Studies in Low-Order Spectral Systems

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Technical report to The National Science Foundation
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ABSTRACT

A low-order system of spectral equations--representing in some measure the physical properties of the atmosphere--has been shown to apply to both barotropic and baroclinic flows. The system allows for an arbitrary zonal flow and one planetary wave with no approximations for spherical geometry. The analytic solutions in terms of elliptic functions have been described and the required initial conditions have been clarified. The energetics of both barotropic and baroclinic flows for the low-order truncation have been discussed and a Fjortoft (1953) type theorem for baroclinic energy exchange has been developed. The applicability of linear analysis to the nonlinear equations has been shown and correspondence to the true solution established. Some barotropic calculations are described and show a remarkable variety of energy exchanges, depending on initial conditions and truncation.

1. Introduction

Over the last two decades, developments in the prediction of atmospheric events on the planetary scale have followed two trends of thought which are by no means mutually exclusive. On the one hand, the more pragmatic view toward prediction--which is ultimately the purpose of science--has stimulated the development of models, their numerical integration, and the determination of their success relative to observation. On the other hand, detailed investigations of the individual characteristics of the atmosphere which determine the observed flow have been studied by a number of techniques, principal among which is the method of linearization. There is indeed no question that studies of the latter type will provide valuable information for the direction in which the former modelling developments should proceed.

The present study is designed to provide both a method whereby the details of the nonlinear exchange process which occurs in the atmosphere may be considered in a simplified form, and also to indicate from some calculations the complexity of those exchanges. Thus this study falls into the second category described above; however, since it deals with nonlinear properties, the linearization procedure is clearly inapplicable. The nonlinear exchange under consideration is the physical process whereby atmospheric variables (momentum, heat, etc.) are transferred from one scale to another by the flow field, and is described mathematically by the so-called "advection terms" in the equations of motion. The exchange process is envisioned more clearly (at least by the writer) from a wave space representation; i.e., by allowing the space dependence of the dependent variables involved to be expressed in a series of orthogonal polynomials. Such a representation, which also necessitates a set of time dependent expansion coefficients, will result in the ultimate expression for the "advective" exchange terms as a sum of quadratic products of the expansion coefficients. The number of such products will be established by the series truncation, in effect the number of degrees of freedom desired for representing any space dependent variable in series form. The multiplicity of such products, however, again obscures the exchange process.

It was Lorenz (1960a) who first recognized that the equations representing the motion, when expressed in terms of a series expansion[†], could be significantly simplified and yet retain their nonlinearity. The simplifying procedure was to truncate the expansion series to a few terms. Remarkably, Lorenz found that with only three active components (expansion coefficients) the exchange properties derived showed characteristics in common with atmospheric events;

[†]Equations in this form are now commonly called "spectral equations" and this terminology will be utilized in the sequel.

moreover, the time dependent solutions to the truncated (herein called "low-order") equations were given analytically. Unfortunately, Lorenz' system was expressed in cartesian geometry and had limited applicability to the geometry of the atmosphere. He had, furthermore, applied the analysis only to barotropic flow.

An expansion which has more general applicability to the geometry of the earth-atmosphere involves the solid spherical harmonics, and had been utilized even before Lorenz' effort to represent the barotropic vorticity equation in spectral (expansion) form by Silberman (1954). Following further studies wherein this method was applied to the barotropic vorticity equation (BVE) with many degrees of freedom, (Platzman, 1960; Baer and Platzman, 1961), Platzman (1962), in common with Lorenz, discussed the various low-order systems which could be solved analytically in spherical coordinates. The advantage of these systems--the low-order expression of the BVE--is the accurate inclusion of the curvature of the geopotential surfaces and the exact incorporation of the Coriolis effect. It is one such system, categorized by Platzman as "Class L3" and involving interactions between an arbitrary zonal field and one planetary wave, which will be discussed in detail in this study.

Although both Lorenz and Platzman considered the low-order systems with reference to the BVE, it is a simple matter to extend the systems to a quasi-geostrophic, baroclinic model with fixed static stability and applicable to two layers (or levels). In this event, one may represent the flow by a vertical mean and shear (following Lorenz, 1960b) wherein the interacting zonal flow will describe the shear flow (the zonal mean flow will be seen to be inactive) and the wave flow will be a combination of both mean and shear.

We shall present, then, a set of low-order equations which will be applicable to both barotropic or baroclinic flow, depending on the definition of the variables. The solutions to these equations will be given in detail and their elliptic characteristics made evident. Since we deal here with an initial value problem, the specification of initial conditions and its possible variation over atmospheric extremes will be discussed. Furthermore, because of the profound truncation applied, truncation must also be considered as a pertinent variable in the solutions. One of the principal exchange characteristics, and one which cannot be determined from linearized equations, is the energy exchange between the zonal flow and the planetary wave. As the solutions to the low-order systems are periodic, the energy exchange properties over any nonlinear period may be--and are--discussed in detail. A particularly interesting feature of this energy exchange is the observation that an exchange theorem as stated by Fjortoft (1953) for barotropic motion involving the exchange between different scale components is also applicable to the baroclinic model herein considered.

Although it has been emphasized that the essential properties to be described by the low-order systems are nonlinear, linear analysis does give some indication of the behavior of the systems when the initial wave energy is of perturbation amplitude, and stable solutions prevail. The potentialities of linear analysis to the problem at hand are therefore investigated both with regard to the information concerning the motions which may be forthcoming, and also as a check on the nonlinear solutions. Finally, a number of calculations with the barotropic vorticity equation are described and indicate the variety of energy exchange which the low-order systems are capable of representing.

2. The Low-Order Equations

From the viewpoint of the earth's atmosphere, perhaps the most interesting energy exchange process is between the planetary waves and the zonal flow, both in the vertical mean and shear flow which are frequently defined to describe barotropic and baroclinic effects respectively. Investigations of these exchanges by analysis of real data leave strong indications as to the direction and magnitude of such atmospheric processes; see, for example, Saltzman (1958), Kung (1966a, b) and Wiin-Nielsen (1967). From a theoretical outlook, such exchanges may be conveniently isolated (but clearly not completely described) by the low-order systems and the corresponding exact solutions obtained therefrom. Furthermore, by proper selection of the modelling equations, the "barotropic" and "baroclinic" effects may be completely isolated. The low-order system obtained for exchange between the zonal flow and one planetary wave over a spherical surface is denoted as the "Class L3" by Platzman (1962) in his discussion of the spectral vorticity equation.

In the following development, all equations will be represented in their spectral form. Since we are interested in the exchange between the zonal flow and a planetary wave, we shall also break the dependent variables into their zonal and wave components. This may be easily accomplished by the definition,

$$\psi(\lambda,\mu,t) = \bar{\psi} + \psi'$$
$$\bar{\psi}(\mu,t) \equiv \frac{1}{2\pi}\int_0^{2\pi}\psi d\lambda \qquad (2.1)$$

where ψ is any dependent variable, λ is longitude, and the bar operator denotes longitudinal averaging. All variables will be non-dimensionalized, in space by the mean radius of the earth and in time by the earth's rotation rate. The Coriolis parameter will be defined by the variable f which is proportional to $\mu \equiv \sin\phi$, the sin of latitude.

The procedure whereby the physical equations are converted to spectral form has been discussed in detail by Platzman (1960) and Baer and Platzman (1961) and will not be repeated. Very briefly, the technique is as follows: each dependent variable is expanded in a series of solid harmonics with time dependent coefficients. Because of the orthogonality of the harmonics over a spherical surface, a set of ordinary, nonlinear differential equations may be developed with time as the independent variable, and with the expansion coefficients as the dependent variables. Although both the barotropic and baroclinic equations to be discussed may be expressed immediately in identical spectral form, (see Baer and King, 1967), we shall expand somewhat on the development for added clarity.

Barotropic exchange:

The barotropic processes may be determined from the barotropic vorticity equation (BVE) which states that in any horizontal (or pressure) surface, the absolute vorticity associated with a particle is conserved;

$$\frac{\partial}{\partial t} \nabla^2\psi = J(\nabla^2\psi + f, \psi) \qquad (2.2)$$

In this equation, ψ represents the stream function and the Jacobian operator is taken with regard to λ and μ respectively. The spectral form of the equation for the time rate of change of the expansion coefficients of the stream function may be written immediately (Baer, 1964) as,

$$\frac{d}{dt} \psi_\gamma = -i\ell_\gamma \omega_\gamma \psi_\gamma + \sum_{\alpha,\beta} \psi_\beta \psi_\alpha I_{\gamma\beta\alpha} \qquad (2.3)$$

where γ ranges over all allowed index values. The indices (γ,α,β) may be considered as vectors or complex numbers describing the ordinal and planetary number of any component; thus, for example,

$$\gamma \equiv n_\gamma + i\ell_\gamma \ .$$

For the problem under consideration here, we choose an arbitrary zonal current,

$$\bar{\psi}(\mu,t) = \sum_{\gamma}^{N} \psi_\gamma(t) Y_\gamma(\mu) \qquad (2.4)$$

and two components in one planetary wave for which we shall use the index

$$\psi'(\lambda,\mu,t) = \psi_\alpha(t)Y_\alpha(\mu,\lambda) + \psi_\beta(t)Y_\beta(\mu,\lambda) \qquad (2.5)$$

where α, β have the unique values,

$$\alpha \equiv n_\alpha + i\ell; \quad \beta \equiv n_\beta + i\ell \qquad (2.6)$$

It must be recognized that ψ' is a real function whereas the expansion coefficients (ψ_α,ψ_β) are complex; therefore in the expansion (2.5), the conjuga values of the expansion terms must also be included.

Given the representations (2.4) and (2.5), we may extract the required equations from the more complete set expressed by (2.3). These have been written by Platzman (1962) and Baer (1964), and are,

$$\frac{d\psi_\gamma}{dt} = 2a_\gamma \mathrm{Im}\ \psi_\alpha\psi_\beta^*$$

$$\frac{d\psi_\beta}{dt} = -i\ell\omega_\beta\psi_\beta + \sum_{\gamma\neq 1} b_\gamma\psi_\gamma\psi_\beta + \sum_\gamma e_\gamma\psi_\gamma\psi_\alpha \qquad (2.7)$$

$$\frac{d\psi_\alpha}{dt} = -i\ell\omega_\alpha\psi_\alpha + \sum_{\gamma\neq 1} g_\gamma\psi_\gamma\psi_\alpha + \sum_\gamma f_\gamma\psi_\gamma\psi_\beta \ .$$

The first of Eqs. (2.7) represents a set of equations and there are as many as there are elements in the sum specified by (2.4). The solution of these equations will be discussed in the next section, and the coefficients (constant in time and space) will be defined subsequently.

Baroclinic exchange:

The simplest model which will describe baroclinic energy exchange is a representation of the vertical structure of the atmosphere by two layers with constant stability. Such a model, in which the quasi-geostrophic approximation has been made, and in which the product of the divergence times vorticity as it appears in the vorticity equation is approximated by a mean vorticity (Coriolis parameter) has been given by Lorenz (1960b) in terms of the vertical mean wind and shear wind (actually stream functions) and is represented as follows;

$$\frac{\partial}{\partial t} \nabla^2\psi = J(\nabla^2\psi + f,\psi) + J(\nabla^2\tau,\tau)$$

$$\frac{\partial}{\partial t} (\nabla^2-r^2)\tau = J\left((\nabla^2-r^2)\tau,\psi\right) + J(\nabla^2\psi + f,\tau) \tag{2.8}$$

In these equations, ψ represents the mean stream field and τ the shear field, both of which may be determined from the stream fields in the two layers by addition and subtraction respectively. The Jacobian operator is the same as for the BVE and r^2 is a parameter which depends on the mean Coriolis parameter and stability in the form,

$$r^2 \equiv \frac{f_0^2}{\sigma}$$

where σ is a stability parameter and maintained constant. It is an easy matter to show that in the model described by (2.8), the mean potential temperature is linearly proportional to the shear (τ) by virtue of the thermal wind equation.

As in (2.1) we again consider the flow to be given averaged flow and in one planetary wave component. Rather than allowing two components in one wave as was done for the BVE, Eq. (2.5), we allow only one wave component for the mean flow (pressure averaged) and one component for the shear flow. The zonal distributions of both the mean and shear flow may still be arbitrary. The dependent variables are thus expressed as follows:

$$\psi = \bar{\psi} + \psi' \qquad \tau = \bar{\tau} + \tau'$$

$$\bar{\psi} = \sum_{\gamma}^{N} \Psi_\gamma(t)Y_\gamma(\mu) \qquad \bar{\tau} = \sum_{\gamma}^{N} \psi_\gamma(t)Y_\gamma(\mu) \tag{2.9}$$

$$\psi' = \psi_\alpha(t)Y_\alpha(\lambda,\mu) \qquad \tau' = \psi_\beta(t)Y_\beta(\lambda,\mu)$$

The space dependent functions, Y, are the same as those used in the expansions for the BVE, but the time dependent expansion coefficients clearly have a different meaning, although they have intentionally been represented by identical symbols; the significance of this choice will soon become apparent.

Substitution of Eqs. (2.9) into (2.8), multiplication by an arbitrary polynomial Y_σ and integrating over the unit spherical surface yields the set of equations for the expansion coefficients Ψ_γ, ψ_γ, ψ_α, ψ_β.

Let us first consider the equations for the coefficients Ψ_γ. Any change of such a component must arise from the interaction of the wave components. However, from the definition of the wave components (Eq. 2.9), the Jacobian of the wave components vanishes; i.e.,

$$J(\nabla^2\psi',\psi') \propto J(\psi',\psi') = 0$$

$$J(\nabla^2\tau',\tau') \propto J(\tau',\tau') = 0$$

by virtue of the fact that (see for example Hobson, 1955)

$$\nabla^2 Y_\alpha = -c_\alpha Y_\alpha$$

where c_α is a constant to be defined subsequently. This result implies that the zonal components of the vertical mean flow will remain constant with time;

$$\Psi_\gamma = \text{constant} \tag{2.10}$$

The time variation of the zonal components of the shear flow will also be determined by the wave component interactions; but it can be seen from (2.8) that these interactions involve Jacobians of the form $J(\nabla^2\tau',\psi')$ which do not necessarily vanish. The resulting equations for the zonal shear flow are, from (2.8) and (2.9),

$$\frac{d}{dt}\psi_\gamma = 2\hat{a}_\gamma \operatorname{Im} \psi_\alpha\psi_\beta^* \tag{2.11}$$

The similarity between this equation and the first of (2.7) should indicate the intent in the choice of symbol notation.

The time rate of change of the components ψ_α and ψ_β will depend on the nonlinear interaction between the components themselves and the coefficients of the zonal mean and shear flows. Since the zonal mean coefficients are constants (2.10 above), interactions with them will have only a linear effect. Moreover, because of the earth's rotation, linear effects of the Rossby-Haurwitz type are allowed. We may therefore consolidate these interactions and write the following predictive equations:

$$\frac{d\psi_\beta}{dt} = -i\ell\hat{\omega}_\beta\psi_\beta + \sum_\gamma \hat{b}_\gamma\psi_\gamma\psi_\beta + \sum_\gamma e_\gamma\psi_\gamma\psi_\alpha$$

$$\frac{d\psi_\alpha}{dt} = -i\ell\hat{\omega}_\alpha\psi_\alpha + \sum_\gamma \hat{g}_\gamma\psi_\gamma\psi_\alpha + \sum_\gamma \hat{f}_\gamma\psi_\gamma\psi_\beta \qquad (2.12)$$

Again we note the similarity of Eqs. (2.12) with the corresponding Eqs. (2.7). The coefficients of these equations, as those of (2.7) will be listed below.

In summary, then, we see from (2.7), (2.11) and (2.12) that the equations governing the motion of a low-order barotropic or baroclinic system, with variables defined respectively by (2.4-2.5) and (2.9), may be written formally by identical expressions which are,

$$\dot{\psi}_\gamma = 2a_\gamma \mathrm{Im}\ \psi_\alpha\psi_\beta^*$$

$$\dot{\psi}_\beta = -i\ell\omega_\beta\psi_\beta + \sum_\gamma b_\gamma\psi_\gamma\psi_\beta + \sum_\gamma e_\gamma\psi_\gamma\psi_\alpha \qquad (2.13)$$

$$\dot{\psi}_\alpha = -i\ell\omega_\alpha\psi_\alpha + \sum_\gamma g_\gamma\psi_\gamma\psi_\alpha + \sum_\gamma f_\gamma\psi_\gamma\psi_\beta$$

The dot notation has been used to indicate time differentiation and the asterisk represents conjugation. Since ψ_α and ψ_β are complex numbers, two additional equations can be generated from (2.13) by taking the conjugates of the last two equations. The significance of the dependent variables and the definitions of the coefficients are listed in Table 1. Some of the quantities listed in Table 1 require definition. The eigenvalues of the solid harmonics described above Eq. (2.10) and defined as c_α are calculated as follows:

$$c_\alpha = n_\alpha(n_\alpha+1) \ .$$

In the baroclinic equations which involve divergence, the appropriate eigenvalue is given by d_α where

$$d_\alpha = c_\alpha + r^2 \ .$$

The quantity Ω represents the mean angular velocity of the model atmosphere above that imparted by the earth's rotation. The parameters ω_α, ω_β describe, for the barotropic atmosphere, the Rossby-Haurwitz wave speed. Similar expressions could be developed for the baroclinic model, but the term representing Ω has not been extracted from sum of the constant Ψ_γ components.

Table 1. Definition of variables in Eqs. (2.13)

Parameter	Barotropic	Baroclinic
γ	$2j+1;\ 0 \le j \le J$	Same
α	$n_\alpha + i\ell$	Same
β	$n_\beta + i\ell(n_\beta \ne n_\alpha)$	$n_\beta + i\ell(n_\beta =$ or $\ne n_\alpha)$
ψ_γ	Zonal expansion coefs.	Zonal shear expansion coefs.
Ψ_γ	---	Zonal mean expansion coefs.
ψ_α	Wave expansion coef.	Mean wave expansion coef.
ψ_β	Wave expansion coef.	Shear wave expansion coef.
a_γ	$-c_\gamma^{-1}(c_\beta - c_\alpha)K_{\alpha\beta\gamma}$	$-d_\gamma^{-1}(d_\beta - c_\alpha)K_{\alpha\beta\gamma}$
ω_α	$-2c_\alpha^{-1}(1+\Omega) + \Omega$	$-c_\alpha^{-1}\left(2 + \frac{1}{\ell}\sum_\gamma (c_\alpha - c_\gamma)K_{\alpha\alpha\gamma}\Psi_\gamma\right)$
ω_β	$-2c_\beta^{-1}(1+\Omega) + \Omega$	$-d_\beta^{-1}\left(2 + \frac{1}{\ell}\sum_\gamma (d_\beta - c_\gamma)K_{\beta\beta\gamma}\Psi_\gamma\right)$
b_γ	$ic_\beta^{-1}(c_\beta - c_\gamma)K_{\beta\beta\gamma}$	0
e_γ	$ic_\beta^{-1}(c_\alpha - c_\gamma)K_{\beta\alpha\gamma}$	$id_\beta^{-1}(c_\alpha - d_\gamma)K_{\alpha\beta\gamma}$
g_γ	$ic_\alpha^{-1}(c_\alpha - c_\gamma)K_{\alpha\alpha\gamma}$	0
f_γ	$ic_\alpha^{-1}(c_\beta - c_\gamma)K_{\alpha\beta\gamma}$	Same

Since we deal with complex numbers, $i = \sqrt{-1}$, and the constants $K_{\alpha\beta\gamma}$ are proportional to the interaction coefficients $I_{\alpha\beta\gamma}$ (Eq. 2.3) and have been termed "coupling integrals" by Baer and Platzman (1961). The integrals have the form

$$K_{\alpha\beta\gamma} \equiv \tfrac{1}{2}\int_{-1}^{1} P_\alpha \left(\ell_\beta P_\beta \frac{dP_\gamma}{d\mu} - \ell_\gamma P_\gamma \frac{dP_\beta}{d\mu}\right) d\mu$$

where the functions $P_\alpha(\mu)$ are the normalized Legendre polynomials defined by Platzman (1960). It should be noted that although the indices n_α, n_β may be chosen arbitrarily subject to the constraint that $n_\alpha \pm \ell$ be odd, they should nevertheless be so chosen that ψ_α and ψ_β are time dependent; otherwise, the solutions become trivial. The basis for this choice is that the appropriate coupling integrals do not vanish. The conditions for non-vanishing of the coupling integrals are detailed by Baer and Platzman (1961).

The formal identity of the equations representing the two models under evaluation in this investigation is not a unique occurrence. Many physical systems may be represented in this way (see Baer and King, 1967) and thus the solution for one will be applicable to all with a simple reinterpretation of the variables.

Before proceeding to discuss the solutions of the system (2.13), it will be of interest to see how the divergence in the baroclinic system is related to the mean and shear variables of the flow. The Eqs. (2.8) have been written in their compact form by the elimination of divergence from the thermodynamic equation for convenience in arriving at the solution (2.13). In expanded form, the equation for shear vorticity and potential temperature (written in terms of shear) are, respectively,

$$\frac{\partial}{\partial t} \nabla^2 \tau + J(\psi, \nabla^2 \tau) + J(\tau, \nabla^2 \psi + f) = f_0 X$$

$$\frac{\partial \tau}{\partial t} + J(\psi, \tau) = \frac{f_0}{r^2} X \qquad (2.14)$$

$$X \equiv \nabla^2 \chi$$

where χ represents the velocity potential of the lower layer. Since X occurs linearly in (2.14) as does τ, we may assume the spectral expansion,

$$X = \bar{X} + X'$$

$$\bar{X} = \sum X_\gamma Y_\gamma, \quad X' = X_\beta Y_\beta + X_\beta^* Y_\beta^*$$

The simplest procedure for establishing the values of X_γ and X_β are to substitute the expansions (2.9) into (2.14), multiply (2.14) by a given Y_σ and integrate both equations over the unit sphere. The resulting equations will then determine the time variation of a given shear coefficient (since $\frac{\partial \tau}{\partial t}$ exists linearly) which may be eliminated between the two equations, yielding an equation in a coefficient of X. This procedure, using the orthogonality properties previously discussed and the properties of the Jacobians, yields the following results:

$$X_\gamma = \frac{2r^2(c_\gamma + c_\alpha - c_\beta)}{f_0 d_\gamma} K_{\alpha\beta\gamma} \operatorname{Im} \psi_\alpha \psi_\beta^*$$

$$X_\beta = \frac{ir^2}{d_\beta f_0} \Big(\sum_\gamma (c_\beta + c_\alpha - c_\gamma) K_{\alpha\beta\gamma} \Psi_\gamma \psi_\alpha - \big(\sum_\gamma c_\gamma K_{\beta\beta\gamma} \Psi_\gamma - 2\ell \big) \psi_\beta \Big) \tag{2.15}$$

It will become apparent in the next section, wherein we shall determine the solution of system (2.13), how (2.15) may be solved as a function of time.

3. Exact Solutions

The general solution to the "Class L3" 3-component system has been shown by Platzman (1962) to be given in terms of elliptic integrals. In our discussion, the system has been somewhat expanded in terms of degrees of freedom by allowing for an arbitrary zonal field. Thus, since the wave components (α,β) are both complex, they yield four real dependent variables, whereas the zonal field (represented by (2.4) or (2.9)) yields N dependent variables. All the N zonal variables need not be active, however, a fact which depends on the non-zero character of a_γ as may be seen from the first of Eqs. (2.13). On the assumption that there are $\bar{N} \leq N$ variable zonal coefficients, we see that the system under consideration has $\bar{N} + 4$ degrees of freedom. We shall show that by a simple integration of the zonal equations, this system can be reduced to a 3-component system similar to that described by Platzman.

A solution of system (2.13) may be determined through the use of real variables by describing the complex wave coefficients using amplitude and phase. Noting that the zonal coefficients are real and thus are represented by amplitude only, we may make the following definitions:

$$\nu_{\alpha,\beta} \equiv \ell\omega_{\alpha,\beta}; \quad \nu \equiv \nu_\alpha - \nu_\beta$$

$$\psi_{\alpha,\beta} \equiv \frac{B_{\alpha,\beta}}{\sqrt{2}} e^{i\Theta_{\alpha,\beta}}; \quad \psi_\gamma \equiv B_\gamma \tag{3.1}$$

$$\theta \equiv \Theta_\alpha - \Theta_\beta$$

Using the variables $G_{\delta\varepsilon}$ as defined in Table 2 and the definitions (3.1), we may rewrite the system (2.13), after taking real and imaginary parts, in terms of real variables alone.

$$\dot{B}_\gamma = a_\gamma B_\alpha B_\beta \sin\theta$$

$$\dot{B}_\beta = -G_{\beta\alpha} B_\alpha \sin\theta$$

$$\dot{B}_\alpha = G_{\alpha\beta} B_\beta \sin\theta \qquad (3.2)$$

$$\dot{\theta}+\nu = G_{\alpha\alpha} - G_{\beta\beta} + \left(\frac{G_{\alpha\beta} B_\beta}{B_\alpha} - \frac{G_{\beta\alpha} B_\alpha}{B_\beta}\right) \cos\theta$$

It is interesting to note that the solution does not depend on the individual phase of each wave, but only on the phase difference of the two waves. Thus, the degrees of freedom in the system are actually one less than previously specified, or $\bar{N} + 3$.

Table 2. Evaluation of the variables $G_{\delta\varepsilon}$, $g_{\delta\varepsilon}$, $h_{\delta\varepsilon}$

$\delta\varepsilon$	$G_{\delta\varepsilon}$	$g_{\delta\varepsilon}$	$h_{\delta\varepsilon}$
$\beta\beta$	$-i\sum_\gamma b_\gamma B_\gamma$	$-i\sum_\gamma \frac{a_\gamma}{a_n} b_\gamma$	$-i\sum_\gamma s_\gamma b_\gamma$
$\alpha\alpha$	$-i\sum_\gamma g_\gamma B_\gamma$	$-i\sum_\gamma \frac{a_\gamma}{a_n} g_\gamma$	$-i\sum_\gamma s_\gamma g_\gamma$
$\alpha\beta$	$-i\sum_\gamma f_\gamma B_\gamma$	$-i\sum_\gamma \frac{a_\gamma}{a_n} f_\gamma$	$-i\sum_\gamma s_\gamma f_\gamma$
$\beta\alpha$	$-i\sum_\gamma e_\gamma B_\gamma$	$-i\sum_\gamma \frac{a_\gamma}{a_n} e_\gamma$	$-i\sum_\gamma s_\gamma e_\gamma$

Consider now the zonal component with the smallest index γ (because the zonal coefficients are real, the indices γ are also real) for which $a_\gamma \neq 0$, and denote this value of γ by n. This choice is completely arbitrary, but as will be seen in the sequel, it is also completely general. We then note from the first of Eqs. (3.2) that each γ-equation can be related to the n-equation by the differential equation,

$$\frac{\dot{B}_\gamma}{a_\gamma} = \frac{\dot{B}_n}{a_n}$$

or, on integration,

$$B_\gamma = \frac{a_\gamma}{a_n} B_n + s_\gamma \tag{3.3}$$

The quantities s_γ depend on the initial values of the variables. Since all amplitudes and phases are known initially, the constants of integration may be uniquely determined; we shall defer the detailed dependence of these constants on the initial conditions until the end of this section.

From (3.3) we note the linear dependence of B_γ on B_n. Substituting this relationship into the equations for $G_{\delta\varepsilon}$ (Table 2), we have the linear dependence of $G_{\delta\varepsilon}$ on B_n as follows:

$$G_{\delta\varepsilon} = g_{\delta\varepsilon} B_n + h_{\delta\varepsilon} \tag{3.4}$$

where the dependence of $g_{\delta\varepsilon}$ and $h_{\delta\varepsilon}$ on the physical parameters of the system have been defined in Table 2. For a complete evaluation of these parameters, reference must be made to Table 1.

Substitution of (3.4) and the relations in Table 2 into (3.2) yields the closed system of first order differential equations in time in the four variables B_n, B_α, B_β and θ as follows:

$$\begin{aligned}
\dot{B}_n &= a_n B_\alpha B_\beta \sin\theta \\
\dot{B}_\beta &= -(h_{\beta\alpha}+g_{\beta\alpha}B_n)B_\alpha \sin\theta \\
\dot{B}_\alpha &= (h_{\alpha\beta}+g_{\alpha\beta}B_n)B_\beta \sin\theta \\
\dot{\theta} &= -\bar{\nu}+\bar{g}\, B_n + \left((h_{\alpha\beta}+g_{\alpha\beta}B_n)\frac{B_\beta}{B_\alpha} - (h_{\beta\alpha}+g_{\beta\alpha}B_n)\frac{B_\alpha}{B_\beta}\right)\cos\theta
\end{aligned} \tag{3.5}$$

where

$$\bar{\nu} \equiv \nu - h_{\alpha\alpha}+h_{\beta\beta}$$

$$\bar{g} \equiv g_{\alpha\alpha}-g_{\beta\beta}$$

If this system is soluble, we should be able, in principle, to reduce it to a differential equation in one dependent variable alone. Thus, without loss of generality, we may say that

$$B_\beta = B_\beta(B_n), \quad B_\alpha = B_\alpha(B_n), \quad \theta = \theta(B_n).$$

We then note, for example,

$$\dot{B}_\beta = B_\beta' \dot{B}_n = B_\beta' a_n B_\alpha B_\beta \sin\theta$$

where the prime denotes differentiation with respect to B_n and we have substituted from the first of Eqs. (3.5). Applying a similar differentiation to B_α, substituting into the second and third of Eqs. (3.5) and integrating with respect to B_n, we have the quadratic dependence of B_α, B_β on B_n as follows;

$$B_\beta^2 = -(2h_{\beta\alpha}+g_{\beta\alpha}B_n)\frac{B_n}{a_n} + D_\beta$$

$$B_\alpha^2 = (2h_{\alpha\beta}+g_{\alpha\beta}B_n)\frac{B_n}{a_n} + D_\alpha \tag{3.6}$$

In (3.6) the quantities D_β, D_α are constants of integration depending on initial values and will be evaluated subsequently. If we can now establish $\theta(B_n)$ explicitly, we may derive a differential equation in B_n alone from the first of (3.5). Multiplying the last of (3.5) by $\sin\theta$, and substituting from the second and third of these equations we get,

$$(\dot{\cos\theta}) = \dot{B}_n(\cos\theta)' = \left((\bar{\nu}-\bar{g}\,B_n) - a_n(B_\alpha B_\beta)'\cos\theta\right)\sin\theta$$

Using the first of (3.5) and combining terms, we arrive at the equation,

$$(B_\alpha B_\beta \cos\theta)' = \frac{\bar{\nu}-\bar{g}\,B_n}{a_n}$$

which may be integrated to yield a solution for θ in terms of B_n as follows;

$$a_n B_\alpha B_\beta \cos\theta = G(B_n)$$

$$G(B_n) \equiv (\bar{\nu} - \frac{\bar{g}}{2}B_n)B_n + K \tag{3.7}$$

where K is a constant of integration.

If we now square the first of Eqs. (3.5), replace $\sin^2\theta = 1-\cos^2\theta$ and substitute for $\cos\theta$ from (3.7), we arrive at the differential equation for $B_n(t)$ in the form,

$$\dot{B}_n^2 = a_n^2 B_\alpha^2 B_\beta^2 - G^2 = \sum_{i=0}^{4} b_i B_n^i \tag{3.8}$$

Since the right-hand side of (3.8) is a quartic function in B_n, the solution may be determined in terms of elliptic functions. With the use of (3.6) and (3.7), the coefficients b_i may be defined as follows:

$$\begin{aligned}
b_0 &\equiv a_n^2 D_\alpha D_\beta - K^2 \\
b_1 &\equiv 2a_n (h_{\alpha\beta} D_\beta - h_{\beta\alpha} D_\alpha) - 2\bar{\nu}K \\
b_2 &\equiv -4h_{\alpha\beta} h_{\beta\alpha} + a_n (g_{\alpha\beta} D_\beta - g_{\beta\alpha} D_\alpha) - \bar{\nu}^2 + \bar{g}K \\
b_3 &\equiv \bar{g}\,\bar{\nu} - 2(h_{\alpha\beta} g_{\beta\alpha} + h_{\beta\alpha} g_{\alpha\beta}) \\
b_4 &\equiv -\frac{\bar{g}^2}{4} - g_{\alpha\beta} g_{\beta\alpha} \quad .
\end{aligned} \tag{3.9}$$

The constants of integration necessary for evaluation of the above coefficients are determined from the problem constants (Tables 1 and 2) and initial values of the dependent variables denoted by zero subscripts, and are listed below:

$$\begin{aligned}
s_\gamma &= B_{\gamma 0} - \frac{a_\gamma}{a_n} B_{n0} \\
D_\beta &= B_{\beta 0}^2 + \frac{B_{n0}}{a_n} (g_{\beta\alpha} B_{n0} + 2h_{\beta\alpha}) \\
D_\alpha &= B_{\alpha 0}^2 - \frac{B_{n0}}{a_n} (g_{\alpha\beta} B_{n0} + 2h_{\alpha\beta}) \\
K &= G_0 - B_{n0} (\bar{\nu} - \frac{\bar{g}}{2} B_{n0}) \\
G_0 &= a_n B_{\alpha 0} B_{\beta 0} \cos \theta_0
\end{aligned} \tag{3.10}$$

The general solution of (3.8) depends on the values of the coefficients given in (3.9). The complete solution is presented in Appendix A. For purposes of the text, it is only necessary to know that the solution $B_n(t)$ may be written

$$B_n(t) = B_n(\lambda,\mu,a,B_{n_0},\mathrm{sn}\,\omega t) \qquad (3.11)$$

The constants λ, μ, ω depend on the problem characteristics and the initial conditions and are defined in Appendix A; ω represents the nonlinear frequency of the periodic elliptic sine functions (sn) whose properties are described in many texts (see, for example, Caley, 1961). The solution to Eqs. (3.5) is completed by solving (3.7) for θ,

$$\theta = \tan^{-1}\left(\frac{(\sum b_i B_n^i)^{\frac{1}{2}}}{G}\right) \qquad (3.12)$$

It may be interesting, however, to see the solution to the individual phase angles θ_α, θ_β, respectively. Since θ is known, only one of these solutions is necessary. Writing the imaginary part of the second of Eqs. (2.13) after substitution of (3.1), we have for Θ_β,

$$\dot{\theta}_\beta = -\nu_\beta + h_{\beta\beta} + g_{\beta\beta}B_n + \frac{(g_{\beta\alpha}B_n + h_{\beta\alpha})G}{a_n B_\beta^2} \qquad (3.13)$$

Since both G and B_β are known functions of B_n, and B_n is given by (3.11) as a function of time, this equation may be integrated formally. However, the integration involves elliptic integrals of the third kind and a detailed discussion is therefore deferred to Appendix B; the solution may be written,

$$\Theta_\beta(t) = \Theta_{\beta 0} + A_1 + \sum_{j=1}^{3}\left(c_{Fj}F(\zeta_j,\omega t) + c_{Hj}H(\zeta_j,\omega t)\right) \qquad (3.14)$$

In (3.14), the coefficients are described in Appendix B in terms of initial values and problem constants, the functions F are elementary integrals, and the functions H are elliptic integrals of the third kind (see Caley, 1961). It is of particular interest to note from (3.14) that the phase angles Θ_α, Θ_β, do not necessarily have the same period as the amplitude oscillations (characterized by the frequency ω) but merely have this oscillation superimposed on them. If we denote the characteristic period of the amplitude fluctuation as T, where

$$T = \frac{4K}{\omega}$$

and K is the complete elliptic integral of the first kind, then F (ωT) and H (ωT) do not necessarily vanish, and the phase periods may be calculated from these values and A_1. Furthermore, because of the probable disparity between these periods and T, the spacial distribution of the flow field need never be repeated although the energy in each component is repeated with period T. This point will be discussed in further detail subsequently.

For reference to the ensuing discussion, we now reiterate that the complete solution to Eqs. (2.13) for either the barotropic or baroclinic model are given by (3.11), (3.3), (3.6), (3.12) and (3.14).

4. Initial Conditions and Truncation

Characteristic properties of the solution (3.11) and consequently the behavior of the system (2.13) as it reflects the true atmosphere will be described in terms of the initial configuration of the system to be studied, since (2.13) is represented as an initial value problem. However, since we are dealing with a highly truncated system, the character of the truncation applied will also affect the solution. We may therefore state that the following parameters are necessary as initial conditions for solution of (2.13).

(a) Total available kinetic energy
(b) Kinetic energy in the wave
(c) Latitudinal distribution of zonal flow
(d) Wave number and profile

These parameters may be varied, thereby yielding solutions to (2.13) for different atmospheric conditions. Aside from the specification of these initial parameters, however, it is necessary--as indicated above--to establish truncation by determining the wave vectors α, β and the allowed range of γ. For consistency in establishing different solutions, we shall always use for truncation the wave vectors given by the initial representation together with a zero initial phase angle for both wave components α and β. The latter condition may be made completely general if the initial specification of the zonal field includes components which may be active but begin with zero amplitude. The wave truncation is directly related to the initial specification, parameter (d), and cannot be altered.

Let us now consider the initial representation in somewhat more detail. Following the presentation used by the writer (1964), we will define the stream field as,

$$\begin{aligned} \psi_\ell &= g_\ell F_\ell(\mu) \cos \ell\lambda \\ &= \sum_\gamma \psi_\gamma P_\gamma(\mu) && \ell=0 \qquad (4.1) \\ &= 2(\psi_\alpha P_\alpha + \psi_\beta P_\beta) \cos \ell\lambda && \ell\neq 0 \end{aligned}$$

where g_ℓ represents an amplitude factor for the zonal motion or the wave, and $F_\ell(\mu)$ describes the initial latitudinal distribution. The stream coefficients in (4.1) may be determined by applying the orthogonality condition for the Legendre Polynomials and result in the values,

$$\psi_\gamma = \frac{g_\ell A_\gamma}{2(2-\delta_{\ell,0})}$$

$$A_\gamma \equiv \int_{-1}^{1} P_\gamma F_\ell(\mu)\, d\mu \tag{4.2}$$

where γ will represent α, β for $\ell \neq 0$. The kinetic energy in the zonal field and the wave may be written

$$K_\ell = \sum c_\gamma \psi_\gamma^2 \qquad \ell=0$$

$$= 2c_\alpha \psi_\alpha^2 + 2c_\beta \psi_\beta^2 \qquad \ell \neq 0 \tag{4.3}$$

The details of the energetics are discussed in Section 5. Finally, if the energy and truncation are specified, the amplitude factors may be computed as follows:

$$g_0 = 2\left(\frac{K_0}{\sum\limits_\gamma c_\gamma A_\gamma^2}\right)^{\frac{1}{2}}$$

$$g_\ell = 2\left(\frac{2K_\ell}{c_\alpha A_\alpha^2 + c_\beta A_\beta^2}\right)^{\frac{1}{2}} \tag{4.4}$$

It should be noted that the zonal energy represents the shear flow and the α and β coefficients represent mean and shear in the wave for the baroclinic problem, whereas the α and β coefficients represent two separate wave components for barotropic flow.

In terms of (4.1) - (4.4), we may now discuss the details of the initialization. The process outlined is chosen so that systematic variation of parameters is easily achieved. It should be noted, however, that the numerical specification of all the required stream coefficients is sufficient as initial conditions.

(a) Total available energy

We have chosen to represent the total available energy to the system by assuming that all the initial energy resides in the zonal flow described by a specified profile and an amplitude such that

$$u(t=0) = \bar{u}_0 G(\mu) \tag{4.5}$$

The distribution G in effect determines the factors A_γ from the relationship between the stream field and the zonal wind

$$F_0(\mu) = \int_0^\mu (1-\mu^2)^{-\frac{1}{2}} G(\mu) d\mu$$

and the second of Eqs. (4.2). If we now set

$$g_0 = -\bar{u}_0$$

we may compute the initial zonal stream coefficients from the first of (4.2) and the zonal kinetic energy (here the total energy) from the first of (4.3). Thus we state that the kinetic energy of the system will be given as,

$$K = K(\bar{u}_0, G) \tag{4.6}$$

following the calculation procedure outlined above. For the baroclinic problem, the profile and amplitude specified by (4.5) are for the shear flow, and the zonal mean flow (not active in time) may be arbitrarily specified. Depending on the function G, some of the energy computed from (4.6) may not be available for energy exchange, as may be seen from (2.13) when $a_\gamma=0$. Thus we may define the initial available kinetic energy,

$$\begin{aligned} K_a &= K - K_u \\ K_u &= \sum_\gamma c_\gamma \psi_\gamma^2 \qquad \gamma \text{ only for } a_\gamma=0 \end{aligned} \tag{4.7}$$

For the baroclinic problem, it would be more appropriate to include the available potential energy also, since the available kinetic energy may change by transfer of potential energy (see Section 5). Such a computation may be easily achieved with the information presented above.

(b) Kinetic energy in the wave

With the determination of the total available energy computed from the specification of the zonal wind amplitude $\bar{u}_0$ and profile G, we may describe the kinetic energy in the wave relative to the total energy by the parameter ρ,

$$\rho = \frac{K_\ell}{K} \tag{4.8}$$

For fixed K the energy may be partitioned initially between the zonal flow and the wave, differently for different values of ρ. Once ρ is specified, K_ℓ is easily calculated from (4.8). Given the wave profile (A_α, A_β) we may then calculate the wave amplitude from (4.4) and finally the wave stream coefficients from (4.2). Furthermore, since the zonal kinetic energy is given from the total energy and that in the wave,

$$K_0 = K(\bar{u}_0, G) - K_\ell$$

the zonal amplitude $g_0(\rho)$ and the zonal stream coefficients are computed in a manner identical to the calculation of the wave coefficients, using the known values of the A_γ.

(c) Latitudinal distribution of zonal flow

The latitudinal distribution of zonal flow implies a specification of the function $G(\mu)$ as may be seen from (4.5). There are a number of ways in which this may be accomplished, but perhaps the most general form would be a polynomial in $\mu = \sin\phi$. Since, however, we wish the zonal field to be symmetric about the equator (an even function) and since we furthermore wish to have a series which is easily represented by a finite series of Legendre Polynomials, we choose the form,

$$G(\mu) = (1-\mu^2)^{\frac{1}{2}} \sum_{i=0}^{M} \hat{a}_i \mu^{2i} \tag{4.9}$$

The coefficients $\hat{a}_i$ may be chosen so that the series represents the desired zonal flow data. Integrating G as indicated in the equation following (4.5), we find

$$F_0 = \sum_{i=0}^{M} \hat{a}_i \frac{\mu^{2i+1}}{2i+1}$$

Now since μ^{2i+1} may be represented in terms of a series of normalized Legendre Polynomials (see Platzman, 1960),

$$\mu^{2i+1} = (2i+1)! \sum_{j=0}^{i} 2^{2j+1} \frac{\sqrt{4j+3}\ \ (i+j+1)!}{(i-j)!(2i+2j+3)!} P_{2i+1}(\mu)$$

the function F_0 may also be represented as a series of Legendre Polynomials of the form

$$F_0(\mu) = \sum_{j=0}^{M} \hat{b}_j P_{\gamma_j}$$

$$\gamma_j \equiv 2j+1 \tag{4.10}$$

$$\hat{b}_j \equiv 2^{\gamma_j} \sqrt{2\gamma_j+1} \sum_{i=j}^{M} (2i)! \frac{(j+i+1)!}{(i-j)!(2i+2j+3)!} \hat{a}_i$$

Finally we have from (4.2) that the parameters which specify profile, are given as

$$A_{\gamma_j} = 2\hat{b}_j \tag{4.11}$$

Two zonal profiles have been studied and both apply for the barotropic problem. One has been designed to conform to the observed, normal wind profile for January at 500 mb and the other represents a split-jet. The observed jet and the representations used are presented on Figure 1. To establish the observed jet, it has been more convenient to use a less general representation of the form,

$$G(\mu)_{A-J} = (\sin^4 3\phi \cos^2\phi + \sin^4 2\phi - \alpha\cos^2\phi)\cos\phi \tag{4.12}$$

where the conversion of (4.12) from (4.9) may be achieved by direct expansion†. The distribution (4.12) allows for a jet with maximum around 30°N, and a choice of $\alpha=0.2$ will establish the equatorial easterly with magnitude one-fifth that of the maximum westerly jet, comparable to the observed profile. The coefficients $\hat{a}_i$ for this profile (denoted A-J) are listed in Table 3.

The double jet (D-J) profile has been chosen from the representation

$$G(\mu)_{D-J} = (\sin^2 4\phi + 2\sin^2\phi) \cos\phi \tag{4.13}$$

and is described in Figure 1. The coefficients $\hat{a}_j$ for this profile are listed in Table 3.

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†The writer has developed a general formula for conversion of involved trigonometric formulas to series of the form (4.9), but the details of this conversion are irrelevant here.

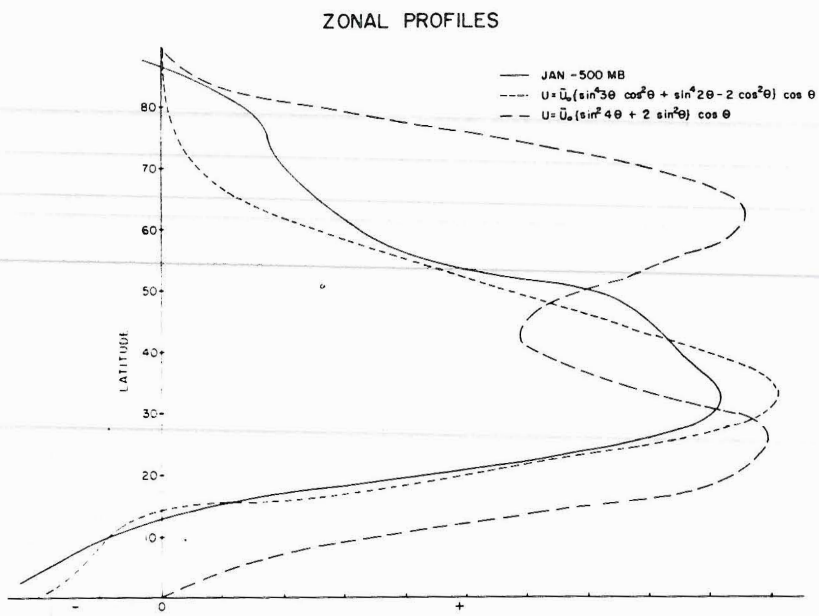

Fig. 1 Zonal wind profiles for a 500 mb distribution and two distributions selected for computation, plotted on a relative scale vs. latitude.

Table 3. Coefficients $\hat{a}_j$ for the atmospheric-jet and split-jet profiles.

j	A-J	D-J
0	- 0.2	0
1	0.2	18
2	97	-80
3	- 545	128
4	1312	-64
5	-1632	
6	1024	
7	- 256	

(d) Wave number and profile

The constraint imposed by the low-order system allows interaction between only one planetary wave and a zonal flow. However, the selection of planetary wave number is arbitrary and must be set as an initial condition. We must furthermore specify the profile parameters A_α and A_β either directly or through a profile function $F_\ell(\mu)$; their interpretation with regard to the barotropic or baroclinic problem has already been made evident. The profile function may be interpreted physically as the latitudinal distribution of the meridional wind component at the longitude where the wave zonal wind component vanishes.

It is immediately apparent that the wave representation is considerably more constrained than that for the zonal wind field. Moreover, since the components for the baroclinic problem are essentially independent, one needs only to determine the components A_α and A_β for the profiles which are to be investigated. A somewhat greater degree of freedom is available for the barotropic problem, wherein one may adjust profiles by a combination of the α and β contributions. Consider, for example, the profile

$$F_\ell(\mu) = \sin^{r_\ell} 2\phi \cos^{q_\ell} \phi \tag{4.14}$$

a representation used by the writer in a previous study (Baer, 1964). Such a profile, with specification of r_ℓ and q_ℓ, may be easily varied for different planetary wave numbers. On the assumption that we wish to investigate the non-linear interaction of the longest allowed waves in the latitudinal direction (given the planetary wave number ℓ), that the wave should interact with the lowest active component of the zonal field (γ=3), and that, furthermore, (4.14) can indeed be represented by no more or less than two associated Legendre Polynomials, we find from the definition of the polynomials (Jahnke and Emde, 1945) that the following restrictions must be imposed;

$$\begin{aligned} r_\ell + q_\ell &= \ell + 2s_\ell \\ 2r_\ell + q_\ell &= \ell + 3 \end{aligned} \tag{4.15}$$

where s_ℓ may take on the values 0 or 1, as desired. Solving for r_ℓ and q_ℓ from (4.15) in terms of the known quantities s_ℓ and ℓ, we find for the profile function,

$$F_\ell = 2^{(3-2s_\ell)} \mu(1-\mu^2)^{\ell/2} (s_\ell + (-)^{s_\ell} \mu^2) \tag{4.16}$$

Expressing (4.16) in terms of P_α and P_β and substituting into the second of (4.2), the integration yields

$$A_\alpha = p_\ell\left((2\ell+5)s_\ell + (-)^{s_\ell}3\right)\left(\frac{3(2\ell+7)}{\ell+1}\right)^{\frac{1}{2}}$$

$$A_\beta = (-)^{s_\ell}6p_\ell \tag{4.17}$$

$$p_\ell = 2^{(\ell+5-2s_\ell)}\ \frac{(\ell+1)!}{2\ell+5}\ \left(6(2\ell+7)(2\ell+3)!\right)^{-\frac{1}{2}}$$

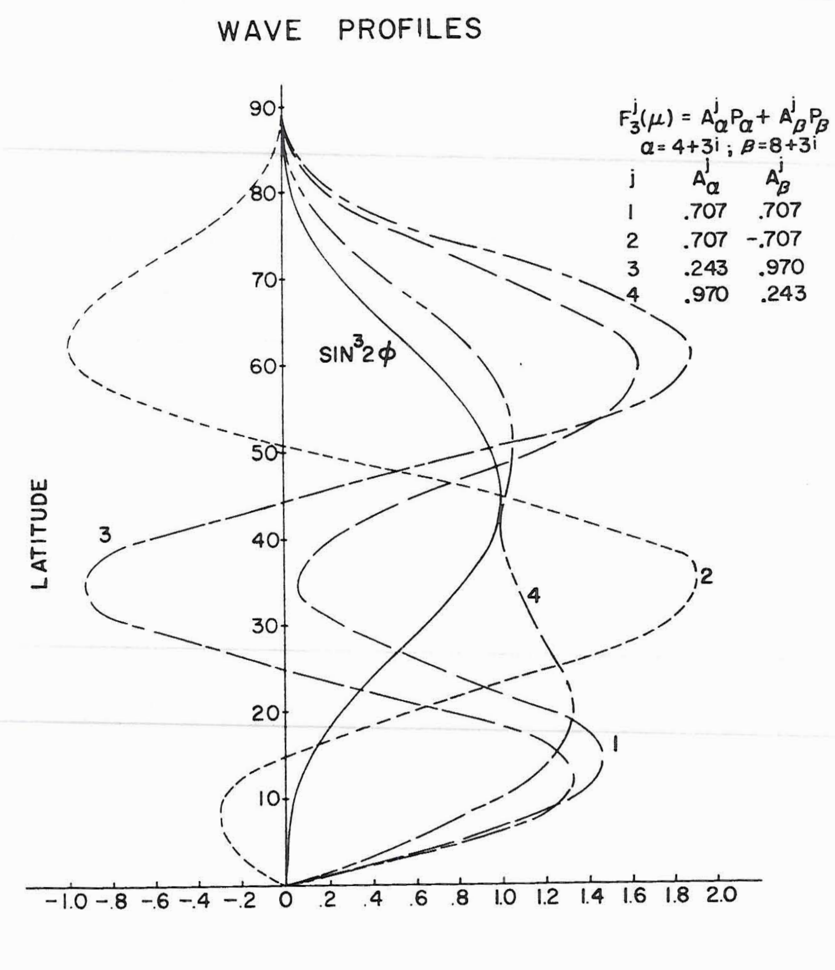

Fig. 2 Wave profiles for wave $\ell=3$ including different combinations of the polynomials P_4^3, P_8^3, presented on a relative scale vs. latitude.

To understand the limitations placed on the profile (4.16), several other profiles have been tested by barotropic calculation. Figure 2 describes various profiles for wave $\ell=3$ including F_ℓ (4.16) for $s_\ell=0$, and variable coefficients for $n_\alpha=4$, $n_\beta=8$. The amplitudes of the coefficients are listed on the figure and need not be duplicated in the text. It is noteworthy, however, that (4.16) describes the longest wave--as intended--and that other allowed profiles can be remarkably different, possibly leading to different solutions of (2.13). Other profiles have also been used for calculation and will be discussed subsequently.

5. Energy Considerations

We have seen from Section 3 that the entire solution is dependent only on the time variation of the lowest active zonal component, B_n. Thus it should be possible to describe the energy (both kinetic and potential) as well as the energy changes and the transfer processes in terms of this variable also, and consequently in terms of time.

The kinetic energy per unit mass for the models considered herein and described by (2.2) and (2.8) may be written as,

$$K = -\int\psi\nabla^2\psi dA - \int\tau\nabla^2\tau dA \qquad (5.1)$$

where dA represents an element of area on the unit sphere. Although τ, which describes the shear stream field for the baroclinic problem (see also 2.9), is undefined for the barotropic problem, we may redefine the barotropic variables so that (5.1) will be applicable to both cases. With reference to (2.4) and (2.5), let ψ in (5.1) represent the α-wave and the inactive zonal components (those components for which $a_\gamma=0$). If we now let τ represent the β-wave and the remaining (active) zonal components, (5.1) will describe also the energy in the barotropic case except for $n_\alpha=n_\beta$, a disallowed truncation (see Table 1).

The available potential energy necessary to satisfy the energy conservation conditions applicable to (2.8) is given as

$$P = r^2\int\tau^2 dA \qquad (5.2)$$

where r^2 has been defined in Section 2. Since there is no available potential energy in the barotropic case, we define r^2 (barotropic) = 0.

Not only is the energy distributed between potential and kinetic, but in each of these categories some of the energy is in the zonal flow and some in the planetary wave. The distribution of the energy in all categories may be determined by introducing the expansions for the stream field, given by (2.4), (2.5) or (2.9) into (5.1) and (5.2) and performing the integration. The results are as follows:

$$K = \bar{K} + K_z + K_\alpha + K_\beta$$

$$P = P_z + P_\beta \tag{5.3}$$

$$K + P = \text{const}$$

In (5.3), $\bar{K}$ represents the kinetic energy in the zonal flow which does not change with time (the mean zonal flow for the baroclinic case), K_z represents the KE of the changing zonal flow (the zonal shear flow for the baroclinic case), K_α and K_β represent the KE in the α- and β-waves respectively, P_z represents the PE in the zonal shear flow and P_β is the PE in the β-wave (shear flow). Clearly the last two quantities do not exist for the barotropic case. Applying the relations for amplitude dependence on B_n from (3.3) and (3.6), we have,

$$\bar{K} = \sum_\gamma c_\gamma \Psi_\gamma^2 \quad (\gamma\text{'s for which } a = 0\text{, the barotropic case})$$

$$K_z = \sum_\gamma c_\gamma B_\gamma^2 = \bar{c}_\gamma(t) \sum B_\gamma^2 = \bar{c}_\gamma (k_{2z} B_n^2 + k_{1z} B_n + k_{0z}) .$$

$$P_z = r^2 \sum B_\gamma^2 = \frac{r^2}{\bar{c}_\gamma} K_z = r^2 (k_{2z} B_n^2 + k_{1z} B_n + k_{0z})$$

$$K_\beta = c_\beta B_\beta^2 = c_\beta (k_{2\beta} B_n^2 + k_{1\beta} B_n + k_{0\beta}) \tag{5.4}$$

$$P_\beta = r^2 B_\beta = \frac{r^2}{c_\beta} K_\beta$$

$$K_\alpha = c_\alpha B_\alpha^2 = c_\alpha (k_{2\alpha} B_n^2 + k_{1\alpha} B_n + k_{0\alpha})$$

The coefficients used in (5.4) which depend on the truncation and initial conditions are listed in Table 4.

We note, immediately, that the potential and kinetic energies in the β-wave (wave energy in the shear flow) are related to each other by a fixed constant, $r^2 c_\beta^{-1}$, for all time. Similarly, by defining a mean eigenvalue for the zonal flow, $\bar{c}(t)$, we may show that there is a relationship between the zonal kinetic and potential energies. Although this relationship is time dependent, we may establish limits on the ratio. From the definition of the eigenvalues c_γ

(Section 2), we note that they are positive and monotonically increasing for increasing n_γ. Thus $\bar{c}_\gamma$ is bounded by the values

$$c_\gamma(\min) \leq \bar{c}_\gamma(t) \leq c_\gamma(\max)$$

which is readily verified from (5.4). Thus the potential energy in the zonal shear flow is confined by the kinetic energy in the shear flow. We shall discuss the significance of this result later in this section.

Table 4. Evaluation of the coefficients $k_{\delta\varepsilon}$

$\varepsilon \backslash \delta$	2	1	0
z	$\sum_\gamma a_\gamma^2/a_n^2$	$2\sum_\gamma s_\gamma a_\gamma/a_n$	$\sum_\gamma s_\gamma^2$
β	$-g_{\beta\alpha}/a_n$	$-2h_{\beta\alpha}/a_n$	D_β
α	$g_{\alpha\beta}/a_n$	$2h_{\alpha\beta}/a_n$	D_α

If we are interested in the combined energy in different components (zonal or wave) it is simply necessary to replace the eigenvalue c_γ by d_γ. Thus, for example,

$$K_z+P_z = \sum_\gamma d_\gamma B_\gamma^2 = \bar{d}_\gamma \sum_\gamma B_\gamma^2$$

The formulas for the energy components (5.4) all depend on the dependent variable B_n which in turn has been described in Section 3 as a periodic function of time. It can be seen that the energy components have their maxima and minima at the same times as the function B_n. They have, moreover, other possible extremes which are listed in Table 5.

For initial conditions appropriate to the earth's atmosphere, the extremes listed in Table 5 do not seem to be attainable. Therefore, in the following discussion, we shall concentrate our attention on the energy extremes which correspond to the extremes of B_n. Since we shall suggest in Section 6 that the initial phase angle between the α- and β-wave may be chosen as $\theta_0=0$ without loss of generality, for this condition the first of (5.2) shows that B_n

must have an extreme at $t=0$ and, since it is periodic with period $4K/\omega=T$, it will also have an extreme at $t=T/2$ (see Appendix A for a detailed discussion of the constants).

Table 5. Extremes of energy components.

Component	B_n for Extreme
K_z	$-a_n \sum c_\gamma s_\gamma a_\gamma / \sum c_\gamma a_\gamma^2$
P_z	$-k_{1z}/2k_{2z}$
K_β, P_β	$-k_{1\beta}/2k_{2\beta}$
K_α	$-k_{1\alpha}/2k_{2\alpha}$

The unique property of the nonlinear solution available here is the time variation of the amplitudes of the individual components (or, equivalently, energy components), a property not available from the linearized equations. The maximum changes of these amplitudes may be found by calculating the difference between their maximum and minimum points. With reference to energy components, such a calculation will specify the amount of energy which any component can exchange with the other active components. For precision we shall define energy range of any component as the value at its first extreme minus the value at its second extreme. Since we have already seen that the energy components have their extremes at the same times as B_n, and furthermore that B_n has extremes at $t=0$ and $t=T/2$, we have for any energy component E,

$$\Delta E = E\big(B_n(t=0)\big) - E\big(B_n(t=T/2)\big) \tag{5.5}$$

Let us now define the mean and difference of the variable B_n at its extremes, as well as similar quantities for the time dependent eigenvalue $\bar{c}_\gamma(t)$ as

$$\begin{aligned} \bar{B} &\equiv B_n(0) + B_n(T/2) \\ \Delta B &\equiv B_n(0) - B_n(T/2) \\ \bar{c} &\equiv \bar{c}_\gamma(0) + \bar{c}_\gamma(T/2) \\ \Delta c &\equiv \bar{c}_\gamma(0) - \bar{c}_\gamma(T/2) \end{aligned} \tag{5.6}$$

The latter two definitions are necessary to describe the range of the zonal kinetic energy. If we now apply the range operator (5.5) to the different energy components (5.4), we may express the maximum energy exchange in each component in terms of the mean and difference of the basic variable B_n (5.6) as follows:

$$\Delta K_z = \frac{\bar{c}\Delta B}{2}(k_{2z}\bar{B} + k_{1z}) + \frac{\Delta c}{2}\left(\frac{k_{2z}}{2}\left((\Delta B)^2 + \bar{B}^2\right) + k_{1z}\bar{B} + k_{0z}\right)$$

$$\begin{aligned} \Delta P_z &= r^2 \Delta B(k_{2z}\bar{B} + k_{1z}) \\ \Delta K_\beta &= c_\beta \Delta B(k_{2\beta}\bar{B} + k_{1\beta}) = \frac{c_\beta}{r^2}\Delta P_\beta \\ \Delta K_\alpha &= c_\alpha \Delta B(k_{2\alpha}\bar{B} + k_{1\alpha}) \end{aligned} \qquad (5.7)$$

Since the maximum exchange of energy described by (5.7) depends on the coefficients $k_{\delta\varepsilon}$ defined in Table 4, we see that the initial values and the system truncation are of primary importance in determining what these exchanges will actually be. Furthermore, one may normalize the exchange values listed above by the initial value of the time-dependent energy, thereby determining the percent of activity of each component. Because of the relatively different magnitudes of the kinetic and potential energies, it is preferable to normalize the kinetic energies by the total initial KE and the potential energies by the total initial PE. The interchange between the different components of energy and its magnitude may then be readily studied as a function of the initial values and the spectral truncation.

As an example, as ΔB approaches zero, we see that the energy ranges approach zero (except for the zonal kinetic) and thus the system under which this condition prevails ($\Delta B \to 0$) may be described by linear processes. The non-vanishing of the zonal kinetic energy implies a nonlinear transfer amongst the individual zonal components. If, however, only one zonal component is allowed to exist, Δc will vanish and ΔK_z will be proportional to ΔB. Furthermore, under this last assumption, the kinetic and potential energies of the zonal flow will be proportional by a time independent constant since $\bar{c}_\gamma(t) = c_\gamma =$ constant.

By using standard atmosphere temperature values, one finds that the non-dimensional constant $r^2 \approx 200$. To measure the relationship of kinetic to potential energy, we must compare r^2 with c_β for the wave energy and with $\bar{c}_\gamma$ for the zonal energy. Recalling that the eigenvalues (c_γ) are given approximately by the square of the latitudinal index of the solid harmonics (n_γ), we see that the two forms of energy are roughly equivalent for indices n_r 13-14.

Characteristic zonal profiles of the atmosphere including jets are well represented by coefficients with eigenvalues less than $c_r = n_r(n_r+1)$ and hence we may conclude that since $\bar{c}_\gamma < c_r$, the zonal potential energy will always have more energy than the zonal kinetic. More specifically, it is rarely necessary to include zonal indices $n_\gamma > 9$. Thus we may say that in the models considered in this paper,

$$K_z < \tfrac{1}{2} P_z$$

which is also observed in the atmosphere (Wiin-Nielsen, 1967) and in less truncated general circulation models (Phillips, 1956; Smagorinsky, 1965, etc.).

For the planetary waves, we see that the potential energy of the longest waves will be considerably greater than the kinetic, whereas for shorter waves, $\ell > 12$, the kinetic energy will exceed the potential. For very short waves, the kinetic energy will dominate, but the approximations made in deriving this system (Section 2) may not be appropriate to such small scales.

The average properties of the energy exchange are described by (5.7). However, since we know the solution for all time, it is possible to describe the instantaneous exchange at any time. For convenience of notation, we shall define a time dependent variable, R_γ such that

$$\begin{aligned} R_\gamma &\equiv 2K_{\alpha\beta\gamma} B_\alpha B_\beta B_\gamma \sin\theta = \frac{2K_{\alpha\beta\gamma}}{a_n} B_\gamma \dot{B}_n \\ &= \frac{-2\dot{B}_n}{d_\beta - c_\alpha} \left(\frac{d_\gamma a_\gamma^2}{a_n^2} B_n + \frac{d_\gamma a_\gamma s_\gamma}{a_n} \right) \end{aligned} \tag{5.8}$$

where the dependence of R_γ on B_n has been established by use of (3.3). On summation over all γ we have that

$$A(t) \equiv \sum_\gamma R_\gamma = \frac{-2\dot{B}_n}{d_\beta - c_\alpha} \sum_\gamma \frac{d_\gamma a_\gamma^2}{a_n^2} (B_n - B_{zex}) \tag{5.9}$$

where B_{zex} represents the value of B_n at the time when the total zonal energy (PE plus KE) would have an extreme value, not including the points $\dot{B}_n = 0$. This extreme value,

$$B_{zex} \equiv \frac{a_n \sum_\gamma d_\gamma a_\gamma s_\gamma}{\sum_\gamma d_\gamma a_\gamma^2}$$

has already been discussed (see Table [illegible]) [illegible] of energy extremes.

The function A(t) is clearly a periodic function of time with period T and changes sign where $\dot{B}_n=0$. From limited computations which show that the energy does not reach extremes other than at extremes of B_n, we may assume for the subsequent discussion that B_n does not generally take on the value B_{ex} and hence that A(t) changes sign only once during a period (provided that $\dot{B}_n(0)=0$). Let us further define the sum,

$$\sum_\gamma c_\gamma R_\gamma = c^*_\gamma(t)A(t) \tag{5.10}$$

where $c^*_\gamma(t)$ is an averaged value of the c_γ weighted by the R_γ and is not necessarily bounded. We shall, however, concern ourselves primarily with times at which

$$c_{\gamma\ min} \le c^*_\gamma(t) \le c_{\gamma\ max} \tag{5.11}$$

To discuss the instantaneous energy exchange, we now write the time rate of change of the energy components listed in (5.4):

$$\begin{array}{llllll}
\dot{K}_z = & c_\alpha A & - c_\beta A & & & -r^2\sum_\gamma \dfrac{c_\gamma+c_\alpha-c_\beta}{d_\gamma} R_\gamma \\[2ex]
\dot{P}_z = & & & -r^2 A & & +r^2\sum_\gamma \dfrac{c_\gamma+c_\alpha-c_\beta}{d_\gamma} R_\gamma \\[2ex]
\dot{K}_\beta = & -c_\alpha A & & & +c^*_\gamma A & +r^2 \dfrac{(c_\beta-c_\alpha-c^*_\gamma)}{d_\beta} A \\[2ex]
\dot{P}_\beta = & & & r^2 A & & -r^2 \dfrac{(c_\beta-c_\alpha-c^*_\gamma)}{d_\beta} A \\[2ex]
\dot{K}_\alpha = & & c_\beta A & & -c^*_\gamma A &
\end{array} \tag{5.12}$$

These equations are derived by differentiation of (5.4) and substitution of (3.2). The barotropic exchanges may be seen clearly simply by setting r=0.

The contributions from divergence listed as the last terms in (5.12) and describing the exchange between potential and kinetic energies in the zonal and wave separately, may be calculated from the vorticity equation and thermodynamic equation as represented by (2.14). If the first of (2.14) is multiplied by τ, the second by $r^2\tau$, both integrated over the unit sphere and (2.15) substituted for the divergence coefficients, the divergence terms of (5.12) become apparent. To examine the nature of this exchange in more detail, let us assume that

$$r^2 \sum_{\gamma} \frac{(c_\gamma + c_\alpha - c_\beta)}{d_\gamma} R_\gamma \approx (c^*_\gamma + c_\alpha - c_\beta) A \qquad (5.13)$$

based on our earlier observation that we may truncate γ with reasonable fidelity to atmospheric zonal wind profiles such that $c_{\gamma\ max} < r^2/2$. Recalling the constraint on active γ-components given by Baer and Platzman (1961),

$$|n_\alpha - n_\beta| \leq n_\gamma \leq n_\alpha + n_\beta$$

thereby establishing the maximum and minimum values of n_γ as

$$n_{\gamma\ min} = |n_\alpha - n_\beta|$$

$$n_{\gamma\ max} = n_\alpha + n_\beta \quad .$$

we may write

$$c_\alpha - c_\beta \approx \pm (c_{\gamma\ max}\, c_{\gamma\ min})^{\frac{1}{2}} \quad \begin{array}{l} + \text{ for } n_\alpha > n_\beta \\ - \text{ for } n_\alpha < n_\beta \end{array}$$

$$c_\alpha + c_\beta \approx \tfrac{1}{2}(c_{\gamma\ max} + c_{\gamma\ min}) \qquad (5.14)$$

Substitution of (5.14) and (5.13) into the divergence terms of (5.12) now allows us to establish the exchange of energy within the zonal and wave separately. Subject to the condition that $c^*_\gamma > 0$ (considerably weaker than 5.11) and $A(t) > 0$ which occurs during half the time period, Table 6 describes the direction of this energy flow. Clearly for the other half period when $A(t) < 0$, these directions are reversed.

We note from this table that if the shear wave is longer than the mean wave ($n_\alpha > n_\beta$), the direction of flow does not depend on the magnitude of c^*_γ, but only on the sign of $A(t)$. If, however, the shear wave is shorter, we see that the direction of flow from PE to KE is the same for the zonal and wave except for the restricted range where

$$\sqrt{c_{\gamma\ min}\, c_{\gamma\ max}} < c^*_\gamma < \tfrac{1}{2}(c_{\gamma\ max} + c_{\gamma\ min}).$$

Table 6. Energy flow within zonal and wave for different values of $c^*_\gamma>0$ and $A(t)>0$

Range	Zonal		β-wave
	$n_\alpha>n_\beta$	$n_\beta>n_\alpha$	
$c^*_\gamma<\sqrt{c_{\gamma\,min}c_{\gamma\,max}}$	$K_z\rightarrow P_z$	$K_z\leftarrow P_z$	$K_\beta\leftarrow P_\beta$
$\sqrt{c_{\gamma\,max}c_{\gamma\,min}}<c^*_\gamma<\frac{1}{2}(c_{\gamma\,max}+c_{\gamma\,min})$	$K_z\rightarrow P_z$	$K_z\rightarrow P_z$	$K_\beta\leftarrow P_\beta$
$c^*_\gamma>\frac{1}{2}(c_{\gamma\,max}+c_{\gamma\,min})$	$K_z\rightarrow P_z$	$K_z\rightarrow P_z$	$K_\beta\rightarrow P_\beta$

The entire energy flow diagram, listing all five energy terms defined in (5.4) may be seen from Fig. 3. The direction of flow shown in the figure is based on the condition that $A(t)>0$, $c^*_\gamma>0$, and the choice of transfer within the zonal and wave can be established from Table 6. The barotropic exchange is described by simply ignoring the boxes for available potential energy.

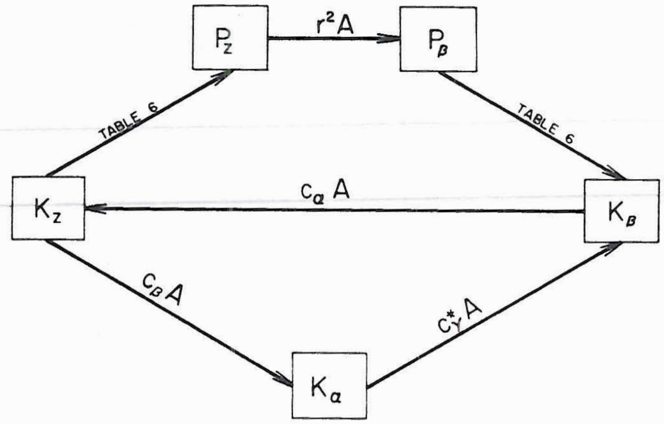

Fig. 3 Energy transfer diagram for all possible energy forms based on the conditions that $A(t) > 0$ and $c^*_\gamma > 0$ (see text).

Applying the following replacements to Fig. 3,

$$\begin{aligned} K_z &\rightarrow K_z + P_z \\ K_\beta &\rightarrow K_\beta + P_\beta \\ c_\alpha &\rightarrow c^*_\alpha \equiv c_\alpha - r^2 \end{aligned} \qquad (5.15)$$

the lower three boxes will also describe the exchange of total energy without distinguishing between potential or kinetic.

When approximation (5.13) is substituted into (5.12) and one considers the cycle during which $A(t)>0$, it is easily seen that the zonal kinetic energy decreases whereas the shear wave kinetic energy (K_β) increases. By combining the kinetic and potential energies, the following equations result:

$$\begin{aligned} \dot{E}_z &\equiv \dot{K}_z + \dot{P}_z = (c^*_\alpha - c_\beta)A \; : \; (c^*_\gamma) \\ \dot{E}_\beta &\equiv \dot{K}_\beta + \dot{P}_\beta = (c^*_\gamma - c^*_\alpha)A \; : \; (c_\beta) \\ \dot{E}_\alpha &\equiv \dot{K}_\alpha \qquad\quad\;\; = (c_\beta - c^*_\gamma)A \; : \; (c^*_\alpha) \end{aligned} \qquad (5.16)$$

The quantities in parentheses after the colon in (5.16) may be considered as "modified eigenvalues" associated with the energy function which each follows. Equations (5.16) then state that energy will flow into or out of the component whose "modified eigenvalue" is of intermediate value, at the expense of the other two components. This result is identical to one derived by Fjortoft (1953) for the two dimensional barotropic model and compares here to the case where r=0 and $c^*_\alpha = c_\alpha$. Certain observations may be drawn from this result.

(a) When the energy of the wave mean flow is in a long wave, i.e., $c_\alpha < r^2$, c^*_α will be negative and on the assumption that $c^*_\gamma > 0$, the mean wave energy will never be the sole recipient of energy from the other two (shear energy) sources, or vice versa.

(b) If the shear wave energy resides in a wave shorter than or equal to the mean wave energy, the "modified eigenvalue" of the shear wave will always be greater than that of the mean wave, $c_\beta > c^*_\alpha$.

(c) In application to typical atmospheric zonal wind profiles, if the energy in the shear wave is not in too long a wave, both the mean wave and the shear wave will feed the zonal energy and vice versa; i.e., $c^*_\alpha < c^*_\gamma < c_\beta$.

The above results, especially with regard to maximum energy transfer during a half period between extremes of B(t), may also be derived by invoking the concept of conservation of potential vorticity appropriate to the baroclinic model discussed herein and following the technique utilized by Fjortoft (1953).

6. Linear Analysis

One means of linearizing system (2.13) which has real physical significance, and whereby some of the characteristics of the system may be simplified, is to assume that the energy in the wave components (α,β) is of perturbation amplitude in comparison with the zonal energy. Since we have shown in Section 5 that the potential energy is roughly proportional to the kinetic energy, no loss of generality will ensue if we confine our discussion to kinetic energy. In symbolic form, we may say that

$$
\begin{aligned}
\psi_\gamma &= \bar{\psi}_\gamma + \psi_\gamma' \qquad & \psi_\gamma' &<< \bar{\psi}_\gamma \\
\psi_\alpha &= \psi_\alpha' & \psi_\alpha' &<< \bar{\psi}_\gamma \qquad (6.1)\\
\psi_\beta &= \psi_\beta' & \psi_\beta' &<< \bar{\psi}_\gamma
\end{aligned}
$$

where the barred zonal coefficients may be represented by the initial configuration and the primed quantities will be considered perturbations. It is important to note that the perturbation wave coefficients are small with regard to any of the zonal coefficients allowed by the truncation (see Section 4). Following customary linearizing procedures, we shall neglect all second order or higher terms (products of primed quantities) on substitution of (6.1) into (2.13). It is immediately evident that the perturbation changes of the zonal coefficients cannot be calculated directly from the differential equations, since the change is of second order, but we shall see that the first order changes (implied) can be established from the energy relations. The wave perturbation equation may be written

$$
\begin{aligned}
\dot{\psi}_\beta &= -i\eta_\beta\psi_\beta + i\bar{G}_{\beta\alpha}\psi_\alpha \\
\dot{\psi}_\alpha &= -i\eta_\alpha\psi_\alpha + i\bar{G}_{\alpha\beta}\psi_\beta \qquad (6.2)\\
\eta_\beta &\equiv \nu_\beta - \bar{G}_{\beta\beta}, \text{ etc.}
\end{aligned}
$$

The barred quantities, $\bar{G}_{\beta\beta}$, etc. can be established from Table 2 by substituting the initial values of the zonal coefficients and holding the terms constant.

System (6.2) is a linear homogeneous set having two frequencies $\psi \propto e^{i\sigma t}$) which satisfy the quadratic equation,

$$\sigma^2 + (n_\alpha + n_\beta)\sigma + n_\alpha n_\beta - \bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha} = 0 \qquad (6.3)$$

he two frequencies are therefore

$$\sigma_{a,b} = \sigma_1 \pm \sigma_2$$
$$\sigma_1 \equiv -\frac{n_\alpha + n_\beta}{2} \; ; \quad \sigma_2 \equiv \tfrac{1}{2}\left((n_\alpha - n_\beta)^2 + 4\bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha}\right)^{\frac{1}{2}} \qquad (6.4)$$

le first observe that this system is capable of instability which depends on :he initial zonal configuration and the wave number and wave profile. The :ondition in terms of the problem parameters is

$$4\bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha} \gtrless -(n_\alpha - n_\beta)^2 \quad \begin{cases} \text{stable} \\ \text{unstable} \end{cases}$$

:or the barotropic problem, this stability criteria should be contrasted to :hat given by Kuo (1949), which may be written in terms of spherical geometry ıs

$$\frac{d^2}{d\mu^2}\left((1-\mu^2)^{\frac{1}{2}}u\right) - 2 = 0$$

vhere μ represents the initial zonal wind distribution. The disparity be- :ween the two conditions is apparent in that Kuo's condition depends only on :he zonal configuration and not on the wave.

The condition for instability with regard to the baroclinic problem may be compared with that established by Phillips (1954) for a 2-layer mode. and written

$$V^2(\alpha\alpha^*)^2\left(4 - \left(\frac{\alpha\alpha^*}{r^2}\right)^2\right) > \left(\frac{\partial f}{\partial \phi}\right)^2$$

where V represents zero-order value of the zonal wind shear and α is a characteristic wave vector as defined in Table 1 (Phillips deals with only one wave number in each of the two horizontal dimensions).† The stability

---

†The asterisk denotes conjugation throughout this manuscript unless otherwise defined.

properties of the low-order systems (both linear and nonlinear) are currently under investigation and will be discussed in a subsequent report.

We shall, for the remainder of this section consider only stable solutions with the implied condition that σ_2 is real. The complete solutions for the wave components ψ_α, ψ_β require a specification of their initial values and must depend on both frequencies; they may be written as

$$\begin{aligned}\psi_\alpha &= e^{i\sigma_1 t}\left(Ae^{i\sigma_2 t}+Be^{-i\sigma_2 t}\right)\\ \psi_\beta &= e^{i\sigma_1 t}\left(q_1 Ae^{i\sigma_2 t}+q_2 Be^{-i\sigma_2 t}\right)\end{aligned} \tag{6.5}$$

Substituting these equations into (6.2) and evaluating the equations at t=0 when $\psi_\alpha = \psi_{\alpha 0}$, $\psi_\beta = \psi_{\beta 0}$, we find that the coefficients q do not depend on the initial values of the dependent variables,

$$\begin{aligned}q_1 &= \frac{\bar{G}_{\beta\alpha}}{\sigma_a+n_\beta} = \frac{\sigma_a+n_\alpha}{\bar{G}_{\alpha\beta}}\\ q_2 &= \frac{\bar{G}_{\beta\alpha}}{\sigma_b+n_\beta} = \frac{\sigma_b+n_\alpha}{\bar{G}_{\alpha\beta}}\end{aligned} \tag{6.6}$$

whereas the amplitude factors A and B depend on the initial values as follows;

$$\begin{aligned}A &= \frac{q_2\psi_{\alpha 0}-\psi_{\beta 0}}{q_2-q_1} \equiv |A|e^{i\theta_a}\\ B &= \frac{q_1\psi_{\alpha 0}-\psi_{\beta 0}}{q_2-q_1} \equiv |B|e^{i\theta_b}\end{aligned} \tag{6.7}$$

The latter definitions for A and B imply that the initial phase angles need not vanish. More specifically,

$$\tan\theta_a = \frac{\mathrm{Im}(q_2\psi_{\alpha 0} - \psi_{\beta 0})}{\mathrm{Re}(q_2\psi_{\alpha 0} - \psi_{\beta 0})}$$

$$\tan\theta_b = \frac{\mathrm{Im}(q_1\psi_{\alpha 0} - \psi_{\beta 0})}{\mathrm{Re}(q_1\psi_{\alpha 0} - \psi_{\beta 0})}$$

Let us now use (6.5) to establish the energy variations with time for the linearized system. We have for the α-component,

$$K_\alpha = 2c_\alpha \psi_\alpha \psi_\alpha^* \tag{6.8}$$

$$= 2c_\alpha\left(|A|^2+|B|^2+2|A||B| \cos(2\sigma_2 t + \theta_a - \theta_b)\right)$$

If we now create a new time variable which is linearly related to t by the equation

$$t' = t - T$$

$$T = \frac{\theta_b - \theta_a}{2\sigma_2}$$

we find the equation for the energy in the α-wave,

$$K_\alpha = 2c_\alpha\left(|A|^2+|B|^2+2|A||B| \cos 2\sigma_2 t'\right) \tag{6.8'}$$

Since the frequency ($2\sigma_2$) is the same in both (6.8) and (6.8'), and since (6.8) has an identical form to (6.9) for zero initial phase angles, we must conclude that except for a shift in time of the maximum and minimum of the α-energy, the selection of non-zero phase angles does not contribute information to the solution of (6.2) different from the more simple condition of zero initial phase angles. We shall see subsequently that the other energy components (β,γ) have the same time dependence as (6.8) and thus need not be considered in detail in this discussion. Our selection of zero initial phase angles as initial conditions to the nonlinear problem (Section 4) and our confidence in the generality of those conditions is based on the above results from the linear solution. A complete proof would, of course, require direct analysis of the nonlinear equations.

Returning to the discussion of energy variations, we note from the definition of the β-wave energy in terms of ψ_β, Eqs. (5.4) and (6.5), that the time dependence is identical to (6.8) which describes the α-wave energy. For the zonal energy, multiplying the first of (2.13) by $2\psi_\gamma$, ignoring the variable part on the right-hand side ($\psi_\gamma \rightarrow \bar{\psi}_\gamma$), and integrating, the variation may be given as

$$K_\gamma = 4c_\gamma a_\gamma \bar{\psi}_\gamma \int \mathrm{Im}\ \psi_\alpha \psi_\beta^* dt + \text{const.} \tag{6.9}$$

which is a second order variation. Assuming now that A and B are real--based on our observation that zero initial phase angles are completely general--and substituting (6.5), (6.6) and (6.7) into (6.9), we find

$$K_z = 4\sum_\gamma c_\gamma a_\gamma \bar{\psi}_\gamma \frac{AB}{\bar{G}_{\alpha\beta}} \cos 2\sigma_2 t + \text{const.} \tag{6.10}$$

With this presentation of the different energy components, we may investigate the range of energy variation (maximum energy exchange) as defined by (5.5). Since the extremes occur at $t=0$ and $t=\pi/2\sigma_2$, we have from (6.8), (6.10) and the β-energy relation,

$$\Delta K_z = -8 \frac{\sum c_\gamma \bar{\psi}_\gamma a_\gamma}{\bar{G}_{\alpha\beta}} AB$$

$$\Delta K_\beta = 8c_\beta \frac{\bar{G}_{\beta\alpha}}{\bar{G}_{\alpha\beta}} AB \tag{6.11}$$

$$\Delta K_\alpha = -8c_\alpha AB$$

The dependence of the exchange on the level of total energy ($\bar{u}_0$) and the relative energy in the wave (ρ) may now be established from the product AB. The initial stream coefficients used in A and B may be represented in terms of an amplitude g_ℓ and profile parameters A_α, A_β as seen from (4.2). Combining this form with the definition of ρ, Eq. (4.8), and the dependence of K_ℓ on g_ℓ from (4.4), we find

$$\rho = \frac{K_\ell}{K} = \frac{g_\ell^2}{8K} (c_\alpha A_\alpha^2 + c_\beta A_\beta^2)$$

such that

$$AB = -\rho S \tag{6.12}$$

$$S \equiv \frac{\bar{G}_{\alpha\beta}^2 \, K}{8\sigma_2^2 (c_\alpha A_\alpha^2 + c_\beta A_\beta^2)} \left(q_1 q_2 A_\alpha^2 + A_\beta^2 - (q_1+q_2) A_\alpha A_\beta\right)$$

In (6.12), K represents the zero order energy initially placed into the system and does not change for variation in ρ. Therefore we see that the maximum energy exchange among the components is linearly proportional to the relative amount of energy in the wave components. In the limiting case of no initial nergy in the wave--i.e., $\rho=0$--no motion would occur, a result which is evilent also from Eqs. (2.13).

Whereas the energy range (ΔK) is linearly proportional to ρ with zero ntercept, the slope of the line is dependent on the amplitude $\bar{u}_0$. Noting rom (4.2) that the initial zonal coefficients ($\bar{\psi}_\gamma$) are proportional to $\bar{u}_0$ nd the profile parameters A_γ, and from Table 2 that $\bar{G}_{\alpha\beta}$, $\bar{G}_{\beta\alpha}$ are proporional to the $\bar{\psi}_\gamma$, we may observe from (6.11) that the variation of slope with) is incorporated entirely in the function S (Eq. 6.12). The definitions of

q_1 and q_2 from (6.6) yield required coefficients in the function S, (6.12), in terms of known quantities as follows:

$$q_1 + q_2 = \frac{\nu+\bar{G}_{\beta\beta}-\bar{G}_{\alpha\alpha}}{\bar{G}_{\alpha\beta}} = \frac{s_1+ s_2\bar{u}_0}{\bar{u}_0}$$

$$q_1q_2 = - \frac{\bar{G}_{\beta\alpha}}{\bar{G}_{\alpha\beta}} = - \frac{\sum e_\gamma A_\gamma}{\sum f_\gamma A_\gamma}$$

$$\sigma_2^2 = \tfrac{1}{4}(n_\alpha-n_\beta)^2+ \bar{G}_{\alpha\beta}\bar{G}_{\beta\alpha}= \frac{\bar{G}_{\alpha\beta}^2}{4\bar{u}_0^2} \left((s_1+s_2\bar{u}_0)^2- 4q_1q_2\bar{u}_0^2\right) \tag{6.13}$$

$$K = \frac{\bar{u}_0^3}{4} \sum c_\gamma A_\gamma^2$$

where

$$s_1 \equiv \frac{2\nu}{\sum if_\gamma A_\gamma} \; ; \quad s_2 \equiv \frac{\sum (g_\gamma-b_\gamma)A_\gamma}{\sum f_\gamma A_\gamma} \tag{6.14}$$

We consider ν, as defined by (3.1), to be independent of $\bar{u}_0$ because it does not depend on the energy available for exchange. Substituting the results of (6.13) into (6.12), the $\bar{u}_0$ dependence of S is shown to be

$$S = \left(\frac{\sum c_\gamma A_\gamma^2}{2(c_\alpha A_\alpha^2 + c_\beta A_\beta^2)}\right)\bar{u}_0^3 \; \frac{(q_1q_2A_\alpha^2 + A_\beta^2 - s_2A_\alpha A_\beta)\bar{u}_0- s_1A_\alpha A_\beta}{(s_1+s_2\bar{u}_0)^2 - 4q_1q_2\bar{u}_0^2} \tag{6.15}$$

This is not a simple formula. We may, however, note that since we have chosen to consider only stable solutions, σ_2^2 will not change sign and therefore the denominator will also be of one sign for all allowed values of $\bar{u}_0$. The slope will change sign, however, about the critical value of $\bar{u}_0$,

$$\bar{u}_0(c) \equiv \frac{s_1A_\alpha A_\beta}{q_1q_2A_\alpha^2 + A_\beta^2 - s_2A_\alpha A_\beta} \; .$$

When $\bar{u}_0=\bar{u}_0(c)$, the linear system is inactive, independent of the value of ρ. Such initial conditions have been observed in calculations of the nonlinear system and will be discussed in the sequel.

To describe the comparison between linear and nonlinear solutions, Table 7 has been prepared, based on calculations with a given initial state but variable ρ. The initial configuration applies to the barotropic problem with

Table 7. Comparison between exact nonlinear solution and linear solution for several amplitude values. Listed as a function of ρ--the ratio of perturbation energy to total energy--are the exact energy range $\Delta K_\alpha/K_a$ and the percent error of the linear value to the exact.

	$\bar{u}_0 = .075$		$\bar{u}_0 = .135$		$\bar{u}_0 = .275$	
ρ	$\Delta K_\alpha/K_a$ exact	% of error	$\Delta K_\alpha/K_a$ exact	% of error	$\Delta K_\alpha/K_a$ exact	% of error
.02	-.01917	- 1.26	-.02249	- 1.23	-.02358	- 1.23
.05	-.04614	- 5.20	-.05419	- 5.01	-.05687	- 4.94
.10	-.08681	-11.8	-.1022	-11.3	-.1074	-11.1
.15	-.1228	-18.6	-.1451	-17.7	-.1527	-17.3
.20	-.1549	-25.4	-.1835	-24.1	-.1934	-23.5
.30	-.2087	-39.4	-.2493	-37.0	-.2637	-35.8
.40	-.2518	-54.2	-.3033	-50.1	-.3223	-48.1
Linear Slope $=\left(\frac{8c_\alpha S}{K_a}\right)$	-.9708		-1.138		-1.194	

the atmospheric jet described in Fig. 1, for wave $\ell=3$ and profile given by (1.17) with $s=0$. Three amplitude values $(\bar{u}_0)$† have been chosen to show the variation of solutions both with ρ and $\bar{u}_0$. We have chosen to describe the energy range in the α-component; the ranges of the other components are available (and computable from 6.11) and show similar variations. The linear slope described in the Table is proportional to S and is given as

$$\text{Linear Slope} = \frac{8c_\alpha S}{K_a}$$

The error in the linear solution, described as the difference between the exact and linear values normalized by the exact value and represented as a percent, follows an expected pattern. For increasing values of ρ, the linearization hypothesis becomes less and less valid. Although the activity of the α-wave increases for increased energy in the system--larger $\bar{u}$ --the errors decrease slightly. We may conclude, therefore, that for $\rho \ll 1$, linear calculations give a reasonable approximation to the nonlinear solutions.

7. Phase Characteristics and Other Flow Properties

The phase angles, Θ_α and Θ_β, may be calculated as functions of time, as described in Appendix B. These solutions indicate that the angles need not be periodic with period T, although we can see from (3.12) that their difference (θ) must. If we assume that the phase angle periods differ from T (the nonlinear energy exchange period), then as already proposed, the spatial distribution of a wave component added to the zonal field need not show a repeating pattern until a modulation cycle between the energy period and the wave period is completed. This is especially true in the barotropic problem where the wave configuration may be more complicated.

For the baroclinic problem, the wave tilt with height is immediately calculated from the phase angle of the shear wave, Θ_β. If, moreover, one assumes that the wave vectors α and β are equal--which does not imply that Θ_α and Θ_β are equal--then the wave has only one component, $P_\alpha = P_\beta$, and no horizontal exchanges will exist. Such a condition applied to the barotropic problem would yield trivial solutions.

Although the equations for the individual phase angles are reasonably complex, we may draw certain conclusions from θ, their difference. For convenience we repeat here the variation of θ with time,

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†All variables in this report have been nondimensionalized using the earth's rotation rate for time and the mean radius of the earth for space.

$$\tan\theta = \frac{\dot{B}_n}{G} \tag{3.12}$$

The variation of B_n has already been established, and since it is periodic with period T, it has a limited range. There are three possibilities for the variation of $G(B_n)$ in the range $mT \leq t \leq (m+1)T$[†] and they depend on whether G has zero, one, or two roots in the allowed range of B_n. Since we have selected $\theta = 0$ at $t = mT$, it will also be zero at $t = (m+\frac{1}{2})T$. The range of θ may be summarized in terms of the number of roots of G and is listed in Table 8. We have noted that θ must pass through zero at $t = (m+\frac{1}{2})T$; thus θ remains either positive or negative during one half period and reverses its sign during the other half. The direction of θ at the beginning of the period is also specified in the Table. Aside from the information on the position of the wave components relative to each other during the period which Table 8 yields, we shall see how the variation of θ is related to the variation of other flow characteristics.

Table 8. Range of θ in terms of the number of roots of G in one nonlinear period and direction of θ as t increased from mT.

No. of Roots	0	1	2
Range	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$0 \leq \theta \leq 2\pi$	$-\pi < \theta < \pi$
$\theta(mT + \varepsilon) > 0$		$\left\{ \begin{array}{l} B_n(mT)\ \text{min.};\quad G(mT) > 0 \\ B_n(mT)\ \text{max.};\quad G(mT) < 0 \end{array} \right.$	
$\theta(mT + \varepsilon) < 0$		$\left\{ \begin{array}{l} B_n(mT)\ \text{max.};\quad G(mT) > 0 \\ B_n(mT)\ \text{min.};\quad G(mT) < 0 \end{array} \right.$	
$\varepsilon << T$			

Phase frequencies may be calculated directly from (3.13) once B_n is known. These frequencies include the Rossby-Haurwitz contribution, $\nu_{\alpha,\beta}$, a linear contribution, $h_{\alpha\alpha,\beta\beta}$, based on the jet configuration and a nonlinear term. The frequencies clearly undergo a periodic variation with the same period as B_n. They have maximum and minimum values at the same times as B_n, but they may have other relative extremes depending on whether or not $\ddot{\theta}_{\alpha,\beta} = 0$ in the allowed range

[†]In this discussion, m may take on any integral value.

of B_n. Differentiation of (3.13) with respect to B_n and setting the resulting equation to zero will yield a quartic equation, the roots of which will represent the values of B_n (or T) at which the phase frequency will have extremes. These values may then be compared with the allowed range of B_n to determine whether extra maxima or minima do indeed exist.

Let us now consider the horizontal distribution of the wave at any time. We shall focus our attention on the barotropic problem since the wave distribution is more complex, but the results can easily be extended to the mean wave field of the baroclinic model. We have from (2.5) and (3.1) that

$$\psi'(\lambda,\mu,t) = \sqrt{2}\left(B_\alpha P_\alpha \cos(\ell\lambda+\theta_\alpha) + B_\beta P_\beta \cos(\ell\lambda+\theta_\beta)\right) \tag{7.1}$$

from which we may determine the meridional velocity component after substitution of the definition,

$$\varepsilon \equiv \ell\lambda + \theta_\beta \tag{7.2}$$

and recalling the definition of θ from (3.1). The resulting equation for meridional velocity v is,

$$\begin{aligned} v &= (1-\mu^2)^{-\frac{1}{2}} \frac{\partial\psi'}{\partial\lambda} \\ &= -\sqrt{2}\,\ell(1-\mu^2)^{-\frac{1}{2}}\left((B_\beta P_\beta + B_\alpha P_\alpha \cos\theta)\sin\varepsilon + B_\alpha P_\alpha \sin\theta\cos\varepsilon\right) \end{aligned} \tag{7.3}$$

Since the trough or ridge line is defined as the longitude at which the motion is zonal, the trough (ridge) longitude, λ_{tr}, is given from (7.3) by

$$\tan\varepsilon_{tr}(\mu,t) = -\frac{B_\alpha P_\alpha \sin\theta}{B_\alpha P_\alpha \cos\theta + B_\beta P_\beta} \tag{7.4}$$

We see immediately that the latitudinal dependence of λ_{tr} is related to θ, for when B_n is at an extreme, and hence $\theta = 0$ or π, the trough (ridge) is north-south. The slope of the line is related both to the magnitude and sign of θ and can be evaluated as follows. Differentiating (7.4) with respect to μ, solving for the variation of λ_{tr} and using the first of (3.2), we find

$$\frac{d\lambda_{tr}}{d\mu} = \frac{1}{\ell(1+\tan^2 \epsilon_{tr})} \frac{d \tan \epsilon_{tr}}{d\mu}$$

$$= \frac{\hat{B}_n}{\ell a_n} \frac{P_\alpha P_\beta' - P_\beta P_\alpha'}{|B_\alpha P_\alpha e^{i\Theta_\alpha} + B_\beta P_\beta e^{i\Theta_\beta}|^2} \tag{7.5}$$

Eq. (7.5) indicates that the trough slope goes through a periodic cycle, beginning with no slope at $t = mT$ (north-south), bending to one side for one-half period, returning to zero at $t = (m+\frac{1}{2})T$, and then bending to the other side for the second half period. Since the numerator of (7.5) may have roots in the allowed range of μ, the direction of the trough line at any time will be a function of latitude and may change sign.

The transport of momentum across a latitude circle has been shown to depend closely on the slope of troughs and ridges (Starr, 1948) and thus to the development of jets and barotropic instability. Continuing our discussion of the barotropic problem, the momentum transport may be defined as

$$M = \frac{1}{2\pi} \int_0^{2\pi} uv(1-\mu^2)^{\frac{1}{2}} \, d\lambda$$

$$= - \frac{(1-\mu^2)^{\frac{1}{2}}}{2\pi} \int \frac{\partial\psi}{\partial\lambda} \frac{\partial\psi}{\partial\mu} \, d\lambda \tag{7.6}$$

We again use the wave stream function ψ' from (7.1) since it is apparent that the zonal components will make no contribution to the integral. Substituting ϵ and θ for Θ_α and Θ_β and performing the required differentiations and integration as indicated by (7.6) we find

$$M = \ell \frac{\hat{B}_n}{a_n} (P_\alpha P_\beta' - P_\beta P_\alpha') \, (1-\mu^2)^{\frac{1}{2}} \tag{7.7}$$

The momentum transport is therefore periodic with period T, zero at $t = mT$, $(m+\frac{1}{2})T$, and opposite in direction during the two half cycles. Rather than discuss the details of (7.7), we shall show that the momentum transport is directly related to the tilt of the trough. By combining (7.7) with (7.5) we have,

$$M = \frac{d\lambda_{tr}}{d\mu} \left\{\ell^2 (1-\mu^2)^{\frac{1}{2}} |B_\alpha P_\alpha e^{i\theta_\alpha} + B_\beta P_\beta e^{i\theta_\beta}|^2\right\} \tag{7.8}$$

The results of this analysis, which show the momentum transport linearly related to trough slope with the same sign, are in complete agreement with general theory.

Finally, we may consider the mean square vorticity for both the barotropic and baroclinic models. For the baroclinic problem, the conservation condition must be converted to conservation of potential vorticity. In spectral terms, this modification is trivial, implying merely that we substitute d_γ for c_γ as eigenvalues. From the basic Eqs. (2.13) we find, therefore, the mean square vorticity (the squared vorticity integrated over the entire field) to be given by the relation,

$$F = \sum d_\gamma^2 \; B_\gamma^2 + c_\alpha^2 \; B_\alpha^2 + d_\beta^2 \; B_\beta^2 \tag{7.9}$$

This parameter must be conserved for the flows considered herein, and the constraint thus imposed has already been implied in our discussion of energy exchanges in Section 5.

8. Some Barotropic Calculations

The properties of the low-order system (2.13) have been discussed in general terms by various methods in the preceeding sections. The exact variation of the system will be described only by applying specific initial conditions to the solution as outlined in Section 3. There are, however, an infinite variety of initial conditions which could be considered. We consequently confine our attention to a discussion of the barotropic problem with a limited range of initial conditions. Those conditions, as we shall see, confine the energy of the system to the range of atmospheric possibility, the zonal wind configuration to realistic flows, and the wave configurations to simple distributions for the long and medium scale waves. In all calculations, we shall investigate the changes in the system due to variations in ρ (the relative energy in the perturbation) while maintaining the other initial conditions invariant. We may then expect to observe, over the range of atmospheric conditions, the variability of the model solutions.

In the calculations to be discussed, five different amplitudes ($\bar{u}_0$) were used, three with the A-J (see Section 4 for definitions) and two with the D-J. These amplitudes have been used together with the jet profiles to give an average energy K (see 4.6) for a layer of 50 mb in units of 10^5Joules/m^2. The choice of these units was established to compare with the values given by Kung (1966) for winter, summer and annual mean of 1962, all of which have been listed together in Table 9.

Table 9. Mean kinetic energy used in calculations and observed atmospheric values (Kung, 1966) for 50 mb layer in units of 10^5Joules/m^2. Observed values have been averaged between 900-100 mb.

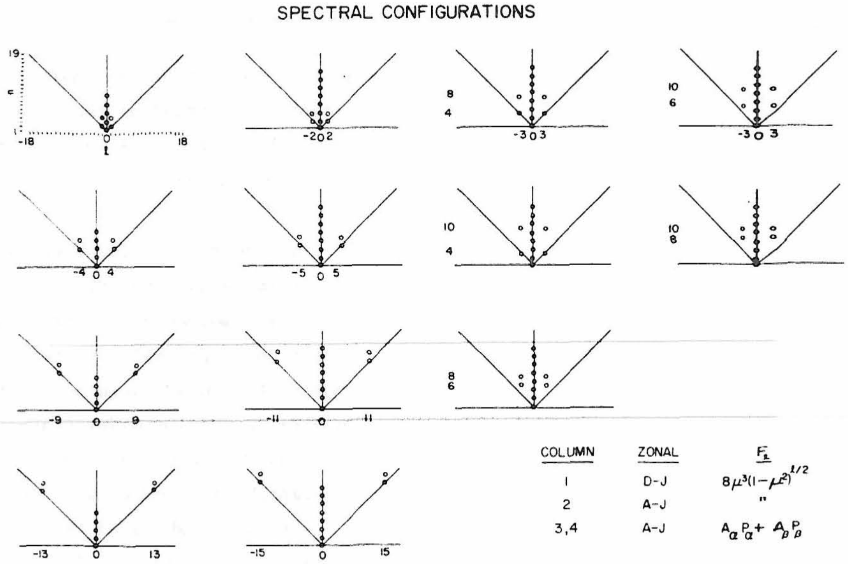
$\bar{u}_0$	Jet	K
.075	A-J	.340
.135	A-J	1.10
.275	A-J	4.57
.067	D-J	1.34
.133	D-J	5.33
Observed values: Kung (1966b)		
Summer		.527
Winter		1.46
Annual Mean		.948
Layer Extremes		.130 - 3.01

It is evident from the Table that the range of energy chosen for calculation effectively straddles the values observed in the atmosphere. One could reasonably extend the range of the calculations, but the results from the values of energy actually used may be meaningfully compared to atmospheric events.

The zonal profiles used in the computations have already been described in Section 4 and depicted on Figs. 1 and 2. The A-J is clearly applicable to the real atmosphere by definition, whereas the D-J was chosen more for simplicity of representation than for reality. A more realistic D-J would require too much resolution in the zonal coefficients and consequently work counter to the low-order concept. Nevertheless, the computations with the selected D-J should be indicative of the interactions which might take place in atmospheric flow with a double jet.

Having established the variations on the zonal flow, we now consider the initial wave profiles. The wave energy will be considered, as suggested previously, by allowing ρ to vary over its entire range ($0<\rho\leq 1$). We first select the simple profile given by (4.16) and allow ℓ to vary over the range $1\leq\ell\leq 18$; this profile has the virtue of involving only the lowest allowed modes in the latitudinal direction. From calculations with this function we are able to establish how the nonlinear properties of our system vary with regard to wave number. Lest we generalize on these solutions, however, we have selected to investigate a number of different wave profiles for a given wave number (ℓ=3) to determine the variability of solutions in terms of wave profiles. These profiles allow for arbitrary specification of A_α, A_β, n_β and n_α (where we require, with complete generality, $n_\beta > n_\alpha$). Some of these profiles have been presented in Fig. 2, together with F_ℓ of Eq. (4.16) for ℓ=3.

The low-order truncation for the initial configurations discussed above may be conveniently represented on an n-ℓ diagram in which the active components (represented by γ, α, and β values) are described by circles. Figure 4 includes a number of such diagrams and we shall denote the allowed (circled) components in each diagram as a spectral configuration. Reference to Table 3 indicates that the A-J has eight zonal components whereas the D-J has only four. Columns one and two of Fig. 4 show some possible configurations with either of the allowed zonal jets, using the wave profile given by (4.16) for several different wave numbers. The last two columns of the figure show the configurations for the A-J with different wave profiles of wave ℓ=3.



COLUMN	ZONAL	F_ℓ
1	D-J	$8\mu^3(1-\mu^2)^{1/2}$
2	A-J	"
3,4	A-J	$A_\alpha P_\alpha + A_\beta P_\beta$

Fig. 4 Spectral configurations for a number of truncations used for calculation. Circled points indicate allowed wave vectors.

We shall now consider some of the features of the low-order system which result from the solutions for specific initial conditions and truncation. Foremost among these features is the nonlinear energy exchange period. Since the conventional linear theory cannot predict nonlinear periods, their prediction by the low-order equations should add significantly to our understanding of the more general system. Nevertheless, unless the atmosphere is constituted as a highly truncated system (not a frequent occurrence) one should not anticipate--nor do we find--exact periodicity in nonlinear flow. There are indications that some quasi-periodicity may exist in the atmosphere (Namias, 1954; Eliasen, 1958) although data limitations must be considered

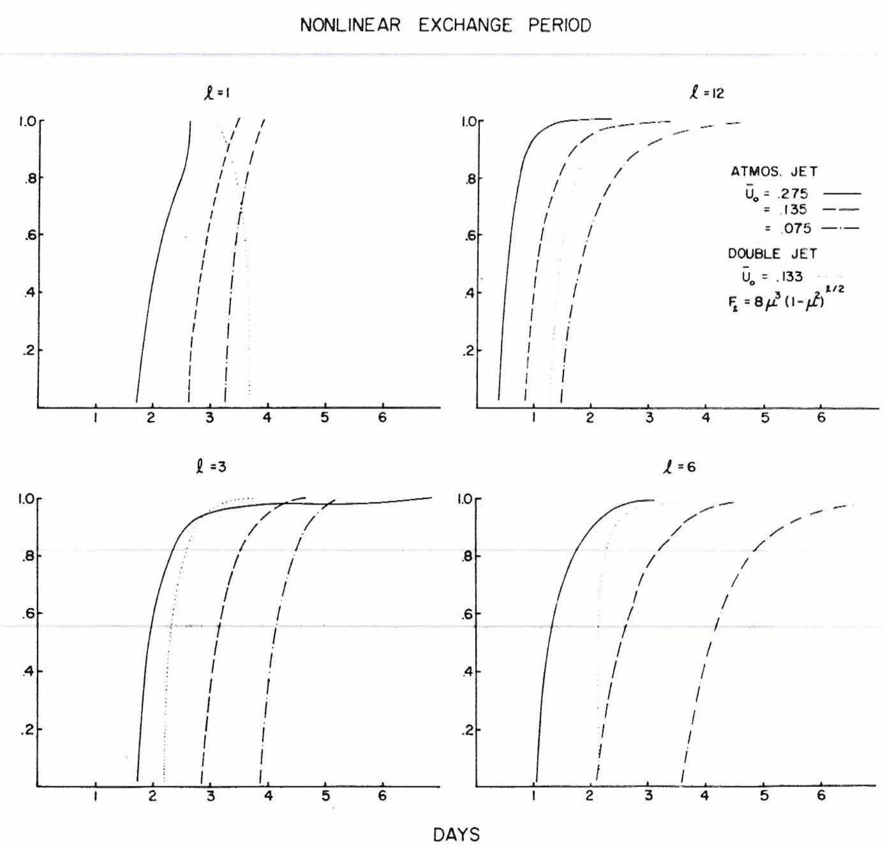


Fig. 5 Nonlinear energy exchange periods for different initial configurations for a variety of wave numbers, plotted against ρ (ordinate).

in evaluating such observations. Furthermore, integrations of less truncated models (Baer, 1964) also indicate periodicities, although the number of such calculations is severely limited. We must therefore interpret the observation of exact periods as representing atmospheric behavior with a mixture of optimism and suspicion.

Figure 5 describes the nonlinear exchange period, T, as defined by (A-9) in days for various energy levels, both jets, and a selection of wave numbers (ℓ = 1,3,6,12) for the wave configuration given by (4.16); the periods are plotted against ρ (ordinate). We first observe that for all realistic values of ρ--values of ρ>8 are rarely, if ever, observed in the atmosphere--the periods are relatively insensitive to the initial partitioning of energy between the zonal flow and the wave. For all waves there appears to be a systematic decrease in the period with increasing total energy; the curves for those waves not described in the figure show no erratic behavior and may be interpolated from the given data. The periods for the D-J appear somewhat longer than for the A-J, but this variation is not of great significance. We also note a maximum in the period for the wave three, after which the periods gradually become shorter for shorter wave lengths. The magnitudes of the periods, however, which run from less than one to four days are not in close agreement with previous calculation and observation. Whereas the three-day period for wave number one is in agreement with a previous calculation by the writer (Baer, 1964), the value for wave number three does not compare favorably (previously calculated as ≈6 days) nor does it correspond with the observed value of 6-7 days (Namias, 1954, etc.). Although direct observations or calculations are not available for shorter waves, it is unlikely that waves in the primary energy input region (ℓ = 5-7) would undergo periodicities as short as four days or less.

Lest one conclude that the low-order system will yield periods only of the type described by Fig. 5, we consider the nonlinear periods generated by alternate wave profiles. In Fig. 6 we describe four different wave truncations for wave number three with four independent choices of the profile parameters A_α and A_β for each truncation. The corresponding periods are again plotted against ρ. The first two truncations do not show striking differences from the results of Fig. 5; however, the latter two, especially for truncation n_α=6, n_β=8, show remarkably different periods, which appear to depend considerably more on the truncation than on the selection of profile parameters. Whereas our previous observation suggested that the nonlinear periods are only weakly dependent on ρ, for the latter two truncations on Fig. 6 the period decreases rapidly with increasing ρ. We furthermore note that the periods may be as long as fifteen days.

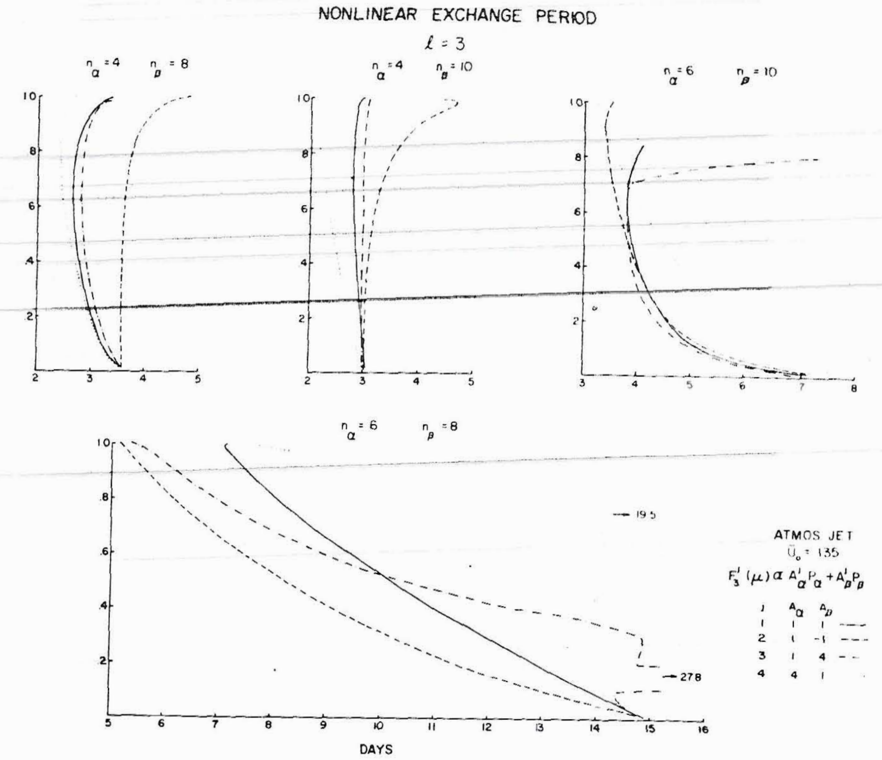


Fig. 6 Nonlinear energy exchange periods for wave ℓ=3 with different initial wave configurations, plotted against ρ (ordinate).

The existence of a nonlinear period does not, of course, give an indication of the nonlinear activity of the system, measured by the amount of energy actually exchanged during the period. Linear theory indicates (Section 6) that for small ρ the exchange must also be small, provided that the initial configuration is stable (a condition which is generally satisfied in our calculations). Long periods for small values of ρ (little energy initially in the wave) therefore indicate a very slowly and weakly changing system. The amount of energy exchanged per unit time under this condition must indeed be small since it will be proportional to ρ normalized by the period.

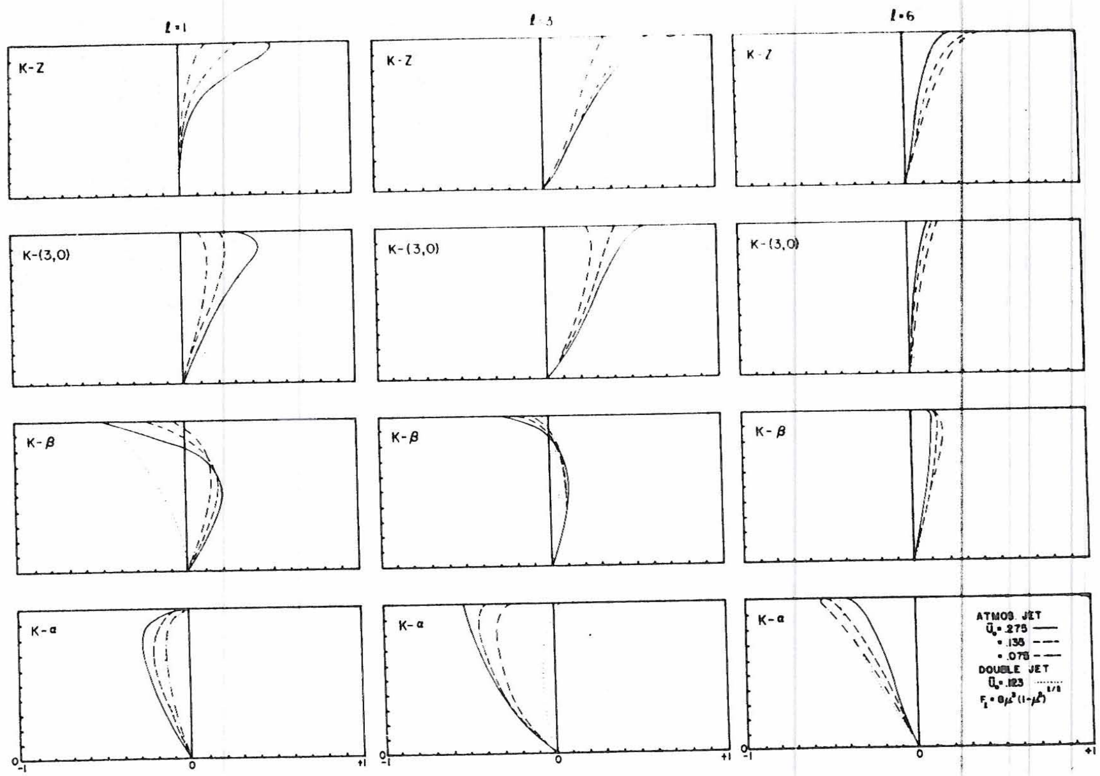


Fig. 7 Maximum energy exchange for three different truncations (different waves) including energy of the zonal current and the wave components normalized by the available energy K_α and plotted against ρ (ordinate).

In order to discuss the energy exchanged during the period more knowledgeably, we now present the energy ranges as defined by (5.7) but normalized by the available energy K_a (see Eq. 4.7). Figure 7 describes the maximum energy exchange on a scale $-1 \leq \Delta K/K_a \leq 1$ against ρ for the total zonal energy, the lowest active zonal component energy ($\gamma=3$), the β-wave energy and the α-wave energy. Our presentation is limited to the results for $\ell = 1,3,6$ since the variations become reasonably small for shorter waves. The initial conditions applicable for the calculations described in the figure are those for which the periods were shown in Fig. 5. The purpose of presenting the $\gamma=3$ zonal mode is to determine the importance of the lowest mode in the zonal field. For the conditions represented in Fig. 7, it is evident, on comparing the upper two diagrams for each wave, that the $\gamma=3$ mode dominates the exchange for the total zonal energy. This observation is consistent with the previous calculation by the writer (Baer, 1964) for a less truncated model. Moreover, from an observational point of view, it is reasonable to expect the least variable (in latitude) part of the zonal field to vary most with time in order to maintain the moderately smooth zonal profiles which are actually measured.

A second pronounced feature of the exchange as described in Fig. 7 is the compensation between the two wave components. In all cases, except for very large and unrealistic values of ρ, when the β-energy is increasing the α-energy is decreasing, which results in a smaller exchange of the zonal energy. The activity of the system is therefore not uniquely defined simply by the exchange between the zonal flow and the wave, since the wave may undergo significant modifications without involving the zonal energy. One obvious exception to the above conclusion is the behavior of the D-J profile for $\ell=1$, wherein both the α- and β-wave energies change with the same sign. The D-J profile seems, furthermore, to be almost completely inactive for the case $\ell=3$, thereafter following the behavior of the A-J profile for shorter waves.

Although the exchanges are not large for reasonable values of ρ, a somewhat surprising feature is the influence of the total energy on the exchange process. Whereas for wave number one the activity of the system increases for increased energy amplitude--as might be anticipated--this tendency is reversed as the waves get shorter. Thus for wave number six, as the energy amplitude in the system increases, the exchange process is inhibited.

The variety of energy exchange which may occur for a single wave (here chosen as $\ell=3$) due to modifications in wave truncation and profile parameters is exemplified by Fig. 8. Many possible exchange properties may be noted. In one case ($n_\alpha=6$, $n_\beta=8$) the exchange is small and not strongly dependent on ρ; in the others there is a strong dependence on ρ with a tendency for the

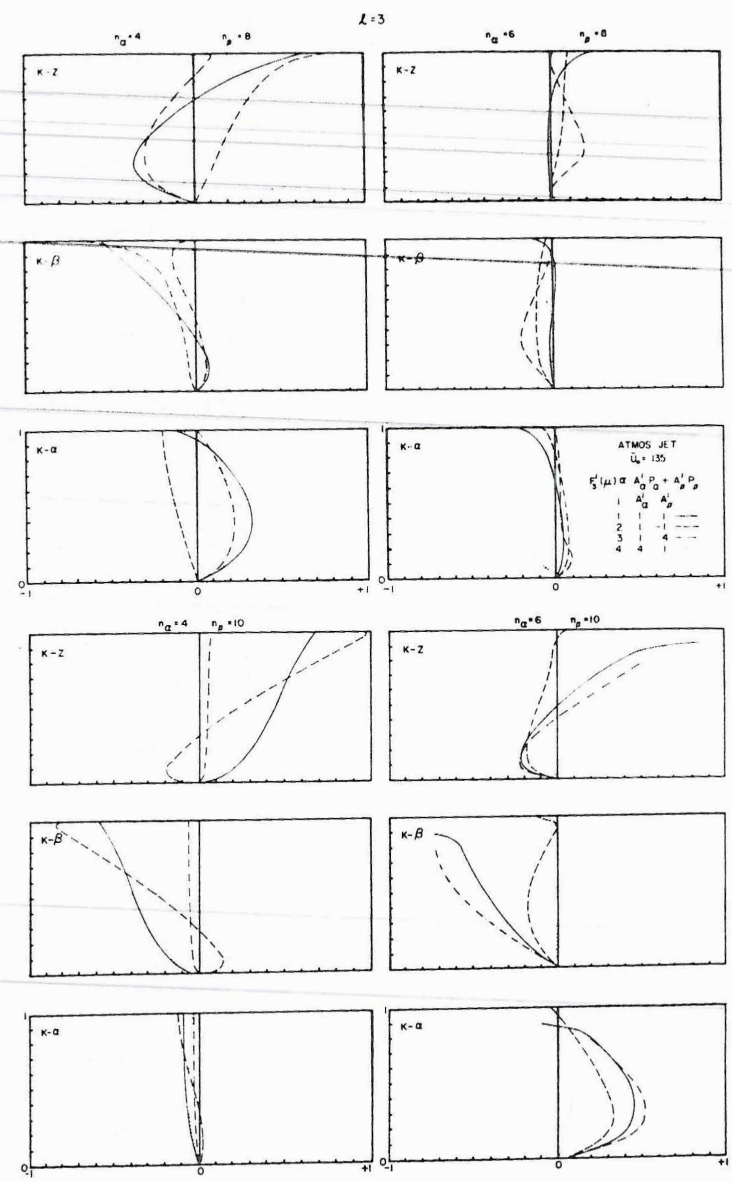


Fig. 8 Maximum energy exchange for wave ℓ=3 with different wave configurations and profile parameters, including normalized energy in the zonal flow and both wave components plotted against ρ (ordinate).

exchange of zonal energy during the period to change direction as ρ increases. In one case (n_α=4, n_β=8), where the zonal energy does not have any range (ρ≈.7), the wave components are reasonably active; in case n_α=4, n_β=10, however, at ρ≈.3 the entire system appears to be inert. There does seem to be an increase of activity with the addition of the component n_β=10; however, in one case the α-wave energy is almost completely inactive whereas in the other it shows significant activity. The inevitable conclusion from this figure is clearly the great variety of exchange possibilities for the low-order system depending on truncation.

The previous discussion has dealt with properties of the flow independent of their detailed time characteristics. Since the exact time variations are known, we now present the time fluctuations of the normalized energy components during an entire period. Fig. 9 shows the solutions in time for the initial configuration given by the A-J and the wave profile taken from (4.16) for selected values of ρ. Since each value of ρ constitutes a separate problem, the solutions are shown graphically with ρ increasing to the right. For each ρ value, two charts are shown; the upper one describes the behavior of the zonal energy and its individual components, and the lower one describes the total wave energy and the α- and β-wave components. The upper set of charts applies for wave ℓ=10 and energy amplitude $\bar{u}_0$=.135 whereas the lower set applies for ℓ=3 and $\bar{u}_0$=.275. Since both initial states are stable by linear analysis (see Section 6) the lack of activity with time for small values of ρ is to be anticipated. We see for the case ℓ=10 that the individual wave components are quite active, but their activity tends to cancel such that the zonal energy does not have large time fluctuations. This tendency for the short waves has already been indicated from Fig. 7. On the contrary, for the ℓ=3 case there is little tendency for cancellation between the wave components and the consequent zonal energy variations are large. Since the β-wave energy is reasonably uniform throughout the period for all ρ values, the dominant exchange is carried on by the α-wave. We also note the dominant influence of the γ=3 zonal component in the total zonal wave energy.

The changes in time variation which arise due to the choice of different profile parameters and/or different wave truncation are made evident from Fig. 10. We have selected to describe three different truncations, with three different sets of profile parameters for each truncation. In all cases, the initial conditions involve interaction of wave ℓ=3 with the A-J, for an energy amplitude of $\bar{u}_0$=.135 and for initial wave energy given by ρ=.6. Several interesting features appear on this figure. Because of the interaction of the wave components, we note that a double period in the total zonal energy

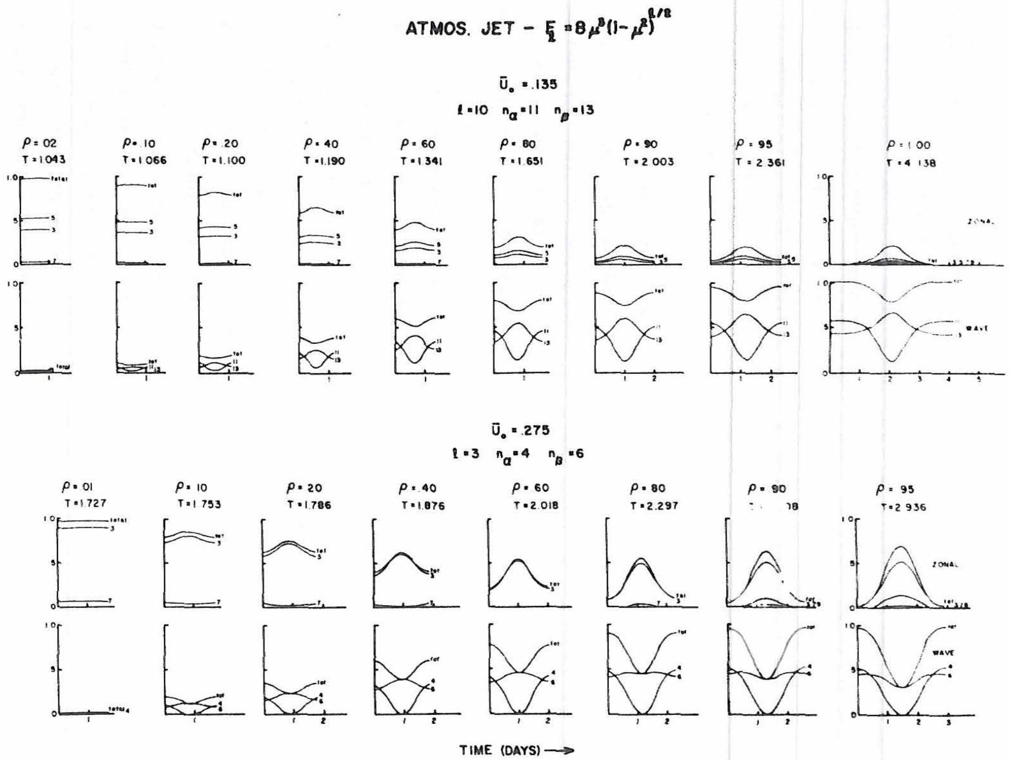


Fig. 9 Time variations of the different energy components during a complete nonlinear period for two waves (ℓ = 3,10) shown for increasing values of ρ.

(also total wave energy) evolves. At least one of the cases is almost entirely inactive, a remarkable result for a specification with more than half of the energy in the wave initially. Finally, the elliptic nature of the solution is clearly evident in the case $n_\alpha=8$, $n_\beta=10$ and $A_\alpha=4$, $A_\beta=1$.

In summary, the most evident observation from these calculations is the variability of the solutions. Depending on the initial specification, the solutions may range from a completely inactive system to one with strongly elliptic variations. They may show highly variable wave components which may or may not interact to cancel the variations in the total wave energy. The response of the system does not necessarily depend on its total energy amplitude. In order to make a satisfactory comparison of the low-order system to the atmosphere, therefore, it is essential to know very precisely the configuration of the atmosphere. Should the atmosphere not be in a configuration which is representable by the low-order truncation, any comparison between reality and the model may be seriously questioned. The model and the representative calculations herein described nevertheless give an indication of the variety of barotropic motions which might be experienced in the atmosphere.

9. Conclusion

The detailed nonlinear exchange processes (energy and others) are difficult to isolate and hence to comprehend in any general model of the atmosphere. By severely truncating atmospheric models, it is possible to bring these exchange processes into focus. We have chosen to truncate the spectral form of the barotropic vorticity equation and the two level potential vorticity equation with constant stability to allow exchange only between an arbitrary zonal flow and one planetary wave. The resulting equations have been termed "low-order spectral equations", although other truncations lead to equations which may be included in this terminology.

The solutions to the low-order equations considered herein, which are identical in form for both the barotropic and baroclinic cases, are analytic in time, and have been detailed in the text. Their dependence on intial conditions has been shown and requires a specification of the total energy in the system (clearly not a condition for linear problems), the initial zonal profile, the relative initial energy in the wave and the wave profile. Since the solutions are periodic, the energy exchanges amongst the components (both kinetic and potential) may be established at any time during the cycle. Furthermore, because of the periodicity, the maximum exchanges for any component may be ascertained. An interesting feature of this system is the fact

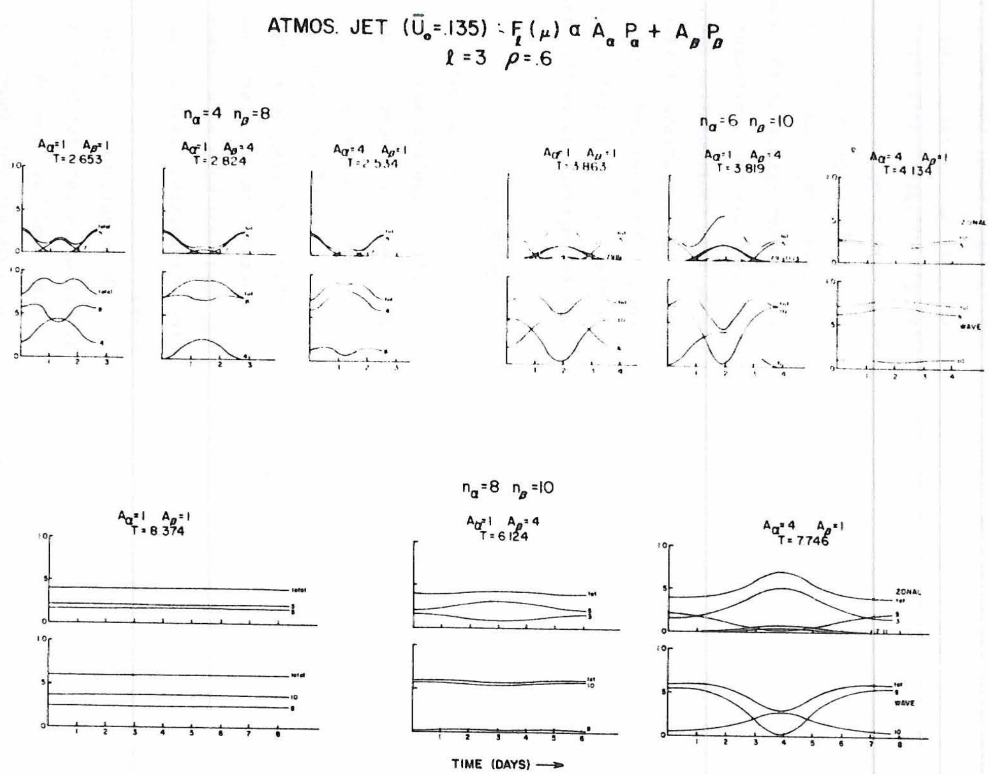


Fig. 10 Time variations of the different energy components during a complete nonlinear period for ℓ=3, ρ=.6 shown for various wave profiles and profile parameters.

that the energy exchange in the baroclinic case follows a law similar to one derived by Fjortoft (1953) for barotropic flow.

Although the equations which are solved are nonlinear, if the linearized form of the equations shows stability (either barotropic or baroclinic), the linear solutions will give a good indication of the behavior of the system when the initial energy in the wave is small. Indeed, for small initial wave energy, the linear solution shows that the maximum exchange over one period is linearly proportional to the wave energy, a result which is substantiated by the nonlinear solution.

A number of properties of the system other than energy exchange may be determined, which include the phase angle variations with time, the wave speed, and momentum transport across latitude circles. However, all these calculations require the specification of initial conditions for solution. To determine how the system responds to variable initial conditions, therefore, we calculated for the barotropic model with two zonal jets and a variety of wave profiles and wave numbers. The results of these calculations indicate, although they cannot be directly compared to atmospheric motions, that a wide variety of nonlinear processes may be expected. The reader is referred to Section 8 for details of these calculations.

We may now ask what utility the low-order system serves, since the results of computation do not appear to compare favorably with observations of atmospheric flow. To begin, atmospheric flow involves a composition of waves, all interacting, and therefore shows a picture of a more complex phenomenon. The individual exchanges, as described by the low-order system, must exist, however. Therefore, we should attempt to expand the low-order system to more than one wave, and interpret the consequences of such a modification in terms of the low-order behavior of each wave independently. The low-order system studied in this report is capable of giving information on the interaction of more than one wave, but only as that interaction affects the zonal flow. Thus we may build up more complex systems which will approach observed motions.

The problem of stability (both barotropic and baroclinic) which has received wide attention by renown scientists is also amenable to study by the low-order systems. Whereas previous analyses required linear equations --thereby limiting the conclusions to the determination of the stability criterion and possibly growth rates--the nonlinear systems maintain their conservation conditions and therefore have no such limitations. Cursory calculations with the low-order equations through the linear stability point indicate that no sharp transition in processes takes place, contrary to the indications of linear theory. Analysis of this problem is being undertaken currently.

One might expect that linear effects added to the low-order system should not alter drastically the nonlinear character of the system and consequently analytic solutions should ensue from their inclusion. Based on this assumption, one could then investigate the influence of orography, friction and heating on a low-order system. Although the conservation conditions would be lost for the latter two effects, it may be possible to isolate the waves and profiles which are most profoundly influenced by these most important of atmospheric forces.

Finally, the availability of analytic solutions to nonlinear systems should prove valuable as an aid in studying computational stability and truncation. Here again we have the opportunity of following the exact solution in comparison with those determined from different truncation schemes, rather than relying only on the linear forms of the basic equations. A comprehensive study of this problem is currently being completed.

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Appendix A

Eq. (3.8) may be solved in terms of elliptic integrals since it involves a general quartic equation on the right-hand side. The solution to such integrals is well known (see, for example, Bowman, 1961) and we shall therefore give only a brief review here, outlining the method for reduction to standard form. For convenience, we rewrite (3.8);

$$\int dt = \int \left(\sum_i^4 b_i B_n^i\right)^{-\frac{1}{2}} dB_n \qquad (3.8)$$

If we denote the roots of the equation

$$\sum_i b_i B_n^i / b_4 = 0$$

as α, β, γ, δ respectively, where we shall take the conjugate pairs (if they exist) as (α,β), (γ,δ), one may define the following quantities:

$$b \equiv -\frac{\alpha+\beta}{\nu} \qquad b' \equiv -\frac{\gamma+\delta}{\nu}$$

$$c \equiv \alpha\beta \qquad c' \equiv \gamma\delta$$

Now choosing $\lambda > \mu$ as the two roots of the equation

$$(b-b')\theta^2 - (c'-c)\theta + cb'-bc' = 0,$$

we may, by first introducing the transformation

$$B_n = \frac{\lambda x+\mu}{1+x} \qquad (A.1)$$

write the quartic as,

$$\sum_i^4 b_i B^i = (1+x)^{-4}(px^2+q)(p'x^2+q') \qquad (A.2)$$

where

$$p \equiv b_4(\lambda^2+2\lambda b+c) \qquad p' \equiv \lambda^2+2\lambda b'+c'$$

$$q \equiv b_4(\mu^2+2\mu b+c) \qquad q' \equiv \mu^2+2\mu b'+c'$$

Substituting (A.2) into (3.8), we have

$$\int dt = (\lambda-\mu) \int \left((px^2+ q)(p'x^2+ q')\right)^{-\frac{1}{2}}dx \tag{A.3}$$

(A.3) may be further simplified by defining the quantities

$$a^2 \equiv \min \left(\frac{|q|}{|p|}, \frac{|q'|}{|p'|} \right)$$

$$b^2 \equiv \max \left(\frac{|q|}{|p|}, \frac{|q'|}{|p'|} \right) \tag{A.4}$$

and the indices

$$j_p \equiv \tfrac{1}{2}(1- \frac{p}{|p|}), \text{ etc.}$$

such that

$$\left.\begin{array}{l} s = j_q \\ r = j_p \end{array}\right\} \text{ if } b^2= \frac{|q|}{|p|} \qquad \left.\begin{array}{l} s = j_{q'} \\ r = j_{p'} \end{array}\right\} \text{ if } a^2= \frac{|q|}{|p|} \tag{A.5}$$

and the primed quantities take on the remaining values. Substituting (A.4) and (A.5) into (A.3) we have, finally,

$$\int dt = \frac{\lambda-\mu}{\sqrt{|pp'|}} \int \left\{ \left((-)^r x^2+(-)^s b^2\right) \left((-)^{r'} x^2+(-)^{s'} a^2\right) \right\}^{-\frac{1}{2}}dx \tag{A.6}$$

The final reduction of (A.6) to normal form requires a knowledge of the distribution of the indices which we shall denote as group m,

$$m \equiv (r,s,r',s') \tag{A.7}$$

In terms of m, we may express x(t) as

$$x^2 = \frac{\eta_1+\eta_2 y_m^2 \ (\omega t+\alpha)}{\eta_3+\eta_4 y_m^2 \ (\omega t+\alpha)} \tag{A.8}$$

$$(\eta_1,\ \eta_2,\ \eta_3,\ \eta_4) = f(m)$$

Table A1. Value of the function B_n for different allowed values of m, and related quantities.

m=(r,s,r´,s´)	x_0	α	(n_1,n_2,n_3,n_4)	Λ_m	k_m^2	$B_n(t)$	B_n(extremes); $\theta_0=0$
(0,0,0,1) (1,1,1,0)	+a -a	0 2K	$(a^2,0,1,-1)$	$(a^2+b^2)^{\frac{1}{2}}$	$b^2/(a^2+b^2)$	$\frac{\lambda x_0+\mu cn\omega t}{x_0+cn\omega t}$	B_{n0}; $B_{n0}-\frac{2(B_{n0}-\lambda)(B_{n0}-\mu)}{2B_{n0}-\lambda-\mu}$
(0,1,0,0) (1,0,1,1)	+b -b	0 2K	$(b^2,0,1,-1)$	$(a^2+b^2)^{\frac{1}{2}}$	$a^2/(a^2+b^2)$	Same	Same
(0,0,1,0) (1,1,0,0)	+a -a	0 2K	$(a^2,-a^2,1,0)$	$-(a^2+b^2)^{\frac{1}{2}}$	$a^2/(a^2+b^2)$	$\frac{\mu+\lambda x_0 cn\omega t}{1+x_0 cn\omega t}$	Same
(1,0,0,0) (0,1,1,1)	+b -b	0 2K	$(b^2,-b^2,1,0)$	$-(a^2+b^2)^{\frac{1}{2}}$	$b^2/(a^2+b^2)$	Same	Same
(1,0,1,0) (0,1,0,1)	+a -a	K -K	$(0,a^2,1,0)$	b	a^2/b^2	$\frac{\mu dn\omega t+\lambda x_0 cn\omega t}{dn\omega t+x_0 cn\omega t}$	Same
	+b -b	K -K	$(b^2,0,0,1)$	-b	a^2/b^2	$\frac{\lambda x_0 dn\omega t+\mu cn\omega t}{x_0 dn\omega t+cn\omega t}$	Same
(1,0,0,1) (0,1,1,0)	+a -a	+K -K	$(b^2,a^2-b^2,1,0)$	-b	$(b^2-a^2)/b^2$	$\frac{\lambda x_0+\mu dn\omega t}{x_0+dn\omega t}$	B_{n0}; $B_{n0}+\frac{(b-a)(B_{n0}-\mu)(B_{n0}-\lambda)}{a(B_{n0}-\lambda)-b(B_{n0}-\mu)}$
	±b	0	Same	Same	Same	$\frac{\mu+\lambda x_0 dn\omega t}{1+x_0 dn\omega t}$	B_{n0}; $B_{n0}+\frac{(a-b)(B_{n0}-\lambda)(B_{n0}-\mu)}{b(B_{n0}-\lambda)-a(B_{n0}-\mu)}$

$$y_m(\omega t) = sn(\omega_m t)$$

$$\omega_m \equiv \frac{A_m \sqrt{|P||P'|}}{\lambda - \mu} \qquad (A.8)$$

The variables above, together with their dependence on the allowed values of the set m are listed in Table A1 together with the dependence of B_n on $\omega_m t$. Here α_m denotes the phase angle which is established from initial values described by a zero subscript. For completeness, we note that

$$\omega_m t = \int_0^{y_m} \left((1-z^2)(1-k_m^2 z^2)\right)^{-\frac{1}{2}} dz$$

where the quarter period (K) is given for $y_m=1$. Thus the period of the elliptic function in days is given as,

$$T_m = \frac{2K}{\pi\omega_m} \quad . \qquad (A.9)$$

Appendix B

To complete the solution of system (2.13), it is necessary to establish the dependence of one of the phase angles, say θ_β, on time. By use of (3.12), the other angle (θ_α) will be determined. We shall therefore concentrate in this discussion on θ_β, although by merely reversing notation (letting $\alpha \to \beta$ and $\beta \to \alpha$) the equations will be applicable to θ_α.

The equation of interest is (3.13) which we shall repeat here for clarity:

$$\dot{\theta}_\beta = -\nu_\beta + h_{\beta\beta} + g_{\beta\beta} B_n + \frac{(g_{\beta\alpha} B_n + h_{\beta\alpha}) G}{a_n B_\beta^2} \tag{3.13}$$

where all the terms have been defined in Section 3. If we now make the substitution $B_n = B_n(x)$ from (A.1) and arrange terms, we have the equation for $\dot{\theta}_\beta$;

$$\dot{\theta}_\beta = -\nu_\beta + h_{\beta\beta} + g_{\beta\beta} \frac{\lambda(x+\mu/\lambda)}{1+x} - s_1 \frac{x+m_3}{1+x} \frac{(x^2+m_1 x+m_0)}{(x^2+2n_1 x+n_0)} \tag{B.1}$$

Eq. (B.1) requires the following definitions:

$$d_1 \equiv \frac{h_{\beta\alpha}}{g_{\beta\alpha}} \; ; \quad d_2 \equiv \frac{a_n D_\beta}{g_{\beta\alpha}} \; ; \quad d_3 \equiv -\tfrac{1}{2}\bar{g}$$

$$n_0 \equiv \frac{\mu^2+2d_1\mu-d_2}{\lambda^2+2d_1\lambda-d_2} \; ; \qquad n_1 \equiv \frac{\lambda\mu+d_1(\mu+\lambda)-d_2}{\lambda^2+2d_1\lambda-d_2}$$

$$m_0 \equiv \frac{d_3\mu^2+\mu\bar{\nu}+K}{d_3\lambda^2+\bar{\nu}\lambda+K} \; ; \qquad m_1 \equiv \frac{2d_3\lambda\mu+\bar{\nu}(\mu+\lambda)+2K}{d_3\lambda^2+\bar{\nu}\lambda+K}$$

$$m_3 \equiv \frac{\mu+d_1}{\lambda+d_1} \; ; \qquad s_1 \equiv \frac{(\lambda+d_1)(d_3\lambda^2+\bar{\nu}\lambda+K)}{\lambda^2+2d_1\lambda-d_2}$$

In these definitions, λ and μ have been determined in Appendix A and the other constants have been established in Section 3. The quantity K used here comes from (3.7) and should not be confused with the complete elliptic integral of the first kind.

The last term of (B.1) may be reduced to simple fractions if we first note that the roots of the quadratic in the denominator are given as

$$x^2+2n_1x+n_0 = (x-n_2)(x-n_3)$$

$$n_{2,3} = -n_1 \pm (n_1^2 - n_0)^{\frac{1}{2}} \tag{B.2}$$

We shall assume the roots real; if they are not, the simplest procedure is to calculate θ_α. The poles of (B.1) exist at the points $(-1,n_2,n_3)$ and we may therefore compute the residues about those points, which are given by the relations

$$\Phi_1(-1) = \frac{(m_3-1)(1-m_1+m_0)}{(n_2+1)(n_3+1)}$$

$$\Phi_2(n_2) = \frac{(m_3+n_2)(n_2^2+m_1n_2+m_0)}{(n_2+1)(n_2-n_3)}$$

$$\Phi_3(n_3) = \frac{(m_3+n_3)(n_3^2+m_1n_3+m_0)}{(n_3+1)(n_3-n_2)}$$

Introducing these residues into (B.1) and combining terms, we now find that the equation may be written in the form,

$$\dot{\theta}_\beta = A_1 + \frac{c_1}{x+1} - \frac{c_2}{x-n_2} - \frac{c_3}{x-n_3} \tag{B.3}$$

when

$$A_1 \equiv -\nu_\beta + h_{\beta\beta} + g_{\beta\beta}\lambda - s_1$$

$$c_1 \equiv g_{\beta\beta}(\mu-\lambda) - s_1\Phi_1$$

$$c_2 \equiv s_1\Phi_2$$

$$c_3 \equiv s_1\Phi_3$$

If we now complete the squares of the terms in (B.3) and integrate, we arrive at Eq. (3.14),

$$\theta_\beta(t) = \theta_{\beta 0} + A_1t + \sum_{j=1}^{3} \left(c_{Fj}F(\zeta_j,\omega t) + c_{Hj}H(\zeta_j,\omega t)\right) \tag{B.4}$$

Table B1. Values of c_{Fj}, c_{Hj} and ζ_j as used in Eq. (B.4).

j	ζ_j	c_{Fj}	c_{Hj}
1	1	c_1	$-c_1$
2	n_2^2	$-c_2$	$-n_2c_2$
3	n_3^2	$-c_3$	$-n_3c_3$

The values of c_{Fj}, c_{Hj} and ζ_j are listed in Table B1. The integrals F and H, taken from t=0 to some arbitrary time τ may be expressed as

$$F(\zeta_j,\omega\tau) = \int_0^\tau \frac{x\,dt}{x^2-\zeta_j} = \int_0^{\frac{iK+\omega t}{\omega}} \frac{x\,dt}{x^2-\zeta_j} + NF(\zeta_j,4K)$$

$$H(\zeta_j,\omega\tau) = \int_0^\tau \frac{dt}{x^2-\zeta_j} = \int_0^{\frac{iK+\omega t}{\omega}} \frac{dt}{x^2-\zeta_j} + NH(\zeta_j,4K) \tag{B.5}$$

where

$$\omega\tau = (4N+i)K + \omega t$$

$$0 \le i \le 3 \tag{B.6}$$

$$0 \le \omega t \le K$$

The integer N specifies the number of complete nonlinear periods (T) included in τ and the index i represents the quadrant of the last period. Therefore, t as defined in the limits of (B.5) will range only over a quarter period. With this representation of F and H, we now see why θ_β need not have periodicity of period T; for by substituting the right-hand side of (B.5) into (B.4), we have

$$\theta_\beta(\tau) = \bar{A} + \frac{iK+\omega t}{\omega} A_1 + \sum_{j=1}^{3} \left(c_{Fj}F(\zeta_j, \frac{\omega t+iK}{\omega}) + c_{Hj}H(\zeta_j, \frac{\omega t+iK}{\omega})\right)$$

$$\bar{A} \equiv \theta_{\beta 0} + \frac{4KA_1N}{\omega} + N\sum_{j=1}^{3} \left(c_{Fj}F(\zeta_j,4K) + c_{Hj}H(\zeta_j,4K)\right) \tag{B.7}$$

For each full nonlinear period T, $\bar{A}$ is incremented as N increases by an integer.

The remainder of this Appendix will be devoted to the evaluation of the integrals F and H.

Functional form of $F(\omega t)$:

In differential form, we note from (B.5) that F may be written,

$$dF(\zeta_j, \omega t) = \frac{xdt}{x^2 - \zeta_j} \tag{B.8}$$

where x has been defined in (A.1) and ζ_j in Table B1. The variation of t with respect to x is given by (A.3) or (A.6) and depends on x^2; therefore, on substitution of (A.6) into (B.8) we see immediately that F is soluble in terms of elementary integrals. To elucidate this possibility, let us define the variable Z such that

$$Z^2 \equiv \frac{(-)^{r'} x^2 + (-)^{s'} a^2}{(-)^r x^2 + (-)^s b^2} \tag{B.9}$$

We shall attempt to make $Z^2 < 1$ and have therefore chosen form (B.9) since $b^2 > a^2$ by definition (A.4). However, depending on the range of x, it may be necessary to invert (B.9). Substituting for dt from (A.6) and for x^2 from (B.9) into (B.8), we have for dF,

$$dF = -\frac{A_m}{\omega_m} \frac{dZ}{\bar{a}Z^2 + \bar{c}}$$

$$\bar{a} \equiv -\left((-)^s b^2 + (-)^r \zeta_j\right) \tag{B.10}$$

$$\bar{c} \equiv (-)^{s'} a^2 + (-)^{r'} \zeta_j$$

If we now recall the dependence of x on $y_m = \mathrm{sn}\,\omega_m t$ from (A.8), we may establish the dependence of Z on $\omega_m t$. This dependence has been listed in Table B2 as a function of the index set, m. The values of $\bar{a}$ and $\bar{c}$, which also depend on m, are listed in the same Table, together with some limits on their values which can be determined from known constants. It should be noted that the quantity k'_m used in Table B2 is the complementary modulus defined as $k'^2_m = 1 - k^2_m$, where the values of the modulus k^2_m are listed in Table A1.

Proceeding with the integration of (B.10) we arrive at the two possible solutions;

Table B2. Values of $\bar{a}$, $\bar{c}$, and $Z(\omega t)$ relevant to $F(\omega t)$ and values of F in the different quadrants. The subscript m has been suppressed on ω_m, although applicable.

$m=(r,s,r^1,s^1)$	$\bar{a}$	$\bar{c}$	Limits		$Z(\omega t)$	$F(\iota K+\omega t)$				$F(4K)$
						$\iota=0$	$\iota=1$	$\iota=2$	$\iota=3$	
(0,0,0,1) (1,0,0,0) }	$-(b^2+\zeta_j)$	(ζ_j-a^2)	$\zeta_j<a^2$	$\frac{\bar{a}}{\bar{c}}>1$	$k'sn\omega t/dn\omega t$	$F(\omega t)$	$F(K-\omega t)$	$-F(\omega t)$	$-F(K-\omega t)$	0
(0,1,0,0) (1,0,1,1) }	$-(\zeta_j+a^2)$	$-(b^2-\zeta_j)$	$\zeta_j<b^2$	$\bar{a}\bar{c}>0$	"	"	"	"	"	0
(0,0,1,0) (1,1,0,1) }	$-(b^2+\zeta_j)$	$a^2-\zeta_j$	$\zeta_j>a^2$	$\frac{\bar{a}}{\bar{c}}>1$	$ksn\omega t/dn\omega t$	"	"	"	"	0
(1,0,0,0) (0,1,1,1) }	$-(a^2+\zeta_j)$	$-(\zeta_j-b^2)$	$\zeta_j>b^2$	$\frac{\bar{a}}{\bar{c}}>1$	"	"	"	"	"	0
(1,0,1,0) } (0,1,0,1)	ζ_j-b^2 $a^2-\zeta_j$	$a^2-\zeta_j$ ζ_j-b^2	$\zeta_j>a^2$ $\zeta_j<b^2$	$\bar{c}<0$ $\bar{c}<0$	$kcn\omega t/dn\omega t$ $cn\omega t/dn\omega t$	$F(\omega t)$ ( ) $-F(0)$	$-F(K-\omega t)$ ( ) $-F(0)$	$-F(\omega t)$ ( ) $-F(0)$	$F(K-\omega t)$ ( ) $-F(0)$	0
(1,0,0,1) (0,1,1,0) }	ζ_j-b^2	ζ_j-a^2	$\zeta_j<a^2$ or $\zeta_j>b^2$	$\bar{a}\bar{c}>0$	$cn\omega t/sn\omega t$	$F(\omega t)$ ( ) $-F(0)$	$-F(K-\omega t)$ ( ) $-F(0)$	$F(\omega t)$ ( ) $-3F(0)$	$-F(K-\omega t)$ ( ) $-3F(0)$	$-4F(0)$

$$F = -\frac{A_m}{\omega_m \sqrt{\bar{a}\bar{c}}} \tan^{-1} \frac{\bar{a}Z}{\sqrt{\bar{a}\bar{c}}} \qquad \text{if } \bar{a}\bar{c}>0$$

$$= \frac{-A_m}{2\omega_m \sqrt{-\bar{a}\bar{c}}} \cdot \ln \left| \frac{\bar{a}Z-\sqrt{-\bar{a}\bar{c}}}{\bar{a}Z+\sqrt{-\bar{a}\bar{c}}} \right| \qquad \text{if } \bar{a}\bar{c}<0 \tag{B.11}$$

where the dependence on ωt is apparent from Table B2. In general, the behavior of F during a nonlinear period given by $4K=\omega T$ is similar to trigonometric functions over a complete cycle. However, this is not the case for all values of m; therefore we list in Table B2 the values of F during one cycle in which, as in (B.6), $0\leq\omega t\leq K$ and $0\leq i\leq 3$ for $F=F(iK+\omega t)$. In the event that F is not purely periodic, we list also the value for F(4K).

Functional form of $H(\omega t)$:

The differential form of H, from (B.5) may be written

$$dH(\zeta_j,\omega t) = \frac{dt}{x^2-\zeta_j} \tag{B.12}$$

If we substitute for x in terms of y given by (A.8), (B.12) becomes,

$$dH = A_H dt + B_H(1-\xi_j y^2)^{-1}dt$$

$$A_H \equiv \frac{\eta_4}{\eta_2-\zeta_j\eta_4}$$

$$B_H \equiv \frac{\eta_3\eta_2-\eta_1\eta_4}{\eta_2-\zeta_j\eta_4} \tag{B.13}$$

$$\xi_j \equiv \frac{\zeta_j\eta_4-\eta_2}{\eta_1-\zeta_j\eta_3}$$

The variables A_H, B_H and ξ_j are listed in Table B3.for the different index group m. In the subsequent discussion, we shall suppress the j index, although it will be implied, as we have already suppressed the m subscript. Let us now define two angles θ and α (not related to the variables used in the main text) as follows;

Table B3. Values of A_Π, B_Π, ξ_j and Π relevant to $\Pi(\omega t)$.

$m = (r,s,r',s')$	A_Π	B_Π	ξ_j	Π	Conditions
(0,0,0,1) (1,1,1,0) }	$-1/\zeta_j$	$-\dfrac{a^2}{\bar{c}\zeta_j}$	$\zeta_j/\bar{c}$	Π_1	
(0,1,0,0) (1,0,1,1) }	$-1/\zeta_j$	$-b^2/\bar{c}\zeta_j$	$\zeta_j/\bar{c}$	Π_1	
(0,0,1,0) (1,1,0,1) }	0	$1/\bar{c}$	$a^2/\bar{c}$	Π_1	
(1,0,0,0) (0,1,1,1) }	0	$1/\bar{c}$	$b^2/\bar{c}$	Π_1	
(1,0,1,0) (0,1,0,1) }	0	$-1/\zeta_j$	a^2/ζ_j	Π_2	$\zeta_j<b^2$; $k^2<\xi_j<1$
	"	"	"	Π_3	$\zeta_j>b^2$; $0<\xi_j<k^2$
	$-1/\zeta_j$	$1/\zeta_j$	ζ_j/b^2	Π_2	$\zeta_j>a^2$; $k^2<\xi_j<1$
	"	"	"	Π_3	$\zeta_j<a^2$; $0<\xi_j<k^2$
(1,0,0,1) (0,1,1,0) }	0	$-1/\bar{a}$	$-kb^2/\bar{a}$	Π_1	$\zeta_j>b^2$
	"	"	"	Π_2	$\zeta_j<a^2$

$$\sin\theta \equiv y = \mathrm{sn}\,\omega t$$

$$\sin^2\alpha \equiv k^2 \tag{B.14}$$

$$\Delta\theta \equiv \sqrt{1-k^2\sin^2\theta} = \sqrt{1-k^2\mathrm{sn}^2\omega t} = \mathrm{dn}\,\omega t$$

The relationship between θ and ωt is evident from (A.8); α is simply computed from the modulus k listed in Table A1. Integrating (B.13) after transforming dt to dθ from (B.14) we have,

$$\omega\int_0^{\omega t} \frac{dt}{1-\xi y^2} = \int_0^{\phi} \frac{d\theta}{(1-\xi\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} = \int_0^{\phi} \frac{d\theta}{\Delta\theta(1-\xi\sin^2\theta)}$$

$$\equiv \Pi(\xi;\phi\backslash\alpha) \tag{B.15}$$

The function Π which represents the required integral in the evaluation of H is called the elliptic integral of the third kind, and has been evaluated over the range $0\leq\phi\leq\pi/2$ (Abramowitz and Stegun, 1964). Since ωt ranges over a quarter period (K) in this range of ϕ (B.14) we may define the integral

$$\Pi(\xi\backslash\alpha) \equiv \Pi(\xi;\pi/2\backslash\alpha)$$

The value of the integral may be determined in any quadrant from its value in the first quadrant (where it has been defined) as follows:

$$\Pi(\xi;\phi\backslash\alpha) = 2\Pi(\xi\backslash\alpha) - \Pi(\xi;\pi-\phi\backslash\alpha) \left\{\begin{matrix}\omega\tau=\omega t+K\\ \pi/2\leq\phi\leq\pi\end{matrix}\right.$$

$$= 2\Pi(\xi\backslash\alpha) + \Pi(\xi;\phi-\pi\backslash\alpha) \left\{\begin{matrix}\omega\tau=\omega t+2K\\ \pi\leq\phi\leq 3\pi/2\end{matrix}\right.$$

$$= 4\Pi(\xi\backslash\alpha) - \Pi(\xi;2\pi-\phi\backslash\alpha) \left\{\begin{matrix}\omega\tau=\omega t+3K\\ 3\pi/2\leq\phi\leq 2\pi\end{matrix}\right.$$

For the complete period, T, in which case $\phi=2\pi$,

$$\Pi(\xi;2\pi\backslash\alpha) = 4\Pi(\xi\backslash\alpha)$$

On the assumption that Π as given in (B.15) may be evaluated--utilizing the above information to extend the integral to arbitrary time--the function H may be expressed in the form,

$$H(\zeta_j,\omega\tau) = \frac{1}{\omega}\{A_H\omega\tau + B_H\Pi(\xi_j,\omega\tau\backslash\alpha)\} \tag{B.16}$$

when, from (B.14), $\sin\phi = \mathrm{sn}\omega\tau$. Unfortunately the computation of Π varies, depending on the sign and magnitude of ξ_j, and three different functions must be calculated: Π_1, Π_2 and Π_3. The function appropriate to any m value is listed in Table B3.

For completeness, we conclude this Appendix with the formal representation of the three functions ($\Pi_{1,2,3}$) together with the necessary definitions.

$$\Pi_1(\xi;\omega t\backslash\alpha) = -2q_1\delta_2\sum_{s=1}^{\infty}(-)^s\frac{q^{2s}}{s(1-q^{2s})}\sin\frac{\pi s}{K}\omega t\ \sinh 2s\beta$$

$$+q_1\delta_2\tan^{-1}(\tanh\beta\tan\frac{\pi}{2K}\omega t)$$

$$+q_3\tan^{-1}(p_2\frac{\mathrm{sn}\omega t\,\mathrm{cn}\omega t}{\mathrm{dn}\omega t})$$

$$+(q_2 - \frac{q_1\mu\delta_2 2\pi}{K})\omega t$$

$$\Pi_2(\xi;\omega t\backslash\alpha) = -2\delta_3\sum_{s=1}^{\infty}(-)^s\frac{q^{2s}}{s(1-q^{2s})}\sin\frac{s\pi}{K}\omega t\ \sinh 2s\beta_1$$

$$+\delta_3\tan^{-1}(\tanh\beta_1\tan\frac{\pi}{2K}\omega t)$$

$$-4\mu\delta_3\frac{\pi}{2K}\omega t$$

$$\Pi_3(\xi;\omega t\backslash\alpha) = -2\delta_3\sum_{s=1}^{\infty}\frac{q^s}{s(1-q^{2s})}\sin\frac{s\pi}{K}\omega t\sin 2s\beta_3$$

$$+\frac{\pi}{2K}W(\beta_3)\omega t$$

$$\varepsilon \equiv \sin^{-1}\sqrt{1/(1-\xi)} \;;\; \varepsilon_1 \equiv \sin^{-1}\sqrt{(1-\xi)/k'^2} \;;\; \varepsilon_3 \equiv \sin^{-1}\sqrt{\xi/k^2}$$

$$\beta \equiv \frac{\pi}{2K} F(\varepsilon,k') \;;\; \beta_1 \equiv \frac{\pi}{2K} F(\varepsilon_1,k') \;;\; \beta_3 \equiv \frac{\pi}{2K} F(\varepsilon_3,k)$$

$$q_1 \equiv \frac{-\xi k'^2}{(1-\xi)(k^2-\xi)} \;;\; q_2 \equiv \frac{k^2}{k^2-\xi} \;;\; q_3 \equiv \frac{\sqrt{q_1}}{k'}$$

$$p_2 \equiv \left(\frac{\xi(\xi-k^2)}{1-\xi}\right)^{\frac{1}{2}} \;;\; \delta_2 \equiv \frac{1}{k'^2 q_3} \;;\; \delta_3 \equiv |q_3^2|^{\frac{1}{2}}$$

$$\mu \equiv \frac{\sum\limits_{s=1}^{\infty} s q^{s^2} \sinh 2s\beta}{1 + 2\sum\limits_{s=1}^{\infty} q^{s^2} \cosh 2s\beta}$$

$$W(\beta_3) \equiv \cot\beta_3 + 4\sum_{s=1}^{\infty} \frac{q^{2s}}{1 - 2q^{2s}\cos 2\beta_3 + q^{4s}} \sin 2\beta_3$$

$$q \equiv \exp(-\pi K'/K)$$

The latter quantity q is defined as the nome. The function $F(\varepsilon,k)$ is the incomplete elliptic integral of the first kind, whereas K is the complete elliptic integral for the modulus k. The complementary complete integral K' is evaluated using k'. Thus we have,

$$K = F(\pi/2,k);\; K' = F(\pi/2,k').$$

All the above relations have been programmed for digital computation.