 4.45) and the present solution for subcritical disturbances is quite interesting and suggests that more generally, the present solution (4.43) may be stated as:

$$K_{max}(x) \propto \frac{B_1 B_4 \overset{*}{\omega} A \exp(\xi_2)}{1 + M(1-\omega)\exp(\xi_2)} \,. \tag{4.46a}$$

Then the amplification factor becomes:

$$\left(\frac{K(x)}{K_o}\right)_{max} = a*^2 \propto \frac{[MB_2\omega^{1/2}\exp(\xi_2)]}{[1 + M(1-\omega)\exp(\xi_2)]} \tag{4.46b}$$

where K_o is the value of K at the initial point of interest. In subcritical cases, $K_o \equiv K_{ci}$, but in supercritical cases, K_o corresponds to the value of the input supercritical disturbance kinetic energy.

In the region very close to the point of neutral stability (i.e., $\omega \to 1$) the solution (4.46b) reduces to:

$$a*^2_{max} \propto MB_2 \quad \omega^{1/2} \exp(\xi_2) \tag{4.47}$$

$$\doteq \exp(\xi_2) \text{ , for the constants as defined below.}$$

This is the linearized solution given in equation (4.28). Far away from the point of neutral stability equation (4.46a) reduces to the following steady or equilibrium state

$$(K_{max})_e \propto B_4 \quad \overset{*}{\omega}/(1-\omega) \tag{4.48}$$

which is independent of the initial free stream disturbance.

By a bounding value argument in which it is demanded that in a flat plate case, the linearized form of equation (4.46b) give a value in the neighborhood of e^{12}, the constants in equation (4.43) have been estimated as follows:

$B_1 \doteq 0.33$

$B \doteq 0.003$

$B_4 \doteq 90$.

(iv) The next critical point of interest is the transition point corresponding to SN_{ct}. The force field hypothesis indicates that transition is incipient when the magnitude of K_{ct}, the total kinetic energy of the disturbance reaches a certain critical value. For both subcritical and supercritical disturbances it has been noted above that

$$K \propto \frac{B_1 B_4 \omega^{1/2} A \exp(\xi_2)}{1 + M(1-\omega) \exp(\xi_2)}$$

$$\text{i.e.,} \quad \underset{\text{(Constant)}}{K_{ct}} \propto \left(\frac{B_1 B_4 \omega^{1/2} A \exp(\xi_2)}{1 + M(1-\omega) \exp(\xi_2)}\right)_{ct}, \text{ at incipient transition} \qquad (4.49)$$

The relation (4.49) may be reduced to the following form

$$\frac{K_{ct}\, SN_{ct}^{1/2}}{\overline{B}_1 I_1 SN_{ci}^{1/2}} = \left[1 - \frac{K_{ct}}{B_4 SN_{ci}} (SN_{ct} - SN_{ci})\right] \exp\left[B(SN_{ct} - SN_{ci})\right] \qquad (4.50)$$

where $\overline{B}_1 \doteq B_1 B_4$. Except for nearly zero initial free stream turbulence, $K_{ct}(SN_{ct} - SN_{ci})/(B_4 SN_{ci})$ is << 1. Hence, from (4.50),

$$(SN_{ct} - SN_{ci}) \propto - \left[\frac{1}{B} \ln \left(\overline{B}_1 I_1 \frac{SN_{ci}^{1/2}}{K_{ct}}\right) - \frac{1}{B} \ln SN_{ct}^{1/2}\right] \qquad (4.51a)$$

Since SN_{ct} is large, $> 10^3$, $\ell n\ SN_{ct}$ is approximately constant for all SN_{ct}. Substituting for the numerical constants in (4.51a) with $K_{ct} \approx 0.3$, one obtains that:

$$(SN_{ct} - SN_{ci}) \doteq -1.3 \times 10^3 \{\ell n\ (33\ I_1)\} . \qquad (4.51b)$$

Equation (4.51b) thus gives a simple relation between the stability number for incipient transition and that for incipient laminar flow instability. This relationship is compared against flat plate experimental results in Figure (9) and the agreement is very good. Obviously the logarithmic form for the function $(SN_{ct} - SN_{ci})$ is valid only for values of $I_1 \leq 0.03$, since a negative $(SN_{ct} - SN_{ci})$ is physically meaningless. For $I_1 > 0.03$, the points of instability and transition are virtually coincident, so that, physically, negative $(SN_{ct} - SN_{ci})$ may be equated to zero.

The above discrepancy was anticipated, for in the solution of the equations of Appendix C, complicated functions were always replaced by much simpler functions which behaved functionally similarly in the region of interest. Hence, the solutions (4.46 - 4.51) represent only the simplest functional approximations of the appropriate exact solutions, in the region in which they are physically meaningful.

4.3 A Flow Describing Function

A major implication of the force field hypothesis is that there exists an explicit non linear (or linear) transfer function for all flows. It was obtained early in section (4.1) that with reference to Figure (3a), a flow variable at a section B downstream of a section A is given by the following function:

$$q_B = \phi(\eta^*, SN_D, q_A) \quad . \tag{4.52}$$

where $SN_D = (SN_B - SN_A)/SN_B$. The function ϕ is the non-linear (or linear) transfer (or describing) function for the flow. In attempting to derive an analytical formulation for ϕ, one may resort to the indications of experimental results and intuition. Such an effort may, however, be fruitless since it would require an ingenius interpretation of a very large collection of experimental results to avoid the frustrating conclusion that fluid flow constitutes a very complicated non-linear system the mathematical formulation of which had better be avoided. Alternatively one may attempt to derive ϕ directly from the differential equations of fluid flow. This latter approach would practically involve the analytical solution of the boundary layer equations which would make the present discussion superfluous. In view of these difficulties, it is obvious that a meaningful and useable flow describing function can be derived only semi-empirically using plausible physical assumptions and some conclusions from the differential equations of fluid flow.

Let ϕ be assumed to be of the following form:

$$\phi = \sum_{i=0}^{\infty} G_i \, (\eta^*, SN_D) \, q_A^i \quad . \tag{4.53}$$

The transferable quantity, q, must be an invariant quantity (a scalar or second order tensor) for equation (4.53) to be both mathematically and physically meaningful. Since in the force field hypothesis, energy is a primary variable, q will always represent some total energy. In the present case, $q = u_i u_i$, the total kinetic energy. If one defines a dimensionless $\tilde{q}(=q/q_{max} \leq 1)$ then the infinite series of equations (4.53) must converge. That is,

$$\phi = G_o + G_1\tilde{q}_A + G_2\tilde{q}_A^2 + \ldots G_N\tilde{q}_A^N$$

where N is some finite integer. (4.54)

Obviously, G_o represents the energy sources and/or sinks between the positions A and B.

The boundary conditions are

$$\begin{aligned} &\text{At } \eta^* = 0 && \tilde{q} = 0 \\ &\eta^* \to \infty && \tilde{q} \to 1 \\ & && \frac{\partial^n q}{\partial \eta^{*n}} = 0 \qquad n = 1, 2, \ldots \infty \end{aligned} \tag{4.55}$$

where the tilde denote dimensionless mean quantities.

The governing equation for the convection of a transferable scalar quantity in a steady flow is written as follows: (Hinze, 1959 p. 296)

$$U_i \frac{\partial \Gamma}{\partial x_i} = \frac{\partial}{\partial x_i} (\varepsilon \frac{\partial \Gamma}{\partial x_i}) + F_\gamma \tag{4.56}$$

where Γ is the scalar quantity to be transfered,

ε is a transfer coefficient which may be a constant or some function of η^*

F_γ is the driving force or source term.

For two-dimensional incompressible steady flow, equation (4.56) after discretizing in the x-direction according to the Crank-Nicholson method discussed in Chapter III, becomes approximately:

$$\Gamma_i = \Gamma_{i-1} + \left(\frac{\partial \Gamma_i}{\partial \eta^*} + \frac{\partial \Gamma_{i-1}}{\partial \eta^*}\right)\left(\frac{\partial \varepsilon}{\partial \eta^*} - \tilde{v}\right)\frac{d}{\tilde{u}} + \left(\frac{\partial^2 \Gamma_i}{\partial \eta^{*2}} + \frac{\partial^2 \Gamma_{i-1}}{\partial \eta^{*2}}\right)\frac{\varepsilon d}{\tilde{u}} + \overline{F} \tag{4.57}$$

where $\tilde{v}$, $\tilde{u}$ are half way point dimensionless local mean velocities, d is the dimensionless x-increment and $\overline{F}$ is some dimensionless function of the sources and $\tilde{u}$, d, etc.

One may now quasi-linearize equation (4.57) by making the following assumptions:

(i) $\Gamma_i = A(x+d)\ \eta^{*N(x+d)}$

$\Gamma_{i-1} = A(x)\ \eta^{*N(x)}$

(ii) $N(x)$ is a slowly varying function of x such that $\frac{d^2}{dx^2}(N) \approx 0$.

(iii) d, the x-spacing is small enough for terms in d^2 and higher order to be negligible in comparison to other terms.

Equation (4.57) then reduces to the following:

$$\Gamma_i = \Gamma_{i-1}\left\{\left[\frac{2 - \varepsilon d.N/(\eta^{*2}\tilde{u})}{\left(1 - \frac{d.N(\varepsilon' - \tilde{v})}{\eta^*\tilde{u}} - \frac{N^2 d\varepsilon}{\eta^{*2}\tilde{u}}\right)}\right] - 1\right\} + F^*$$

where F^* comprises all the non-linear and source terms. The coefficient of Γ_{i-1} in equation (4.58) goes from $0 \to 1$ as the parameter η^*/SN_D goes from $0 \to \infty$. As a first approximation, this coefficient will be represented by the hyperbolic tangent of $a\eta^*/SN_D$.

$$\text{i.e.,} \quad \Gamma_i = \Gamma_{i-1} \text{ Tanh } (a\ \eta^*/SN_D) + F^* \tag{4.59}$$

where a , is a numerical constant

$$\text{i.e.,} \quad G_1(\eta^*,\ SN_D) = \text{ Tanh } (a\ \eta^*/SN_D). \tag{4.60}$$

Equation (4.60) implies correctly that the force field is inversely proportional to the boundary layer relative stability number for small arguments of tanh (.).

Equation (4.54) now becomes

$$\tilde{q}_B = \phi = G_o + \text{Tanh } (a\ \eta^*/SN_D)\ \tilde{q}_A + \sum_{i=2}^{\overline{N}} G_1 \tilde{q}_A^{\,i} \tag{4.61}$$

where $\overline{N}$ is a finite integer. Since in fully developed laminar and turbulent boundary layers similarity solutions, which signify a linear transfer, are usually possible, it should be obvious that the terms

$\sum_{i=2}^{\overline{N}} G_i \tilde{q}_A^{\,i}$ in equation (4.61) are substantially different from zero

only in developing boundary layers, such as the entrance region and transitional boundary layers, where the complete Navier-Stokes equations must replace the boundary layer equations as the deductive tool. Further, physical reasoning dictates that if q_A is negative, q_B must also be negative so long as the incremental distance Δx is taken reasonably small. This reasoning implies in the absence of a source/sink term, G_o, that ϕ is an odd function in q_A. Since, however, G_o is not always negligible, one must not jump to the above conclusion. Nevertheless, it seems that if q is normalized with its maximum value as herein, ϕ may be approximated by the following simple polynominal:

$$\phi_M = G_o + \text{Tanh}\ (a\ \eta^*/SN_D)\ \tilde{q}_A + G_2\tilde{q}_A^2 + G_3\tilde{q}_A^3\ . \tag{4.62}$$

Substituting in the unsimplified equation for the transport of scalar quantities (4.56) and repeating the procedure used to obtain G_1, one finds that the coefficients G_2 and G_3 satisfy the following equations:

$$\frac{\partial^2 G_2}{\partial \eta^2} + f_3(\eta^*;SN_D)\ \frac{\partial G_2}{\partial \eta^*} + f_4(\eta^*;\ SN_D)\ G_2 = 0 \tag{4.63}$$

$$\frac{\partial^2 G_3}{\partial \eta^{*2}} + f_1(SN_D,\ \eta^*)\ \frac{\partial G_3}{\partial \eta^*} + f_2(SN_D,\ \eta^*)\ G_3 = 0 \tag{4.64}$$

where f_1 and f_2 behave functionally as

$$f_1 \simeq a\ [\exp(-hb\eta^*/SN_D) - \exp\ (-\eta^* b/SN_D)]$$

$$f_2 \simeq d\ \exp\ (-\ \eta^* b/SN_D)$$

and f_3 and f_4 differ from f_1 and f_2 respectively only by constant factors

i.e. $f_3 = g_1 f_1$ $(g_1 < 1)$

$f_4 = -g_2 f_2$ $(g_2 = 0(1))$.

and b, d and h are numerical constants. If the following transformations are made

$$x = \exp\ (-\ b\eta^*/SN_D)$$

$$G_3 = Y(x)$$

equation (4.64) may be expressed as follows:

$$xY'' + (b_1 - mx)\ Y' - d_1 Y = 0 \tag{4.65}$$

where $b_1(>1)$ and d_1 are numerical constants.

Equation (4.65) is identical to Kummer's equation. The general solution of Kummer's equation is given in terms of confluent hypergeometric functions. For detailed consideration of this type of equations, the reader is referred to Abramowitz M. and Stegun, A. I. (1965). Kamke (1948) lists the following solution:

$$Y = x^{-1/2b} \exp\ (1/2\ Mx)\ y\left(\frac{-2d_1 + mb_1}{-2m}\ ;\ 1/2\ (b_1 - 1)\ ;\ -\ mx\right)$$

$$\text{where}\quad y(A;B;X) \quad 1 + \sum_{i=1}^{\infty} \frac{A(A+1)\ldots.(A+i-1)X^i}{B(B+1)\ldots.(B+i-1)i!} \tag{4.66}$$

which is Kummer's function $M(a,b,z)$ and in the present problem is absolutely convergent since $0 < |mx| < 1$,

$$y(A;B;X) \equiv \frac{\Gamma(B)}{\Gamma(A)} \exp(X)\, X^{A-B} [1 + O(|X|^{-1})]$$

for real $X > 0$, where $\Gamma(.)$ is the Gamma function and $O(.)$ is "order of (.)". Therefore G_3 is given as follows:

$$G_3(\eta^*, SN_D) \equiv \frac{\Gamma(1/2b_1 - 1/2)}{\Gamma(a_1 SN_D - \frac{b_1}{2})} \exp\{-\eta^*(a_2 - a_3/SN_D)\} \exp(-\xi_3)\, \beta^* \tag{4.67}$$

where $\xi_3 = a_4 \exp(-a_5\, \eta^*/SN_D)$

$$\beta^* = (-a_6 SN_D)(a_7 SN_D - a_8)$$

and the a's are numerical constants. For reasonably short steps (i.e., $SN_D \approx O(0.1)$, the constants in equation (4.67) have been roughly estimated by inspection to yield the following form of the G_3 function:

$$G_3 \approx 8 \exp(20\, SN_D)[\exp(-4.5\, \eta^*/SN_D) - 0.26 \exp(-5\, \eta^*/SN_D)] \, . \tag{4.68}$$

The solution of equation (4.63) is identical to that for equation (4.64) except for the values of the constants. Thus, one obtains by arguments similar to those used above that:

$$G_2 \approx 2 \exp (5\ SN_D)[\exp(-4.5\ \eta^*/SN_D) - 1.3 \exp(-4\ \eta^*/SN_D)].$$

(4.69)

For the zero pressure gradient case with no heat or mass transfer at the wall, the source/sink term G_o is proportional to $\tau^* \equiv \varepsilon \left(\frac{\partial \bar{u}_i}{\partial x_j}\right)^2$, for incompressible flows. If one applies this approximation to the transfer equation (4.56) and analyses the equation in the manner discussed earlier in this section, it is easy to come to the conclusion that for kinetic energy G_o may be represented functionally as:

$$G_o \approx - \frac{A}{\overline{SN}} \exp(-\ \alpha\eta^*/SN_D) \tag{4.70}$$

where α and A are numerical constants, and $\overline{SN} = (SN(x) + SN(x+d))/2$.

The functions G_2 and G_3 which are the coefficients of the non-linear terms of ϕ are evidently not very simple functions. In the region of the wall boundary layer where the boundary layer characteristics are not changing very fast, $SN_D \rightarrow 0$, and the transfer function ϕ is approximately a linear function.

In summary, it is noted that a fluid transfer function may be defined as follows for scalar quantities, $\tilde{q}(-q/q_{max})$:

$$\tilde{q}(x+\ x) = G_o + G_1\tilde{q}(x) + G_2\tilde{q}^2(x) + G_3\tilde{q}^3(x)$$

where the coefficients G_o, G_1, G_2 and G_3 appear to have the following forms:

$$G_o \doteq \frac{-A}{\overline{SN}} \exp(-\ \alpha\eta^*/SN_D)\ ,\ (\alpha \approx 1, \text{ and } A \approx 0.001 \text{ for kinetic energy})$$

$G_1 \doteq \text{Tanh}\ (a\ \eta^*/SN_D)$, $(a \approx 0.07)$

$G_2 \doteq 2\ \exp(5\ SN_D)[\exp(-4.5\eta^*/SN_D) - 1.3\ \exp\ (-4\eta^*/SN_D)]$

$G_3 \doteq 8\ \exp(20\ SN_D)[\exp(-4.5\eta^*/SN_D) - 0.26\ \exp(-5\eta^*/SN_D)]$

and the quantities are as defined earlier in this section. It must be noted that G_o , G_1 , G_2 and G_3 as written above represent only a first approximation. Figure (17) shows the approximate functional forms of these coefficients, across a wall boundary layer, for fully-developed flows.

4.4 Use of a Describing Function Method for Continuous Solution of the Wall Boundary Layer

By virtue of the force field hypothesis it was possible to deduce a simplified flow transfer or describing function, ϕ, with which invariant flow quantities could be transferred from an upstream to a downstream position. An attempt will now be made to demonstrate how this elementary transfer function could be used to ease the computation of at least the simple zero pressure gradient two-dimensional wall boundary layer.

The appropriate governing equations for the two-dimensional incompressible flow reduce to just two, the continuity equation and the quasi-invariant transfer equation, namely:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (4.71)$$

$$\tilde{q}(x+d) = G_o + G_1\tilde{q}(x) + G_2\tilde{q}^2(x) + G_3\tilde{q}^3(x) \qquad (4.72)$$

where $d = \Delta x$, G_o , G_1, G_2 and G_3 are as defined in section (4.3). Equation (4.72) has essentially combined the boundary layer momentum and kinetic energy equations in a form which makes it unnecessary to make any explicit statements concerning the turbulence action.

It must be noted that q in (4.72) is a total dimensionless quantity (e.g., $u^2 + v^2 \equiv \bar{u}^2 + \overline{u'^2} + \bar{v}^2 + \overline{v'^2}$). In terms of mean and fluctuating quantities the continuity equation may be split into two, and the transfer equation expanded to include the influence of the fluctuating quantity at the upstream position, as follows:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \tag{4.73a}$$

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \tag{4.73b}$$

$$\tilde{q}(x+d) = G_o + G_1 [q(x) + q'(x)] + G_2[q^2(x) + 2q(x)q'(x) + q'^2(x)]$$

$$+ G_3 [q^3(x) + 3q^2(x)\, q'(x) + 3q(x)\, q'^2(x) + q'^3(x)] \tag{4.73c}$$

where $d = (\Delta x)$.

To make computations by the describing function method one must know an upstream position exactly or at least closely. These demands are no different from those basically required by conventional prediction methods. One may then set up the prediction problem downstream of the given position as follows:

(i) Assume that

$$SN(x+d) \equiv SN(x) + \frac{d}{dx}(SN(x)).d$$

$$\text{i.e.} \quad SN_D \equiv \frac{d}{dx}(SN(x)).d/\ SN(x+d)$$

(ii) Compute $q(x+d)$ and $q'(x+d)$ from equation (4.73c).

(iii) By definition, for the kinetic energy

$$q(x+d) = \tilde{u}^2 + \tilde{v}^2 = Q, \tag{4.74}$$

where the tilde denote dimensionless mean quantities. From equation (4.73a) when normalized by scaling length Δ one obtains that:

$$\Delta \frac{\partial \tilde{u}}{\partial x} + \tilde{u} F\, Re_{\Delta} + \frac{\partial \tilde{v}}{\partial \eta^*} = 0 \tag{4.75}$$

where $F = \frac{\nu}{U_1^2} \frac{dU_1}{dx}$

From equations (4.74) and (4.75) one obtains that:

$$\tilde{u} \frac{\partial \tilde{u}}{\partial \eta^*} \doteq \tilde{u}^2 f_1 + \tilde{u} f_2 + f_3 \tag{4.76}$$

where $f_1 = (\frac{1}{4Q} \frac{\partial Q}{\partial \eta^*})$

$$f_2 = (F\, Re_{\Delta} + \frac{\Delta}{d})\, Q^{1/2}$$

$$f_3 = (\frac{1}{2} \frac{\partial Q}{\partial \eta^*} - \frac{\Delta}{d} \tilde{u}(x)\, Q^{1/2}) .$$

Equation (4.76) has the following solution for $\eta^* \neq 0$: - (Kamke (1948) Abel's equation of the second type).

$$\tilde{u} = [Q - \frac{\Delta}{d} Q^{1/2} \int_0^{\eta^*} u(x)\, d\eta^* + f_2 \int_0^{\eta^*} Q^{1/4}\, d\eta^*]^{1/2} \qquad (4.77)$$

Then $$\tilde{v} = (Q - \tilde{u}^2)^{1/2} \qquad (4.78)$$

(iv) Sn(x+d) may now be computed from the velocity profiles at the (x+d) position.

(v) $SN_D = 1 - SN(x)/SN(x+d)$ is computed and (ii) through (v) repeated until convergent solution is obtained.

No explicit convergence analysis has been made yet, as a rigid proof of the convergence of the above scheme. Convergence is accepted, a posteriori, on the basis of examples tried. Indeed, the marching scheme used here is no different from that used in most conventional numerical schemes.

Starting from the same initial input profiles and specification, the Schubauer and Klebanoff (1955) boundary layer has been computed. The results are displayed in Figures (10) through (12), and are reasonably good, for the fully laminar and fully turbulent regimes. However, the describing function method as presented in this work is much too simple and does not predict the transitional boundary layer, satisfactorily.

Chapter V

5. A TRANSITION FUNCTION METHOD FOR CONTINUOUS SOLUTION OF THE WALL BOUNDARY LAYER

Differential and integral methods of mean velocity field closure used to calculate laminar and turbulent boundary layers give very good results for some practical cases. Quite a lot of work has gone into developing efficient computer programs for such calculations. Although the more recent methods of boundary computation, for instance, the method of invariant modelling discussed in Appendix A of this work, are undoubtedly superior to the differential and integral methods of mean velocity field closure, the latter methods should not be completely forgotten especially where computational facilities are limited. One major advantage of the method of invariant modelling, say, over the older methods is that the former is not restricted in its basic form to any special regime of the boundary layer, laminar or turbulent, that is. Hence the method of invariant modelling can be used for continuous computation of the complete boundary layer, laminar through transitional to turbulent, without additional assumptions or modifications. With the older computational methods, however, the user has to know precisely where to stop using the laminar flow assumptions and start using the turbulent flow assumptions. Moreover, since a finite length of transitional (neither exactly laminar nor exactly turbulent) boundary layer may exist between the laminar and the turbulent boundary layers, a third set of assumptions are needed if one wishes to calculate a boundary layer from laminar through transition to turbulent flow, as in airfoil design. Persh (1957) has suggested a method of calculating through the transitional boundary layer which could be applied to provide the desired continuous solution of the boundary layer.

In this section, an attempt will be made to evaluate and use an intermittency factor (or transition function) which is zero for fully laminar flows and unity for fully turbulent flows, to link the laminar and turbulent boundary layers. Implicit in the concept of the intermittency factor is the experimentally confirmed supposition that the transitional boundary layer is a simple mixture of a laminar and a turbulent boundary layer. Such an intermittency factor had been mentioned by Herring and Mellor (1970) and partially evaluated by Emmons (1951), Schubauer and Klebanoff (1955). The function evaluated in the present work is thought, however, to be more general and more useful.

The ways in which this intermittency function may be used in differential and integral relation methods of boundary layer computation are discussed in principle, below. The actual application of this linking mechanism, however, is tested only for the differential method. The results, as shown in Figures (10) through (12), are very good.

5.1 The Form of the Transition Function

A derivation of an intermittency factor (the transition function) which describes quantitatively the transition process will now be presented. Once again, the function derived here is not necessarily the exact description of the transition process but is considered a sufficient description, for general cases.

It was pointed out in the review of boundary layer research, Chapter 2, that the transitional boundary layer appears to be statistically similar for all cases irrespective of the cause of transition. On the basis of this statistical similarity, Emmons (1951) and McCormick (1968) established the nature of the intermittency in the transitional boundary layer, which qualitatively describes the probability that a spatial point in the transitional boundary layer is turbulent. Emmons' derivation is preferred for the purpose of the present work. According to Emmons (1951), if a function $g(x,y,t)$ of position (on the surface) and time is defined to specify the rate of turbulent spot production per unit area, in a boundary layer, then the fraction $f(x)$ of the time during which a point distance x from the leading edge is turbulent is given by

$$f(x) = 1 - \exp(-\sigma g x^3/3U_1) \tag{5.1}$$

where

$$\sigma = \alpha^2 \lambda_t/\beta$$

$$\alpha = \mathrm{Tan}(1/2 \text{ propagation angle})$$

$$\beta = \frac{\text{Velocity of center of spot}}{\text{Free stream velocity, } U_1} \tag{5.2}$$

$$\lambda_t = \frac{\text{Area of spot}}{\text{Square of half-width}}$$

and

$g = g(x,y,t)$ as defined earlier.

The spot characteristic, σ , can be estimated from empirically deduced values of α , β and λ_t and appears to be independent of the process leading to the spot formation. Emmons (1951) evaluated σ to be approximately 0.1. This value will be temporarily retained in the present work and equation (5.1) may then be rewritten as

$$f(x) = 1 - \exp(-0.04gx^3/U_1) \qquad (5.3)$$

The function $g(x,y,t)$ will now be derived. For steady mean flow as considered in this study, g will be independent of time.

According to one of the basic philosophies carried through in this work, there is a critical minimum force necessary to sustain laminar flow. If the flow force field falls below this critical level, the 'locked-in' turbulence begins to manifest itself and, when the constraining force is overwhelmed, burst out. This outburst serves to arrest the tendency for the constraining force to completely disappear, by creating a new force field through the so-called Reynolds shear forces. This behavior represents a phase change of the fluid with respect to its constraining force potential or the fluid cohesive property. That is, through the outburst of the previously constrained turbulence, a fluid of vanishing cohesiveness begins the process of a phase change to a fluid of some finite cohesiveness. The transitional boundary layer represents therefore the region of this phase change.

The rate of turbulence manifestation which corresponds to the conventional amplification rate of disturbances should therefore depend on only by how much the force field has fallen below the critical value for laminar flow. This argument was used in Chapter IV to deduce the amplification characteristics of unstable laminar flows. The quantity of interest in this section is the rate at which the outbursts of turbulence occur. This will be deduced later in this section.

An implication of the above force field description of boundary layer laminar to turbulent transition is that there must exist a finite critical intensity of the manifested turbulence at which turbulence outbursts begin to occur. This latter implication is considerably supported by experimental results, for example, of Elder (1960), Klebanoff et al., (1962), Kovasnay et al., (1962) and Tani and Komoda (1962). These fluid dynamics researchers obtained that turbulence spots begin to form at distance from the wall of about $y/\delta = 0.3$, if the local turbulence intensity $\sqrt{\overline{u'^2}}/U_1 > 18 \pm 2.5\%$. Such high levels of turbulence are generally not found in the free stream and are of the order of magnitude of those encountered in fully turbulent boundary layers. That is, there must exist a powerful amplification process in the laminar boundary layer which contributes substantially to the intensity build-up before local transition takes place. It is interesting to note that Elder (1960), etc., observed that the critical value of the turbulence intensity for the formation of turbulence spots depended on the Reynolds number $(U_1\delta^*/\nu)$ and the profile shape factor, H, (δ^*/θ). The stability number as defined in Chapter IV is a function of $(U_1\delta^*/\nu)$ and H.

If, as discussed above, outbursts of turbulence are the arrestor to vanishing fluid cohesiveness, then as with every other natural system, the occurrence of outbursts should lead to a stable configuration, ultimately. This stable configuration for fluid flow appears to be attained in the fully developed turbulent flow. Hence, soon after the initial bursts of turbulence occur the rate of production of turbulence should tend to some constant value for the flow. With respect to the rate of production of turbulence spots, therefore, the situation is virtually identical to the current in a transient electrical inductive circuit. In this latter case, if a circuit containing resistance, R , and inductance, L^*, in series, is connected across a steady voltage E, the voltage E must supply the iR drop in the circuit and at the same time overcome the emf of self induction. The solution of the governing differential equation is:

$$i = (E/R)[1 - \exp(- Rt/L^*)] \qquad (5.4)$$

and (L^*/R) is the circuit time constant.

In the fluid flow case, soon after the force field falls below the critical level, bursts of turbulence appear. Each burst, however, opposes the production of further bursts, so that as the number of bursts increases, the rate of production of bursts tends toward a finite constant level. That is, the turbulence becomes self-regenerative. By analogy with the electrical system, one may represent the flow situation as follows:

By the force field concept, the steady rate of production of turbulence spots will depend only on the relative stability number between the initial stable position and the position of incipient transition.

$$\text{i.e.,} \quad g_{\text{(steady state)}} = G[SN_{et} - SN_{ct}]/L^2 \text{, per unit time} \qquad (5.5)$$

where

SN_{et} = Stability number at the beginning of the stable state which corresponds approximately to the end of the transition region.

SN_{ct} = Stability number at the position of incipient transition.

L = Some appropriate length scale.

The characteristic time scale for the analogous fluid system is

$$t = (x - x_{ct})/U_1$$

The characteristic length scale is ν_1/U_1 . If it is assumed that the stability number $SN(x)$ is proportional to the dimensionless force quantity, U_1x/ν_1 , then:

$$g = F[\frac{U_1\Delta x_t}{\nu_1}] (\frac{U_1}{\nu_1})^2 [1 - \exp\{ - \alpha_t (\frac{x-x_{ct}}{U_1})\}] \text{ , per unit time} \tag{5.6}$$

where $\Delta x_t = (x_{et} - x_{ct})$, and α_t is the flow time constant which will be assumed in this work to be proportional to $U_1(x-x_{ct})/\nu_1$.

As discussed in Chapter II of this work, the experimental results of Schubauer and Klebanoff (1955) and Schubauer and Skramstad (1948) indicate that whether long or short and independent of the cause of transition, the transitional boundary layer appears to be statistically similar. An extended implication of this behavior would be that the function $F[U_1\Delta x_t/\nu_1]$ is, on the average, a constant independent of the particular flow. That is, the extent or length Δx_t of the transitional boundary layer depends on the unit Reynolds number, U_1/ν_1 . Such an influence is supported by experimental evidence and has been discussed by Morkovin (1968). Thus:

$$g = \beta_t (\frac{U_1}{\nu_1})^2 [1 - \exp\{ - \alpha_t (\frac{x-x_{ct}}{U_1})\}] \text{ , per unit time} \tag{5.7a}$$

where β_t is a generalized constant for flows. By examining the experimental results of Schubauer and Klebanoff (1955), the following form for the rate of production of turbulence spots is quoted:

$$g = 1.53 \times 10^{-10} \left(\frac{U_1}{\nu_1}\right)^2 \left[1 - \exp\left\{- 5.34 \times 10^{-5} (x-x_{ct})^2 / \nu_1\right\}\right] \tag{5.7b}$$

$$\simeq \frac{1}{L^2}, \text{ per unit time.}$$

Dhawan and Narasimha (1958) and Chen and Thyson (1971) obtained equations similar to (5.7b) through different arguments.

The significance of this definition of $g(x,y)$ must be made clear. Equation (5.7b) does not imply that turbulence spots are continuously being formed downstream of the point of initial breakdown. On the contrary, turbulence spots are formed only in the transient region of equation (5.7b). Beyond this region, $g(x,y)$ may be interpreted as being proportional to the production of turbulence, without necessarily formation of discrete turbulence spots.

5.2 A Differential Method

The differential method reduces the partial differential equations of motion to boundary-valued ordinary differential equations with an explicit assumption for the turbulence. Two approaches to closure, the mean velocity field and the turbulence field methods, may be discerned in this method, depending on the treatment of the Reynolds stress terms. The mean velocity field closure method uses the eddy-viscosity and/or the mixing length concept, while the turbulence field closure method relates the Reynolds stress to the turbulence and hence requires the simultaneous calculation of some aspects of the turbulence field. The advantage of the turbulence field closure over the mean velocity field closure is that the latter cannot explicitly include the effects of turbulence history. However, although the turbulence field closure method can follow the turbulence evolution, it cannot be described as a simple analytical method and often requires sophisticated state assumptions for which experimental justification hardly exists. The method to be described in this section employs mean velocity field closure. A turbulence field method is discussed by Donaldson (1971) and was used in this work to study wall boundary layer transition and growth.

The differential method discussed below is essentially that suggested by G. Mellor and J. Herring (1970) but with an improved eddy-viscosity model, discussed in Ref. (3), and a defined transition function, T . The coordinate system used is shown in Figure (6).

The governing equations involve the double correlations $\overline{u'v'}$ and $\overline{v'h'}$. For closure, the following assumptions were made concerning the double correlations:

$$\frac{\mu}{\rho}\frac{\partial u}{\partial y} - \overline{u'v'} = (\nu_a + T\nu_t)\frac{\partial u}{\partial y} = \nu_e \frac{\partial u}{\partial y}$$

$$\frac{\mu_g}{\nu}\frac{\partial h}{\partial y} - \overline{v'h'} = (\nu_g + T_g \nu_{gt})\frac{\partial h}{\partial y} = \nu_{eg}\frac{\partial h}{\partial y} \quad (5.8)$$

where equations (5.8) are valid for all the regimes of the boundary layer if the transition functions T and T_g are defined. The functional forms of Φ and Φ_g are respectively as given by Anyiwo and Meroney (1971) and Herring and Mellor (1970). The detailed consideration of the reduction and solution of the governing equations have already been given by Herring and Mellor (1970).

5.3 An Integral Relation Method

By integrating the boundary layer momentum equation, it becomes possible to avoid the need to make explicit assumptions about the local Reynolds stress. The turbulence terms do not contribute explicitly to the resulting integral equation, but one must now relate the resulting integral parameters, θ , δ^* , and the skin friction, $C_f/2$, through some global assumption about the implicit effects of the turbulence or through further equations. Most integral methods currently used employ the momentum integral equation and a wall friction relation and then any one of the following equations:

i) The energy integral equation;

ii) The entrainment equation; or

iii) The momentum of momentum equation.

If (i) or (iii) is used one must make further assumptions about the turbulent shear stress integral, whereas the choice of (ii) requires some assumption about the entrainment function. Lewkowictz et al. (1970) compared a variety of integral methods and suggested that for flows with adverse pressure gradients the use of the entrainment equation gives the best agreement. For equilibrium and relaxing boundary layers however, the dissipation integral equation or the moment of momentum equation gives better results. In the present study it is aimed to avoid as much as possible any assumptions concerning the turbulent shear stress. The entrainment equation is therefore used in a way which, it is hoped, would not limit it to just adverse pressure gradient flows. Further, the wall skin friction and boundary layer physical thicknesses are calculated from a velocity profile which is made valid for both laminar and turbulent boundary layers. This velocity profile could be obtained by linking a laminar profile to a turbulent profile with a transition function.

The governing equations for the two-dimensional incompressible boundary layer as used in the momentum integral method are:

Momentum integral equation

$$\frac{d\theta}{dx} + \theta[(2+H)\frac{1}{U_1}\frac{dU_1}{dx} + \frac{1}{r}\frac{dr}{dx}] = \frac{v_o}{U_1} + \frac{C_f}{2} \qquad (5.9)$$

where r is the wall radius of curvature.

The velocity profile

$$\frac{u}{U_1} = (1-T)\left(\frac{u}{U_1}\right)_{laminar} + T\left(\frac{u}{U_1}\right)_{turbulent} \qquad (5.10)$$

The entrainment equation:

$$\frac{1}{U_1}\frac{d}{dx}[U_1(\delta-\delta^*)] = \frac{v_o}{U_1} + E \text{ (the entrainment function).} \qquad (5.11)$$

The boundary layer thicknesses are defined as:

$$\delta^* = \int_o^\delta [(1 - \frac{u}{U_1})/(1-y/r_o)]dy \qquad \text{displacement equation}$$

(5.12)

$$\theta = \int_o^{\delta} [\frac{u}{U_1}(1 - \frac{u}{U_1})/(1 - y/r_o)]dy \quad \text{momentum thickness.}$$

The skin friction is calculated from the velocity profile as:

$$\frac{C_f}{2} = \frac{\nu}{U_1^2}[\frac{\partial u}{\partial y}]_{y=0} \tag{5.13}$$

The entrainment equation (5.11) is simply the integral equation of continuity, namely:

$$\frac{1}{U_1}\frac{d}{dx}[U_1(\delta-\delta^*)] = (\frac{d\delta}{dx} - \frac{v_1}{U_1}) + \frac{v_o}{U_1} \tag{5.14}$$

The function $[\frac{d\delta}{dx} - \frac{v_1}{U_1}] = E$, the entrainment function, represents the momentum imbalance due to turbulence action, and was modelled by Head (1958), as $E(H_{\delta-\delta^*})$. The term $\frac{v_o}{U_1}$ is the momentum imbalance due to transpiration. One reason, and perhaps the major one, for the limitation of the entrainment method as cited by Lewkowictz et al. (1970) is that Head's formulation for the entrainment function does not adequately represent the actual function, E, as defined above (equation (5.14)). When Hirst and Reynolds (1968) used a slightly different but more encompassing formulation for E, they obtained good results over a wider class of flows than Head's formulation could predict. As pointed out by Thompson (1965) the function E is not dependent on $H_{\delta-\delta^*}$ only, as Head assumed but depends also on the Reynolds number. In this respect, it will be noted that away from the entrance region of the boundary layer, the function $E = [\frac{d\delta}{dx} - \frac{v_1}{U_1}]$ is proportional to $(\frac{\delta}{2x})$, which is, in fact, a function of $[\frac{U_1(\delta-\delta^*)}{\nu}]$, only. This last quantity is virtually identical to the stability number defined in Chapter 4. Hence, E is some simple function of (1/SN).

With a generalized mean velocity profile as suggested in equation (5.10) the integral relation method using the entrainment equation, easily becomes a continuous method of predicting the general wall boundary layer.

Chapter VI

6. A STATISTICAL ENERGY THEORY OF TURBULENCE

Let it be assumed that any natural system consists of basic particles the combined characteristics of which determine the general characteristics of the system. For continuum flow system, it will be assumed that the basic particle is a conceptual fluid particle defined as a small volume of fluid which behaves as a continuum. With these suppositions, and the force field concept, one may reason that any natural system can be considered as a conglomeration of non-intersecting closed energy volumes in which the basic system particles (active and passive) execute 'chaotic' motions in the absence of a strong enough force field. Turbulence is then viewed as a measure of the instantaneous rate of translocation of particles within and among these energy volumes. It is necessary to use statistical mathematics to describe turbulence since the natural force fields and the subsequent motions of particles are usually only probabilistically deterministic. The remainder of this chapter is devoted to adapting the concept of energy volumes to the boundary layer flow and suggesting preliminary theories that may be used to understand and describe natural flows.

6.1 The Conceptual Energy-Surface Structure of Wall Bounded Flows

It is hypothesized that just as conceptual non-intersecting streamlines are a plausible assumption in conventional two dimensional laminar boundary layer theories so are conceptual non-intersecting energy volumes in a general boundary layer. If this fundamental assumption is acceptable then the following statements may be made concerning the energy volumes.

i) An energy volume is a finite or infinitesimal volume containing particles of identical mean energy. Since in wall boundary layers the contribution of potential energy may usually be neglected, the average energy in an energy volume is determined by the magnitude of the local mean velocity field, that is, is purely kinetic energy. The thickness of the energy volume is directly proportional to the magnitude of the energy excursions about the mean energy level due to random motion of the particles.

ii) The motion of the conceptual fluid particles in an energy volume is determined only by the prevelant force field. The passive particles will execute chaotic motion in the absence of a directional external force field, but in the presence of a directional external force-field will exhibit organized motion.

iii) Turbulence may be defined as proportional to the fluctuational displacement of the particles in the energy volumes, due particle translocation within and among the energy volumes.

A schematic drawing of the conceptual energy volumes in a boundary layer is shown in Figure (8a). Before proceeding further, it is necessary to re-examine the present model of turbulence in the light of certain known characteristics of turbulence, namely:

(i) Turbulence can be sustained only in the presence of a force gradient, as in shear flows.

(ii) Turbulence is a dissipative three dimensional random process which exists only in rotational flows.

(iii) Turbulence is a non-linear phenomenon.

(iv) Turbulence is highly diffusive.

(v) Turbulence is characterized by a time and space scale. That is, the definition of turbulence depends very much on the time and length scales of the flow.

(vi) Turbulence is a continuum problem.

(vii) A turbulent flow is mathematically, an indeterminate flow problem.

It is easy to show that the present model of turbulence does not contradict any of the above characteristics. If there is no local shear in a flow, the mean velocity would be unchanged at all levels so that the whole flow constitutes a single energy volume. The definition of turbulence given in the present model implies that turbulence can exist only in the presence of at least two distinct energy volumes. Translocation of particles in an isolated energy volume are purely enhanced molecular diffusion. If shear exists so that mean velocities at points may be different, definite energy volumes may be distinguished, each containing the particles of identical mean energy. In the wall boundary layer there is a continuous gradient of the mean velocities across the boundary layer. The energy volumes are therefore closely-packed volumes, of thicknesses varying across the boundary layer in accordance with the local magnitude of the permissible fluctuational energy excursion about the local mean energy level. Within each such conceptual energy volume, conceptual fluid particles exhibit varying random motions until some external factor causes a translocation of particles between any two energy volumes. Such translocations cause regenerative collisions and physical collisions are generally dissipative. Hence turbulence can exist only in shear flows, and such turbulence must be regenerative as well as dissipative (ii). Whether or not the turbulence is sustained depends on whether or not the regenerative process is dominant over the dissipative process. The three dimensionality of such a process is not in doubt (ii). It is further obvious that the prediction of the process is only probabilistic, that is, the turbulence process is a random process (ii, vi). Relative to molecular diffusion, turbulence, as defined here, will be highly diffusive (iv). Further, the scale of the motion required to induce the regenerative turbulence process must be relative to the time and length scales of the particular situation in question (vi), that is, to the scales of the thickness of the conceptual energy volumes.

Having now ascertained that the present model of turbulence shows no obvious contradiction to established characteristics of turbulence, it should be safe to try to apply the model to determine for instance the distribution of turbulence in a wall boundary layer.

6.2 A Statistical "Collision" Theory

An attempt will now be made to apply a stochastic model to the concept of energy volumes in order to study the turbulence characteristics of wall boundary layer flows. First, a more detailed description of the concepts involved is given by the following statements:

i) Energy volumes or states extend across the boundary layer from a zero state at the wall to the maximum at the free stream. At each energy state there exist secondary states or energy levels which represent the fluctuations about the mean kinetic energy at that position, in time and space.

ii) The size of a fluid particle is relative to the local length scale and must therefore vary across the boundary layer being smaller in the lower energy states. Since, in the absence of large temperature gradients, the mass density of the fluid must be the same throughout the fluid, it follows that the relative number density with respect to the fluid particles varies across the boundary layer, being larger in the lower energy states.

iii) Fluid particles at every energy state execute fluctuational motions in all directions and suffer collisions among themselves. Such collisions will generally result in energy loss or gain for at least one of the colliding particles. However, there are only a few collisions which are energetic enough to cause a particle to acquire or lose sufficient energy to break through its current state energy barrier into a higher or lower energy state. Such energetic collisions will usually involve those particles in the fringe energy levels of the state. Let the particles in the higher fringe levels be called "hot" particles and those in the lower fringe levels, "cold" particles.

iv) In the absence of shear, the distribution of fluid particles among the energy levels of a given energy state is assumed to be Gaussian. In this respect, the number of "cold" particles is equal to the number of "hot" particles. Shear introduces a skewness in the distribution, such that there is a finite difference between the numbers of "hot" and "cold" particles in a given energy state.

v) The probability of an energetic collision among the "hot" particles is a function of $(N_H - N_C)N_H$, where N_H and N_C are respectively the average number of hot and cold particles. Similarly, the probability of an energetic collision among the "cold" particles is a function of $(N_H - N_C)N_C$.

vi) If a particle leaves an energy state enough particles must enter that energy state to maintain a constant average number density of particles in that state. Particles coming from a higher energy state are larger than the required size for their new energy state and must break down to the appropriate size. Such a process is dissipative in energy. On the other hand, particles arriving from a lower energy state must fuse and release energy thereby. Hence, the rate of production of turbulent energy at an energy state is proportional to the rate of inflow of particles from the lower energy state. Similarly, the rate of dissipation of energy at the state is proportional to the rate of inflow of particles from the higher energy state and the rate of occurrence of energetic collisions within the state, since such collisions must be relatively substantially dissipative.

A schematic drawing of some of the processes discussed above is given in Figure (8b) for a unit volume at an energy state. Let the difference between the average number of "hot" and "cold" particles at time $t = 0$ be N_o^*, and at time t, be N_t^*. Also let $p_1(t)$ and $p_2(t)$ be the rates respectively of occurrence of energetic collisions and of influx of neighboring state particles. A detailed balance yields that:

$$\frac{dP_n}{dt} = p_2(n_o - n + 1)\ P_{n-1}(t) + p_1(n + 1)\ P_{n+1}(t) - [p_2(n_o-n) + p_1 n]P_n(t) \tag{6.1}$$

Equation (6.1) assumes that the probability of transition from energy state x to $(x\pm1)$ in interval $(t,t+\Delta t)$ is $px\Delta t + O(\Delta t^2)$ and the probability of a transition $(x) \rightarrow (x\pm j),\ j > 1$ in the interval $(t,t+\Delta t)$ is at most $O(\Delta t)$. $P_n(t) = \mathrm{Prob}(N_t^* = n)$. The validity of transport equations such as (6.1) has been discussed in some detail by Van Hove (1957), for example. By means of the generating function of $P_n(t)$, namely

$$F(s,t) = \sum_{n=0}^{\infty} P_n(t)s^n \qquad |s| < 1 \tag{6.2}$$

equation (6.1) may be transformed into a partial differential equation as follows:

$$\frac{\partial F}{\partial t} = [p_1 + (p_2-p_1)s - p_1 s^2]\ \frac{\partial F}{\partial s} + N_o^*\ p_2(s-1)F \tag{6.3}$$

Equation (6.3) is similar to equations considered by Ishida (1960) for which the following solution is stated:

$$F(s,t) = \left[\frac{\lambda\exp(-kt)(s-1) + \lambda - s}{\lambda}\right]^{N_o^*} \tag{6.4}$$

where

$$\lambda = \int_o^t p_1\ dt / \int_o^t p_2\ dt$$

$$k = \frac{1}{t} \int_o^t (p_1 + p_2)\, dt$$

Equation (6.4) gives the mean and variance of N_t^* as follows:

$$E[N_t^*] = \frac{N_o^*}{k} \{\frac{1}{t} \int_o^t p_1\, dt \exp (- \int_o^t (p_1+p_2)dt) + \frac{1}{t} \int_o^t p_2 dt\}$$

$$Var[N_t^*] = [N_o^* \omega/(1 + \lambda)](1 = \omega/(1 + \lambda)) \tag{6.5a}$$

where

$$\omega = \lambda \exp (- \int_o^t (p_1+p_2)dt) + 1.$$

This provides the transfer equation for $(N_H - N_C)$ if time is replaced by an appropriate spatial scale. Replacing t by the approximate quantity $\bar{x}/U_1$, $E(N_t^*)$ may be rewritten as follows for transfer in the x-direction:

$$E[N_{\bar{x}}^*] \doteq \frac{N_{xo}^*}{k_{\bar{x}}} \{ \frac{U_1}{\bar{x}} (\int_o^{\bar{x}} p_1(\bar{x},\eta^*)(1-F\, Re_{\bar{x}})/U_1\, d\bar{x}) \exp(- \frac{\bar{x}}{U_1} k_{\bar{x}})$$

$$+ \frac{U_1}{\bar{x}} \int_o^{\bar{x}} p_2(\bar{x}, \eta^*)(1 - F\, Re_{\bar{x}})/U_1\, d\bar{x}\} \tag{6.5b}$$

where

$$k_{\bar{x}} = \frac{U_1}{\bar{x}} \int_o^{\bar{x}} (p_1(\bar{x},\eta^*) + p_2(\bar{x},\eta^*)(1 - F\, Re_{\bar{x}})/U_1\, d\bar{x}$$

$$F = \frac{\nu}{U_1^2} \frac{dU_1}{dx}$$

$\bar{x} = (x-x_o)$.

$p_1(\bar{x},\eta^*)$ = Rate of translocation of particles in current energy state.

= Rate of occurrence of energetic collisions ("hot" and "cold") in the energy state.

$p_2(x,\eta^*)$ = Rate of inflow of out-of-state particles into current energy state.

= Collision rate of "cold" particles at $(\eta^* + \Delta\eta^*)$ x Probability of energetic "cold" collision at $(\eta^* + \Delta\eta^*)$.

+ Collision rate of "hot" collisions at $(\eta^* - \Delta\eta^*)$.

In the absence of any shear the numbers of "hot" and "cold" particles are equal and if one assumes that the particles are Gaussian distributed within the single energy state, one obtains that if the fringe particles are defined as those outside the range of the standard deviation:

$$N_H = N_C \doteq N_T \exp(-1) \tag{6.6}$$

where N_T = Local number density of particles.

If shear exists, one may argue that the modifications of N_H and N_C are proportional to the local shear, for example as follows:

$$\begin{aligned} N_H &\doteq N_T \exp(b_1 \phi - 1) \\ N_C &\doteq N_T \exp(b_2 \phi - 1) \end{aligned} \tag{6.7}$$

where ϕ is a characteristic local shear quantity (e.g., $\phi = \tau/\rho U_1^2$), and b_1 and b_2 are numerical constants. From equations (6.7) one obtains that:

$$N_H + N_C \doteq N_T[1 + \exp(b_4\phi)]\ \exp(b_1\phi - 1)$$

$$N_H - N_C \doteq N_T[1 - \exp(b_4\phi)]\exp(b_1\phi - 1)$$

where b_4 is a numerical constant.

The rate quantities p_1 and p_2 may now be written as follows:

$$p_1 \doteq Z_C\ (\eta^*)\ f_C + Z_H(\eta^*)\ f_H$$

$$p_2 \doteq Z_C(\eta^* + \Delta\eta^*)\ f_C + Z_H(\eta^* - \Delta\eta^*)\ f_H \tag{6.9}$$

where f_C and f_H are the local probabilities of energetic collisions,

Z = The local collision rate $\simeq 2N^2d^2(\pi kT/m)^{1/2}$

N = The local number of particles involved

d = The local diameter of particles ($\propto (\eta^* + \bar{a})\Delta$)

m = The local mass of particles

$\bar{a}$ is proportional to the diameter of the smallest volume of fluid that may be considered a continuum, non-dimensionalized by Δ. (Approximately, $\bar{a}$ is equal to 0.002Δ, for air)

$$f_H \doteq b_4\ \phi\ \exp\ (b_1\phi - 1)$$

$$f_C \doteq b_4\ \phi\ \exp(b_2\ \phi - 1)$$

$$Z \doteq A\,N^2\,(\eta^* + \bar{a})^2\,(\frac{\pi kT}{m})^{1/2} .$$

Therefore,

$$p_1 \doteq A_1 N_T^2\,(\frac{\pi kT}{m})^{1/2}\quad \Delta^2(\eta^* + a)^2\ \phi[1 - b_5\phi]\ \exp 3(b_1\phi - 1)$$

$$p_2 \doteq A_2 N_T^2\,(\frac{\pi kT}{m})^{1/2}\quad \Delta^2(\eta^* + \bar{a})^2\ \phi[1 - b_5\phi]\ \exp 3(b_1\phi - 1) \qquad (6.10)$$

N_T $(=\rho/m)$ is the total number of particles per local unit volume, $(\pi kT/m)$ is proportional to the local pressure so that $N_T^2(\pi kT/m)^{1/2}$ $\Delta^2(\eta^* + \bar{a})^2$ may be written as proportional to $U_1/(\Delta(\quad * + \bar{a}))$.

The local rate per unit volume of production of turbulence kinetic energy, Z_p, is proportional to the rate of inflow of particles from the lower energy state,

$$\text{i.e.,}\quad \frac{Z_p}{\rho U_1^3} \propto Z_H\,(\eta^* - \Delta\eta^*)\,f_H \doteq \frac{A_1}{(\eta^*+\bar{a})}\ \phi\ \exp\,(b_1\phi - 1) \qquad (6.11)$$

But at the wall which corresponds to the relative zero energy state, there is no inflow of particles from a lower energy state. Hence, turbulence production rate must be zero at the wall. That is, equation (6.11) is valid for all but the relative zero energy state (i.e., for $\eta^* \geq \bar{a}$). The formulation of $Z_p/\rho U_1^3$ for $\eta^* < \bar{a}$ is at the moment uncertain. By comparison with some measurements of Klebanoff (1954) it seems that $A_1 \approx 2$ and $b_1 \approx 1.2$.

The local rate per unit volume of dissipation of turbulence kinetic energy Z_D , is proportional to the local rate of inflow of particles from the higher energy state plus the local rate of energetic collisions

$$\text{i.e.,} \quad \frac{Z_D}{\rho U_1^3} \propto Z_C(\eta^* + \Delta\eta^*) f_C + p_1 \doteq \frac{A_2}{(\eta^*+\bar{a})} \phi \quad \exp 3(b_6\phi - 1) \qquad (6.12)$$

Unlike the situation with turbulence production, inflow of particles from a higher energy state is quite possible at the wall state. Hence, there is no bounding condition that $Z_D/\rho U_1^3$ be zero at the wall. Equation (6.12) is therefore valid for all η^* . Again by comparison with measurements of Klebanoff (1954) it seems that A_2 is about the order of magnitude of 10 and b_6 of magnitude about 1.2. Some plots of equations (6.11) and (6.12) are shown in Figures (14) and (15). Although these plots cannot be directly compared with the experimental points of Klebanoff (1954) the trend is quite encouraging.

From equation (6.5b),

$$\frac{E[N^*_{\bar{x}}]}{N^*_{xo}} \propto \frac{I(\bar{x})}{I(o)} = B_1[1 + \exp(\xi)] \equiv \bar{a}^{*2} \qquad (6.13)$$

where

$$\xi = \frac{B_2}{(\eta^*+\bar{a})} \int_o^{\bar{z}} [\phi(1 - b_5\phi)(1-F\, Re_{\bar{x}}) \exp 3(b_1\phi - 1)]\, d\bar{z} \quad .$$

$I(\bar{x})$ is the local turbulence intensity, $I(0)$ is the turbulence intensity at the initial point, x_o . $\bar{z} = \bar{x}/\Delta$ and b_1 , b_5 , B_1 , B_2

are numerical constants. Equation (6.13) gives the local disturbance amplification factor relative to an initial position of interest, x_o, as well as the local intensity of the turbulence. The constants B_1, B_2, and b_5 may be estimated from the following plausibility argument. As $\eta^* \to \infty$, it is conventional to assume that $I \to 0$. In actual flows, I at $\eta^* \to \infty$ is the free stream turbulence intensity which is observed not to change appreciably downstream. That is, $I(\bar{x})/I(o) \simeq 1$ as $\eta^* \to \infty$. This boundary condition suggests that $B_1 = 0.5$. The boundary condition at $\eta^* = 0$ cannot be considered as equation (6.13) is valid for $\eta^* \geq a$. In the upstream region of the boundary layer the argument of the integration is positive, decreasing in the downstream direction in line with the streamwise variation of the shear in boundary layers. This behavior is controlled by the function $\phi(1 - b_5\phi)$. Hence, b_5 must be of the order of $(1/\phi)$ at the position of maximum amplification. On the basis of this argument, transition data for flat plate flows were generated by the method discussed in Chapter III of this work, and b_5 has been estimated as about 660. Further, the requirement that for a flat plate boundary layer, natural transition from laminar to turbulent flow occurs when the amplification factor is of the order of e^9, yields that the constant B_2 is of the order of 300.

Hence, if one knows the distribution of the shear force in a flow problem, the collision theory gives through equations (6.11, 6.12, and 6.13) the appropriate characteristics of the turbulence in the flow. The above requirement is tantamount to knowing the local mean velocity distribution in the flow. In the earlier chapters of this report a few techniques for computing mean velocity profiles were discussed. The difference-differential and the integral relation methods seem to be very commonly used because of their relative simplicity. These latter methods however cannot describe the turbulence characteristics of the flow they predict. If however, the results of equations (6.11, 6.12, and 6.13) are coupled to the difference-differential or the integral relation prediction technique, they supplement each other and provide a very powerful but simple technique for calculating both laminar and turbulent flows of the most general type. The results shown in Figures (13 through 15) are quite encouraging toward the adoption and improvement of the statistical "collision" theory.

Chapter VII

7. A DISCUSSION OF PRESENT RESULTS

Three basic techniques for predicting wall boundary layer flows have been presented in this work. The first technique is a differential method in which the boundary layer equations for the mean and perturbation quantities are closed by the method of invariant modelling and solved directly by computer. The approximation in this method are those incurred in the modelling for third and higher order velocity correlations, and of course the boundary layer approximations used to simplify the equations. The assumption that the local disturbance length scales are proportional to the vertical distance from the wall, does not seem to introduce appreciable error in the results. The numerical method used to compute the results does, however, impose some limitations on the computer program. Small steps in x, of the order of 10 to 20 times the local boundary layer displacement thickness must be made if convergence is to be ensured. This limitation arises from the fact that by solving for f'' at each x-step one must iterate on two functions, f' and f for simultaneous convergence. Moreover the higher derivatives f''' and f^{iv} become important functions making the solutions very sensitive to inaccurracies in the numerical computation of f''' and f^{iv}. Better alternatives to the current numerical method are being sought. The predictions are nevertheless, very good in the laminar, transitional and turbulent regimes.

The second computational technique is another differential method but in which closure of the mean flow equations is attained through an eddy-viscosity concept. This method computes laminar and turbulent boundary layers. By assuming that the transitional boundary layer is a simple mixture of a laminar and a turbulent boundary layer, an intermittency factor was established to define what proportion of the time, a given location in the transitional boundary layer is turbulent. This linking mechanism ideally empowers this second computational technique to predict the entire boundary layer from laminar through transitional to the turbulent regimes. The primary deficiency of this computational method include all the defects of the eddy viscosity concept as well as those of the assumption of a simple intermittency nature for the transitional boundary layer. In most practical situations these defects, especially the latter, are not serious and as shown in Figures (10) through (12) the predictions are very good. This modified mean velocity field closure technique now has an advantage over the turbulence field closure discussed above, by being simpler and less demanding on Computer time. However, the mean velocity field closure method still has the major disadvantage of being unable to predict the turbulence characteristics of the flow it computes.

By suggesting that the transfer of any scalar variable in the boundary layer is governed solely by the fluid property of "cohesiveness" and by establishing that the cohesiveness of the fluid is determined solely by the flow stability number and the local position, it seems that fluid flow can be described as a non-linear system with an explicit transfer function. An attempt was made to evaluate this transfer function and to provide a simple step transfer of energy in the boundary layer streamwise direction. Predictions of simple two dimensional zero pressure gradient boundary layers made with this describing function method give satisfactory results in the fully laminar and fully turbulent boundary layers but it seems that one must go to more complicated transfer functions in order to predict the transitional and more complicated boundary layers satisfactorily with this method. However, the present results using this method are quite gratifying especially with regard to the infancy of the describing function method. The important thing is to note that such a transfer function does exist and that the parameters of importance are η^* and SN, only. At this stage, one cannot fairly compare the describing function method against other prediction methods in view of the former's present limitation to very simple flows. Nevertheless, the simplicity of the describing function method, at least for simple flows, gives reason for the optimism that it can soon be developed to provide a simpler and faster alternative to other prediction methods for two-dimensional flows.

Most theoretical work using a statistical formulation for turbulence have been done for spatially homogeneous or spatially homogeneous and isotropic turbulence. The practical problems however are neither spatially homogeneous nor isotropic. In this work a statistical collision theory which is not restricted to homogeneous or isotropic turbulence but which is valid for all types of shear flows has been developed to predict turbulence characteristics. This prediction technique requires prior knowledge of the local mean shear. Thus, coupled with a simple technique for predicting the mean flow shear, a simple iterative method could be developed to predict both the mean and the fluctuational characteristics of shear flows. This circumvents the necessity to solve complicated boundary layer equations for the fluctuational motion, with their inherent and often experimentally uncomfirmed approximations. The results of the present work have been coupled to the second differential method discussed here, and predictions have been made of the turbulence characteristics of a flat plate flow. The results, Figures (13) through (15) exhibit the proper trends. With this latter modification of the mean velocity field closure method of boundary layer prediction, the method becomes preferable over turbulence field methods. It was noted in preliminary computations that at practically no extra expense in computational time, the modified mean velocity field closure method, coupled to the statistical collision theory results, predicts both the mean and fluctuational motions of two dimensional flows just as well as turbulence field methods.

Extensions of the force field hypothesis have also yielded very good predictions of the points of incipient laminar instability and laminar to turbulent transition as well as the non-linear amplification characteristics of disturbances in a basic flow. Some of these predictions are shown in Figures (9) and (16).

Chapter VIII

8. CONCLUDING REMARKS

This work consists of three main parts. In the first part, a numerical experimental technique has been developed to satisfactorily generate boundary layer experimental results. The main advantages of this numerical data generating process lie in its capability to provide simulated physical experimental data at a fast rate and for a wide variety of combinations of desirable boundary conditions which may be impossible to attain in physical experimentation. Data generated by the numerical technique show no appreciable inferiority to equivalent physical experimental data. Moreover, there seems to be greater hope of improving mathematical models of physical phenomena than of improving measurement hardware and technique particularly for fluctuating flow variables. Also, a linking mechanism in the form of an intermittency factor (or transition function) has been developed to extend the power of differential and integral relation methods to the complete prediction of boundary layers from the laminar through the transitional to the turbulent regimes.

The second part of this work is based on a force-field hypothesis, which emphasizes a dynamic fluid property called the "cohesiveness" or fluid "tenacity". This fluid property defines the ability of the fluid to resist perturbations, and appears to be the primary factor in the determination of flow characteristics. All the preliminary results derived on the basis of the force-field hypothesis show no significant deviation from reality. In fact, the incompleteness of some previous boundary layer theories become very obvious in view of the force-field idea. Boundary layer phenomena and, indeed, general natural phenomena seem to follow exactly according to this force-field concept. A general force-field theory is therefore stated as follows:

"Particles in any system in nature will tend to execute independent behavior or motions in accordance with their separate internal force fields except in as constrained by the prevelant external force-field. It requires a steady force field above a certain critical magnitude to establish 'order' among the particles. The magnitude of this critical force field is determined by the average internal force field of the particles."

For the special case of wall boundary layers, the following conclusions are drawn on the basis of the force-field theory:

(i) The laminar, transitional and turbulent boundary layers represent different phases of the basic boundary layer fluid at least with respect to the "cohesive" fluid property.

(ii) The general characteristics of the wall boundary layer may be described, completely, solely by the local "cohesive" fluid property which is a function of position and section stability number, only. The section stability number is given by the following relation:

$$SN \doteq [Re_{\Delta}^2 + a_1(Ra + Ta + A_t)]^{1/2} .$$

(iii) The position of incipient laminar instability is given approximately by:

$$SN_{ci} \approx 1000.$$

for a length scale $\Delta \equiv \delta^* \, 10^{-2.34 \tanh(\frac{10}{H} - 4)}$.

This corresponds to:

$$Re_{\delta^* ci} \doteq \frac{\{1 - 0.012\ (Ra + Ta + A_t\}^{1/2}}{\{1 + 280\ I_1^2\}\ \{1 + v_o/U_1\}} \; 10^{\{3 + 2.34 \tanh(\frac{10}{H} - 4)\}}$$

(iv) The frequency of the most favored local disturbance is given approximately by:

$$f_p \approx 0.016 \frac{U_1}{\nu_1 \eta^*} \cdot \frac{1}{Re_\Delta}$$

where $\Delta \equiv \delta^* \, 10^{-2.34 \tanh(\frac{10}{H} - 4)}$

is a modified boundary layer thickness.

(v) The position of incipient natural laminar to turbulent transition is given approximately by:

$$SN_{ct} \approx SN_{ci} - 1.3 \times 10^3 \;\; \ell n(33 I_1); \;\; 0 < I_1 < 0.03$$

$$= SN_{ci} \;\; ; \text{otherwise}$$

i.e., $Re_{\delta^*\, ct} = Re_{\delta^* ci} - 674.2 \;\; \ell n(33 I_1)$.

(vi) The amplitude, K, for supercritical and subcritical disturbances, of the total disturbance kinetic energy at the critical layer is given approximately by:

$$K(x)_{max} \propto \frac{30\omega^{1/2} I_1 \exp[0.003(SN-SN_{ci})]}{1 + 0.33\ I_1\ (1-\omega)\ \exp[0.003(SN-SN_{ci})]}$$

and the maximum amplification factor is given by

$$a^{*2}_{max} = \frac{\omega^{1/2} \exp[0.003(SN-SN_{ci})]}{1 + 0.33\ I_1(1-\omega)\ \exp[0.003(SN-SN_{ci})]} .$$

(vii) The rate of production of turbulence (spots) may be given approximately as:

$$g \approx 1.53 \times 10^{-10} \left(\frac{U_1}{\nu_1}\right)^2 \{1 - \exp[-5.34 \times 10^{-5} (x-x_{ct})^2/\nu_1]\}.$$

An intermittency factor (or transition function) may then be defined to indicate what proportion, f, of the transitional boundary layer is turbulent at any instant.

$$f = 1 - \exp[-0.04\ g\ x^3/U_1]$$

(viii) A flow transfer function may be defined for a boundary layer scalar variable, q , such that:

$$q(x+\Delta x) \approx G_o(\eta^*,SN_D) + G_1(\eta^*,SN_D)q(x) + G_2(\eta^*,SN_D)q^2(x)$$

$$+ G_3(\eta^*, SN_D)q^3(x)$$

$$0 \leq q \leq 1 .$$

The boundary layer does indeed behave closely to a linear system except in the developing regions such as the entrance region and the transitional boundary layer. Even in these developing regions, the non-linearity is not very strong except when the external and wall influences such as pressure gradient, heat and mass transfer and wall curvature, are appreciable.

The third portion of this work assumes that conceptual fluid particles in a boundary layer fluid are arranged in continuous mean energy levels according to their mean energy content. The subsequent application of a statistical "collision theory" with these conceptual fluid particles as the basic particles has yielded qualitatively very good predictions of the characteristics of turbulence in the boundary layer, such as the rates of production and dissipation of turbulence energy, and the amplification characteristics of disturbances in shear flows.

IX

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APPENDICES

Appendix A

THE METHOD OF INVARIANT MODELLING

This technique for general shear flow computation due to C. Donaldson et al. (1968) is essentially a method of closing the equations of the Reynolds stress tensor by means of physical models of the higher order terms in these equations such that the models satisfy the following criteria:

i) They must exhibit all the tensor properties and properties of symmetry of the original terms.

ii) They must be dimensionally correct.

iii) They must be invariant under a Galilean transformation.

iv) They must be such as to satisfy all the general conservation laws.

When all these requirements are met, it is found that, if one wishes to consider the simplest models, the choice of models is not large. Details of the method of invariant modelling have been discussed by Donaldson et al. (1968, 1970, 1971), and will not be reproduced in this work. It can only be discussed here briefly why the method of invariant modelling is necessary and what are the final results.

The method of invariant modelling arose out of a desire to utilize the formidable power of modern computing machines by introducing more computation, if necessary, into a formulation for shear flows, just so as to get rid of some of the shortcomings of previous techniques while still retaining a method which will be useful for practical engineering problems. The eddy viscosity methods based on Prandtl's mixing length idea have been very useful in flow computation but because they really do not consider the dynamics of the development of the various components of the Reynolds stress tensor, they are restricted to only some types of shear flows. The methods due to Glushko (1965) and Bradshaw et al. (1967) which may be considered the forerunners of the method of invariant modelling attempt to keep track of the dynamics of the turbulence itself by computing the development of the local turbulent kinetic energy per unit mass,

$$E = \frac{1}{2}\ \overline{u_i' u_j'}$$

as a shear layer develops and relating the local Reynolds stress to this quantity. The shortcoming of these latter methods lies in the explicit assumption that must be made concerning the nature of the shear stress. The only assumptions of the method of invariant modelling are in the derivation of the models for the modelled terms. Donaldson and Rosenbaum (1968) have suggested the following models:

The velocity diffusion term: $-\ \rho(\overline{u'^j u_i' u_k'})_{,j}$

$$\equiv \rho\{\Lambda\sqrt{\overline{u'^m u_m'}}\ [g^{j\ell}(\overline{u_i' u_k'})_{,\ell} + (\overline{u'^j u_i'})_{,k} + (\overline{u'^j u_k'})_{,i}]\ \}_{,j} \qquad \text{(A-1)}$$

The tendency-toward-isotropy term: $\overline{p'(u'_{i,k} + u'_{k,i})}$

$$\equiv \frac{\rho\sqrt{\overline{u'^m u'_m}}}{\Lambda}\ \left(g_{ik}\ \frac{\overline{u'^m u'_m}}{3} - \overline{u_i' u_k'}\ \right)\ . \qquad \text{(A-2)}$$

The dissipation term: $\mu \overline{g^{mn} \frac{\partial u_i'}{\partial x^m} \frac{\partial u_k'}{\partial x^n}}$

$$\equiv \mu \frac{\overline{u_i'u_k'}}{\lambda^2} \tag{A-3}$$

The pressure diffusion term: $\overline{(p'u_k')}_{,i} + \overline{(p'u_i')}_{,k}$

$$\equiv -[\rho\sqrt{\overline{u'^m u_m'}}\ \Lambda\ (\overline{u'^\ell u_k'})_{,\ell}]_{,i} - [\rho\sqrt{\overline{u'^m u_m'}}\ \Lambda\ (\overline{u'^\ell u_i'})_{,\ell}]_{,k} \tag{A-4}$$

These models have been obtained using only the second-order correlations and two length scales Λ and λ which are related as follows:

$$\lambda = \frac{\Lambda}{\sqrt{a+bR}}$$

where

$$R = \frac{\rho\sqrt{\overline{u'^m u'_m}}\ \Lambda}{\mu} \tag{A-5}$$

and a and b are numerical constants. The quantities, a, b, λ and Λ have been sufficiently discussed in Chapter 3 of this work. Using these results, the set of equations necessary to describe the characteristics of the most general type of incompressible shear flow are:

$$\bar{u}^j_{,j} = 0 \tag{A-6}$$

$$\rho \frac{\partial \bar{u}}{\partial t} + \rho \bar{u}^j \bar{u}_{i,j} = - \frac{\partial \bar{p}}{\partial x^i} + (\bar{\tau}_i^j - \rho\overline{u'^j u_i'})_{,j} \tag{A-7}$$

$$\rho \frac{\partial}{\partial t} (\overline{u_i'u_k'}) + \rho \bar{u}^j (\overline{u_i'u_k'})_{,j} = - \overline{\rho u'^j u_k'}\ \bar{u}_{i,j} - \overline{\rho u'^j u_i'}\ \bar{u}_{k,j}$$

$$\rho\{\Lambda\sqrt{\overline{u'^m u_m'}}\ [g^{j\ell}(\overline{u_i'u_k'})_{,\ell} + (\overline{u'^j u_i'})_{,k} + (\overline{u'^j u_k'})_{,i}]\}_{,j}$$

$$+ \rho\{[\Lambda\sqrt{\overline{u'^m u'_m}}\ (\overline{u'^\ell u'_i})_{,\ell}]_{,k} + [\Lambda\sqrt{\overline{u'^m u'_m}}\ (\overline{u'^\ell u_k'})_{,\ell}]_{,i}\}$$

$$+ \frac{\rho\sqrt{\overline{u'^m u_m'}}}{\Lambda}\ g_{ik} \frac{\overline{u'^m u_m'}}{3} - \overline{u_i'u_k'}$$

$$+ \mu g^{mn}(\overline{u_i'u_k'})_{,mn} - 2\mu \frac{\overline{u_i'u_k'}}{\lambda^2} \quad . \tag{A-8}$$

In these equations,

$$\bar{\tau}_{ij} = \mu(\bar{u}_{i,j} + \bar{u}_{j,i}) \tag{A-9}$$

and

g_{ij} is the metric tensor.

For time-independent incompressible boundary layer flows with velocity components $(\bar{u}, \bar{v}, 0)$ and (u', v', w'), the equations are, in the orthogonal curvilinear coordinate system (x,y,z) as shown below.

Continuity:

$$\frac{1}{(1-cy)} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} - \frac{cv}{(1-cy)} = 0 \tag{A-10}$$

x-momentum:

$$\frac{1}{(1-cy)}\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - \frac{cuv}{(1-cy)} = -\frac{1}{\rho(1-cy)}\frac{\partial p}{\partial x}$$

$$+ \nu[\nabla^2 u - \frac{2c}{(1-cy)^2}\frac{\partial v}{\partial x} - \frac{c}{(1-cy)}\frac{\partial u}{\partial y} - \frac{v\ dc/dx}{(1-cy)^3}]$$

$$+ \frac{\partial \sigma}{\partial y} - \frac{c\sigma}{(1-cy)} - \frac{1}{(1-cy)}\frac{\partial k_1}{\partial x} \qquad \text{(A-11)}$$

y-momentum:

$$\frac{u}{(1-cy)}\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{cu^2}{(1-cy)} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$+ \nu[\nabla^2 v + \frac{2c}{(1-cy)^2}\frac{\partial u}{\partial x} - \frac{c^2 v}{(1-cy)} - \frac{c}{(1-cy)}\frac{\partial v}{\partial y} + \frac{u\ dc/dx}{(1-cy)^3}]$$

$$+ \frac{\partial \sigma}{\partial x} - \frac{\partial k_2}{\partial y} - \frac{ck_1}{(1-cy)} + g(\frac{T-T_1}{T_1}). \qquad \text{(A-12)}$$

Energy:

$$\frac{u}{(1-cy)}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\nabla^2 T - \frac{\partial \overline{v'T'}}{\partial y}$$

$$+ \{\nu\ (\frac{\partial u}{\partial y}) + 2\nu(\frac{\partial v}{\partial y})^2 + 2\nu[\frac{1}{(1-cy)}\frac{\partial u}{\partial x} + \frac{cv}{(1-cy)}]^2\}/C_p$$

(A-13)

where

$$\nabla^2 = \frac{1}{(1-cy)^2} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{y \frac{dc}{dx} \frac{\partial}{\partial x}}{(1-cy)^3}$$

$$\alpha = \frac{k}{\rho c_p} \text{ (the thermal diffusivity)}$$

Streamwise velocity correlation: $\overline{u'u'} = k_1$

$$\frac{u}{(1-cy)} \frac{\partial k_1}{\partial x} + v \frac{\partial k_1}{\partial y} = 2\sigma \frac{\partial u}{\partial y} - \frac{2k_1}{(1-cy)} \frac{\partial u}{\partial x} + \frac{2k_1 \, cv}{(1-cy)}$$

$$+ \frac{\partial \nu}{\partial y} \frac{\partial}{\partial y} [2k_1 - \frac{K}{3}] + \frac{2}{(1-cy)} \frac{\partial}{\partial y} \frac{\partial \sigma}{\partial x} + [50 \frac{\partial u}{\partial y} + \frac{\sqrt{K}}{y}](\frac{K}{3} - k_1)$$

$$+ \nu \frac{\partial^2 k_1}{\partial y^2} - \frac{2\nu k_1}{y^2} + \frac{1}{(1-cy)} \frac{\partial}{\partial x} [\frac{y\sqrt{K}}{(1-cy)} \frac{\partial k_1}{\partial x} + 2y\sqrt{K} \frac{\partial \sigma}{\partial y}]$$

$$+ \frac{\partial}{\partial y} [y\sqrt{K} \frac{\partial k_1}{\partial y}] \qquad \text{(A-14)}$$

Transverse velocity correlation: $\overline{v'v'} = k_2$

$$\frac{u}{(1-cy)} \frac{\partial k_2}{\partial x} + v \frac{\partial k_2}{\partial y} = -2k_2 \frac{\partial v}{\partial y} + \frac{\partial^2 k_2}{\partial y^2} - \frac{2\nu k_2}{y^2}$$

$$+ \frac{\partial}{\partial y} [\frac{2y\sqrt{K}}{(1-cy)} \frac{\partial \sigma}{\partial x} + y\sqrt{K} \frac{\partial k_2}{\partial y}] + [50 \frac{\partial u}{\partial y} + \frac{\overline{K}}{y}](\frac{K}{3} - k_2)$$

$$+ \frac{1}{(1-cy)} \frac{\partial}{\partial x} [\frac{y\sqrt{K}}{(1-cy)} \frac{\partial k_2}{\partial x}] + \frac{\partial \nu}{\partial y} \frac{\partial}{\partial y} [4k_2 - \frac{K}{3}]$$

$$\text{(A-15)}$$

Lateral velocity correlation: $\overline{w'w'} = k_3$

$$\frac{u}{(1-cy)}\frac{\partial k_3}{\partial x} + v\frac{\partial k_3}{\partial y} = \frac{\partial \nu}{\partial y}\frac{\partial}{\partial y}\left[2k_3 - \frac{K}{3}\right] + \left[50\frac{\partial u}{\partial y} + \frac{\sqrt{K}}{y}\right]\left(\frac{K}{3} - k_3\right) + \nu\frac{\partial^2 k_3}{\partial y^2} - \frac{2\nu k_3}{y^2} + \frac{1}{(1-cy)}\frac{\partial}{\partial x}\left[\frac{y\sqrt{K}}{(1-cy)}\frac{\partial k_3}{\partial x}\right] + \frac{\partial}{\partial y}\left[y\sqrt{K}\frac{\partial k_3}{\partial y}\right] \qquad \text{(A-16)}$$

Cross correlation: $\overline{u'v'} = -\sigma$

$$\frac{u}{(1-cy)}\frac{\partial \sigma}{\partial x} + v\frac{\partial \sigma}{\partial y} = 3\frac{\partial \nu}{\partial y}\frac{\partial \sigma}{\partial y} + k_2\frac{\partial u}{\partial y} + \frac{\partial}{\partial y}\left[\frac{y\sqrt{K}}{(1-cy)}\frac{\partial k_1}{\partial x}\right] + \nu\frac{\partial^2 \sigma}{\partial y^2} - \frac{2\nu\sigma}{y^2} + \frac{1}{(1-cy)}\frac{\partial}{\partial x}\left[y\sqrt{K}\frac{\partial k_2}{\partial y} - \frac{y\sqrt{K}}{(1-cy)}\frac{\partial \sigma}{\partial x} - y\sqrt{K}\frac{\partial k_1}{\partial y}\right] - \left(50\frac{\partial u}{\partial y} + \frac{\sqrt{K}}{y}\right). \qquad \text{(A-17)}$$

It would be desirable to eliminate the pressure terms in the momentum equations. This may be achieved by combining the x- and y-momentum equations as follows: differentiate the x-momentum equation with respect to y and the y-momentum equation with respect to x and subtract. The combined momentum equation, the so-called vorticity equation is:

$$\frac{\partial}{\partial y}\left[\frac{u}{(1-cy)}\frac{\partial u}{\partial x}\right] + \frac{\partial}{\partial y}\left[v\frac{\partial u}{\partial y}\right] - \frac{\partial}{\partial y}\left[\frac{cuv}{1-cy}\right] - \frac{\partial}{\partial x}\left[\frac{u}{(1-cy)}\frac{\partial v}{\partial x}\right]$$

$$- \frac{\partial}{\partial x}\left[v\frac{\partial v}{\partial y}\right] - \frac{\partial}{\partial x}\left[\frac{cu^2}{1-cy}\right] = \frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y} - \frac{1}{\rho}\frac{\partial}{\partial y}\left[\frac{1}{(1-cy)}\frac{\partial p}{\partial x}\right]$$

$$+ \frac{\partial}{\partial y}\{\nu[\nabla^2 u - \frac{2c}{(1-cy)^2}\frac{\partial v}{\partial x} - \frac{c}{(1-cy)}\frac{\partial u}{\partial y} - \frac{v\ dc/dx}{(1-cy)^3}]\} + \frac{\partial^2\sigma}{\partial y^2}$$

$$- \nu\frac{\partial}{\partial x}[\nabla^2 v + \frac{2c}{(1-cy)^2}\frac{\partial u}{\partial x} - \frac{c^2 v}{(1-cy)^2} - \frac{c}{(1-cy)}\frac{\partial v}{\partial y}$$

$$+ \frac{u\ dc/dx}{(1-cy)^3}]$$

$$- \frac{\partial^2\sigma}{\partial x^2} - \frac{\partial}{\partial x}[g(\frac{T-T_1}{T_1})] - \frac{\partial}{\partial y}\left[\frac{c\sigma}{(1-cy)}\right] - \frac{\partial}{\partial y}\left[\frac{1}{1-cy}\frac{\partial k_1}{\partial x}\right]$$

$$+ \frac{\partial^2 k_2}{\partial x\partial y} + \frac{\partial}{\partial x}\left[\frac{c\ k_1}{1-cy}\right]\ . \tag{A-18}$$

To compute a given boundary layer flow, an initial boundary layer profile at some point in the flow $(x = x_o)$ must be given together with the distributions of $\overline{u'u'}$, $\overline{v'v'}$, $\overline{w'w'}$ and $\overline{u'v'}$, at this station. The pressure gradient on the surface, that is, the mean velocity external to the boundary layer, must be given. In general, it is assumed that $\overline{u'u'} = \overline{v'v'} = \overline{w'w'} = \overline{u'v'} = 0$ at $y = \infty$ and at $y = 0$. If the initial boundary layer is chosen laminar but contains some small initial disturbance, say $\overline{u'u'} = \overline{v'v'} = \varepsilon_o(y)$, a kind of transition to a turbulent flow takes place. The character of this turbulent boundary layer is independent of the initial conditions which started the transition at sufficient distances downstream from $x = x_o$.

Appendix B

APPENDIX TO CHAPTER III OF TEXT

In so far as possible, consistency is maintained between the notations in the text and in the computer program. The last letter of a program notation is usually related to the subscript or superscript of the quantity it symbolizes in the text. For example,

Endings (or subscripts)	Usual significance
M	Quantities evaluated at mean position (i + 1/2), say
x	Derivatives with respect to x
W(o)	Value at the wall
E(1)	Value at the free stream
R	Ratios, of local to wall or pushed values
P(') , PP('')	Derivatives with respect to η.
B(i-1)	Value at previous x-stations
MAX (MX)	Maximum value

PROGRAM NOTATION

A1 - A34	Parameters (a_1-a_{34}) in the dimensionless equations of motion
AK1, AK2, AK3	Double velocity correlations for the fluctuation: $\overline{u'u'}/U_1^2 = k_1$, $\overline{v'v'}/U_1^2 = k_2$, $\overline{w'w'}/U_1^2 = k_3$

AK1B, AK3B	Values of AK1, AK2, AK3 at previous x-station
AK1P, AK1PP,	k_1' , k_2''
AK2P, AK2PP,	k_2' , k_2''
AK3P, AK3PP	k_3' , k_3''
ANUX	Local Nusselt Number (based on x); $- x\ (\partial T/\partial y)y{=}0/(T_o-T_1)$
AMPF	Amplification parameter
CF	$2\tau_o/\rho_1 U_1^2$, Skin friction coefficient
CFR	τ/τ_o , Local shear stress ratio
CHL	Characteristic body length, L
CW, (CWM)	Wall curvature $(1/r_o)$, $(CW_{i-1/2})$
CWO	(δ^*/r_o) , wall curvature parameter
CW1	$1 + (\delta^*/r_o)y/\delta^*$
CP	Specific heat at constant pressure C_p
DETO	$(\Delta T)_o = (1 - T_1/T_o)$
DETOX	$[(T_o/T_1)_i - (T_o/T_1)_{i-1}]/\Delta x$
DT	δ^* , displacement thickness
DTS	Value of δ^* before it is altered to conform to integral momentum equation
DUDX	dU_1/dx , x-gradient of freestream velocity
DRDT	$\partial(1/R_{\delta^*})/\partial\eta$,
D2RDT	$\partial^2(1/R_{\delta^*})/\partial\eta^2$
DTX	$d\delta^*/dx$
DTXM	$\frac{1}{2}\ [(d\delta^*/dx)_i + (d\delta^*/dx)_{i-1}]$
DX	$X_i - X_{i-1}$, (ΔX)

DY	$\eta_i - \eta_{i-1}$, $(\Delta\eta)$
E	Eckert Number, $U_1^2/[C_p(T_o - T_1)]$
EPS	ξ , convergence criterion specifying tolerance
F	f } mean velocity field
F1,F2,F3,F4	f', f'', f''', f^{iv} } mean velocity field
FV	u/U_1
FXM	$(f_i - f_{i-1})/\Delta x$
FMY	$(f - \eta)$
FM1	$(1 - f')$
FRDT	$FR_{\delta*}$, Pressure gradient parameter
GAM	$\sqrt{(\tau_o/\rho U_1^2)}$, friction velocity/free stream velocity
GC	g , gravitational constant
GDT	Grashof Number (based on $\delta*$), $g(T_o-T_1)\delta*^3/(T_1\nu_1^2)$
GPW	T_o/T_1
H	Shape factor, $\delta*/\theta$
HQ	$\Delta*/\delta*$
I	x-direction index
ID	Maximum number of x-stations
J	y-direction index
JEY, JEF, JEG, JK1, JK2, JK3, JSG	Maximum number of data points read for η, f', q', k_1, k_2, k_3 and σ, respectively
JD(JE,JY)	Absolute maximum number of data points for any profile

KT, KOUNT	Iteration counter
L	Number of subdivions of the profiles
PF, (PFM)	Pressure gradient parameter $\frac{\nu_1}{U_1^2} \frac{dU_1}{dx}$, (intermediate value)
PR	Prandtl Number, $\mu C_p/k$
Q1, Q2, Q3	q', q", q''' Mean temperature field
RE (REM)	Unit Reynolds Number, U_1/ν_1 (intermediate value)
REDT, (RDT)	$U_1\delta^*/\nu_1$, ($U_1\delta^*/\nu$, local)
REDX	x-Reynolds Number, $U_1 x/\nu_1$
RHO	Density, ρ_1
RUF	Equivalent Nikuradse wall roughness height, K_s
SC	110^oK (for air) . Constant in Sutherland's viscosity formula
SIG, SIGP, SIGPP	σ , σ', σ'' , velocity cross correlation field
SN	Stability Number
ST	Stanton Number, $\alpha/\rho C_p U_1$ = ANUX/(REDX·PR)
STAB	Stability parameter
TAU	$\tau/\rho_1 U_1^2$
TATE	T/T_1
TE	T_1 , free stream ambient temperature
TW	T_o , wall temperature
TK, (STK)	$K = k_1 + k_2 + k_3$, ($\sqrt{K}$)
TIE, (TI)	I_1 , free stream turbulence intensity

TIR	I_1 at initial x-station
THC	k = thermal conductivity
THETAF	θ , momentum thickness
THETAQ	Δ^* , energy thickness
U	Local mean velocity, u
UE	Freestream velocity, U_1
VE, (VEE)	Kinematic viscosity, ν, (ν_1)
VER	ν/ν_1 , ratio of local to freestream kinematic viscosity
VKC	Von-Karman constant
VW	Blowing (or suction) velocity, v_o
VOW, (VOWM)	v_o/U_1 , (intermediate value)
X , (XD)	Local x , ($\overline{x} = x/L$)
Y	$\eta = y/\delta^*$
YPS	Effective roughness scale, K_s/δ^*

The variable coefficients, a_1 - a_{34} for equations (3.22) through (3.28) in the text are as follows:

$$a_1 = \frac{1}{\overline{R}_{\delta*}(1+\overline{C}\eta)}$$

$$a_2 = [\{2 \frac{\partial}{\partial\eta} (\frac{1}{\overline{R}_{\delta*}}) - \overline{v}_o\} \frac{1}{(1+\overline{C}\eta)} - (\overline{f} - \eta)(\overline{\delta}^*_x + \overline{F\,Re}_{\delta*})]$$

$$a_3 = [(\frac{1}{1+\overline{C}\eta}) \{ \overline{C}\,\overline{v}_o + \frac{\partial^2}{\partial\eta^2} (\frac{1}{\overline{R}_{\delta*}}) + (1-\overline{f}')\overline{\delta}^*_x\} + \overline{C}(\overline{f}-\eta)(\overline{\delta}^*_x + \overline{F}\,Re_{\delta*}) - \overline{C}\,\overline{F\,Re}_{\delta*}]$$

$$a_4 = [\overline{C}(2-\overline{f}')\,\overline{F\,Re}_{\delta*} + \overline{f}''\,\overline{F\,Re}_{\delta*}]$$

$$a_5 = (\bar{f}''' - \bar{C}\,\bar{f}'')(\delta^*_i + \delta^*_{i-1})/\Delta x$$

$$a_6 = \bar{C}(1 - \bar{f}')(\delta^*_i + \delta^*_{i-1})/\Delta x$$

$$a_7 = (1 - \bar{f}')(\delta^*_i + \delta^*_{i-1})/\Delta x$$

$$a_8 = \left[\frac{\bar{\sigma}''}{(1+C\eta)} - C\,\bar{\sigma}'\right]$$

$$a_9 = \frac{4\,\overline{F\,Re_{\delta^*}}}{(1+\bar{C}\eta)}\,(\delta^*_i + \delta^*_{i-1})/\Delta x$$

$$a_{10} = \frac{6(\overline{F\,Re_{\delta^*}})^2}{(1+\bar{C}\eta)}$$

$$a_{11} = \frac{\overline{Re}_{\delta^*}}{(1+\bar{C}\eta)}\,(\sigma_i + \sigma_{i-1})\bar{\delta}^*/\Delta x$$

$$a_{12} = 0$$

$$a_{13} = 0$$

$$a_{14} = \left[\frac{\bar{v}_o}{(1+\bar{C}\eta)} + (\bar{f} - \eta)(\bar{\delta}^*_x + \overline{F\,Re_{\delta^*}})\right]$$

$$a_{15} = \left[(1-\bar{f}')\bar{\delta}^*\left(\left.\frac{T_o}{T_1}\right|_i - \left.\frac{T_o}{T_1}\right|_{i-1}\right)/(\Delta x\{\bar{T}_o/T_1 - 1\})\right]$$

$$a_{16} = \frac{\bar{E}}{2\overline{Re}_{\delta^*}}\;\frac{1}{(1+\bar{C}\eta)}$$

$$a_{17} = a_1$$

$$a_{18} = [a_2 + \overline{C}/\overline{R}_{\delta *} (1 + \overline{C}\eta)$$

$$a_{19} = [2(\overline{f}' - 1)\overline{F\ Re}_{\delta *} + 50\overline{f}'' - 2/\eta^2\ \overline{R}_{\delta *}]/(1 + \overline{C}\eta)$$

$$a_{20} = -\ \overline{k}_1' \ (\delta_i^* + \delta_{i-1}^*)/\Delta x$$

$$a_{21} = 0$$

$$a_{22} = 0$$

$$a_{23} = -\ 2\overline{f}''/(1 + \overline{C}\eta)$$

$$a_{24} = [\ \frac{\partial}{\partial \eta} (\frac{1}{\overline{R}_{\delta *}}) \ \ \frac{\partial}{\partial \eta} (\overline{K}) + 50\overline{f}''\ \overline{K}] \ \frac{1}{3(1+\overline{C}\eta)}$$

$$a_{25} = a_{18} + \frac{2}{(1+C\eta)} \frac{\partial}{\partial \eta} (\frac{1}{\overline{R}_{\delta *}})$$

$$a_{26} = a_{19} - 2(\overline{f}'-1)\ \overline{F\ Re}_{\delta *}$$

$$a_{27} = \overline{k}_2'(\delta_i^* + \delta_{i-1}^*)/\Delta x$$

$$a_{28} = 0$$

$$a_{29} = a_{26}$$

$$a_{30} = \overline{k}_3' \, (\delta_i^* + \delta_{i-1}^*)/\Delta x$$

$$a_{31} = [a_2 + \frac{\partial}{\partial \eta} (\frac{1}{\overline{R}_{\delta *}}) \frac{1}{(1+\overline{C}\eta)}]$$

$$a_{32} = a_{26}$$

$$a_{33} = \overline{\sigma}' (\delta_i^* + \delta_{i-1}^*)/\Delta x$$

$$a_{34} = \frac{\overline{k}_2 \, \overline{f}''}{(1+\overline{C}\eta)}$$

$Re_{\delta *}$ $\frac{U_1 \delta^* L}{\nu_1}$, value at free stream.

$R_{\delta *} = \frac{U_1 \, \delta^* L}{\nu}$, local value.

The coefficients A_j , B_j , C_j and D_j for the set of equations (3.23 through 3.28) are as tabulated below:

$j \geq 2$

A_j	B_j	C_j	D_j
$-(\frac{a_1}{\Delta_1} + \frac{a_2}{\Delta_2})$	$(a_3 - a_7) - \frac{2a_1}{\Delta_3}$	$(\frac{a_2}{\Delta_2} - \frac{a_1}{\Delta_4})$	$[-a_1 f''' - a_2 f''' - (a_3 + a_7) f'']_{i-1}$ $+ 2a_8 + a_{10} (\sigma_i + \sigma_{i-1}) - a_9(\sigma_1 - \sigma_{i-1})$ $+ a_{12}(q_i' - q_{i+1}') - a_{11}(F_i - F_{i-1})$ $- a_{14}(f_i' + f_{i-1}' + a_6(f_i' - f_{i-1}')$ $+ a_5(f_i - f_{i-1}) + a_{13}(q_i' + q_{i-1}')$
$-(\frac{a_1}{P_r \Delta_1} + \frac{b_6}{\Delta_2})$	$-(a_{15} + a_7) - \frac{2a_1}{P_r \Delta_3}$	$(\frac{b_6}{\Delta_2} - \frac{a_1}{P_r \Delta_4})$	$[-\frac{a_1}{P_r} q''' - b_6 q'' + (a_{15} - a_7) q']_{i-1}$ $- a_{16}(f_i''^2 + f_{i-1}''^2)$
$-(\frac{a_{17}}{\Delta_1} + \frac{a_{18}}{\Delta_2})$	$(a_{19} - a_7) - \frac{2a_{17}}{\Delta_3}$	$(\frac{a_{18}}{\Delta_2} - \frac{a_{17}}{\Delta_4})$	$[-a_{17} k_1'' - a_{18} k_1' - (a_7 + a_{19}) k_1]_{i-1}$ $- a_{20}(f_i - f_{i-1}) - a_{21}(f_i' - f_{i-1}')$ $- a_{22}(\sigma_i - \sigma_{i-1}) - a_{23}(\sigma_i + \sigma_{i-1}) + 2a_{24}$
$-(\frac{a_{17}}{\Delta_1} + \frac{a_{25}}{\Delta_2})$	$(a_{26} - a_7) - \frac{2a_{19}}{\Delta_3}$	$(\frac{a_{25}}{\Delta_2} - \frac{a_{17}}{\Delta_4})$	$[-a_{17} k_2'' - a_{25} k_2' - (a_{26} + a_7) k_2]_{i-1}$ $- a_{27}(f_i - f_{i-1}) + a_{28}(f_i' - f_{i-1}') + 2a_{24}$
$-(\frac{a_{17}}{\Delta_1} + \frac{a_{18}}{\Delta_2})$	$(a_{29} - a_7) - \frac{2a_{17}}{\Delta_3}$	$(\frac{a_{33}}{\Delta_2} - \frac{a_{17}}{\Delta_4})$	$[-a_{17} k_3'' - a_{18} k_3' - (a_{29} + a_7) k_3]_{i-1}$ $+ a_{30}(f_i - f_{i-1}) + 2a_{24}$
$-(\frac{a_1}{\Delta_1} + \frac{a_{31}}{\Delta_2})$	$(a_{32} - a_7) - \frac{2a_1}{\Delta_3}$	$(\frac{a_{31}}{\Delta_2} - \frac{a_1}{\Delta_4})$	$[-a_1 \sigma'' - a_{31} \sigma' - (a_{32} + a_7) \sigma]_{i-1}$ $+ a_{33} (f_i - f_{i-1}) + 2a_{34}$

APPENDIX C

DERIVATION AND SOLUTION OF EQUATION (4.39) IN THE TEXT

As taken directly from Chapter 3, the equation for the total kinetic energy of the fluctuating flow at a point which lies between the points i and $i-1$, is as follows:

$$a_{17}\,[K''_i + K''_{i-1}] + a_{18}\,[K'_i + K'_{i-1}] + (a_{29} - a_7)\,K_i$$

$$+ (a_{29} - a_7)\,K_{i-1} = (-\,a_{20} - a_{27} + a_{30})\,(f_i - f_{i-1})$$

$$+ (a_{28} - a_{21})\,(f'_i - f'_{i-1})\;. \qquad \text{(C-1)}$$

If K_{i-1} is taken to be always K_{ci} and if

$$\Omega = K/K_{ci}$$

equation C.1 becomes:

$$a_{17}\,K_{ci}\,\Omega'' + \Omega'\,[2a_{17}\,K'_{ci} + a_{18}\,K_{ci}]$$

$$+ \Omega\,[a_{17}\,K''_{ci} + a_{18}\,K'_{ci} + (a_{29} - a_7)\,K_{ci}] + a_{17}\,K''_{ci}$$

$$+ a_{18}\,K'_{ci} + (a_{29} + a_7)\,K_{ci} + (a_{20} + a_{27} - a_{30})(f_i - f_{ci})$$

$$-\,(a_{28} - a_{21})\,(f'_i - f'_{ci}) = 0 \qquad \text{(C-2)}$$

The following assumptions were made in Chapter (4).

$$K_{ci}(\eta) \doteq A\ SN_{ci}^{1/2}\ \eta^{*2}\ \exp\ (-\eta^{*2})$$

$$\frac{u}{U_1} \doteq b_1\ \eta^* + b_2\ SN\ .\ K \qquad \text{(C-3)}$$

where b_1 and b_2 are numerical constants and A is a function of free stream turbulence intensity.

In the neighborhood of the critical layer of the boundary layer ($\eta \simeq 1.0$) one obtains that

$$a_{17}\ K_{ci} \simeq b_3\ A^{3/2}\ SN_{ci}^{3/2}\ \eta^{*4}(1 + \Omega)^{1/2}$$

$$a_{17}\ K'_{ci} \simeq 2b_3\ A^{3/2}\ SN_{ci}^{3/2}\ \eta^{*3}(1 + \Omega)^{1/2}$$

$$a_{18}\ K_{ci} \simeq A^{3/2}\ SN_{ci}^{3/2}\ \eta^{*3}[2 + \Omega'/2(1+\Omega)](1+\Omega)^{1/2}$$

$$a_{18}\ K'_{ci} \simeq 2A^{3/2}\ SN_{ci}^{3/2}\ \eta^{*2}[2 + \Omega'/(2\eta(1+\Omega)\ A\ SN_{ci})]\ (1+\Omega)^{1/2}$$

$$a_{17}\ K''_{ci} \simeq 2b_3 A^{3/2}\ SN^{3/2}\ \eta^{*2}(1+\Omega)^{1/2}$$

$$(a_{29} \pm a_7)\ K_{ci} \simeq \mp\ b_4\ A^2\ SN_{ci}^{5/2}\ \omega^{1/2}\ \eta^{*3} + b_4\ A^2\ SN^{5/2}\omega^2\ \eta^{*3}\Omega$$

$$- b_5\ A\ SN_{ci}\ \eta^{*2} - b_6\ A\ SN^3\ \omega^2\ \eta^{*4}\Omega'$$

$$- b_5\ A^2\ SN^3\omega^2\ \eta^{*3} - b_5\ A^2 SN^3\ \omega^2\ \eta^{*3}\Omega \pm b_4\ A\ SN^{1/2}\ \omega\eta^{*3}$$

$$(a_{20} + a_{27} - a_{30}) \approx d_1 \, ASN_{ci}^{1/2} \, \omega^{1/2} \, \eta^*[\eta^*\Omega' + 2\Omega + 2]$$

$$(a_{28} - a_{21}) \approx 0$$

$$(f_i - f_{ci}) \approx A \, b_2 \, \eta^{*3} SN^2 \, \omega \, [\omega - \Omega]$$

$$(f_i' - f_{ci}') \approx A \, b_2 \, \eta^{*2} SN^2 \, \omega \, (\omega - \Omega)$$

where the b,s and d,s are numerical constants. Making the above substitutions in equation (C.2) one obtains after dividing through by

$$b_3 \, A^{3/2} \, SN_{ci}^{3/2} \, \eta^{*4}(1+\Omega)^{1/2}$$

and simplifying the terms:

$$\Omega'' + \Omega'^2/(\eta^{*2}(1+\Omega)) - \Omega' \, d_1 \, SN^{3/2} \, \omega^{1/2} \, (1+\Omega)^{1/2}/A^{1/2}$$

$$- \Omega^2 \, d_2 \, A^{1/2} \, SN^{3/2} \, \omega^{1/2}/(\eta^*(1+\Omega)^{1/2}) \; -\Omega \, d_3 \, (1/\eta^* + A \, SN^2 \, \omega^{3/2})/\eta^* A^{1/2} SN_{ci}^{1/2}$$

$$+ \; d_4 \, \omega^{3/2}/(\eta^*(1+\Omega)^{1/2} \, A^{1/2} \, SN_{ci}^{1/2}) \approx 0 \qquad \text{(C-4)}$$

where $\omega = SN_{ci}/SN$ and the d,s are numerical constants. Equation C.4 is valid in the region $\eta^* \approx 1.0$, that is, in the neighborhood of the critical layer. For this region, C.4 may be written in the following form: where $\Omega \equiv Y$.

$$Y'' + F_2(Y)\, Y'^2 + F_1(Y)\, Y' + F_o(Y) = 0 \tag{C.5}$$

where

$$F_1(Y) \equiv - d_1 SN^{3/2}\, \omega^{1/2}\, (1+\Omega)^{1/2}/A^{1/2}$$

$$F_2(Y) \equiv 1/(1+\Omega)$$

$$F_o(Y) \equiv - d_2\, A^{1/2}\, SN^{3/2}\, \omega^{1/2}/(1+\Omega)^{1/2}$$

$$- d_3\, (1 + A\, SN^2\, \omega^{3/2})/A^{1/2}\, SN_{ci}^{1/2}$$

$$+ d_4\, \omega^{3/2}/(A^{1/2}\, SN_{ci}^{1/2}\, (1+\Omega)^{1/2})$$

Under the transformations

$$Y'(\eta) \equiv P(Y)$$

$$P(Y) \equiv 1/U(Y)$$

equation C.5 reduces to the following

$$U' = F_2 + F_1 U + F_o U^2 \tag{C-6}$$

which is the classical Abel's equation. Obviously the roots of the right hand side polynomial $f(U) = 0$, of (C-6) are themselves solutions of equation (C-6). The general solution is given, according to Davis (1962) as follows:

$$(U - u_1)^a \; (U - u_2)^b \; (U - u_3)^c = D \exp (F_o Y) \qquad \text{(C-7)}$$

where u_1, u_2 and u_3 are the roots of $f(U) = 0$, a, b, c are fixed constants and D is an arbitrary constant. In the present problem, the roots u_1, u_2, and u_3 are given as:

$$u_1 \, , u_2 = - \frac{F_1}{2F_o} \pm ((\frac{F_1}{2F_o})^2 - \frac{F_2}{F_o}) \qquad \text{(C-8)}$$

$$u_3 = 0$$

Reverting to equation (C-5) with $a = b = 1$ and $c = 0$, one obtains the following solution:

$$Y'^2 \; [D \exp (F_o Y) - u_1 u_2] + Y' \; (u_1 + u_2) - 1 = 0 \, , \qquad \text{(C-9)}$$

which may be solved as a quadratic equation in Y' . One obtains implicitly that:

$$Y' = f(Y)$$

i.e., $\int \frac{dY}{Y'} = \eta + \text{Function } (A, \; \omega, \; SN).$ (C-10)

From equation (C-9), one obtains that, approximately:

$$\int \frac{dY}{Y'} \doteq A \exp[-B.SN(Y-\omega)Y^{1/2} - \omega/Y]$$

where B is a numerical constant. If one assumes that close to the wall, (i.e., $\eta^* \to 0$), $Y(= \Omega/\Omega_{ci})$ tends to unity, then Function (A,ω,SN) in equation (C-10) becomes approximately $B_1A(1-\omega)\exp[B.SN(1-\omega)]$, where B_1 is a numerical constant. Hence, in the neighborhood of the critical layer, the solution to equation (C-5) may be written as:

$$M \exp(\xi)\ B_3 \exp[-B.SN(Y-\omega)Y^{1/2} - \omega/Y - B.SN(Y-\omega)]$$

$$= \quad 1 + M(1-\omega)\exp(\xi) \qquad \text{(C-11)}$$

where

$M = B_1A$

$\xi = B\ SN(1-\omega)$

B_3 is a function of A .

The left hand side of equation (C-11) behaves functionally as

$$M \exp(\xi)\ B_2 \frac{\omega^{1/2}}{y}$$

where B_2 is some simple function of SN, $(\approx B_4/(A.SN_{ci}^{1/2})$. Hence one obtains from (C-11) that

$$Y \doteq \frac{MB_2\omega^{1/2}\exp(\xi)}{1+M(1-\omega)\exp(\xi)} \qquad \text{(C-12)}$$

where B_4 is a numerical constant.

The solution (C-12) has been obtained to be of an identical form to that quoted by Stuart (1958) as due to Landau (1944). From the above discussion, one may get some feel as to what assumptions may have led to the Landau solution.

APPENDIX D

THE COMPUTER PROGRAM FOR TURBULENCE FIELD CLOSURE

```
  ENERGY   FORTRAN EXTENDED VERSION 2.0          11/13/71          10.59.48.

      PROGRAM ENERGY (INPUT,OUTPUT,TAPE5=INPUT,TAPE6=OUTPUT,FILMPL)
C
C*****************************************************************************
C
C     THIS COMPUTER PROGRAM PERFORMS A NUMERICAL INTEGRATION OF THE EQUATIONS
C     OF MOTION FOR A TWO-DIMENSIONAL INCOMPRESSIBLE MEAN FLOW WITH 3-D
C     DISTURBANCES. THE EQUATIONS ARE CLOSED WITH INVARIANT MODELS FOR THE
C     TRIPLE AND HIGHER ORDER CORRELATIONS USING THE DOUBLE VELOCITY CORRELA
C     TIONS AND TWO SCALE LENGTHS, AFTER THE MANNER OF C. DONALDSON.
C     STARTING WITH INITIAL INPUT PROFILES CLOSE TO THE LEADING EDGE OF THE
C     WALL, THE PROGRAM PREDICTS THE BOUNDARY LAYER PROFILES AND PRINCIPAL
C     CHARACTERISTICS FROM THE LAMINAR THRU THE TRANSITIONAL TO THE TURBULENT
C     REGIME. EFFECTS OF STREAMWISE WALL CURVATURE, WALL ROUGHNESS, HEAT AND
C     MASS TRANSFER AT THE WALL AND ARBITRARY PRESSURE GRADIENTS ARE ACCOUNTED
C     FOR. THE OUTPUT INCLUDES PREDICTIONS OF THE POINTS OF INSTABILITY AND
C     TRANSITION AND THE AMPLIFICATION RATES OF THE DISTURBANCES.
C
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C                        COLORADO STATE UNIVERSITY, 1971.
C
C
C*****************************************************************************
      COMMON THTA(200),THTB(200),AY(200),Y(200),F1(200),Q1(200),F2(200),
     1AK1(200),AK2(200),AK3(200),GDT(200),VER(200),ROT(200),THTF(200),
     2  SIG(200),Q2(200),AK1P(200),AK2P(200),AK3P(200),SIGP(200),F4(200)
     3,THTQ(200),FV(200),F(200),SIGPP(200),RTHTA(200),DF(200),DTF(200)
      COMMON AF(200),BF(200),CCF(200),AE(200),BE(200),CE(200),DE(200),
     1AAK2(200),AAK1(200),AAK3(200),ASIG(200),F1B(200),F2B(200),F3B(200)
     2,Q1B(200),Q2B(200),Q3B(200),AK1B(200),F4B(200),AK2B(200),AK3B(200)
     3,BK1(200),CK1(200),BK2(200),CK2(200),BK3(200),CK3(200),BSIG(200)
      COMMON F3(200),Q3(200),AK1PP(200),AK2PP(200),AK3PP(200),DRDT(200),
     1THTQC(200),DTFC(200),FVB(200),SIGB(200),AK1PB(200),AK2PB(200),
     2ZU(200),AK3PB(200),SIGPB(200),CSIG(200),AK3PPB(200),SIGPPB(200),
     3AK1PPB(200),AK2PPB(200),DK1(200),DK2(200),DK3(200),DSIG(200)
      COMMON RDTB(200),THTC(200),THTD(200),THTG(200),THTH(200),THTFC(200
     1),TKB(200),TKPB(200),D2RDT(200),D2RDTB(200),DRDTB(200),GDTB(200),
     2TK(200),TKP(200),FB(200),LABEL(18),FRDT(60),REDT(60)
      COMMON SN(60),VOWX(60),AMPX(60),TAMPX(60),TIE(60),DELTA(60),
     1X(60),UE(60),E(60),DETO(60),GPW(60),PF(60),DETOX(60),VOW(60),
     2XD(60),TW(60),VW(60),CW(60),RUF(60),RE(60),REDX(60),CWO(60),DT(60)
     3,TKMX(60),SIGX(60),THETAF(60),THETAQ(60),H(60),ANUX(60),ST(60),
     4CF(60),GAM(60),DEL(60),THETAFC(60),THETAQC(60),DTC(60),HC(60)
C
      DATA JD,ID,EPS/200,60,0.005/
      DATA CP,PR,SC,VKC/0.434,0.71,110.,0.41/
      CALL SETL (0)                                                 ERC TERM
      READ(5,11) (LABEL(K), K=1,18)
      READ(5,7) JEY,JEF,JEG,JK1,JK2,JK3,JSG
      READ(5,5) TIR,DT(1),CHL
      READ(5,12) RHO,GC,TE,VEE,L
    5 FORMAT(F8.8,5X,F8.4,5X,F10.3)
   11 FORMAT(18A4)
   12 FORMAT(F10.9,F10.3,F10.8,F10.9,10X,I1)
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C
C     READ INITIAL PROFILES
C
      READ(5,2) (Y(J),J=1,JEY)
      READ(5,2) ( F1(J),J=1,JEF)
      READ(5,2) ( G1(J),J=1,JEG)
      READ(5,4) ( AK1(J),J=1,JK1)
      READ(5,4) ( AK2(J),J=1,JK2)
      READ(5,4) ( AK3(J),J=1,JK3)
      READ(5,4) (  SIG(J),J=1,JSG)
    2 FORMAT(10F8.4)
    4 FORMAT(10F8.6)
    6 FORMAT(6F10.3)
    7 FORMAT(7I2)
C
C
C     READ IN PARAMETERS FOR EACH X-STATION
C
      DO 100 I=1,ID
      READ(5,6) X(I),UE(I),TW(I),VW(I),CWO(I),RUF(I)
      IF(X(I) .GT. 1000.) GO TO 90
      RE(I) = UE(I)/VEE
      REDX(I) = RE(I)*X(I)
      XD(I) = X(I)/CHL
      VOW(I) = VW(I)/UE(I)
      DETO(I) = 1. - (TE/TW(I))
      E(I) = (UE(I)**2)/(2.*CP*TE)
      IF(ABS(TW(I) - TE) .GT. 2.) E(I) = (UE(I)**2)/(CP*(TW(I) - TE))
      GPW(I) = TW(I)/TE
  100 CONTINUE
   90 ID = I
      IMX = ID - 1
C
C     CALCULATE THE PRESSURE AND TEMPERATURE GRADIENT PARAMETERS,PF, DETOX
C
      DO 101 I=1,IMX
      II = I - I/IMX
      IP =II+1
      IM =II-1
      IF(I .LE. 1) IM = II
      IF(I .GE. IMX) IP = II
      DUDX = (UE(IP) - UE(IM))/(X(IP) - X(IM))
      PF(I) = DUDX*VEE/(UE(I)**2)
      VOWX(I) = (VOW(IP) - VOW(IM))/(X(IP) - X(IM))
  101 DETOX(I) = (GPW(IP) - GPW(IM))/(XD(IP) - XD(IM))
C
C     SMOOTH THE INITIAL PROFILES
C     SUBDIVIDE PROFILES TO CALCULATE ALL Y-DERIVATIVES
C
      I = 1
      CW(I) = CWO(I)*DT(I)
      CALL DIVIDE(JEY,L,Y,VH,JD)
      CALL DIVIDE(JEF,L,F1,VH,JD)
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      CALL DIVIDE(JEG,L,Q1,VH,JD)
      CALL DIVIDE(JK1,L,AK1,VH,JD)
      CALL DIVIDE(JK2,L,AK2,VH,JD)
      CALL DIVIDE(JK3,L,AK3,VH,JD)
      CALL DIVIDE(JSG,L,SIG,VH,JD)
      JE = MAX0(JEF,JEG)
      JJYL = JE - L
      JY = JE
C
      CALL INTEG(JE,JD,Y,F1,0.0,F)
      DO 1117 J=1,JE
      FV(J) = 1. - F1(J)
      IF((FV(J) - 1.) .LT. 0.) JJK = J
      F(J) = F(J)/F(JE)
 1117 CONTINUE
      DO 51 J=1,JE
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      ZM    = (F(JP)-F(JM))/(Y(JP)-Y(JM))
      F1(J) = (F1(J) + ZM)/2.
   51 CONTINUE
      N = 19
      CALL LEASTSQ(Y,F1,JE,N,F1)
      MMM = JJK + 1
      DO 44 J=MMM,JE
   44 F1(J) = 0.
      DO 102 J=1,JE
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      DY  = Y(JP) - Y(JM)
      F2(J) = (F1(JP) - F1(JM))/DY
      Q2(J) = (Q1(JP) - Q1(JM))/DY
      AK1P(J) = (AK1(JP) - AK1(JM))/DY
      AK2P(J) = (AK2(JP) - AK2(JM))/DY
      AK3P(J) = (AK3(JP) - AK3(JM))/DY
      SIGP(J) = (SIG(JP) - SIG(JM))/DY
      FV(J) = 1. - F1(J)
      IF(FV(J) .LE.0.996) YDEL = Y(J)
      IF((FV(J) - 1.) .LT. 0.) JJK = J
  102 CONTINUE
      JLL = JJK + 1
      JTM = JJK + 2
      JLT = JJK + 3
      N = 19
      CALL LEASTSQ(Y,F2,JE,N,F2)
      DO 43 J=JLL,JE
   43 F2(J) = Q2(J) = 0.
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C
      RDEL = YDEL*DT(I)*RE(I)
      ALAM = PF(I)*(RDEL**2)
      DEMON = 6. + VOW(I)*RDEL + CW(I)
      PAR1 = VOW(I)*RDEL
      PAR2 = CW(I)*YDEL*VOWA(I)
      APO = (12.+ALAM-PAR2)/(DEMON*YDEL)
      BPO = (3.*ALAM-6.*PAR1-3.*PAR2)/(DEMON*(YDEL**2))
      CPO = (12. - 3.*ALAM + 3.5*PAR1+1.5*PAR2)/(DEMON*(YDEL**3))
      DPO = (6. - ALAM + 3.*PAR1 + PAR2)/(DEMON*(YDEL**4))
      DO 110 J=1,JE
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      DY = Y(JP) - Y(JM)
      F3(J) = (F2(JP) - F2(JM))/DY
      Q3(J) = (Q2(JP) - Q2(JM))/DY
      AK1PP(J) = (AK1P(JP) - AK1P(JM))/DY
      AK2PP(J) = (AK2P(JP) - AK2P(JM))/DY
      AK3PP(J) = (AK3P(JP) - AK3P(JM))/DY
      SIGPP(J) = (SIGP(JP) - SIGP(JM))/DY
      IF(Y(J) .LT. 3.) F3(J) = 2.*BPO + 6.*CPO*Y(J) - 12.*DPO*(Y(J)**2)
  110 CONTINUE
      N = 19
      CALL LEASTSQ(Y,F3,JE,N,F3)
      DO 41 J=JTM,JE
   41 F3(J)=Q3(J)=0.
C
      AK1(1)=AK2(1)=AK3(1)=0.
      TK(1) = SIG(1) = 0.
      F1(1) = Q1(1) = 1.
      TKMAX = SIGMAX = -100.
      IF(ABS(TE - TW(I)) .LE. EPS) Q1(1) = 0.
      DO 111 J =1,JE
      IF(DETO(1) .EQ. 0.) Q2(J) = Q3(J) = 0.
      TK(J) = AK1(J) + AK2(J) + AK3(J)
      AK3P(J) = AK3PP(J) = SIGP(J) = SIGPP(J) = 0.
  111 CONTINUE
      UPMAX = 0.
      VPMAX = 0.
      WPMAX = 0.
      F3(1) = 2.*BPO + 6.*CPO*Y(1) - 12.*DPO*(Y(1)**2)
      DO 103 J=1,JE
      TKMAX = AMAX1(TKMAX,TK(J))
      SIGMAX = AMAX1(SIGMAX,SIG(J))
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      TKP(J) = (TK(JP) - TK(JM))/(Y(JP)-Y(JM))
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      F4(J) = (F3(JP) - F3(JM))/(Y(JP) - Y(JM))
  103 CONTINUE
      N = 19
      CALL LEASTSQ(Y,F4,JE,N,F4)
      DO 42 J=JTM,JE
   42 F4(J)=0.
      F4(JE) = 0.
C
C     WRITE OUT SMOOTHED VELOCITY PROFILES
C
      WRITE(6,39)(F(J),F1(J),F2(J),F3(J),F4(J),J=1,JE,L)
   39 FORMAT(5X,5F20.8)
      TKP(JE) = 0.
      AK2P(JE) = AK3P(JE) = SIGP(JE) = 0.
      F2(JE) = Q2(JE) = AK1P(JE) = 0.
      AK1PP(JE) = AK2PP(JE) = 0.
      DEL(1) = DT(1)*YDEL
C
C     END OF INITIALISATION.  BEGINNING OF MARCHING LOOP
C     PREVIOUS STATION HAS BEEN COMPUTED AND NEXT STATION IS BEING STARTED
C     MOVE PREVIOUS PROFILES BACK TO MOVE FORWARD IN X
C
      KOUNT = 0
  300 DO 116 J=1,JE
      FB(J) = F(J)
      F1B(J) = F1(J)
      F2B(J) = F2(J)
      F3B(J) = F3(J)
      F4B(J) = F4(J)
      Q1B(J) = Q1(J)
      Q2B(J) = Q2(J)
      Q3B(J) = Q3(J)
      AK1B(J) = AK1(J)
      AK2B(J) = AK2(J)
      AK3B(J) = AK3(J)
      SIGB(J) = SIG(J)
      AK1PB(J) = AK1P(J)
      AK2PB(J) = AK2P(J)
      AK3PB(J) = AK3P(J)
      SIGPB(J) = SIGP(J)
      AK1PPB(J) = AK1PP(J)
      AK2PPB(J) = AK2PP(J)
      AK3PPB(J) = AK3PP(J)
      SIGPPB(J) = SIGPP(J)
      TKB(J) = TK(J)
      TKPB(J) = TKP(J)
      FVB(J) = FV(J)
  116 CONTINUE
      KAL = 2
      GO TO 777
  555 DO 567 J=1,JE
      DRDTB(J) = DRDT(J)
      D2RDTB(J) = D2RDT(J)
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      GDTB(J) = GDT(J)
      RDTB(J) = RDT(J)
  567 CONTINUE
      KAL = KAL + 1
  777 CONTINUE
      IF(KAL .GT. 2) GO TO 1234
C
C     BEGINNING OF STATION ITERATION LOOP
C     CALCULATE LOCAL X-MEAN VALUES OF THE X-PARAMETERS
C
      KT = 0
      KTK = 0
      IB = I - 1
      IF(I .LE. 1) IB = I
      DX = XD(I) - XD(IB)
      IF(I .LE. 1) DX = XD(2) - XD(1)
      GPWM = (GPW(I) + GPW(IB))/2.
      VOWM = (VW(I) + VW(IB))/(UE(I) + UE(IB))
      REM = (RE(I) + RE(IB))/2.
      EM = (E(I) + E(IB))/2.
  600 CONTINUE
      DT(I) = DT(1)*F(JE)
      CW(I) = CWO(I)*DT(I)
      KT = KT + 1
      REDT(I) = RE(I)*DT(I)
      FRDT(I) = PF(I)*REDT(I)
      CWM = (CW(I) + CW(IB))/2.
      DTM = (DT(I) + DT(IB))/2.
      REDTM = (REDT(I) + REDT(IB))/2.
      FRDTM = (FRDT(I) + FRDT(IB))/2.
      XM = X(I)/DT(I)
      DTXM = 0.86*(1. - FRDT(I)*XM)/SQRT(REDT(I)*XM)
      IF(I .GT. 1) DTXM = (DT(I) - DT(IB))/(X(I) - X(IB))
      DO 113 J=1,JE
      TATE = (1. + (GPW(I)*Q1(J)*DETO(I)))
      VER(J) = (TATE**1.5)*(1. + SC/TE)/(TATE + SC/TE)
      RDT(J) = REDT(I)/VER(J)
      GDT(J) = 0.
      IF(DETO(I) .NE. 0.) GDT(J) = DETO(I)*(DT(I)**3)*GC/((VEE**2)*(1.
     1 + DETO(I)*(Q1(J) - 1.)))
  113 CONTINUE
      DO 117 J=1,JE
      TK(J) = AK1(J) + AK2(J) + AK3(J)
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
  117 DRDT(J) = (1./RDT(JP) - 1./RDT(JM))/(Y(JP) - Y(JM))
       DO 118 J=1,JE
      JJ = J - J/JE
      JP = JJ + 1
      JM = JJ - 1
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      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      DY = Y(JP) - Y(JM)
      TKP(J) = (TK(JP) - TK(JM))/DY
  118 D2RDT(J) = (DRDT(JP) - DRDT(JM))/(Y(JP) - Y(JM))
      GO TO 555
 1234 CONTINUE
      F3(1) = FRDTM*REDTM + F2(1)*(CWM + REDTM*(VOWM - DRDT(1)))
C
C ***** CALCULATE INITIAL PROFILE PARAMETERS ***********************************
C
      IF(I .LE. 1) GO TO 500
C
C
C     CALCULATE ALL THE PROFILE EQUATION COEFFICIENTS, A1 - A34 AND B#S AND ALL
C     THE RIGHT HAND SIDE CONSTANTS.
C
      JJT = JE - 1
  700 DO 114 J=2,JJT
      FM = (F(J) + FB(J))/2.
      F1M = (F1(J) + F1B(J))/2.
      F2M = (F2(J) + F2B(J))/2.
      F3M = (F3(J) + F3B(J))/2.
      FXM = (F(J) - FB(J))/2.
      AK1M = (AK1(J) + AK1B(J))/2.
      AK2M = (AK2(J) + AK2B(J))/2.
      AK3M = (AK3(J) + AK3B(J))/2.
      SIGM = (SIG(J) + SIGB(J))/2.
      AK1PM = (AK1P(J) + AK1PB(J))/2.
      AK2PM = (AK2P(J) + AK2PB(J))/2.
      AK3PM = (AK3P(J) + AK3PB(J))/2.
      SIGPM = (SIGP(J) + SIGPB(J))/2.
      SIGPPM = (SIGPP(J) + SIGPPB(J))/2.
      TKM = (TK(J) + TKB(J))/2.
      TKPM = (TKP(J) + TKPB(J))/2.
      DRDTM = (DRDT(J) + DRDTB(J))/2.
      D2RDTM = (D2RDT(J) + D2RDTB(J))/2.
      GDTM = (GDT(J) + GDTB(J))/2.
      QXM = Q1(J) - Q1B(J)
      F1XM = F1(J) - F1B(J)
      SXM = SIG(J) - SIGB(J)
      F2XM = (F2(J)**2) + (F2B(J)**2)
      RDTM = (RDT(J) + RDTB(J))/2.
      Q1M = (Q1(J) + Q1B(J))/2.
      CW1 = 1. + CWM*Y(J)
      FM1 = 1. - F1M
      FMY = FM - Y(J)
      A35 = CW1*DTM*(VOWM/CW1 + FMY*(DTXM*FRDTM) + DTM*FM*2./DX)
      A36 = A35 *FRDTM*(DTXM + FRDTM)
      A37 = A35*(2.*DTXM + FRDTM + DTM)
      A1 = 1./(RDTM*CW1)
      A2 = (2.*DRDTM-VOWM)/CW1 - FMY*(DTXM*FRDTM)  - CWM/(CW1*RDTM)
      A3 = (CWM*VOWM+D2RDTM+CWM*FM1*DTXM)/CW1 +FMY*CWM*(DTXM+FRDTM)
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 1 - FRDTM*CWM    - CWM*DRDTM/CW1
  A4 =  CWM*(FM1+1.)*FRDTM + FRDTM*F2M   + A36
  A5 = (F3M-CWM*F2M)*DTM*2./DX
  A6 = CWM*FM1*DTM*2./DX     - A37
  A7 = FM1*DTM*2./DX
  A8 = SIGPPM/CW1 - CWM*SIGPM
  A9 = 8.*FRDTM*DTM/(DX*CW1)
  A10 = 6.*(FRDTM**2)/CW1
  A11 = REDTM*SIGM*2.*DTM/(DX*CW1)
  A12 = A13 = 0.
  A14 = VOWM/CW1 + FMY*(DTXM*FRDTM)
  IF(DETO(I) . EQ. 0.)  A15 = 0.
  IF(DETO(I) .NE. 0.) A15 = (FM1*DETOX(I)*DTM)/(GPWM*DETO(I))
  A16 = EM/(2.*REDTM*CW1)
  A17 = A1
  A18 = A2    + CWM/(RDTM*CW1)
  A19 = ( 2.*FM1*FRDTM+50.*F2M-2./((Y(J)**2)*RDTM))/CW1 +2.*FM1*
 1FRDTM
  A20 = (              -AK1PM)*DTM*2./DX
  A21 = A22 = A28 = 0.
  A23 = -2.*F2M/CW1
  A24 = (DRDTM*TKPM+50.*F2M*TKM)/(3.*CW1)
  A25 = A18 + 2.*DRDTM/CW1
  A26 = A19    - 2.*FM1*FRDTM
  A27 = -AK2PM*DTM*2./DX
  A29 = A26
  A30 = AK3PM*DTM*2./DX
  A31 = A2 + DRDTM/CW1
  A32 = A26
  A33 = SIGPM*DTM*2./DX
  A34 = AK2M*F2M/CW1
  B6 = -A14 - DTM*FXM
  JJ = J - J/JE
  JP = JJ + 1
  JM = JJ - 1
  IF(J .LE. 1) JM = JJ
  IF(J .GE. JE) JP = JJ
  DYA = Y(JP) - Y(JM)
  DYAA = (Y(JP)-Y(JJ))*(Y(JP)-Y(JM))/2.
  DYAB = (Y(JP)-Y(JJ))*(Y(JJ)-Y(JM))
  DYAC = (Y(JP)-Y(JM))*(Y(JJ)-Y(JM))/2.
  DY1 = Y(JP) - Y(JJ)
  DY2 = Y(JJ) - Y(JM)
  G1 = G4 = 0.
  G2 = G3 = 1.
  AF(J) = A1/DYAA + A2*G2/DY1
  BF(J) = 2.*A1/DYAB - A2*G1/DY2 + A2*G2/DY1 + A7 - A3
  CCF(J) = A1/DYAC - A2*G1/DY2
  DF(J) =  -(-A1*F4B(J) - A2*F3B(J) - (A3+A7)*F2B(J) + 2.*A8 + A10*2
 1.*SIGM - A9*SXM + A12*GXM - A11*(PF(I) - PF(IB)) - A4*2.*F1M + A6*
 2F1XM + A5*2.*FXM + A13*2.*Q1M )
  AE(J) = A1/(DYAA*PR) + B6*G3/DY1
  BE(J) = 2.*A1/(DYAB*PR) - B6*G4/DY2 + B6*G3/DY1 + A7 + A15
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      CE(J) = A1/(DYAC*PR) - B6*G4/DY2
      DE(J) =-( -A1*Q3B(J)/PR - B6*Q2B(J) + (A15-A7)*Q1B(J) - A16*F2XM )
      AAK1(J) = A17/DYAA + A18*G3/DY1
      BK1(J) = 2.*A17/DYAB - A18*G4/DY2 + A18*G3/DY1 + A7 - A19
      CK1(J) = A17/DYAC - A18*G1/DY2
      DK1(J) =-( -A17*AK1PPB(J) - A18*AK1PB(J) - (A7+A19)*AK1B(J) - A20*
     12.*FXM - A21*F1XM - A22*SXM - A23*2.*SIGM + 2.*A24 )
      AAK2(J) = A17/DYAA + A25*G3/DY1
      BK2(J) = 2.*A17/DYAB + A25*G3/DY1 - A25*G4/DY2 + A7 - A26
      CK2(J) = A17/DYAC - A25*G4/DY2
      DK2(J) =-( -A17*AK2PPB(J) - A25*AK2PB(J) - (A26+A7)*AK2B(J) - A27*
     1 2.*FXM + A28*F1XM + 2.*A24)
      AAK3(J) = A17/DYAA + A18*G3/DY1
      BK3(J) = 2.*A17/DYAB + A18*G3/DY1 - A18*G4/DY2 + A7 - A29
      CK3(J) = A17/DYAC - A18*G4/DY2
      DK3(J) =-( -A17*AK3PPB(J) - A18*AK3PB(J) - (A29+A7)*AK3B(J) + A30*
     12.*FXM + 2.*A24)
      ASIG(J) = A1/DYAA + A31*G3/DY1
      BSIG(J) = 2.*A1/DYAB + A31*G3/DY1 - A31*G4/DY2 + A7 - A32
      CSIG(J) = A1/DYAC - A31*G1/DY2
      DSIG(J) = -(-A1*SIGPPB(J) - A31*SIGPB(J) - (A32+A7)*SIGB(J) + 2.*A
     133*FXM + 2.*A34)
      FV(J) = 1. - F1(J)
      IF(FV(J) .LE. 0.996) YDEL = Y(J)
  114 CONTINUE
C
C  ******  CALCULATE ALL PROFILES  *******************************************
C      RESTATE THE WALL BOUNDARY CONDITIONS
C
      QHW = 1.
      IF(ABS(TE - TW(I)) .LE. EPS) QHW = 0.
      Q1(1) = QHW
      QW = Q1(1)
      AK1(1) = AK2(1) = AK3(1) = 0.
      SIG(1) = 0.
      IF(KT .GT. 1) GO TO 37
      IF(ABS(TE-TW(I)) .LE. EPS)  Q2(1) = 0.
      Q2(1) = -0.332*(PR**.33)*SQRT(REDT(I)*DT(I)/X(I))
   37 CONTINUE
      F2W = F2(1)
      CALL PROFYL(JE,Y,AF,BF,CCF,DF,F2,3,F3,F4,FMAX)
      F3(1) = FRDTM*REDTM + F2(1)*(CWM + REDTM*(VOWM - DRDT(1)))
C
C      TEST FOR FATAL ERROR
C
      IF(F2(1) .GT. 0.) GO TO 888
      GO TO 889
  888 WRITE(6,1000)
      STOP
 1000 FORMAT(10X,*SOLUTION WILL NOT CONVERGE FOR VELOCITY PROFILE*)
C
  889 CONTINUE
      CALL INTEG(JE,JD,Y,F2,0.0,ZU)
```

```
      DO 768 J=1,JE
      F1(J) = 1. + ZU(J)
  768 CONTINUE
      FJEE = 1. - F1(JE)
      CALL INTEG(JE,JD,Y,F1,0.0,F)
      IF(DETO(I) .EQ. 0.) GO TO 3003
      CALL PROFYL(JE,Y,AE,BE,CE,DE,Q1,3,Q2,Q3,QMAX)
      GO TO 3004
 3003 DO 3002 J=1,JE
 3002 Q1(J)=Q2(J)=Q3(J) = 0.
 3004 CONTINUE
C
C     CALCULATE NEW K1, K2, K3, AND SIGMA PROFILES
C
      CALL PROFYL(JE,Y,ASIG,BSIG,CSIG,DSIG,SIG,0,SIGP,SIGPP,SIGMAX)
      CALL PROFYL(JE,Y,AAK1,BK1,CK1,DK1,AK1,0,AK1P,AK1PP,UPMAX)
      CALL PROFYL(JE,Y,AAK2,BK2,CK2,DK2,AK2,0,AK2P,AK2PP,VPMAX)
      CALL PROFYL(JE,Y,AAK3,BK3,CK3,DK3,AK3,0,AK3P,AK3PP,WPMAX)
      TKMAX = UPMAX + VPMAX + WPMAX
      WRITE(6,444) F(JE),F1(JE),F2(1),Q1(1),Q2(1),DT(I),KT
C
C     TEST FOR SIMULTANEOUS CONVERGENCE OF F1 AND Q1 PROFILES
C
      IF(I .LE. 2 .AND. ABS(F1(JE)-0.) .LE. EPS) GO TO 333
      IF(KT .EQ. 1) GO TO 600
      IF(KT.GT.1 .AND. ABS((F2(1) - F2W)/F2W) .GT. EPS) GO TO 999
      IF(ABS(Q1(1) - QHW) .GT. EPS) GO TO 999
      IF(KT.GT.1.AND.ABS(F(JE)-1.).LE. EPS) GO TO 333
  999 CONTINUE
      IF(KT .GT. 20) STOP
      GO TO 600
  444 FORMAT(1X,*F(JE) = *,E9.2,2X,*F1(JE) = *,E9.2,2X,*F2(W) = *,E9.2,
     12X,*Q1(W)=*,E9.2,2X,*Q2(W)=*,E9.2,2X,*DT(I )=*,E9.2,1X,*IT.NO. *,I
     22)
  333 CONTINUE
C
C     PROFILE SOLUTIONS HAVE CONVERGED
C     CALCULATE NEW PROFILE PARAMETERS
C
      CW(I) = CWO(I)*DT(I)
  500 DO 112 J=1,JE
      THTF(J) = F1(J)*(1. - F1(J))
      THTQ(J) =  (1. - Q1(J))
      DTF(J) = DT(I)*F1(J)
      THTFC(J) = THTF(J)*(1. - CW(I)*Y(J))
      THTQC(J) = THTQ(J)*(1. - CW(I)*Y(J))
      DTFC(J) = DTF(J)  *(1. - CW(I)*Y(J))
  112 CONTINUE
      TKMX(I) = TKMAX
      SIGX(I) = SIGMAX
      CALL INTEG(JE,JD,Y,THTF,0.0,THTA)
      H(I) = 1./THTA(JE)
      THETAF(I) = DT(I)/H(I)
```

```
      CALL INTEG(JE,JD,Y,DTFC,0.0,THTC)
      DTC(I) = THTC(JE)/F(JE)
      CALL INTEG(JE,JD,Y,THTFC,0.0,THTD)
      HC(I) = DTC(I)/(DT(I)*THTD(JE))
      THETAFC(I) = DTC(I)/HC(I)
      CALL INTEG(JE,JD,Y,THTQ,0.0,THTB)
      THETAQ(I) = THTB(JE)
      CALL INTEG(JE,JD,Y,THTQC,0.0,THTG)
      THETAQC(I) = THTG(JE)
      HQ = THETAQ(I)/DT(I)
      CW(I) = CWO(I)*DT(I)
      ANUX(I) = -X(I)*Q2(1)/DT(I)
      ST(I) = ANUX(I)/(REDX(I)*PR)
      CF(I) = -2.*VER(1)*F2(1)/REDT(I)
      GAM(I) = SQRT(CF(I)/2.)*UE(I)
      DEL(I) = DT(I)*YDEL
      HF = (10./H(I)) - 4.
      ANUM = -2.34*(EXP(HF) - EXP(-HF))/(EXP(HF)+EXP(-HF))
      YL = 10.**ANUM
      DELTA(I) = YL*DT(I)
C
C     END OF STATION COMPUTATION. WRITE OUT STATION PROFILES AND PARAMETERS.
C
      TIE(I) = TIR
      WRITE(6,1)
      WRITE(6,8)
      WRITE(6,200) X(I),UE(I),TE,TW(I),VOW(I),CW(I),RUF(I)
      WRITE(6,201) REDT(I),REDX(I),ANUX(I),ST(I),PR,E(I),GAM(I)
      WRITE(6,202) TIE(I),TKMAX,UPMAX,VPMAX,WPMAX,SIGMAX,HQ
      WRITE(6,203) THETAF(I),DT(I),THETAQ(I),H(I),CF(I),DEL(I)
      WRITE(6,717) THETAFC(I),THETAQC(I),DTC(I),HC(I)
      WRITE(6,204)
    1 FORMAT(1H1,40X,55HA STATISTICAL ENERGY METHOD FOR STUDYING BOUNDAR
     1Y LAYER)
    8 FORMAT(44X,45HDEVELOPMENT AND LAMINAR/TURBULENT TRANSITIONS)
  200 FORMAT(1H0,10X,*X=*,F7.3,2X,*UE=*,F8.3,2X,*TE=*,F8.3,2X,*TW=*,F8.3
     1,2X,*VW/UE=*,F6.4,2X,*CW=*,F6.4,2X,*RUF=*,F6.4)
  201 FORMAT(1H0,10X,*REDT=*,E9.2,2X,*REDX=*,E9.2,2X,*NUSSELT=*,E9.2,2X,
     1*STANTON=*,E9.2,2X,*PRANDTL=*,F5.3,2X,*ECKERT=*,E9.2,2X,*U(TAU)=*,
     2F6.3)
  202 FORMAT(1H0,1X,*T/INTENSITY=*,F10.8,1X,*K(MAX)=*,F10.8,1X,*U(MAX)=*
     1,F10.8,1X,*V(MAX)=*,F10.8,1X,*W(MAX)=*,F10.8,1X,*UV(MAX)=*,F10.8,1
     2X,*D/D=*,F8.5)
  203 FORMAT(1H0,10X,*THETA=*,F7.5,1X,*DEL(STAR)=*,F7.5,1X,*DEL(THERM)=*
     1,F7.5,1X,*H = *,F5.3,2X,*CF =*,F10.8,1X,*DELTA =*,F7.5)
  204 FORMAT(1H0,1X,1HJ,6X,1HY,8X,2HYY,8X,3HU/U,8X,2HF1,8X,2HQ1,8X,2HK1,
     18X,2HK2,8X,2HK3,8X,3HSIG,9X,3HTAU,4X,3HCFR,7X,2HVE,7X,4HAMPF)
  206 FORMAT(1X,I3,13(1X,1PE9.2))
  717 FORMAT(1H0,10X,*VALUES OF THTA,DTQ,DT,AND H FOR NO CURVATURE ARE *
     1,4F15.5)
      AMPFX = -1000.
      DO 205 J=1,JE,L
      F2M = (F2(J) + F2B(J))/2.
```

```
      F3M = (F3(J) + F3B(J))/2.
      FM = (F(J) + FB(J))/2.
      FMY = FM - Y(J)
      CW1 = 1. + CWM*Y(J)
      GDTM = (GDT(J) + GDTB(J))/2.
      FXM = (F(J) - FB(J))/2.
      YY = Y(J)*DT(I)/DEL(I)
      FV(J) = 1. - F1(J)
      RTHTA(J) = RDT(J)*THETAF(I)/DT(I)
      V    = VOW(I) + FMY*CW1*(DTXM+FRDTM) + 2.*DTM*CW1*FXM/(DX*CHL)
      AMPF = 0.
      IF(TK(J) .NE. 0. .AND. TKB(J) .NE. 0.) AMPF = TK(J)/TKB(J)
      TAU = -VER(J)*F2(J)/REDT(I)    +    SIG(J)
      CFR = TAU/(-VER(1)*F2(1)/REDT(I))
      AMPFX = AMAX1(AMPFX,AMPF)
      IF(AMPF .GE. AMPFX) JT = J
  205 WRITE(6,206) J,Y(J),YY,FV(J),F1(J),Q1(J),AK1(J),AK2(J),AK3(J),SIG(
     1J),TAU,CFR, V,AMPF
      RTHAN = RDT(JE)*YL
      RA = GDT(JE)*PR*((DELTA(I)/DT(I))**3)
      TA = 2.*CW(I)*(RTHAN**2)
      ATY = (VW(I)*DELTA(I)/VEE)**2
      ATX = 1. + VOW(I)
      SN(I) = SQRT(((RTHAN*ATX)**2) + 12000.*(RA+TA+ATY))
      AMPX(I) = AMPFX
      TAMPX(I) = AMPX(I)*AMPX(IB)
      WRITE(6,207) Y(JT),AMPFX
      WRITE(6,609) TAMPX(I),SN(I),RA,TA
      XCT = 0.
      SNR = 1000./SN(I)
      IF(TIR.LE.0.03) SNR = (1000.-1300.*ALOG(33.*TIR))/SN(I)
      IF(ABS(SNR-1).LE.0.01) XCT = X(I)
      IF(XCT.NE.0.) SPFR = 1.53E-10*(RE(I)**2)*(1.-EXP(-5.34E-05*((X(I)-
     1XCT )**2)/VEE))
      IF(XCT .NE. 0.) FINT = 1. - EXP(-0.04*SPFR*(X(I)**3)/UE(I))
      FFM = .016*RE(I)/RTHAN
      CTAMPX = EXP(.003*(SN(I)-1000.))/(1. + .33*(1.-1000./SN(I))*
     1 EXP(.003*(SN(I) - 1000.)))
      WRITE(6,53) SNR,CTAMPX,FFM
      IF(XCT .NE. 0.) WRITE(6,54) SPFR,FINT
      WRITE(6,208)
   53 FORMAT(1H0,1X,*SN(CT)/SN(I) = *,F6.3,2X,*COMPUTED TOTAL AMP. FACTO
     1R = *,E9.2,2X,* MOST FAVORED DISTURBANCE FREQ. = *,F10.5)
   54 FORMAT(1H0,1X,* SPOT FORMATION RATE PER SQ.FT. = *,F10.5,2X,* INTE
     1RMITTENCY FACTOR IN TRANS. BL. = *,F4.2)
  609 FORMAT(1X,*TOTAL AMP. FACTOR = *,E9.2,2X,*STABILITY NO. = *,E9.2,
     12X,*RAYLEIGH NO. = *,E9.2,2X,*TAYLOR NO. = *,E9.2)
      DO 809 J=1,JE,L
      YY = Y(J)*DT(I)/DEL(I)
  809 WRITE(6,209)J,Y(J),YY,GDT(J),F(J),F2(J),F3(J),F4(J),Q2(J),RTHTA(J)
  208 FORMAT(1H1,1X,1HJ,6X,1HY,7X,2HYY,13X,3HGDT,10X,1HF,10X,2HF2,11X,5H
     1F3(J),11X,4HF4-J,5X,5HQ2(J),11X,5HRTHTA)
  209 FORMAT(1X,I3,1X,2(1X,1PE9.2),1X,7(3X,2PE10.2))
```

```
ENERGY   FORTRAN EXTENDED VERSION 2.0              11/13/71              10.59.48.

  207 FORMAT(1H0,5X,*CRITICAL Y(J) = *,F6.3,5X,*WITH MAX. AMPLIFICATION
     1 RATE = *,F10.4)
C
C
C     END OF STATION WRITE OUT. COMPUTE NEXT STATION OR STOP.
C     CORRECT THE DISPLACEMENT THICKNESS WITH THE MOMENTUM INTEGRAL EQUATION
C
      DEX = X(I) - X(IB)
      HB = (H(I) + H(IB))/2.
      COF1 = ((UE(I)/UE(IB))**(2. + HB))*THETAF(I)/THETAF(IB)
      COF2 = EXP(DEX*((CF(IB) + CF(I))/2. + 2.* (VW(I)+VW(IB))/(UE(I)+
     1UE(IB)))/(THETAF(I) + THETAF(IB)))
C
C     PRINT RESULTS AS AN INDICATION OF ACCURACY
C
      WRITE(6,80) COF1,COF2
   80 FORMAT(1H0,10X,*LEFT SIDE MOMENTUM INTEGRAL EQUATION = *,F7.4
     1// 10X,*RIGHT SIDE MOMENTUM INTEGRAL EQUATION = *,F7.4)
C
C     RESET THETA AND DT TO MATCH SKIN FRICTION
C
      DTS = DT(I)
      THETAF(I) = THETAF(I)*COF2/COF1
      DT(I) = H(I)*THETAF(I)
      HQ = THETAQ(I)/DT(I)
      REDT(I) =(REDT(I)/DTS)*DT(I)
      REDX(I) = REDT(I)*X(I)/DT(I)
      ANUX(I) = -X(I)*Q2(I)/DT(I)
      ST(I) =ANUX(I)/(REDX(I)*PR)
      CW(I) = CWO(I)*DT(I)
      CF(I) = -2.*VER(I)*F2(1)/REDT(I)
      GAM(I) = SQRT(CF(I)/2.)*UE(I)
      FRDT(I) = PF(I)*REDT(I)
      DTXM = 0.86*(1. - FRDT(I)*XM)/SQRT(REDT(I)*XM)
      IF(I .GT. 1) DTXM = (DT(I) - DT(IB))/(X(I) - X(IB))
      I = I + 1
      DT(I) = DT(I-1) + DTXM*(X(I) - X(I-1))
      CW(I) = CWO(I)*DT(I)
      IF(I .GT. IMX) GO TO 666
      GO TO 300
C
C     WRITE OUT PRINCIPAL BOUNDARY LAYER PARAMETERS
C
  666 WRITE(6,901) (LABEL(K),K=1,18)
  901 FORMAT(1H1,45X,39HPRINCIPAL BOUNDARY LAYER PARAMETERS FOR/
     1 26X,18A4)
      WRITE(6,902)
  902 FORMAT(//5X,*X*,7X,*DT*,6X,*THTA*,5X,*DTQ*,7X,*H*,7X,*CF*,6X,*ST*,
     16X,*R(DT)*,5X,*NUX*,5X,*TI*,5X,*TKMAX*,5X,*AMPFX*,5X,*U(TAU)*,5X,
     2*SIGMAX*,6X,*E*)
      DO 903 I=1,IMX
  903 WRITE(6,904) X(I),DT(I),THETAF(I),THETAQ(I),H(I),CF(I),ST(I),
     1REDT(I),ANUX(I),TIE(I),TKMX(I),TAMPX(I),GAM(I),SIGX(I),E(I)
  904 FORMAT(/1X,F8.3,1X,3(F8.6,1X),F5.3,1X,2(F7.5,2X),F8.0,3X,F7.4,2X,
     12(E9.2,1X),F6.0,1X,F5.3,3X,E9.2,1X,F7.3)
      STOP
      END
```

```
      SUBROUTINE PROFYL(JE,Y,A,B,C,D,PHI,IBC,PHIP,PH2P,AM)
C
C     COMPUTES THE PROFILE PHI(J) AND ITS FIRST AND SECOND Y-DERIVATIVES
C     COMPUTES ALSO THE MAXIMUM VALUE OF THE PROFILE
C
      DIMENSION Y(300),A(300),B(300),C(300),D(300),PHI(300),PHIP(300)
      DIMENSION PH2P(300),E(300),F(300)
      E(1) = 0.
      F(1) = 0.
      IF(IBC .NE. 3) GO TO 100
      IF(PHI(1) .NE. 0.) E(1) = 1.
      IF(PHI(1) .NE. 0.) F(1) = -PHIP(1)*(Y(2) - Y(1))
  100 CONTINUE
      JYM = JE - 1
      JYMM = JE - 2
      DO 1 J = 2,JYM
      E(J) = 0.
      F(J) = 0.
      BAX = (B(J) - C(J)*E(J-1))
      IF(BAX .NE. 0.) E(J) = A(J)/BAX
      IF(BAX .NE. 0.) F(J) = (D(J) + C(J)*F(J-1))/BAX
    1 CONTINUE
      PHI(JE) = 0.0
      AM = -100.
      DO 2 JJ=1,JYM
      J = JE - JJ
      PHI(J) = E(J)*PHI(J+1) + F(J)
      AM = AMAX1(AM,PHI(J))
    2 CONTINUE
C
C     OBTAIN Y-DERIVATIVES OF PROFILE
C
      DO 3 J=1,JE
      JJ = J - J/JE
      JM = JJ - 1
      JP = JJ + 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      PHIP(J) = (PHI(JP) - PHI(JM))/(Y(JP) - Y(JM))
    3 CONTINUE
      PHIP(JE) = 0.
      DO 4 J=1,JE
      JJ = J - J/JE
      JM = JJ - 1
      JP = JJ + 1
      IF(J .LE. 1) JM = JJ
      IF(J .GE. JE) JP = JJ
      PH2P(J) = (PHIP(JP) - PHIP(JM))/(Y(JP) - Y(JM))
    4 CONTINUE
      PH2P(JE) = 0.
      RETURN
      END
                                   IDENT   PROFYL
```

E DIVIDE FORTRAN EXTENDED VERSION 2.0 11/13/71 10.59.48.

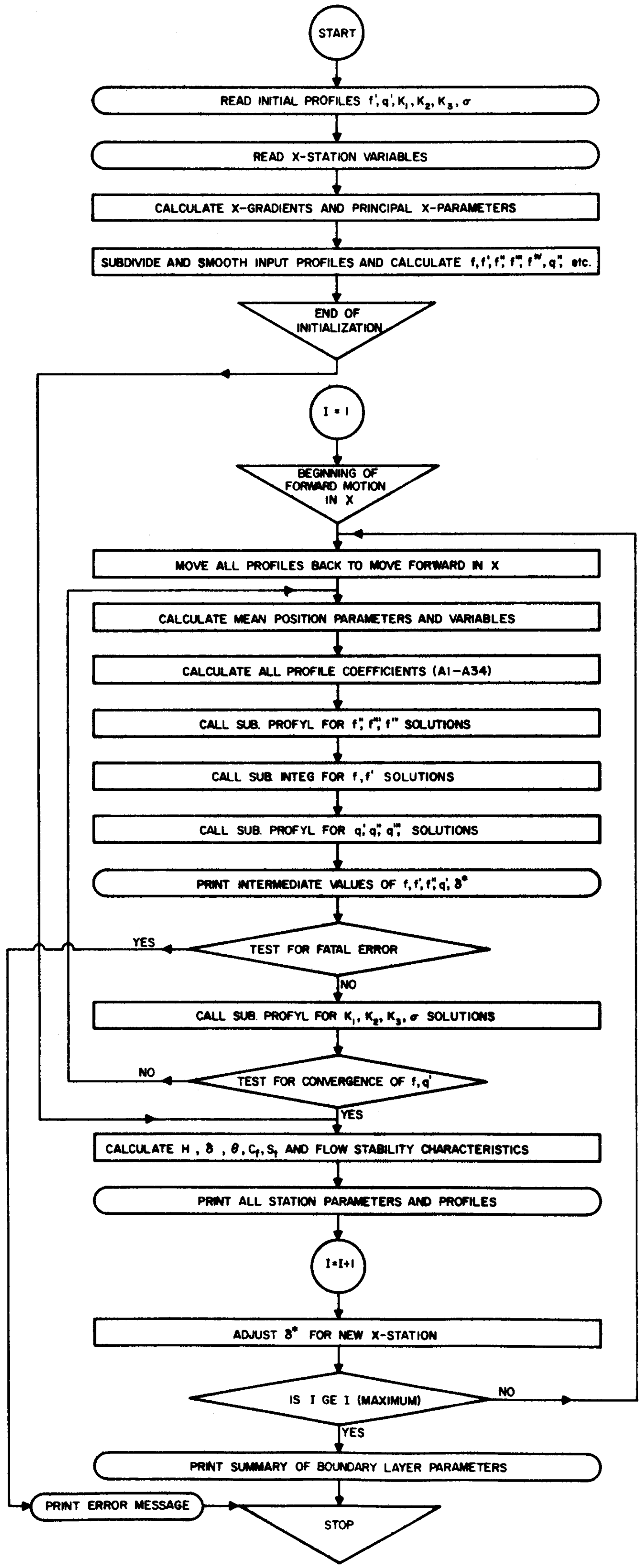
```
      SUBROUTINE DIVIDE (JY, JDIV, W, VH, JD)
      DIMENSION W(300),VH(300)
      DO 110 J=1, JY
      VH(J) = W(J)
  110 CONTINUE
      DIVJ=JDIV
      JYM=JY-1
      JY=JYM*JDIV
      DO 130 J=1, JYM
      DY = (VH(J+1) - VH(J))/DIVJ
      JF=(J-1)*JDIV+2
      JL=J*JDIV
      W(JF-1) = VH(J)
      DO 120 J1=JF, JL
      W(J1) = W(J1-1) + DY
  120 CONTINUE
  130 CONTINUE
      IF(JL+1.LE.JD) W(JY+1) = VH(JYM+1)
      IF (JL+1.GE.JD) RETURN
      JLP=JL+2
      DO 140 J=JLP, JD
      W(J) = 0.
  140 CONTINUE
      RETURN
      END
```

E INTEG FORTRAN EXTENDED VERSION 2.0 11/13/71 10.59.48.

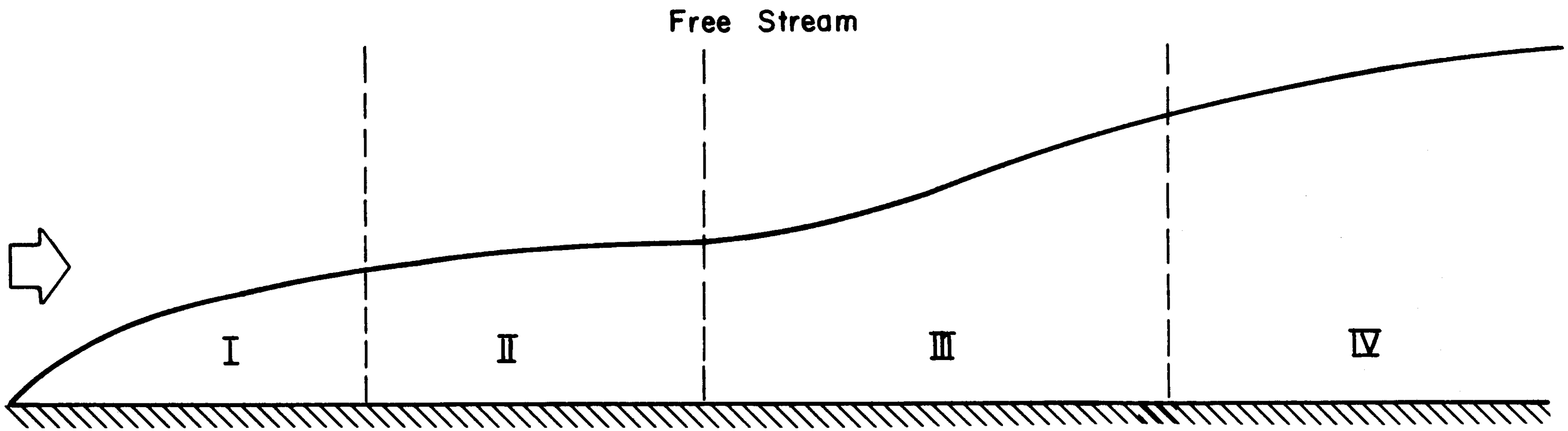
```
      SUBROUTINE INTEG(JE,JD,Y,FP,FIRST,F)
      DIMENSION Y(200),FP(200),F(200)
      JEM = JE - 1
      FP2 = FP(1)
      F1 = FIRST
      F(1) = F1
      DO 110 J=1,JEM
      FP1 = FP2
      FP2 = FP(J+1)
      F1 = F1 + (Y(J+1) - Y(J))*(FP2 + FP1)/2.
      F(J+1) = F1
  110 CONTINUE
      IF(JE .GE. JD) RETURN
      DO 120 J=JE,JD
      F(J) = F(JE)
  120 CONTINUE
      RETURN
      END
```

```
E  LEASTSQ   FORTRAN EXTENDED VERSION 2.0              11/13/71              10.59.48.
      SUBROUTINE LEASTSQ (X,Y,NPOINTS,M,PHI)
      DIMENSION X(200),DUM1(2),Y(200),DUM2(40),A(21,21),DUM3(40),BB(21),
     1B(21),DUM4(2),C(21),DUM5(2),P(40),DUM6(2),AA(21,21),PHI(200)
C LEAST SQUARES POLYNOMIAL FIT
C MCCRAKEN AND DORN PAGE 267
C CORRECTED AND EXPANDED BY WILFORD W. BURT,  C.S.U. ENG. RES. CENTER   3/71
C     GENERATE MATRIX
      NUMBER = NPOINTS
   14 MX2=M*2
      DO 13 I=1,MX2
      P(I)=0.0
      DO 13 J=1,NUMBER
 13   P(I)=P(I)+X(J)**I
      N=M+1
      DO 30 I=1,N
      DO 30 J=1,N
      K=I+J-2
      IF(K) 28, 29
 28   A(I,J)=AA(I,J)=P(K)
      GO TO 30
 29   A(1,1)=AA(1,1)=NUMBER
 30   CONTINUE
C     GENERATE CONSTANT VECTOR
      B(1)=0.0
      DO 21 J=1,NUMBER
 21   B(1)=B(1)+Y(J)
      BB(1)=B(1)
      DO 23 I=2,N
      B(I)=0.0
      DO 22 J=1,NUMBER
 22   B(I)=B(I)+Y(J)*X(J)**(I-1)
   23 BB(I)=B(I)
C     BEGIN MATRIX SOLVE
      NM1=N-1
      DO 300 K=1,NM1
      KP1=K+1
      L=K
C     PARTIAL PIVOTING
      DO 400 I=KP1,N
      IF(ABS(A(I,K)).GT.ABS(A(L,K))) L=I
 400  CONTINUE
      IF(L-K) 405, 500
 405  DO 410 J=K,N
      TEMP=A(K,J)
      A(K,J)=A(L,J)
 410  A(L,J)=TEMP
      TEMP=B(K)
      B(K)=B(L)
      B(L)=TEMP
C     ELIMINATION
 500  DO 300 I=KP1,N
      FACTOR=A(I,K)/A(K,K)
      A(I,K)=0.0
```

```
      DO 301 J=KP1,N
 301  A(I,J)=A(I,J)-FACTOR*A(K,J)
 300  B(I)=B(I)-FACTOR*B(K)
C     BACK SOLVE
      C(N)=B(N)/A(N,N)
      I=NM1
 710  IP1=I+1
      SUM=0.0
      DO 700 J=IP1,N
 700  SUM=SUM+A(I,J)*C(J)
      C(I)=(B(I)-SUM)/A(I,I)
      I=I-1
      IF(I) 710, 800
  800 CONTINUE
      ITER=20
C     PRINT OUT RESULTS
C
C******** OBTAIN FUNCTION AND ITS FIRST TWO DERIVATIVES ***********************
C
C     PRINT X,Y,AND Y(COMPUTED) VALUES
      DO 950 I=1,NUMBER
      YC=C(1)   $   DO 930 J=1,M
  930 YC=YC+C(J+1)*X(I)**J
      PHI(I) = YC
  950 CONTINUE
C     WRITE(6,12)
C  12 FORMAT(1H1)
      RETURN
      END
```


START
READ INITIAL PROFILES f', q', K1, K2, K3, σ
READ X-STATION VARIABLES
CALCULATE X-GRADIENTS AND PRINCIPAL X-PARAMETERS
SUBDIVIDE AND SMOOTH INPUT PROFILES AND CALCULATE f, f', f'', f''', f'''', q'', etc.
END OF INITIALIZATION
I = 1
BEGINNING OF FORWARD MOTION IN X
MOVE ALL PROFILES BACK TO MOVE FORWARD IN X
CALCULATE MEAN POSITION PARAMETERS AND VARIABLES
CALCULATE ALL PROFILE COEFFICIENTS (A1-A34)
CALL SUB. PROFYL FOR f'', f''', f'''' SOLUTIONS
CALL SUB. INTEG FOR f, f' SOLUTIONS
CALL SUB. PROFYL FOR q', q'', q''' SOLUTIONS
PRINT INTERMEDIATE VALUES OF f, f', f'', q', δ*
YES
TEST FOR FATAL ERROR
NO
CALL SUB. PROFYL FOR K1, K2, K3, σ SOLUTIONS
NO
TEST FOR CONVERGENCE OF f, q'
YES
CALCULATE H, δ, θ, Cf, S1 AND FLOW STABILITY CHARACTERISTICS
PRINT ALL STATION PARAMETERS AND PROFILES
I = I + 1
ADJUST δ* FOR NEW X-STATION
IS I GE I (MAXIMUM)
NO
YES
PRINT SUMMARY OF BOUNDARY LAYER PARAMETERS
PRINT ERROR MESSAGE
STOP

FIGURES



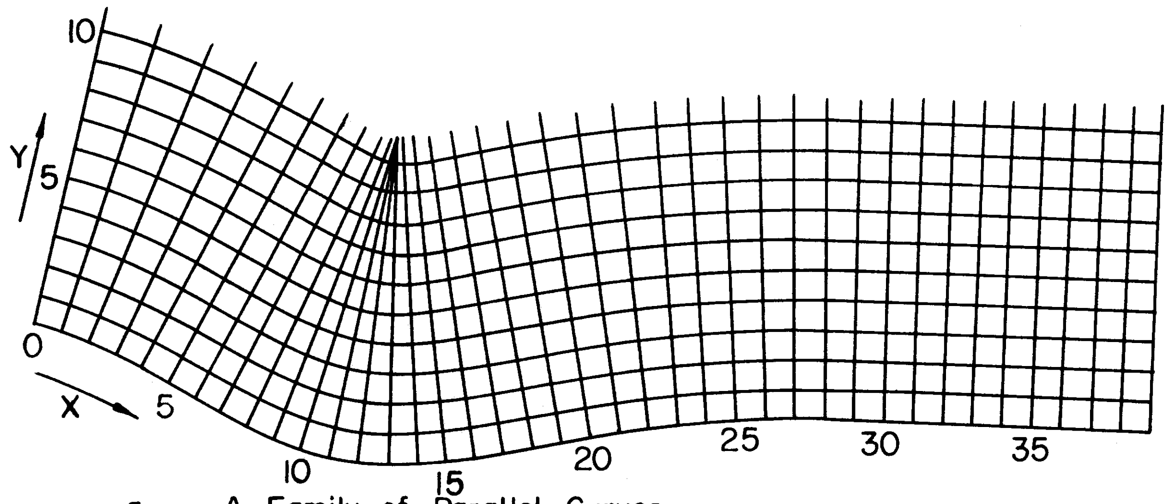
I Entrance Region Laminar Boundary Layer (Stable)

II Unstable Laminar Regime (Linear & Non Linear Amplification)

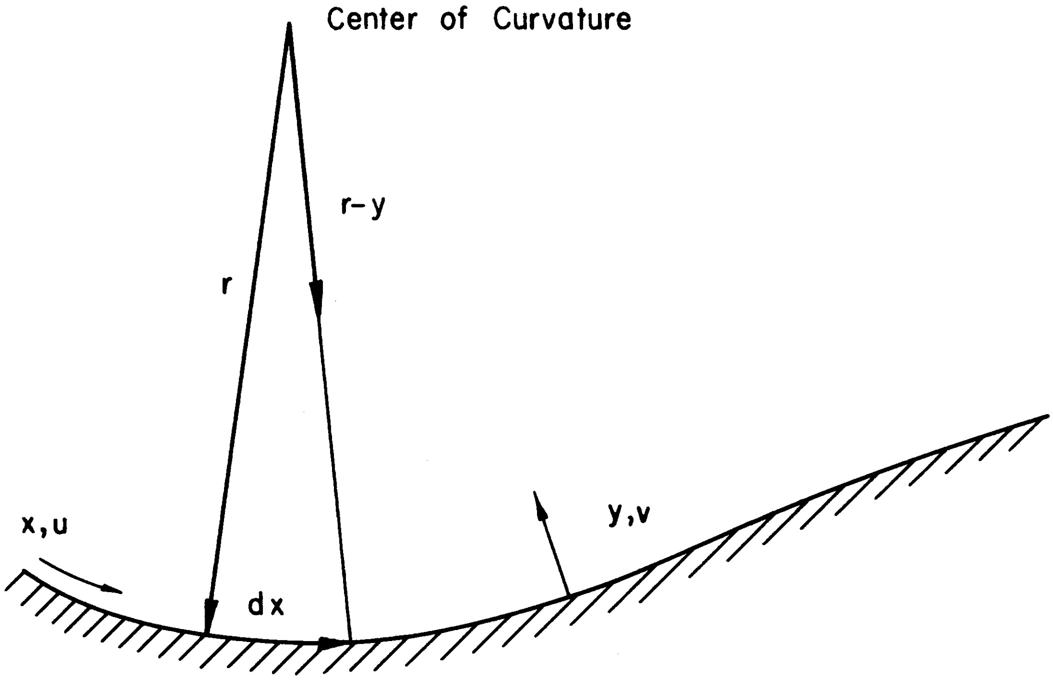
III Transitional Boundary Layer (Intermittently Laminar & Turbulent)

IV Fully Turbulent Regime (Stable)

Fig. 1 Schema of the Conventional Wall Boundary Layer Regimes



a - A Family of Parallel Curves



b - Dimensional Relationships

Fig. 2 Orthogonal Curvilinear Coordinate System.

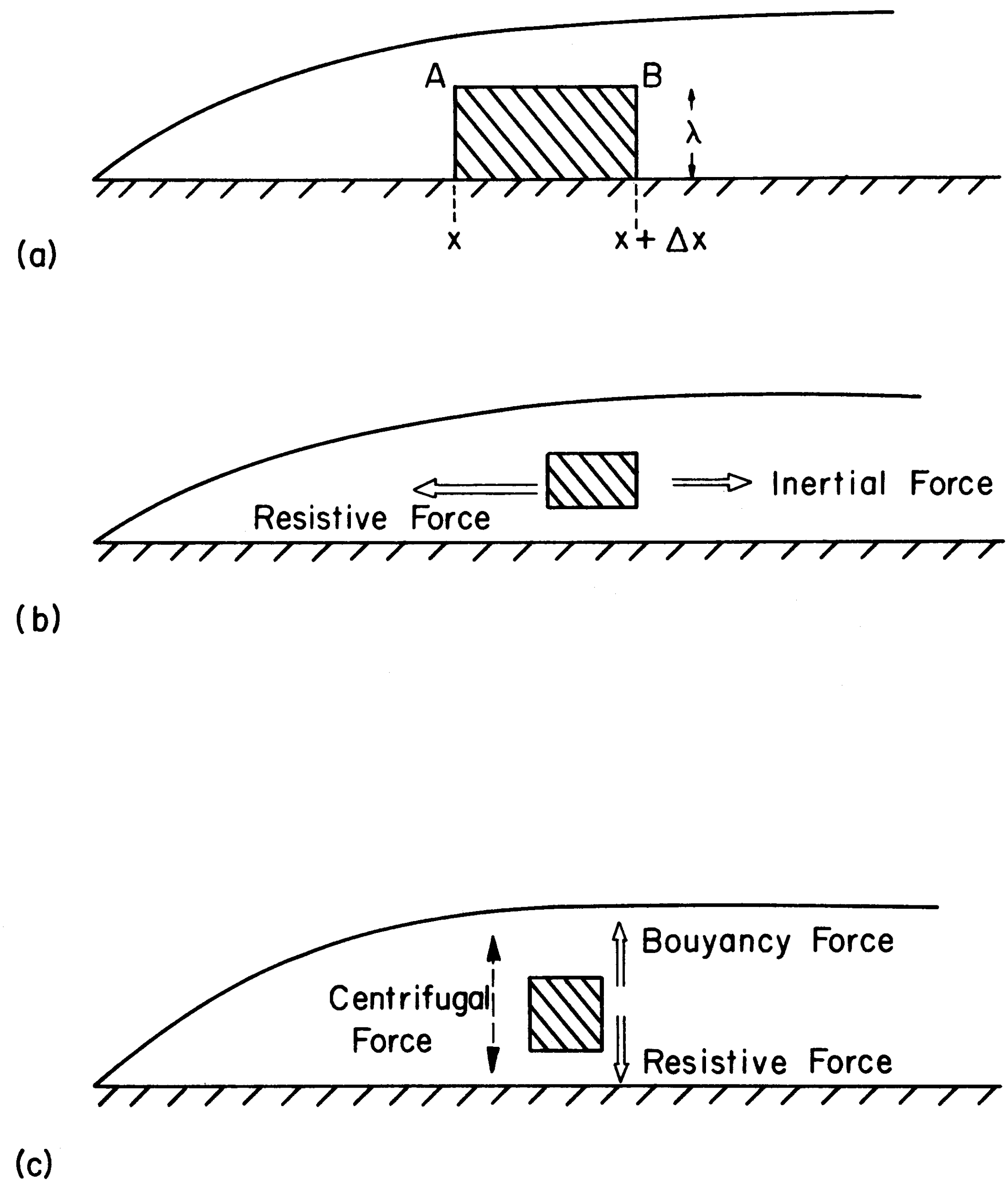
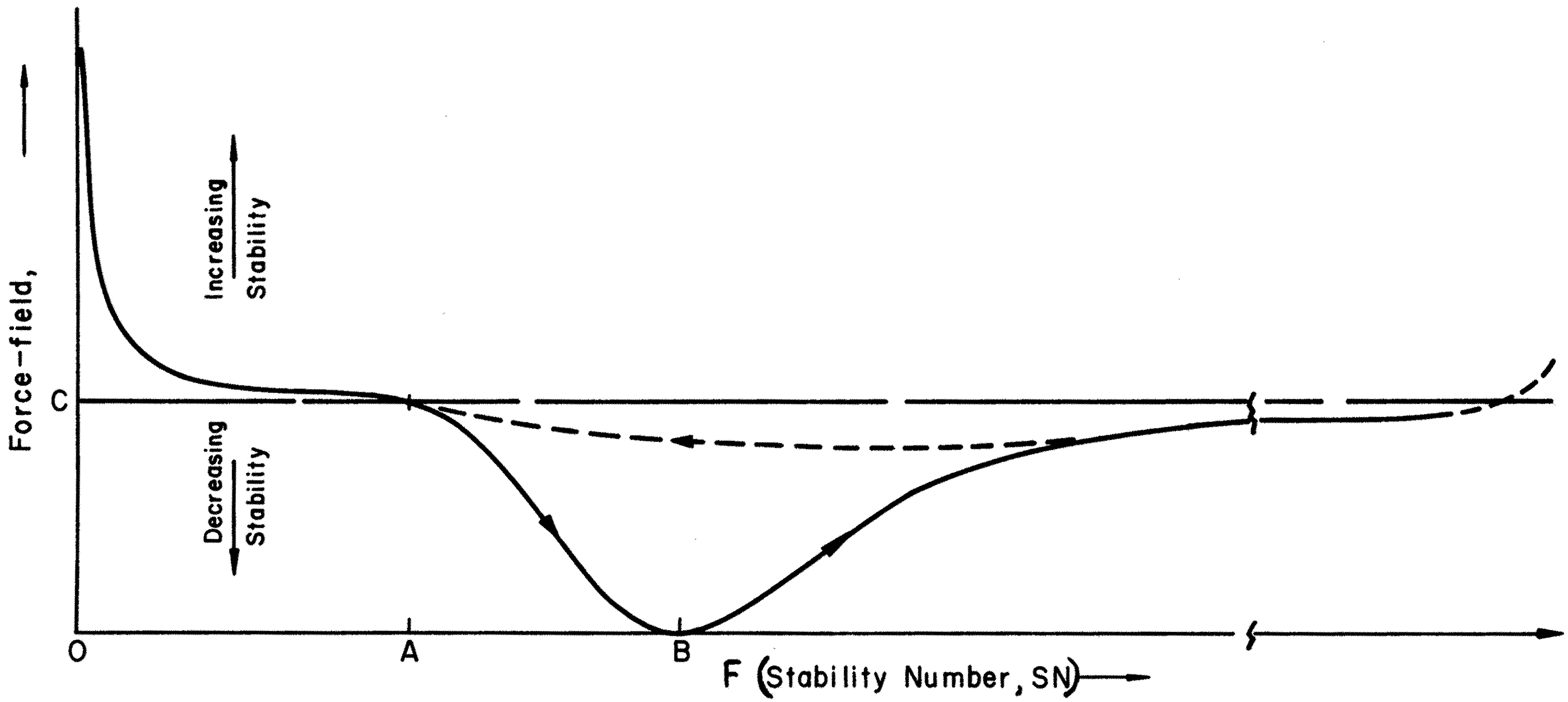


Fig. 3 Schema of Boundary Layer Sections.



A : Point of Incipient Laminar Instability (SN ≈ 10^3)
B : Point of Incipient Laminar/Turbulent Transition
C : Critical Force Level
---- Relaminarization Route

Fig. 4 Schema Of Force-field/Stability Number Relationship ($\bar{\eta} = \bar{\eta}_{crit}$)

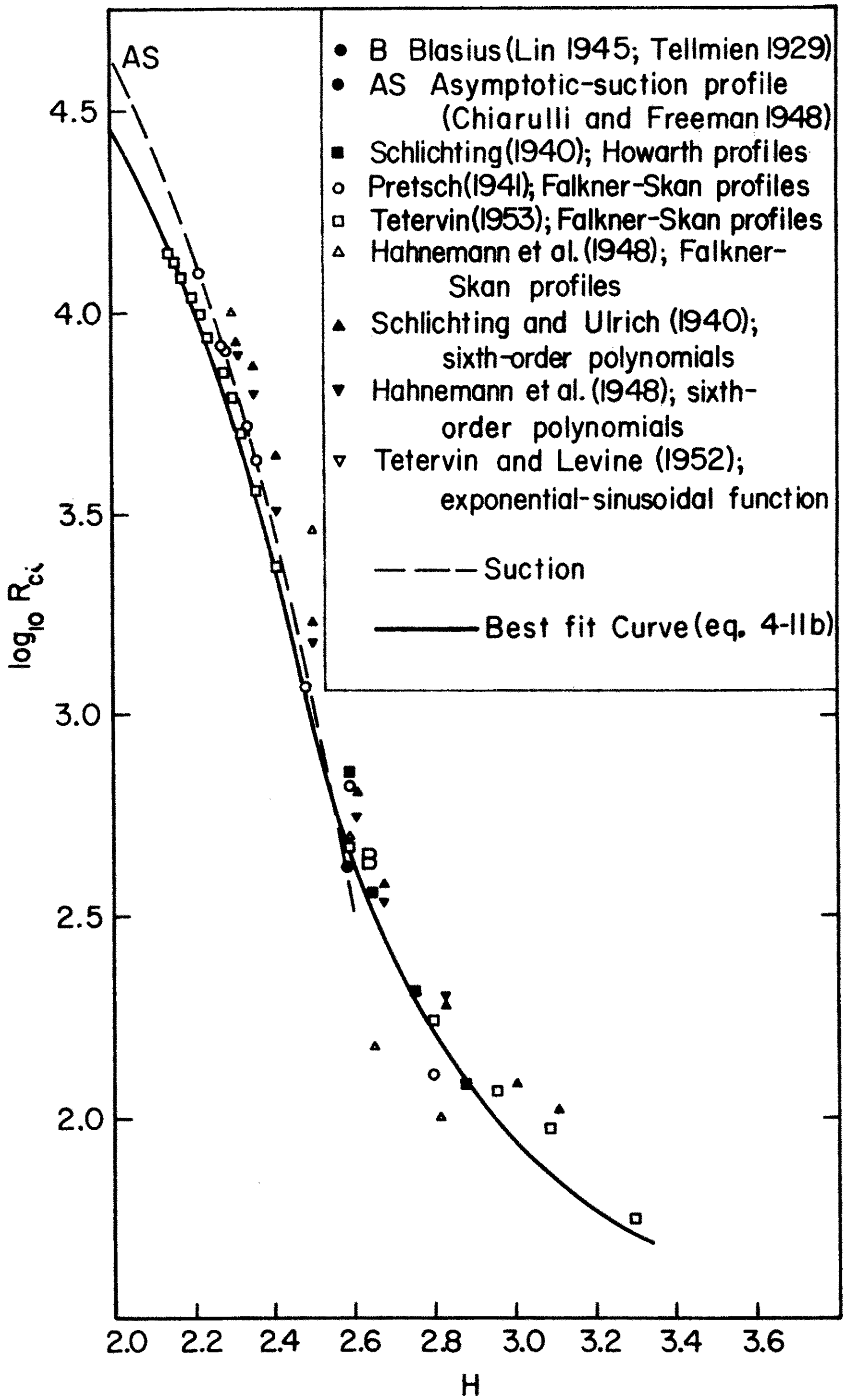
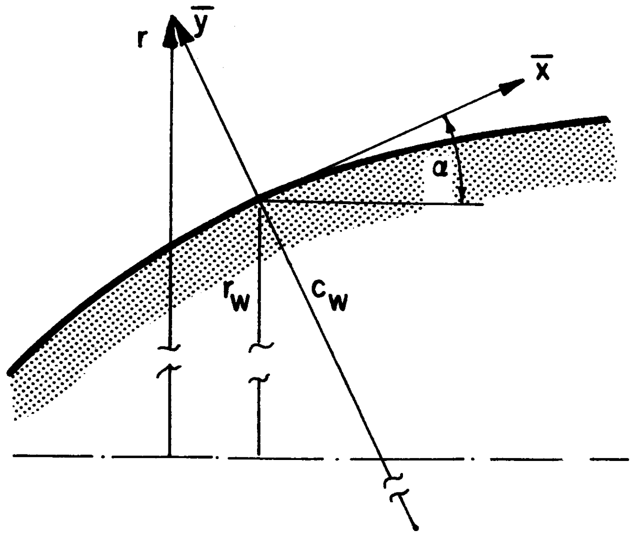
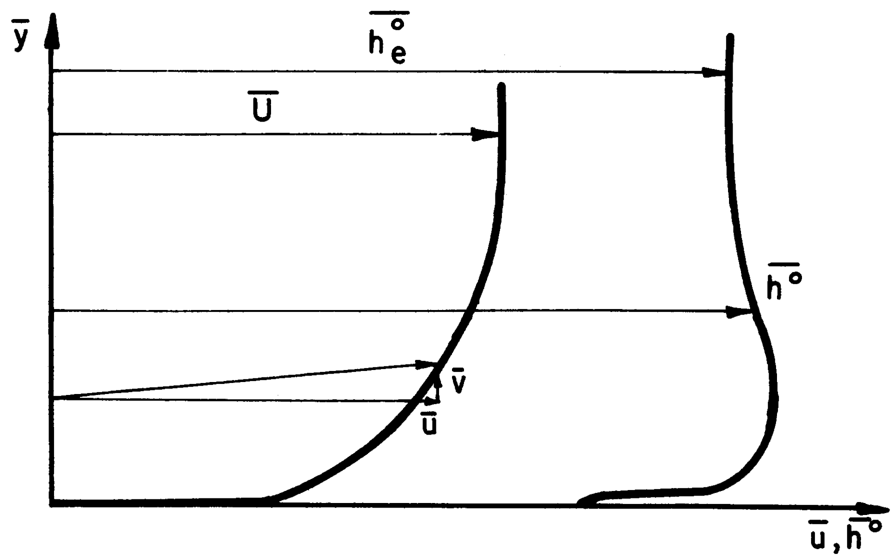


Fig. 5 Functional Relation Between $\left(\frac{U_1 \delta^*}{\nu}\right)_{ci}$ and the Shape Factor H.



a. Coordinate System



b. Description of Velocity and Total Enthalpy Profiles

Fig.6 Herring - Mellor Coordinate System

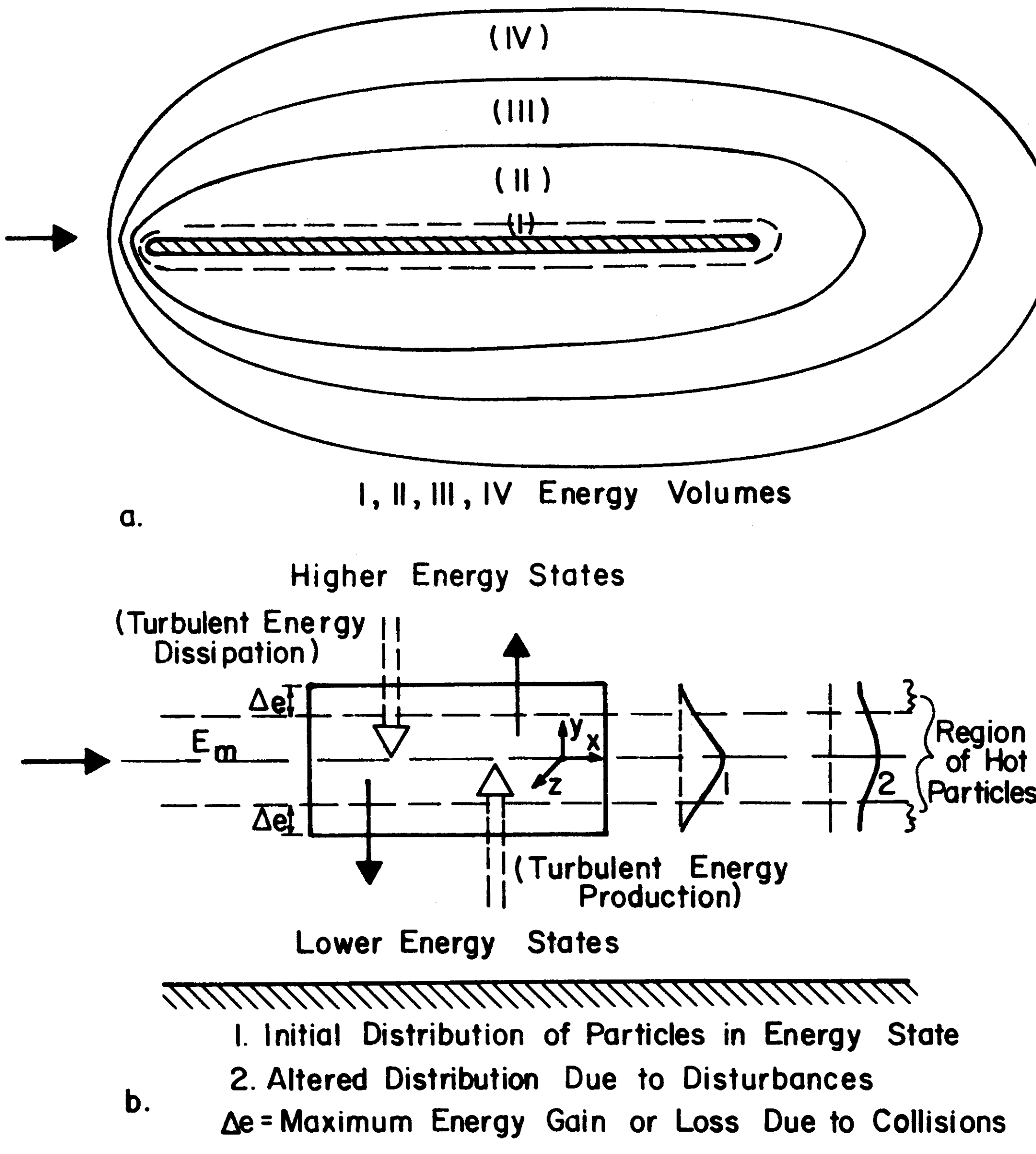


Fig. 8 Schema of Conceptual Energy Volumes on a Flat Plate Boundary Layer.

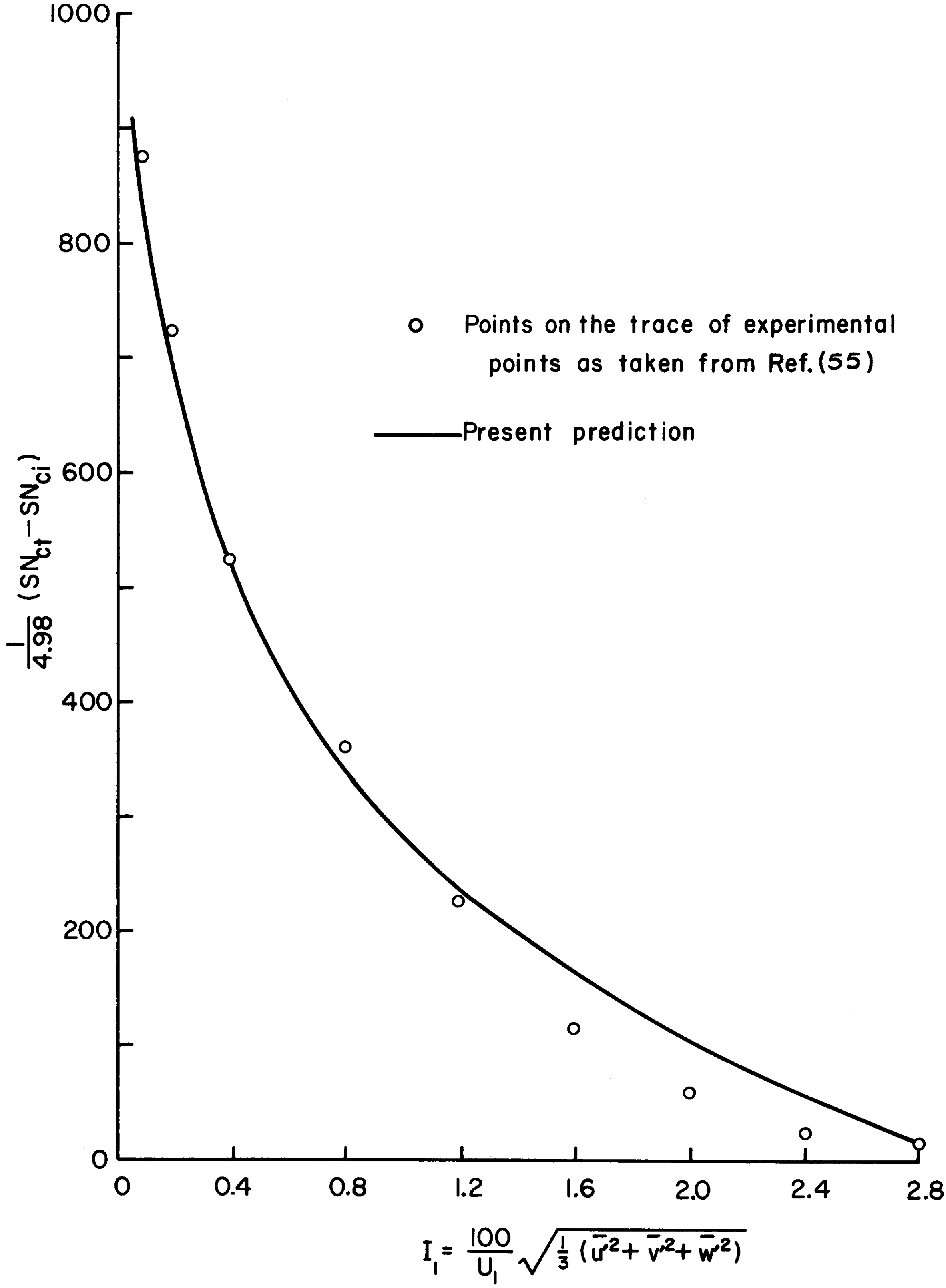


Fig. 9. Effect of Initial Turbulence Intensity on $(SN_{ct} - SN_{ci})$.

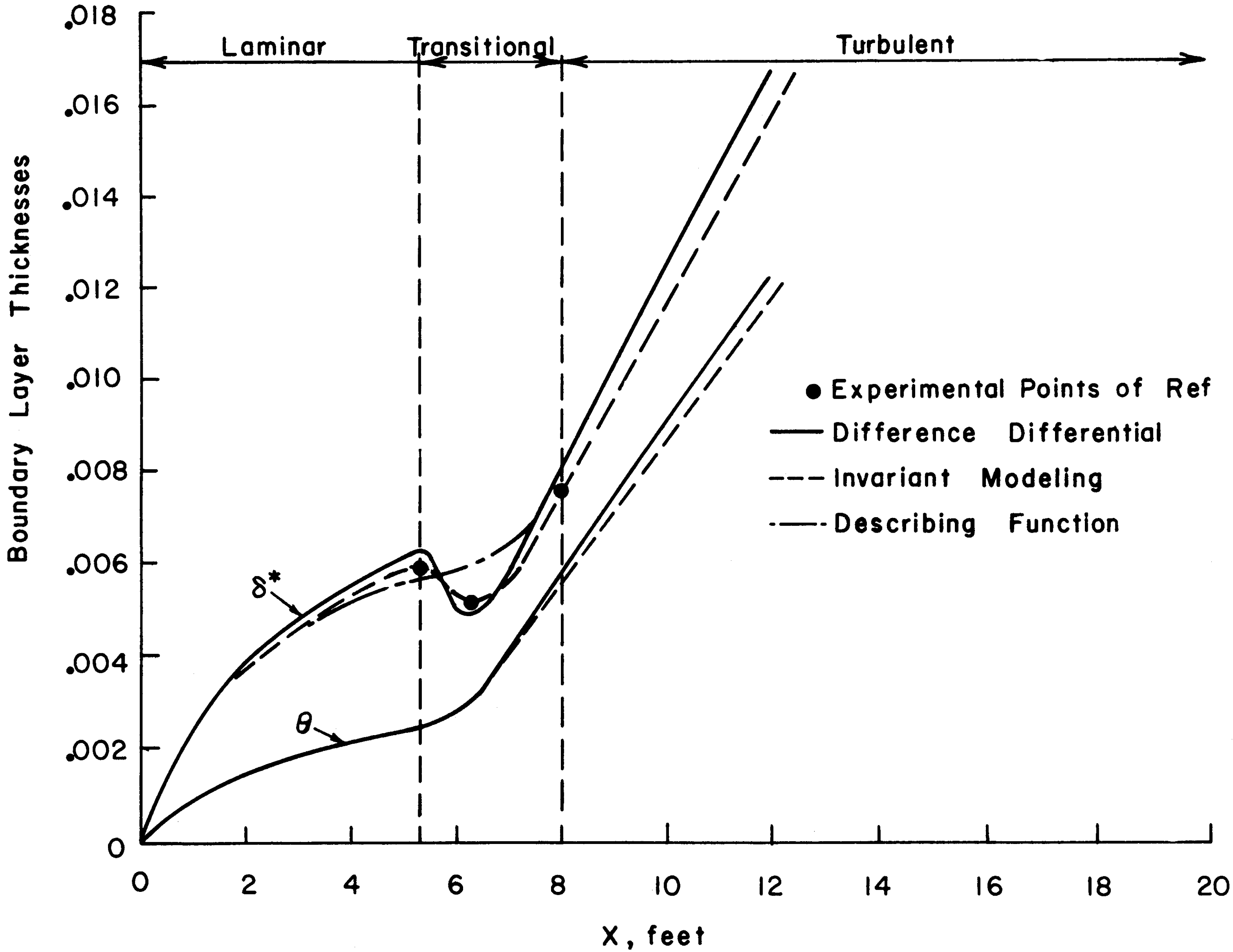


Fig. 10: Continuous Predictions of Boundary Layer Thicknesses

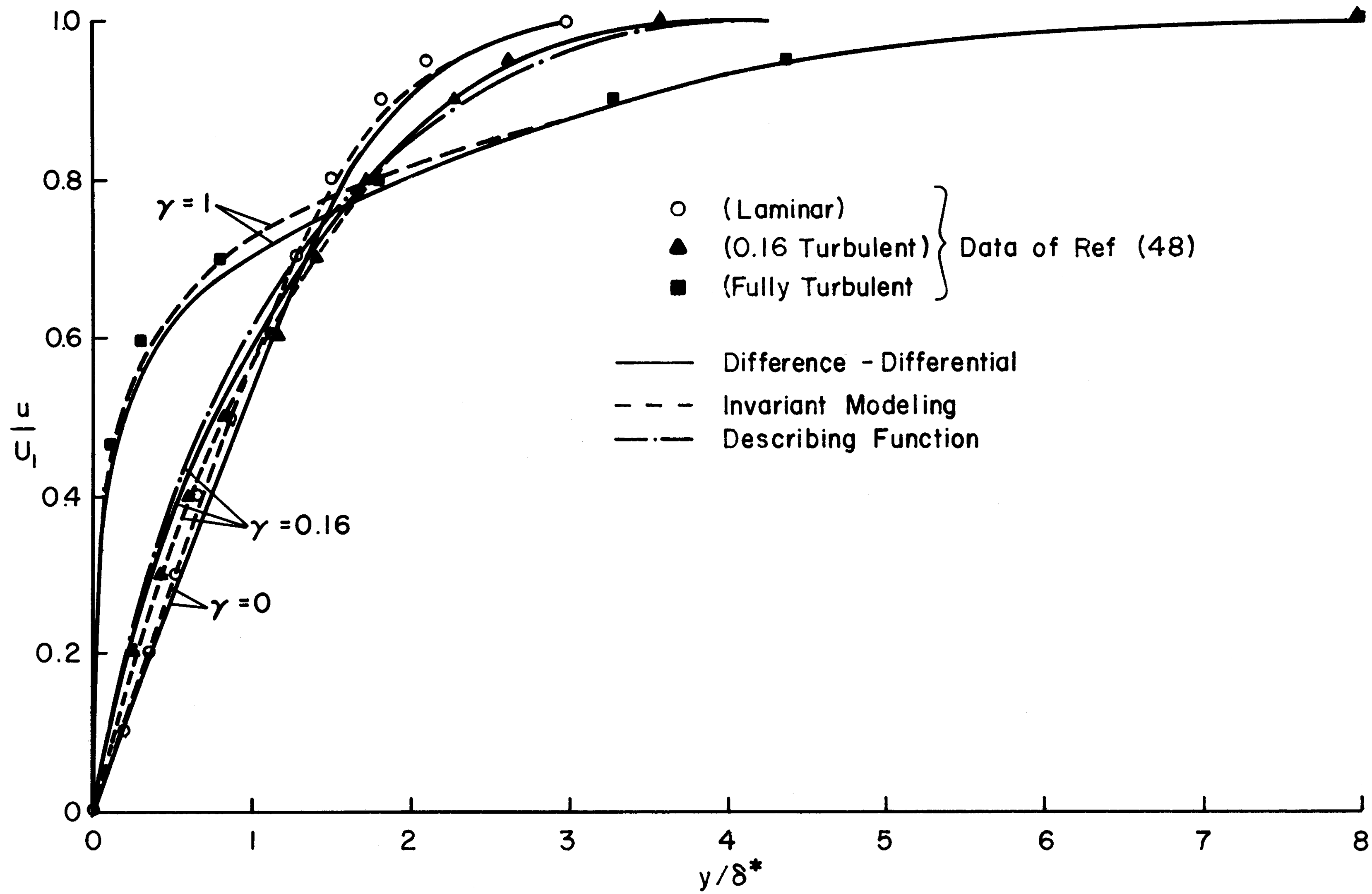


Fig. 11 Continuous Predictions Of Mean Velocity Profiles.

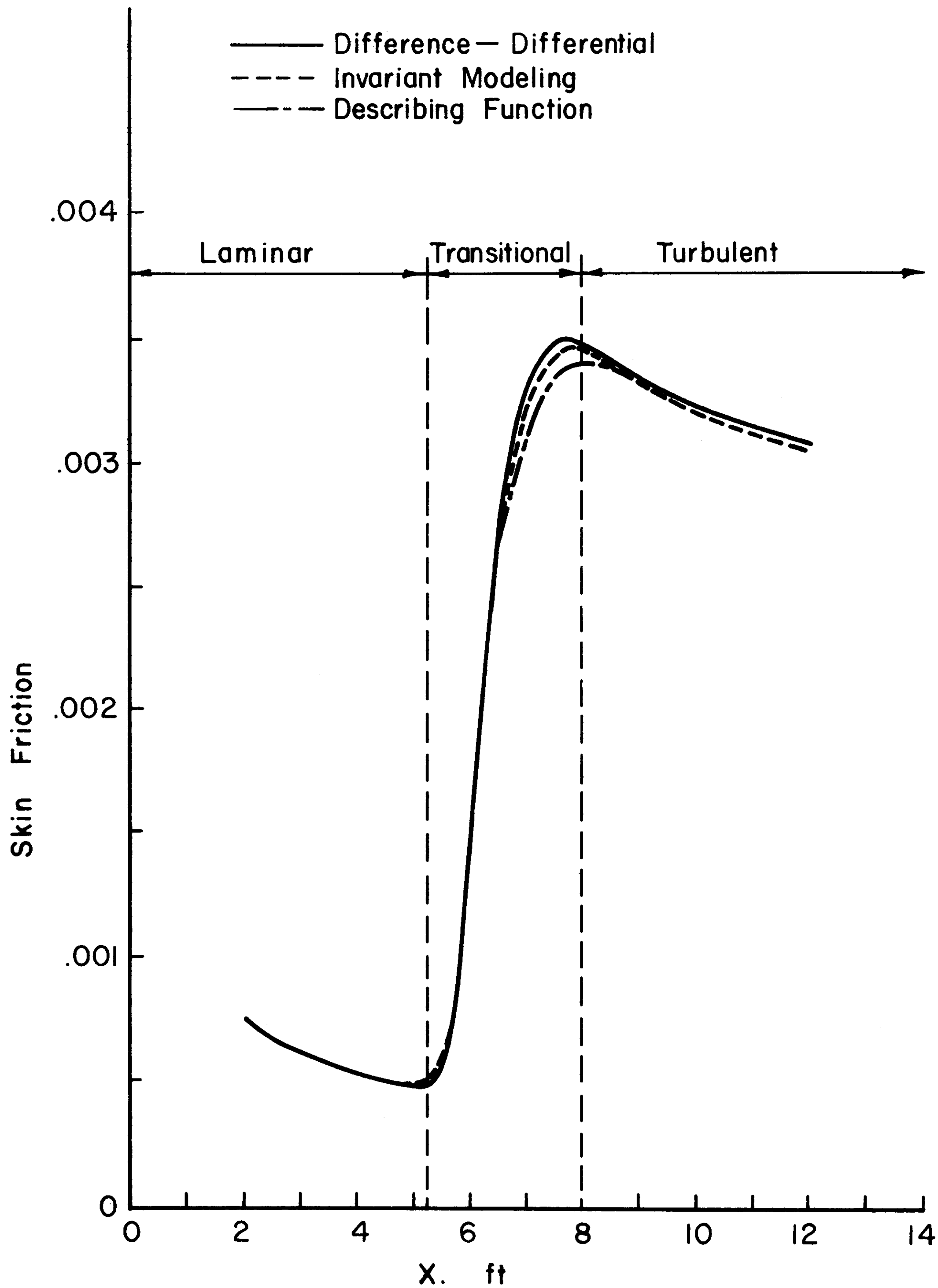


Fig. 12 Continuous Predictions Of Skin Friction.

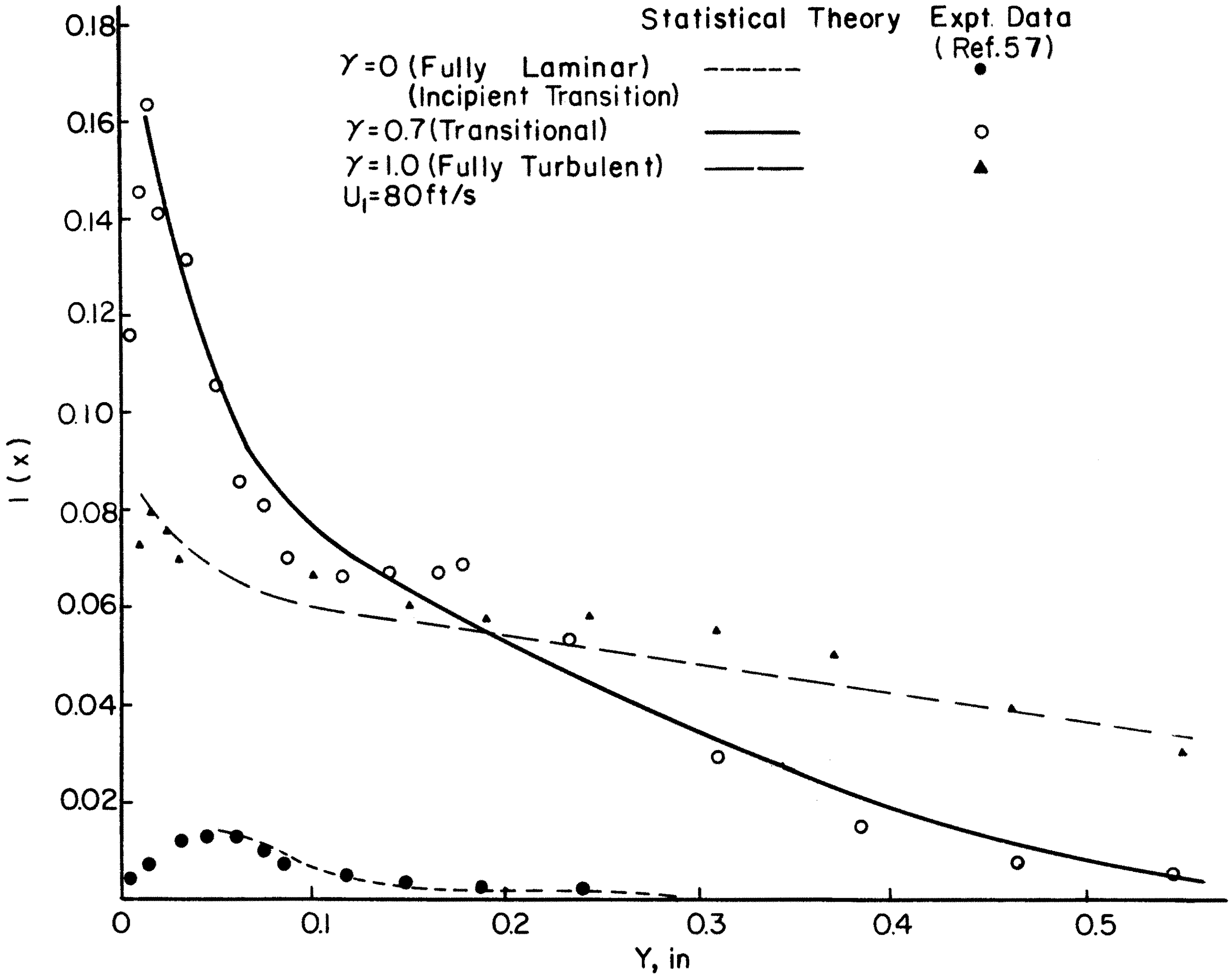


Fig.13 Statistical Theory Prediction of Turbulence Distribution

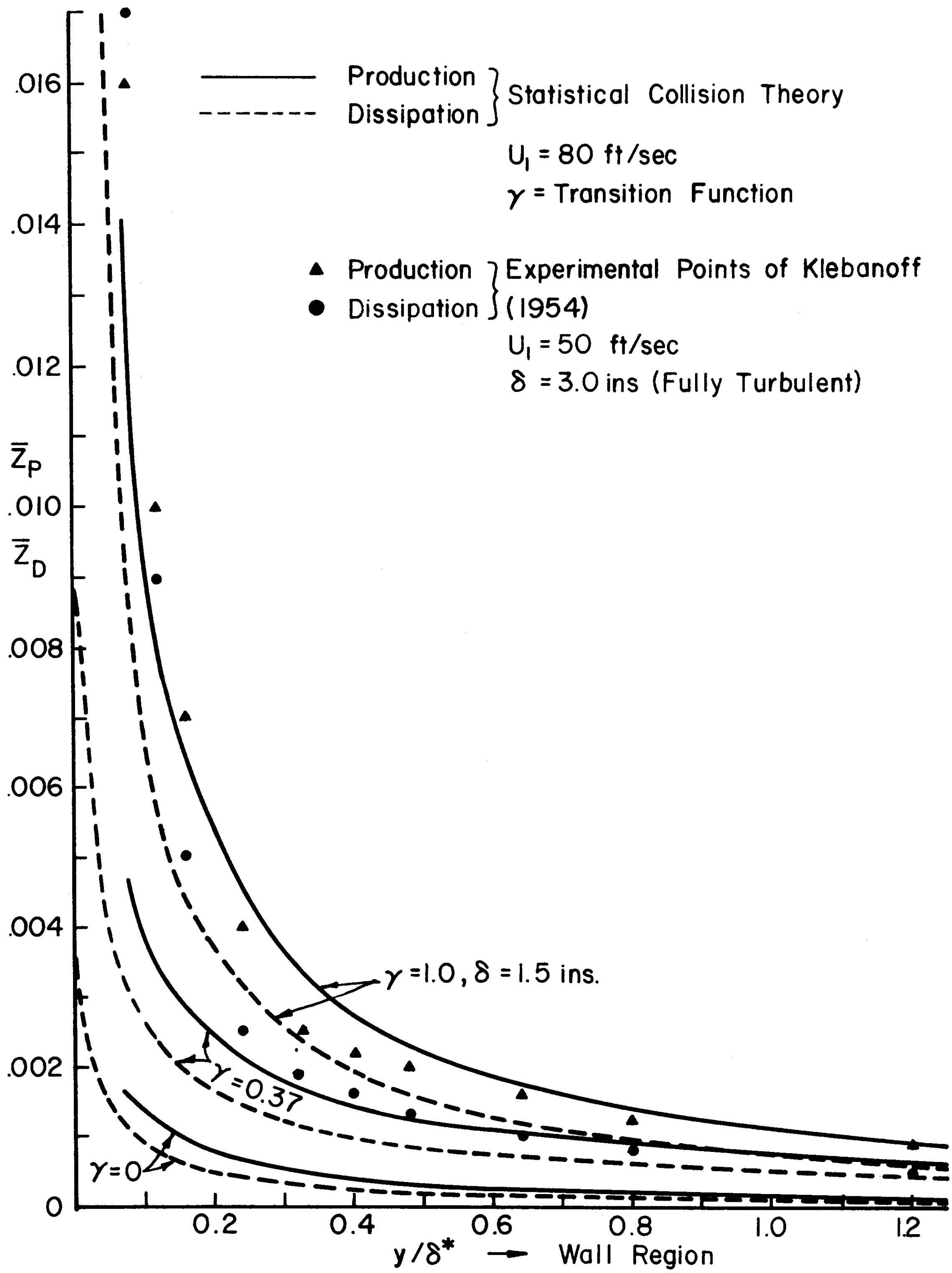


Fig. 14 Turbulence Energy Production and Dissipation Rates In Wall Region.

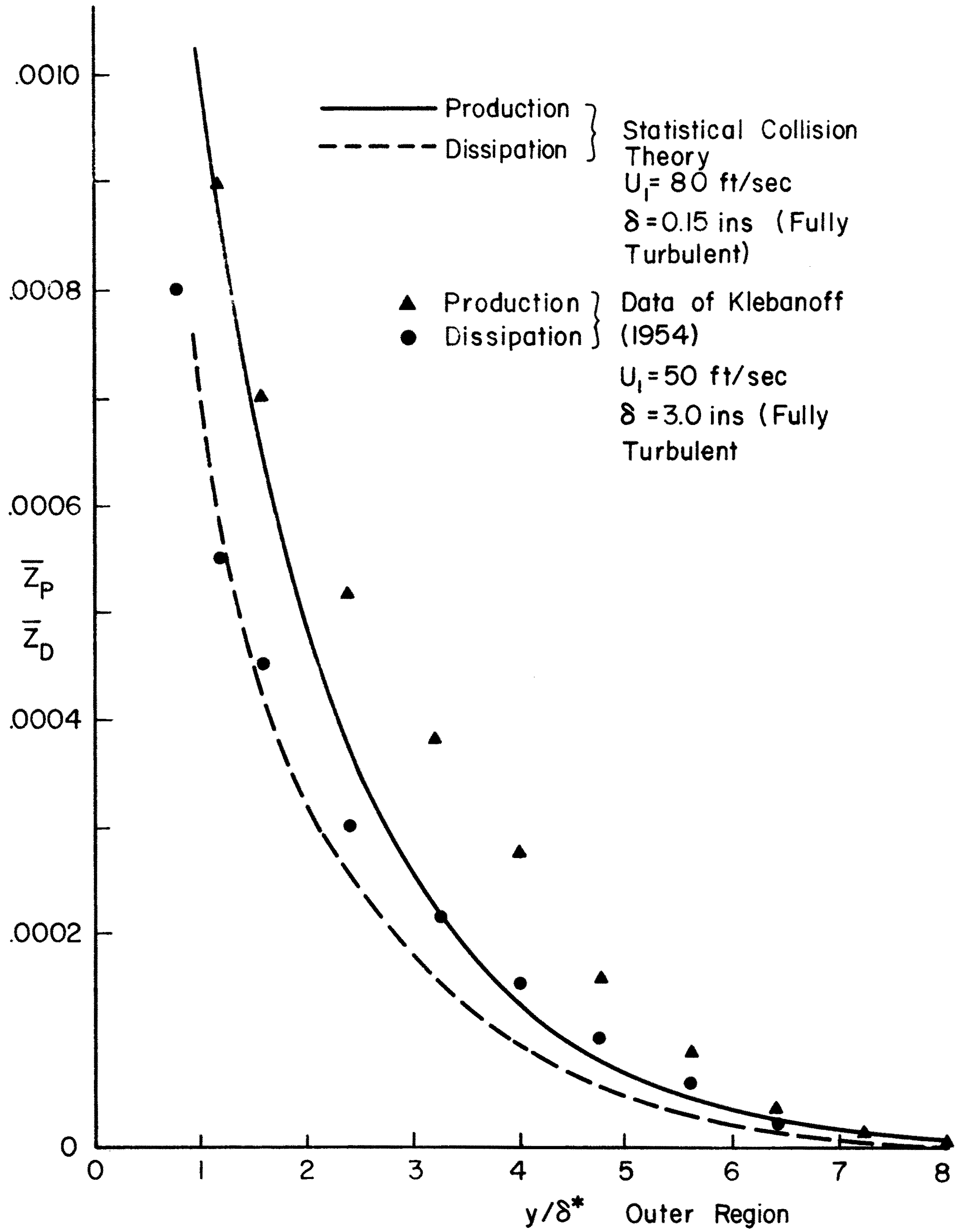


Fig. 15 Turbulence Energy Production and Dissipation Rates Away From Wall.

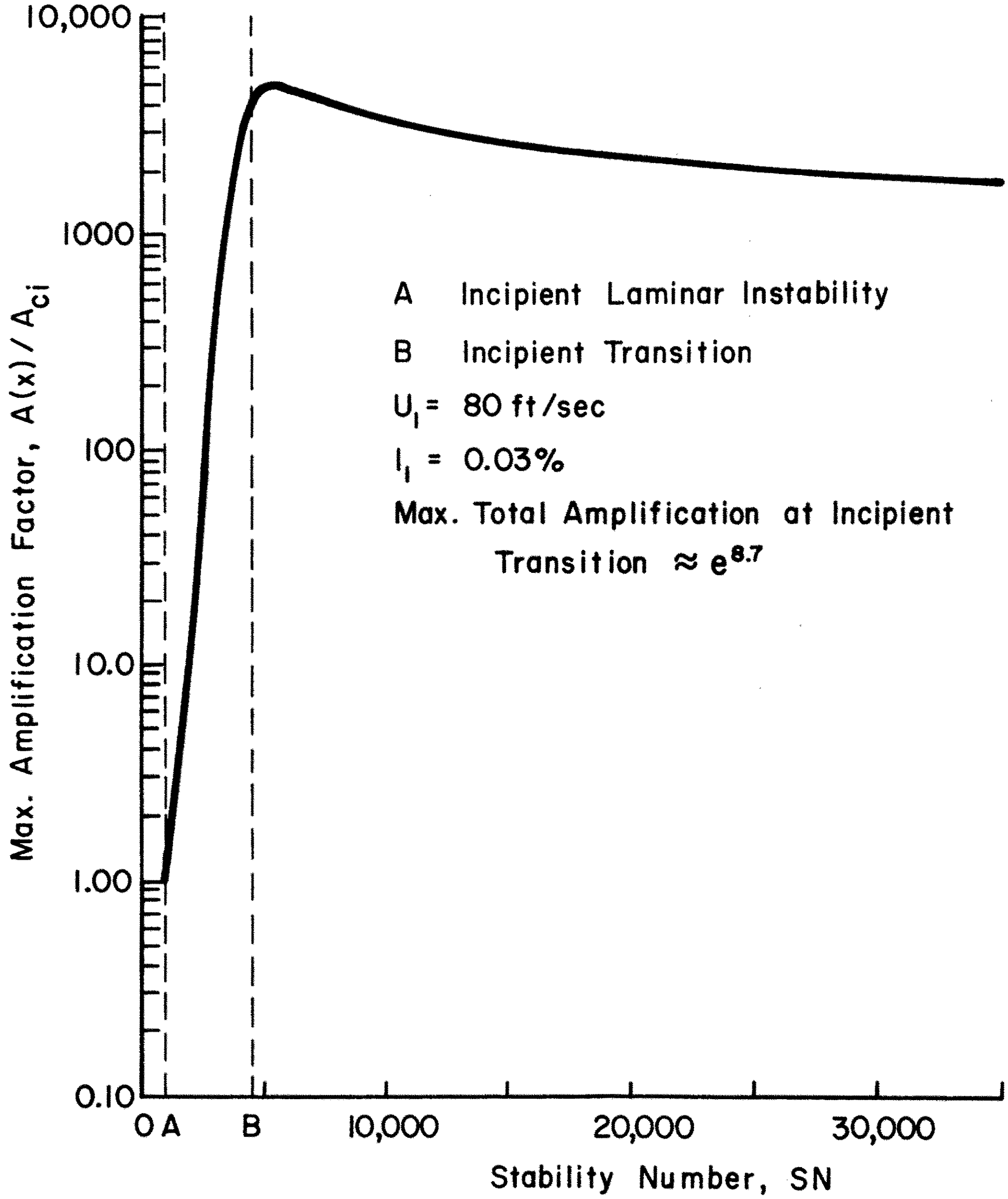


Fig. 16. Typical Streamwise Variation of Max. Amplification Factor.

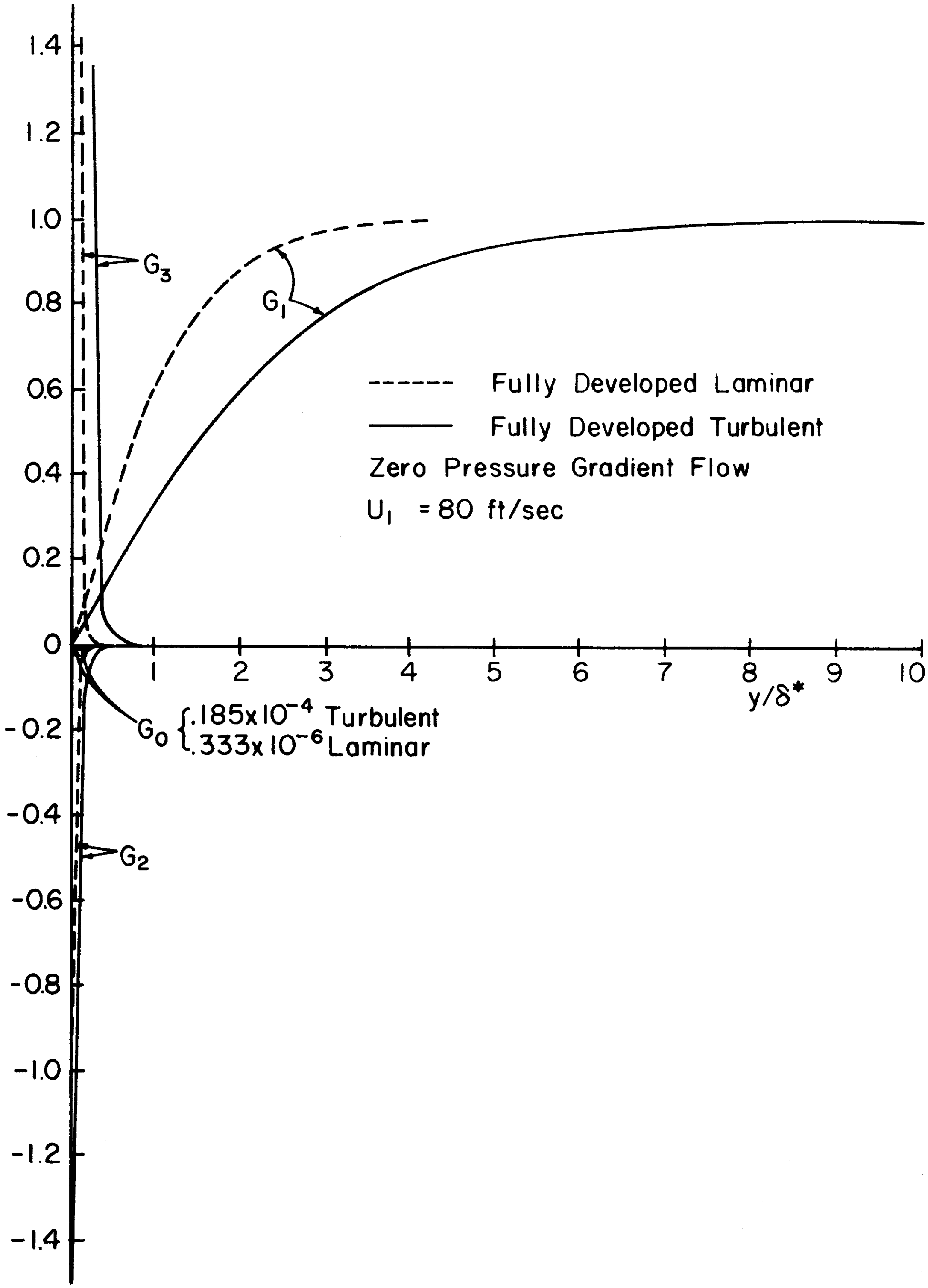


Fig. 17 The Describing Function Coefficients