EAULIQ: THE NEXT GENERATION

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Abstract

This report summarizes the design of a new version of the stratiform cloud parameterization called Eauliq; the new version is called Eauliq NG. The key features of Eauliq NG are:

- a prognostic fractional area covered by stratiform cloudiness, following the approach developed by M. Tiedtke for use in the ECMWF model;
- separate prognostic thermodynamic variables for the clear and cloudy portions of each grid cell;
- separate vertical velocities for the clear and cloudy portions of each grid cell, allowing the model to represent some aspects of observed mesoscale circulations;
- cumulus entrainment from both the clear and cloudy portions of a grid cell, and cumulus detrainment into the cloudy portion only;
- the effects of the cumulus-induced subsidence in the cloudy portion of a grid cell on the cloud water and ice there.

In this paper we present the mathematical framework of Eauliq NG; a discussion of cumulus effects; a new parameterization of lateral mass exchanges between clear and cloudy regions; and a theory to determine the mesoscale mass circulation, based on the hypothesis that the stratiform clouds remain neutrally buoyant through time and that the mesoscale circulations are the mechanism which makes this possible. An appendix also discusses some time-differencing methods.

1. Introduction

What is a stratiform cloud? Virtually all stratiform clouds contain buoyancy-driven turbulence (e.g. Lilly, 1968). On the other hand, many convective clouds extend upward from and/ or grow upward into stratiform cloud layers. For purposes of this paper, *we define stratiform clouds to be clouds which are, in an area-averaged sense, neutrally buoyant with respect to their environments, level-by-level throughout their vertical extents*. In contrast, then, "convective" clouds are those which are either positively or negatively buoyant at some levels, in an area-averaged sense. For example, cumulus or cumulonimbus clouds are positively buoyant through a large fraction of their vertical extents, and they may also contain regions of negatively buoyant convective downdrafts.

Many stratiform clouds are generated by convective clouds. Each day, tens of thousands of cumulonimbus clouds inject enormous quantities of boundary-layer air into the upper troposphere and lower stratosphere (e.g., Riehl and Malkus, 1958). The detrained air forms horizontally extensive and deep "anvil" clouds which contribute as much as 40% of the total precipitation that falls from the convective systems. Observations (e.g., Webster and Stephens, 1980) show that cirrus outflows from tropical convection can extend for many hundreds or even thousands of kilometers downstream from the convective disturbance that generates them. In such cases, the local time rate of change and advection terms of the conservation equation for the large-scale average cirrus ice water concentration must be comparable to the source and sink terms, so that a prognostic approach including advective effects is necessary for accurate predictions of the cloudiness.

The anvil clouds contain mesoscale circulations (e.g., Houze, 1982) and also small-scale moist convective circulations, which influence the evolution of the convective systems and the large-scale circulations in which they develop. Houze (1982) showed that the total heating associated with a mature convective cloud system peaks in the upper troposphere and is quite small in the lower troposphere, whereas the convective heating alone is weaker in the upper troposphere and stronger in the lower troposphere. Similar conclusions have been reached by Johnson and Young (1983), Johnson (1984), and Cheng and Yanai (1989). The heating and drying due to the mesoscale vertical motions are comparable in magnitude to the latent heating and precipitation drying associated with the anvil. The mesoscale heating and drying are not negligible, but neither are they as strong as the convective heating and drying. Rutledge (1986) and Rutledge and Houze (1987) have presented evidence that there are no fundamental differences between the stratiform anvils of midlatitude and tropical convective cloud systems. Houze (1982) has argued that mesoscale vertical motions associated with stratiform anvil clouds make important contributions to the large-scale heat and moisture budgets of convective cloud systems, and he has attempted to determine the typical magnitudes of these effects from observations. On the other hand, Cheng and Yanai (1989) have presented evidence that the effects of mesoscale condensation and evaporation are considerably larger than those of mesoscale vertical drafts. Certainly there is sufficient motivation to develop an accurate physically based parameterization of the effects of mesoscale vertical motions and mesoscale condensation and evaporation on the large-scale heat and moisture budgets.

Anvil and cirrus cloud systems are generally too small to be explicitly represented in large-scale models; this leads us to the problem of "subgrid cloud amount," which can be defined in general terms as *the statistical distribution of cloud water and cloud ice on the subgrid scale.* There are actually three reasons why we are interested in fractional cloudiness. The most obvious is that the small scale distributions of liquid water and ice can have a strong effect on the transfer of radiation. A second reason is that fractional cloudiness matters for cloud microphysics. The importance of this for realistic simulation of large-scale cloudiness has been emphasized by Fowler et al. (1996). As an example, the Colorado State University General Circulation Model (CSU GCM) described by Fowler et al. (1996) predicts the grid-cell averaged mixing ratios of liquid water and ice. Microphysical processes, such as conversion of cloud water to rain water, are *local* processes, and so they must be formulated in terms of the *local* concentrations of cloud water and cloud ice, rather than the predicted large-scale average concentrations. The local concentrations are essentially equal to the large-scale average concentrations divided by the cloud amount. This problem has been discussed by Bechtold et al. (1993). A third reason for interest in fractional cloudiness is that it affects the dynamics of the convective clouds that produce much of the condensation. The effects of fractional cloudiness on cloud dynamics were discussed by Randall (1987).

The earliest cloud amount parameterizations (e.g. Smagorinski, 1960) simply related the fractional cloudiness to the large-scale relative humidity i.e.

$$f = f(\overline{RH}). \tag{1}$$

Xu and Krueger (1991) tested this and other simple cloud amount parameterizations by using a Cloud System Model (CSM)[1], and found that they do not work very well. A similar conclusion was reached by Xu and Randall (1996 a, b).

In a very influential paper, Sundqvist (1978) proposed a stratiform cloud parameterization for large-scale models, in which the large-scale average cloud water mixing ratio was introduced as a prognostic variable, and simple parameterizations of microphysical processes were used to represent the sources and sinks of cloud water. The stratiform cloud parameterization recently incorporated into the Colorado State University General Circulation Model (CSU GCM), as described by Fowler et al. (1996), can be regarded as a recent attempt to follow the trail blazed by Sundqvist, using a somewhat more modern microphysics parameterization. Sundqvist related the cloud amount to the large-scale relative humidity, according to

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[1] We use the term "Cloud System Model" to denote a model with sufficiently high spatial resolution to resolve individual cloud elements, and integrated with a domain size large enough to encompass many individual clouds, and with a time domain long enough to include many cloud life cycles. A second term sometimes used is "Cumulus Ensemble Model," but this seems inappropriate because the models can be applied to stratiform clouds. A third term sometimes used to refer to these models is "Cloud Resolving Models." This name is not very satisfactory because it makes no reference to the large domain size and long integration time characteristic of such models.

$$f = Max\left\{\frac{\overline{RH} - (RH)_0}{1 - (RH)_0}, 0\right\}, \tag{2}$$

where $(RH)_0$ is a "threshold" relative humidity below which the cloud amount is assumed to be zero. Sundqvist further assumed that when stratiform clouds exist, the relative humidity in the cloud-free portion of the grid cell remains constant at $(RH)_0$, while the relative humidity in the saturated portion of the grid cell is equal to the saturation value there. If the area-averaged relative humidity is predicted, then this assumption permits diagnostic determination of the cloud fraction. As explained later, in Eauliq NG we do not assume that the relative humidity in the clear portion of the grid cell is a constant.

A key assumption of Sundqvist (1978), which is not explicitly stated in the paper, is that the potential temperature is horizontally uniform throughout the grid cell. This is roughly consistent with the definition of stratiform clouds given above, i.e. stratiform clouds are neutrally buoyant (in an area-averaged sense) with respect to their environments. The parameterization described in the present paper makes use of this "neutral buoyancy" assumption, but a key point is that *we specify a physical mechanism which maintains this neutral buoyancy over time.*

A different approach was suggested by Sommeria and Deardorff (1977), who proposed a subgrid cloudiness parameterization intended for use in high-resolution, cloud-resolving models. They assumed that a pair of moist conservative variables, such as liquid water potential temperature, θ_l, and total mixing ratio, q_t, undergo subgrid-scale fluctuations and have a joint

Gaussian probability density function (pdf), as sketched in Fig. 1. The shaded region of the sketch

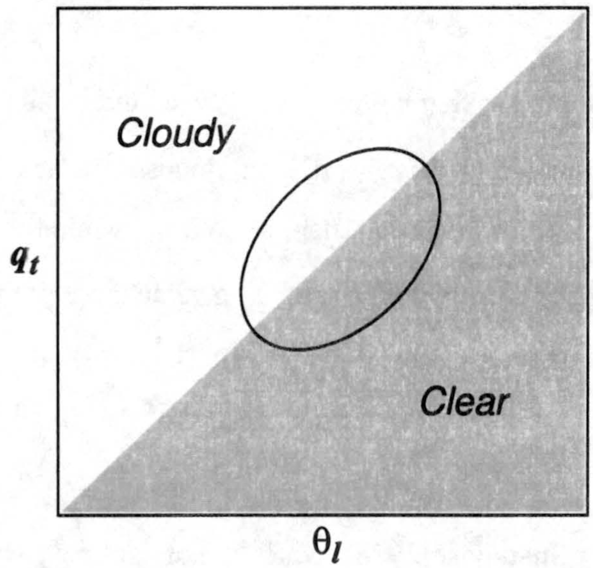

Figure 1: Sketch illustrating the Sommeria-Deardorff approach to cloud amount parameterization. The shaded portion of the figure is unsaturated, and the white portion is saturated. The oval represents an isoline of the joint probability distribution of θ_l and q_t.

represents subsaturated air, and the white region represents saturated air. By integrating the joint pdf over the saturated region, Sommeria and Deardorff were able to determine the cloud amount. In order for this approach to be workable, it is necessary to know the joint pdf of the two moist conservative variables. Le Treut and Li (1988), Smith (1990), and Ricard and Royer (1993) have followed simplified versions of this approach in large-scale models. Le Treut and Li (1988) and Smith (1990) used assumed (rather than predicted) pdfs for the moist conservative variables. In principle, these pdfs should be determined by small-scale and/or mesoscale dynamical processes, such as convective turbulence within the stratiform cloud (Randall 1987; Randall et al. 1992). Ricard and Royer (1993) implemented their parameterization in the context of "level 2" subgrid scale turbulence parameterization (Yamada and Mellor 1979), using 20 layers to represent the vertical structure of the atmosphere. Xu and Randall (1996 a) used a CSM to evaluate the applicability of pdf-based parameterizations to the simulation of large-scale circulations. They found that the coefficients of such parameterizations are cloud-regime-dependent. Nevertheless, pdf-based parameterizations certainly have the potential for further improvement, and in fact the parameterization presented in later this paper can be interpreted as a pdf-based scheme.

Albrecht (1981) considered the gradual evaporation of cloud water produced by detrainment from shallow cumulus clouds. He proposed a parameterization of the form

$$f = \frac{(q_t)_{\text{cld}} - \overline{q_*}}{(q_t)_{\text{cld}} - (q_v)_{\text{clr}}}, \tag{3}$$

where $(q_t)_{\text{cld}}$ is the total *in-cloud* mixing ratio (vapor plus liquid), $\overline{q_*}$ is the large-scale saturation mixing ratio, and $(q_v)_{\text{clr}}$ is the vapor mixing ratio *of the clear air.* Albrecht assumed that the cloudy and clear portions of the grid cell have the same temperature. His parameterization was based on a simple model in which the detrained cloudy air was assumed to "relax" back towards the mean-state mixing ratio. A simple convective cloud model was used to estimate $(q_t)_{\text{cld}}$. The parameterization invoked microphysical processes such as the evaporation of cloud water, but did not explicitly parameterize the microphysics. S. A. Klein (personal communication, 1997) has pointed out that (3) is equivalent to

$$f = \frac{\sqrt{(q_t)_{\text{cld}}}}{\sqrt{(q_t)_{\text{cld}}} + \sqrt{(1 - \overline{RH})\overline{q_*}}}. \tag{4}$$

This means that the cloud amount increases as the in-cloud liquid water mixing ratio increases, and as the large-scale relative humidity increases. Note, however, that (4) can give $f \to 1$ even for $\overline{RH} < 1$. Xu and Randall (1996 b) proposed a semi-empirical cloud parameterization which is somewhat similar in spirit to (4):

$$f = \overline{RH}^p \left\{ 1 - \frac{\exp[-a(q_t)_{\text{cld}}]}{[(1-\overline{RH})\overline{q_*}]^\gamma} \right\}. \tag{5}$$

They used a CSM to evaluate the parameters used in (5), for two different cloud regimes.

Tiedtke (1993) developed a cloud amount parameterization for use in the ECMWF model. He introduced a prognostic equation for the cloud amount in addition to prognostic equations for the mass of cloud water. This prognostic equation was a major advance of Tiedtke's approach; it simply expresses the conservation of mass for the cloudy air, i.e. it is essentially a continuity equation. One of the strengths of Tiedtke's parameterization is that convection acts as a source of stratiform cloud water and cloud amount; this idea was inherent already in the cumulus parameterization of Arakawa and Schubert (1974). Tiedtke also included the effects of the advection of condensed water and cloud amount by the large-scale circulation, and he parameterized the effects of microphysical and turbulent processes on the clouds and the large-scale thermodynamic state. In accord with the definition of stratiform cloudiness given at the beginning of this paper, Tiedtke assumed (tacitly) that the temperature is horizontally uniform across the clear and cloudy sub-regions of each grid cell, but he did not explain how this uniform temperature is maintained.

Fowler et al. (1996) and Fowler and Randall (1996 a, b) developed and tested a bulk cloud microphysics parameterization called "Eauliq," based on the work of Rutledge and Hobbs (1983) and Lin et al. (1983). Eauliq included representations of the microphysical processes responsible for the formation and dissipation of both water and ice clouds. Eauliq included five prognostic variables representing the mixing ratios of water vapor, cloud water, cloud ice, rain, and snow. Graupel and hail were neglected. Cloud water and cloud ice were permitted to form through large-scale condensation and deposition processes. Rain and snow were assumed to be produced through autoconversion of cloud water and cloud ice. Rain drops falling through clouds were assumed to grow by collecting cloud water, and falling snow was assumed to collect both cloud water and cloud ice. These collection processes were formulated using the continuous collection equation. Evaporation of cloud water, cloud ice, rain, and snow were allowed in subsaturated layers. Melting and freezing were considered. Fowler et al. also included a coupling between convective clouds and stratiform anvils through the detrainment of cloud water and cloud ice at the tops of cumulus towers. Interactive cloud optical properties provide the link between the cloud microphysics and radiation parameterizations; the optical depths and infrared emissivities of large-scale stratiform clouds were parameterized in terms of the cloud water and cloud ice paths. Perhaps the most serious weakness of Eauliq is that it does not include a parameterization of cloud amount; Fowler et al. (1996) simply assumed that the cloud amount was either zero (i.e. no cloud in a grid cell) or one (i.e. uniform cloud throughout a grid cell).

The purpose of this paper is to outline a generalization of Eauliq, called "Eauliq NG," which includes a variable cloud amount. We determine the cloud amount prognostically using an equation similar to that proposed by Tiedtke (1993). We also diagnose the thermodynamic properties of the clear and cloudy portions of the grid cell, using the prognostic cloud amount, the prognostic mean-state thermodynamic variables, and the prognostic differences in the thermodynamic variables between the clear and cloudy portions of the grid cell. We separately determine the vertical motions in the clear and cloudy portions of the grid cell, by requiring that the differences in vertical motion act to maintain neutral buoyancy of the stratiform clouds. These same vertical motion differences have additional consequences, of course, which are taken into account in our parameterization.

Section 2 of this paper lays out the conceptual and mathematical framework of Eauliq NG. Section 3 discusses the role of cumulus processes. Section 4 outlines the algorithm used to determine the separate thermodynamic properties of the clear and cloudy sub-regions of each grid cell. Section 5 describes our method to determine the separate vertical motions in the clear and cloudy sub-regions. Section 6 discusses the parameterized lateral mass exchanges between the clear and cloudy sub-regions. Section 7 describes the parameterized microphysical processes. Section 8 gives a summary and conclusions.

2. Framework

2.1 Sub-regions

We divide the horizontal[2] domain of a grid cell into three sub-regions, as shown in the sketch below. These are the clear sub-region, denoted by "clr," the sub-region filled with stratiform cloud, denoted by "cld," and the cumulus sub-region, denoted by "cu." The areas occupied by the three sub-regions are denoted by A^i_{clr}, A^i_{cld}, and A^i_{cu}, respectively. The total area of the grid cell is

$$A^i = A^i_{\mathrm{clr}} + A^i_{\mathrm{cld}} + A^i_{\mathrm{cu}}. \tag{6}$$

Here the superscript i denotes the grid cell under consideration, and subscripts are used to denote a sub-region. We assume that A^i is independent of both time and height.

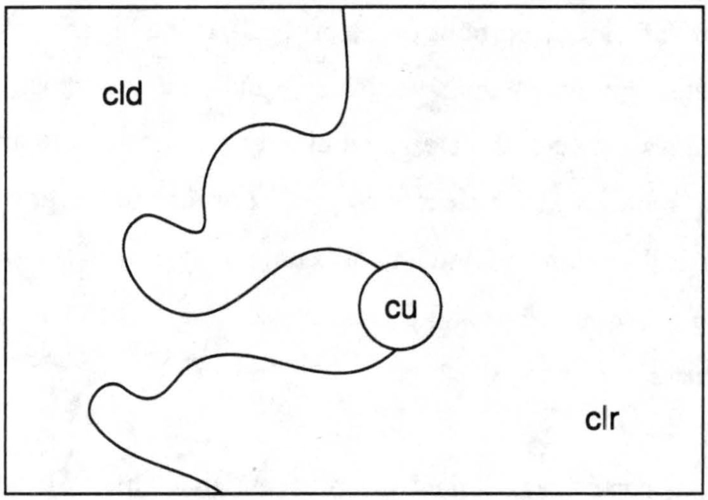

Figure 2: Schematic showing the horizontal cross section of a grid cell containing a clear region (clr), a stratiform-cloudy region (cld), and a cumulus region (cu).

2. We assume that the model has a vertical resolution high enough to capture at least crudely the vertical distribution of the cloudiness; we therefore ignore the possibility of vertically subgrid-scale clouds.

By definition, the cld sub-region contains stratiform cloud water and/or cloud ice, and by definition the clear region contains neither. In other words, cloud water and cloud ice are assumed to occupy the same fractional area, i.e. A^{i}_{cld}. We allow the possibility that rain and snow can exist in either cloudy or cloud-free portions of a grid cell. Obviously rain and snow must originate in cloudy regions, but they can fall into clear regions. The fractional areas occupied by rain and snow are discussed later.

In Fig. 2, the cumulus sub-region is sketched as if it were comparable in size to the others, but this is merely for convenience in making the drawing; in reality we expect the fractional area occupied by the cumulus clouds to be very small in all cases. The "cu" sub-region represents an ensemble of cumulus clouds, which may be further broken down into subensembles, following the "spectral" approach of Arakawa and Schubert (1974). Here we forego the additional notational complexity that would be required to explicitly represent the subensembles, but we do in fact follow the spectral approach.

Following the approach of Margolin et al. (1997), we allow each of the three sub-regions ("clr," "cld," and "cu") to exchange mass laterally with the other two. This mass exchange can occur "inside" the grid cell under consideration, and in addition, each sub-region in grid cell i can exchange mass with neighboring grid cells. We denote lateral mass exchanges inside the cell (hereafter "intra-cell") by E, and lateral mass exchanges with neighboring cells (hereafter "inter-cell") by F.

Consider an arbitrary intensive variable h, and let S_h denote the source or sink of h, which can include the effects of small-scale turbulence and radiation, as well as microphysical processes[3]. We can write the following budget equations for grid box i:

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[3] For now we use h as a generic intensive scalar; later we use the same symbol to denote the generalized moist static energy, which is of course an intensive scalar.

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}) = E^i{}_{\mathrm{cld,clr}} h^i{}_{\mathrm{cld}} + E^i{}_{\mathrm{cu,clr}} h^i{}_{\mathrm{cu}} - (E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{clr,cu}}) h^i{}_{\mathrm{clr}}$$
$$- \sum_{i'} F^{i,i'}{}_{\mathrm{clr}} \hat{h}^{i,i'}{}_{\mathrm{clr}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{clr}} h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}) + (S_h)^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}} , \tag{7}$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) = E^i{}_{\mathrm{clr,cld}} h^i{}_{\mathrm{clr}} + E^i{}_{\mathrm{cu,cld}} h^i{}_{\mathrm{cu}} - (E^i{}_{\mathrm{cld,clr}} + E^i{}_{\mathrm{cld,cu}}) h^i{}_{\mathrm{cld}}$$
$$- \sum_{i'} F^{i,i'}{}_{\mathrm{cld}} \hat{h}^{i,i'}{}_{\mathrm{cld}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{cld}} h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) + (S_h)^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} , \tag{8}$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}}) = E^i{}_{\mathrm{clr,cu}} h^i{}_{\mathrm{clr}} + E^i{}_{\mathrm{cld,cu}} h^i{}_{\mathrm{cld}} - (E^i{}_{\mathrm{cu,clr}} + E^i{}_{\mathrm{cu,cld}}) h^i{}_{\mathrm{cu}}$$
$$- \sum_{i'} F^{i,i'}{}_{\mathrm{cu}} \hat{h}^{i,i'}{}_{\mathrm{cu}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{cu}} h^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}}) + (S_h)^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}} . \tag{9}$$

Here m^i is the mass of dry air, per unit area, in grid cell i. The E's and F's have dimensions of mass per unit time. The terms involving the E's and F's are discussed in the next two sections. When we add (7), (8), and (9), all of the E terms cancel out, but the F terms survive. We obtain

$$\frac{\partial}{\partial t}(m^i \bar{h}^i A^i) =$$
$$- \sum_{i'} (F^{i,i'}{}_{\mathrm{clr}} \hat{h}^{i,i'}{}_{\mathrm{clr}} + F^{i,i'}{}_{\mathrm{cld}} \hat{h}^{i,i'}{}_{\mathrm{cld}} + F^{i,i'}{}_{\mathrm{cu}} \hat{h}^{i,i'}{}_{\mathrm{cu}}) - \frac{\partial}{\partial z}\left(A^i m^i \overline{wh}^i\right) + A^i \overline{(S_h)}^i , \tag{10}$$

where

$$A^i \bar{h}^i = h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}} + h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} + h^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}} , \tag{11}$$

$$A^i \overline{mwh}^i = m^i w^i{}_{\mathrm{clr}} h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}} + m^i w^i{}_{\mathrm{cld}} h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} + m^i w^i{}_{\mathrm{cu}} h^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}} , \tag{12}$$

and

$$A^i\overline{(S_h)}^i = (S_h)^i{}_{\mathrm{clr}}A^i{}_{\mathrm{clr}} + (S_h)^i{}_{\mathrm{cld}}A^i{}_{\mathrm{cld}} + (S_h)^i{}_{\mathrm{cu}}A^i{}_{\mathrm{cu}}. \tag{13}$$

The continuity equations corresponding to (7)-(10) can be obtained by setting $h \equiv 1$ and $S_h = 0$:

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{clr}}) = E^i{}_{\mathrm{cld,clr}} + E^i{}_{\mathrm{cu,clr}} - (E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{clr,cu}}) - \sum_{i'} F^{i,\,i'}{}_{\mathrm{clr}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}), \tag{14}$$

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{cld}}) = E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{cu,cld}} - (E^i{}_{\mathrm{cld,clr}} + E^i{}_{\mathrm{cld,cu}}) - \sum_{i'} F^{i,\,i'}{}_{\mathrm{cld}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}), \tag{15}$$

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{cu}}) = E^i{}_{\mathrm{clr,cu}} + E^i{}_{\mathrm{cld,cu}} - (E^i{}_{\mathrm{cu,clr}} + E^i{}_{\mathrm{cu,cld}}) - \sum_{i'} F^{i,\,i'}{}_{\mathrm{cu}} - \frac{\partial}{\partial z}(m^i w^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}}), \tag{16}$$

$$\frac{\partial}{\partial t}(m^i A^i) = \sum_{i'} (F^{i,\,i'}{}_{\mathrm{clr}} + F^{i,\,i'}{}_{\mathrm{cld}} + F^{i,\,i'}{}_{\mathrm{cu}}) - \frac{\partial}{\partial z}(m^i \overline{w}^i A^i). \tag{17}$$

In (17),

$$A^i\overline{w}^i = w^i{}_{\mathrm{clr}}A^i{}_{\mathrm{clr}} + w^i{}_{\mathrm{cld}}A^i{}_{\mathrm{cld}} + w^i{}_{\mathrm{cu}}A^i{}_{\mathrm{cu}}, \tag{18}$$

where $\overline{w}^i$ is the area-averaged vertical velocity. Eqs. (14)-(16) govern the time change of the mass or area within each sub-region. Eq. (17) is the continuity equation for the whole grid cell. By combining (7)-(9) with (14)-(16), we can derive "advective forms" of the three budget equations:

$$m^i A^i_{\mathrm{clr}} \frac{\partial}{\partial t}(h^i_{\mathrm{clr}}) = E^i_{\mathrm{cld,clr}}(h^i_{\mathrm{cld}} - h^i_{\mathrm{clr}}) + E^i_{\mathrm{cu,clr}}(h^i_{\mathrm{cu}} - h^i_{\mathrm{clr}})$$

$$-\sum_{i'} F^{i,i'}_{\mathrm{clr}}(\hat{h}^{i,i'}_{\mathrm{clr}} - h^i_{\mathrm{clr}}) \tag{19}$$

$$-m^i A^i_{\mathrm{clr}} w^i_{\mathrm{clr}} \frac{\partial}{\partial z}(h^i_{\mathrm{clr}}) + (S_h)^i_{\mathrm{clr}} A^i_{\mathrm{clr}} ,$$

$$m^i A^i_{\mathrm{cld}} \frac{\partial}{\partial t}(h^i_{\mathrm{cld}}) = E^i_{\mathrm{clr,cld}}(h^i_{\mathrm{clr}} - h^i_{\mathrm{cld}}) + E^i_{\mathrm{cu,cld}}(h^i_{\mathrm{cu}} - h^i_{\mathrm{cld}})$$

$$-\sum_{i'} F^{i,i'}_{\mathrm{cld}}(\hat{h}^{i,i'}_{\mathrm{cld}} - h^i_{\mathrm{cld}}) \tag{20}$$

$$-m^i A^i_{\mathrm{cld}} w^i_{\mathrm{cld}} \frac{\partial}{\partial z}(h^i_{\mathrm{cld}}) + (S_h)^i_{\mathrm{cld}} A^i_{\mathrm{cld}} ,$$

$$m^i A^i_{\mathrm{cu}} \frac{\partial}{\partial t}(h^i_{\mathrm{cu}}) = E^i_{\mathrm{clr,cu}}(h^i_{\mathrm{clr}} - h^i_{\mathrm{cu}}) + E^i_{\mathrm{cld,cu}}(h^i_{\mathrm{cld}} - h^i_{\mathrm{cu}})$$

$$-\sum_{i'} F^{i,i'}_{\mathrm{cu}}(\hat{h}^{i,i'}_{\mathrm{cu}} - h^i_{\mathrm{cu}}) \tag{21}$$

$$-m^i A^i_{\mathrm{cu}} w^i_{\mathrm{cu}} \frac{\partial}{\partial z}(h^i_{\mathrm{cu}}) + (S_h)^i_{\mathrm{cu}} A^i_{\mathrm{cu}} .$$

2.2 *Lateral exchanges of mass between subregions*

First, we discuss the mass exchanges between cumulus clouds and the clear and stratiform-cloudy portions of the grid cell. $E^i_{\mathrm{cu,clr}}$ represents the flow of mass from cumulus clouds into the clear portion of the grid cell; here we adopt the convention that the first subscript denotes the sub-region of origin, and the second denotes the destination sub-region. Similarly, $E^i_{\mathrm{cu,cld}}$ represents the source of stratiform cloudy air due to detrainment of air from cumuli into the cloudy part of the grid cell. We assume that air detrained by cumuli is always cloudy, and so always enters the stratiform-cloud sub-region of the grid cell. This means that

$$E^i_{\mathrm{cu,clr}} = 0 . \tag{22}$$

For notational clarity, we write

$$D^i \equiv E^i{}_{\text{cu,cld}} \geq 0\,. \tag{23}$$

$E^i{}_{\text{clr,cu}}$ represents the entrainment of air from the clear part of the box into cumulus clouds. Similarly, $E^i{}_{\text{cld,cu}}$ represents the entrainment of air from the cloudy part of the box into the cumulus clouds. We note that

$$E^i{}_{\text{clr,cu}} \geq 0\,,\ \text{and}\ E^i{}_{\text{cld,cu}} \geq 0\,. \tag{24}$$

Now consider the mass exchanges between the clear and stratiform cloudy portions of the grid cell. $E^i{}_{\text{cld,clr}}$ represents air "moving" from the cloudy area to the clear area. Of course $E^i{}_{\text{cld,clr}}$ amounts to evaporation of the cloud, which is not really motion in the usual sense at all. Similarly, $E^i{}_{\text{clr,cld}}$ represents the transformation of clear air into stratiform cloud air. Again, this is not really "motion;" instead it represents the effects of processes that increase the relative humidity in the clear part of the box, or some portion of it, so as to convert the clear air into cloudy air. Note that, with these definitions,

$$E^i{}_{\text{clr,cld}} \geq 0\,,\ \text{and}\ E^i{}_{\text{cld,clr}} \geq 0\,. \tag{25}$$

In particular, $E^i{}_{\text{clr,cld}}$ is *not* equal to minus $E^i{}_{\text{cld,clr}}$ (unless they both happen to be zero). The two mass exchange processes can occur independently and simultaneously within each grid cell.

When we apply (19) to the cloud water mixing ratio, q_c, which is of course equal to zero in the clear portion of the grid cell, we obtain a very simple result:

$$0 = E^{i}{}_{\mathrm{cld,clr}}(q_c{}^{i}) + (S_{q_c})^{i}{}_{\mathrm{clr}} A^{i}{}_{\mathrm{clr}}. \tag{26}$$

Note that here $q_c{}^{i}$ denotes the value of q_c in the cloudy portion of the grid cell; we omit the subscript "cld" on the grounds that it would be redundant. According to (26), the flux of cloud water from the cloudy to the clear portion of the cell, denoted by $E^{i}{}_{\mathrm{cld,clr}}(q_c{}^{i})$, is balanced by a sink of cloud water (presumably due to evaporation), which prevents any cloud water from accumulating in the clear portion of the grid cell. We can use (26) to diagnose the rate of evaporation in the clear-portion of the grid cell; not only is this needed for such diagnostic purposes, it also represents a source of water vapor in the clear portion of the cell, and it is associated with evaporative cooling there. Similarly, the cloud ice mixing ratio satisfies

$$0 = E^{i}{}_{\mathrm{cld,clr}}(q_i{}^{i}) + (S_{q_i})^{i}{}_{\mathrm{clr}} A^{i}{}_{\mathrm{clr}}. \tag{27}$$

Intra-cell fluxes *between sub-regions of the same type* ("sub-sub-regions") are neglected in this paper, so that, for instance, flows between two clear portions of the same grid cell are assumed to have no effect on quantities of interest. In effect we assume that all subregions *of the same type*, within a given cell, have identical properties, so that exchanges among them are irrelevant.

2.3 Inter-cell exchanges of mass

We assume that when air flows across cell walls, between neighboring grid cells, *it always moves between sub-regions of like type*, so that, for example, air can travel from the cloudy sub-regions of grid box i to the cloudy sub-regions of neighboring grid box i', but not from the cloudy sub-region of grid box i to the clear sub-region of neighboring grid box i'. The rationale is that it would be quite unlikely for the boundary of a sub-region, e.g. a stratiform cloud, to coincide exactly with the wall of a grid cell.

To minimize the number of symbols, we adopt the notation $F^{i,\,i'}{}_{\mathrm{clr}}$ to denote the flow of mass *outward from* grid box i *to* neighboring grid box i', in this case between the clear sub-regions of each. Corresponding conventions are used with $F^{i,\,i'}{}_{\mathrm{cld}}$ and $F^{i,\,i'}{}_{\mathrm{cu}}$. This is why the F terms appear with minus signs on the right-hand sides of (7) - (9). The Fs can have either sign.

$\sum_{i'} F^{i,\,i'}{}_{\mathrm{clr}}$ represents the net flow of air from the clear region in one cell to the clear regions of the surrounding cells. Similarly, $\sum_{i'} F^{i,\,i'}{}_{\mathrm{cld}}$ represents the flow of air from the stratiform cloud region in one cell to the stratiform cloud regions of the surrounding cells, and $\sum_{i'} F^{i,\,i'}{}_{\mathrm{cu}}$ represents the flow of air from the cumulus region in one cell to the cumulus regions of the surrounding cells. Summations such as $\sum_{i'} F^{i,\,i'}{}_{\mathrm{clr}} \hat{h}^{i,\,i'}{}_{\mathrm{clr}}$ represent exchanges between grid box i and all neighboring grid boxes. The "hat" symbol, as in $\hat{h}^{i,\,i'}$, denotes an interpolated value on a cell wall.

For simplicity, we currently neglect any variations of the horizontal velocity between the cloudy and clear regions of the grid cells, although we realize that this must be addressed in the future. We assume that

$$F^{i,\,i'}{}_{\mathrm{clr}} = \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}{}_{\mathrm{clr}}, \tag{28}$$

where $\hat{m}^{i,\,i'}$ is an interpolated mass variable, defined on the cell wall, $v^{i,\,i'}{}_{n}$ is the outward normal velocity component along the cell wall, and $l^{i,\,i'}{}_{\mathrm{clr}}$ is the distance occupied by clear air, along the cell wall. Similarly, we assume that

$$F^{i,i'}{}_{\mathrm{cld}} = \hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'}{}_{\mathrm{cld}}, \tag{29}$$

and

$$F^{i,i'}{}_{\mathrm{cu}} = \hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'}{}_{\mathrm{cu}}. \tag{30}$$

We require that

$$l^{i,i'}{}_{\mathrm{clr}} + l^{i,i'}{}_{\mathrm{cld}} + l^{i,i'}{}_{\mathrm{cu}} = l^{i,i'}, \tag{31}$$

where $l^{i,i'}$ is the total length of the cell wall. We assume that

$$l^{i,i'}{}_{\mathrm{clr}} = l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,i'}}, \tag{32}$$

$$l^{i,i'}{}_{\mathrm{cld}} = l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,i'}}, \tag{33}$$

$$l^{i,i'}{}_{\mathrm{cu}} = l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,i'}}, \tag{34}$$

where $\hat{A}^{i,i'}{}_{\mathrm{clr}}$, $\hat{A}^{i,i'}{}_{\mathrm{cld}}$, and $\hat{A}^{i,i'}{}_{\mathrm{cu}}$ are suitably interpolated "edge" values of the corresponding areas. In view of (31), we must require that

$$\hat{A}^{i,i'}{}_{\mathrm{clr}} + \hat{A}^{i,i'}{}_{\mathrm{cld}} + \hat{A}^{i,i'}{}_{\mathrm{cu}} = \hat{A}^{i,i'}. \tag{35}$$

This requirement is met by either upstream or centered interpolation of the areas, and also by any linear combination of upstream and centered interpolations. Substitution of (32) - (34) into (28) - (30) gives

$$F^{i,\,i'}{}_{\text{clr}} = \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\text{clr}}}{\hat{A}^{i,\,i'}}, \tag{36}$$

$$F^{i,\,i'}{}_{\text{cld}} = \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\text{cld}}}{\hat{A}^{i,\,i'}}, \tag{37}$$

$$F^{i,\,i'}{}_{\text{cu}} = \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\text{cu}}}{\hat{A}^{i,\,i'}}. \tag{38}$$

Substitution of (36)-(38) into (17) leads to

$$\begin{aligned} \frac{\partial}{\partial t}(m^{i}\hat{A}^{i,\,i'}) &= \sum_{i'}\left[\hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\left(\frac{\hat{A}^{i,\,i'}{}_{\text{clr}} + \hat{A}^{i,\,i'}{}_{\text{cld}} + \hat{A}^{i,\,i'}{}_{\text{cu}}}{\hat{A}^{i,\,i'}}\right)\right] - \frac{\partial}{\partial z}(m^{i}\overline{w}^{i}\hat{A}^{i,\,i'}) \\ &= \sum_{i'}\hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'} - \frac{\partial}{\partial z}(m^{i}\overline{w}^{i}\hat{A}^{i,\,i'})\,. \end{aligned} \tag{39}$$

To obtain the second equality in (39), we have used (35). Eq. (39) is just what we would expect from the large-scale point of view. On the other hand, Eqs. (36)-(38) imply a kind of "diffusion" of the mean state properties across cell walls. To see this, substitute (36)-(38) into (10), to obtain

$$\begin{aligned} &\sum_{i'}(F^{i,\,i'}{}_{\text{clr}}\hat{h}^{i,\,i'}{}_{\text{clr}} + F^{i,\,i'}{}_{\text{cld}}\hat{h}^{i,\,i'}{}_{\text{cld}} + F^{i,\,i'}{}_{\text{cu}}\hat{h}^{i,\,i'}{}_{\text{cu}}) \\ &= \sum_{i'}\left(\hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\frac{\hat{A}^{i,\,i'}{}_{\text{clr}}}{\hat{A}^{i,\,i'}}\hat{h}^{i,\,i'}{}_{\text{clr}} + \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\frac{\hat{A}^{i,\,i'}{}_{\text{cld}}}{\hat{A}^{i,\,i'}}\hat{h}^{i,\,i'}{}_{\text{cld}} + \hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\frac{\hat{A}^{i,\,i'}{}_{\text{cu}}}{\hat{A}^{i,\,i'}}\hat{h}^{i,\,i'}{}_{\text{cu}}\right) \\ &= \sum_{i'}\hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\left(\frac{\hat{A}^{i,\,i'}{}_{\text{clr}} + \hat{A}^{i,\,i'}{}_{\text{cld}} + \hat{A}^{i,\,i'}{}_{\text{cu}}}{\hat{A}^{i,\,i'}}\right)\hat{h}^{i,\,i'} - \sum_{i'}G^{i,\,i'}(h) \\ &= \sum_{i'}\hat{m}^{i,\,i'} v^{i,\,i'}{}_{n} l^{i,\,i'}\hat{h}^{i,\,i'} - \sum_{i'}G^{i,\,i'}(h)\ , \end{aligned} \tag{40}$$

$$\frac{\partial}{\partial t}(m^i \bar{h}^i A^i) = -\sum_{i'} \hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,\,i'}} \hat{h}^{i,\,i'}{}_{\mathrm{clr}}$$

$$-\sum_{i'} \hat{m}^{i+i'} v^{i+i'}{}_{nl}^{i+i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,\,i'}} \hat{h}^{i,\,i'}{}_{\mathrm{cld}}$$

$$-\sum_{i'} \hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,\,i'}} \hat{h}^{i,\,i'}{}_{\mathrm{cu}} - \frac{\partial}{\partial z}(m^i \overline{wh}^i A^i) + A^i \overline{(S_h)}^i$$

$$= -\sum_{i'} \hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \left(\frac{\hat{A}^{i,\,i'}{}_{\mathrm{clr}} + \hat{A}^{i,\,i'}{}_{\mathrm{cld}} + \hat{A}^{i,\,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,\,i'}} \right) \hat{h}^{i,\,i'} \tag{41}$$

$$+ \sum_{i'} G^{i,\,i'}(h) - \frac{\partial}{\partial z}(m^i \overline{wh}^i A^i) + A^i \overline{(S_h)}^i$$

$$= -\sum_{i'} \hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \hat{h}^{i,\,i'} + \sum_{i'} G^{i,\,i'}(h) - \frac{\partial}{\partial z}(m^i \overline{wh}^i A^i) + A^i \overline{(S_h)}^i ,$$

where

$$G^{i,\,i'}(h) \equiv -\hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,\,i'}} (\hat{h}^{i,\,i'}{}_{\mathrm{clr}} - \hat{h}^{i,\,i'})$$

$$-\hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,\,i'}} (\hat{h}^{i,\,i'}{}_{\mathrm{cld}} - \hat{h}^{i,\,i'}) \tag{42}$$

$$-\hat{m}^{i,\,i'} v^{i,\,i'}{}_{nl}^{i,\,i'} \frac{\hat{A}^{i,\,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,\,i'}} (\hat{h}^{i,\,i'}{}_{\mathrm{cu}} - \hat{h}^{i,\,i'})$$

represents a diffusive flux that arises from the indicated differences in h. To obtain the third equality in (40) we have again used (35).

2.4 Vertical mass fluxes

It is useful to express all vertical mass exchanges in terms of three mass fluxes:

the large-scale mass flux, $A^i\overline{M}^i \equiv A^i{}_{\mathrm{clr}}M^i{}_{\mathrm{clr}} + A^i{}_{\mathrm{cld}}M^i{}_{\mathrm{cld}} + A^i{}_{\mathrm{cu}}M^i{}_{\mathrm{cu}}$; (43)

the cumulus mass flux, $M^i{}_{\mathrm{c}} \equiv \dfrac{A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - \overline{M})$; and (44)

the mesoscale mass flux, $M^i{}_{\mathrm{meso}} \equiv \dfrac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^iA^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})$. (45)

Here

$$M^i{}_{\mathrm{clr}} \equiv m^i w^i{}_{\mathrm{clr}}; \tag{46}$$

$$M^i{}_{\mathrm{cld}} \equiv m^i w^i{}_{\mathrm{cld}}; \tag{47}$$

$$M^i{}_{\mathrm{cu}} \equiv m^i w^i{}_{\mathrm{cu}}. \tag{48}$$

Straightforward algebraic manipulations, which are summarized in Appendix A, lead to the following results:

$$A^i{}_{\mathrm{clr}}M^i{}_{\mathrm{clr}} = A^i{}_{\mathrm{clr}}\overline{M}^i - \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}) - \frac{A^i{}_{\mathrm{clr}}A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{clr}}), \tag{49}$$

$$A^i{}_{\mathrm{cld}}M^i{}_{\mathrm{cld}} = A^i{}_{\mathrm{cld}}\overline{M}^i + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}) - \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{cld}}), \tag{50}$$

$$\overline{mwh}^i = \overline{M}^i\overline{h}^i + M^i{}_{\mathrm{meso}}(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}}) + M^i{}_{\mathrm{c}}(h^i{}_{\mathrm{cu}} - \overline{h}^i). \tag{51}$$

2.5 Summary of framework

Before making further simplifying assumptions and approximations, we summarize the basic equations of our model, making use of the various results obtained in the preceding sub-sections. The continuity equations can be written as:

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{clr}}) = E^i{}_{\mathrm{cld,clr}} + E^i{}_{\mathrm{cu,clr}} - (E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{clr,cu}}) - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,i'}} - \frac{\partial}{\partial z}(A^i{}_{\mathrm{clr}} M^i{}_{\mathrm{clr}}) \;, \tag{52}$$

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{cld}}) = E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{cu,cld}} - (E^i{}_{\mathrm{cld,clr}} + E^i{}_{\mathrm{cld,cu}}) - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,i'}} - \frac{\partial}{\partial z}(A^i{}_{\mathrm{cld}} M^i{}_{\mathrm{cld}}) \;, \tag{53}$$

$$\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{cu}}) = E^i{}_{\mathrm{clr,cu}} + E^i{}_{\mathrm{cld,cu}} - (E^i{}_{\mathrm{cu,clr}} + E^i{}_{\mathrm{cu,cld}}) - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,i'}} - \frac{\partial}{\partial z}(A^i{}_{\mathrm{cu}} M^i{}_{\mathrm{cu}}) \;, \tag{54}$$

$$\frac{\partial}{\partial t}(m^i A^i) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} - \frac{\partial}{\partial z}(m^i \overline{w}^i A^i) . \tag{55}$$

The conservations equations for h can be written as:

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}) = E^i{}_{\mathrm{cld,clr}} h^i{}_{\mathrm{cld}} + E^i{}_{\mathrm{cu,clr}} h^i{}_{\mathrm{cu}} - (E^i{}_{\mathrm{clr,cld}} + E^i{}_{\mathrm{clr,cu}}) h^i{}_{\mathrm{clr}} - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\mathrm{clr}} - \frac{\partial}{\partial z}\left(A^i{}_{\mathrm{clr}} M^i{}_{\mathrm{clr}} h^i{}_{\mathrm{clr}}\right) + (S_h)^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}} \;, \tag{56}$$

$$\frac{\partial}{\partial t}(m^i h^i_{\mathrm{cld}} A^i_{\mathrm{cld}}) = E^i_{\mathrm{clr,cld}} h^i_{\mathrm{clr}} + E^i_{\mathrm{cu,cld}} h^i_{\mathrm{cu}} - (E^i_{\mathrm{cld,clr}} + E^i_{\mathrm{cld,cu}}) h^i_{\mathrm{cld}}$$

$$- \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\mathrm{cld}} - \frac{\partial}{\partial z}\left(A^i_{\mathrm{cld}} M^i_{\mathrm{cld}} h^i_{\mathrm{cld}}\right) + (S_h)^i_{\mathrm{cld}} A^i_{\mathrm{cld}}, \tag{57}$$

$$\frac{\partial}{\partial t}(m^i h^i_{\mathrm{cu}} A^i_{\mathrm{cu}}) = E^i_{\mathrm{clr,cu}} h^i_{\mathrm{clr}} + E^i_{\mathrm{cld,cu}} h^i_{\mathrm{cld}} - (E^i_{\mathrm{cu,clr}} + E^i_{\mathrm{cu,cld}}) h^i_{\mathrm{cu}}$$

$$- \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cu}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\mathrm{cu}} - \frac{\partial}{\partial z}(A^i_{\mathrm{cu}} M^i_{\mathrm{cu}} h^i_{\mathrm{cu}}) + (S_h)^i_{\mathrm{cu}} A^i_{\mathrm{cu}}, \tag{58}$$

$$\frac{\partial}{\partial t}(m^i \bar{h}^i A^i) = - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \hat{h}^{i,i'} + G^{i,i'}(h)$$

$$- \frac{\partial}{\partial z}\{A^i[\bar{M}^i \bar{h}^i + M^i_{\mathrm{meso}}(h^i_{\mathrm{cld}} - h^i_{\mathrm{clr}}) + M^i_{\mathrm{c}}(h^i_{\mathrm{cu}} - \bar{h}^i)]\} + A^i \overline{(S_h)}^i . \tag{59}$$

Our strategy for the use of these equations is explained later.

3. The cumulus terms

We assume that

$$E^i_{\mathrm{clr,cu}} = \frac{A^i_{\mathrm{clr}}}{A^i} E^i, \tag{60}$$

$$E^i_{\mathrm{cld,cu}} = \frac{A^i_{\mathrm{cld}}}{A^i} E^i. \tag{61}$$

Here E^i is the total rate of entrainment into the cumulus clouds, i.e.

$$E^i = E^i_{\mathrm{clr,cu}} + E^i_{\mathrm{cld,cu}}. \tag{62}$$

Eqs. (60) and (61) mean that the air entrained into cumuli is derived from the clear and cloudy subregions in proportion to the fractional areas covered by those subregions. This assumption is made for simplicity and should be refined in the future.

We also assume that the cumulus clouds detrain only into the cloudy portion of the grid box, on the grounds that the air leaving the cumuli is, after all, cloudy:

$$E^{i}{}_{\mathrm{cu,clr}} = 0\,. \tag{63}$$

To make our notation closer to that used in the earlier literature, we write

$$E^{i}{}_{\mathrm{cu,cld}} \equiv D^{i}\,. \tag{64}$$

Next, we assume, following Arakawa and Schubert (1974), that the fractional area covered by cumulus convection is very small compared with one, i.e.

$$A^{i}{}_{\mathrm{cu}} \ll A^{i} \text{ and } \hat{A}^{i,i'}{}_{\mathrm{cu}} \ll \hat{A}^{i,i'}, \tag{65}$$

and also

$$A^{i}{}_{\mathrm{clr}} + A^{i}{}_{\mathrm{cld}} \cong A^{i} \text{ and } \hat{A}^{i,i'}{}_{\mathrm{clr}} + \hat{A}^{i,i'}{}_{\mathrm{cld}} \cong \hat{A}^{i,i'}. \tag{66}$$

This assumption allows us to simplify our equations as follows:

$$A^{i}\overline{h}^{i} \cong h^{i}{}_{\mathrm{clr}}A^{i}{}_{\mathrm{clr}} + h^{i}{}_{\mathrm{cld}}A^{i}{}_{\mathrm{cld}}, \tag{67}$$

$$M_{c}{}^{i} \cong \frac{A^{i}{}_{\mathrm{cu}}}{A^{i}}M^{i}{}_{\mathrm{cu}}, \tag{68}$$

$$F^{i,i'}{}_{\mathrm{cu}} \cong 0\,, \tag{69}$$

$$\begin{aligned} G^{i,i'}(h) \cong& -\hat{m}^{i,i'} v^{i,i'}{}_{nl}{}^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{clr}}(\hat{h}^{i,i'}{}_{\mathrm{clr}} - \hat{h}^{i,i'}) \\ & -\hat{m}^{i,i'} v^{i,i'}{}_{nl}{}^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{cld}}(\hat{h}^{i,i'}{}_{\mathrm{cld}} - \hat{h}^{i,i'})\,, \end{aligned} \tag{70}$$

$$A^{i}{}_{\mathrm{clr}} M^{i}{}_{\mathrm{clr}} \cong A^{i}{}_{\mathrm{clr}} \overline{M}^{i} - A^{i} M^{i}{}_{\mathrm{meso}} - A^{i}{}_{\mathrm{clr}} M_{c}{}^{i}\,, \tag{71}$$

$$A^{i}{}_{\mathrm{cld}} M^{i}{}_{\mathrm{cld}} \cong A^{i}{}_{\mathrm{cld}} \overline{M}^{i} + A^{i} M^{i}{}_{\mathrm{meso}} - A^{i}{}_{\mathrm{cld}} M_{c}{}^{i}\,. \tag{72}$$

In addition, we can neglect the time-rate-of-change terms in (58) and (54).

Using the various assumptions and definitions discussed above, the continuity equations (52)-(55) can be written as

$$\begin{aligned} \frac{\partial}{\partial t}(m^{i} A^{i}{}_{\mathrm{clr}}) =& -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{nl}{}^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{clr}} + E^{i}{}_{\mathrm{cld,clr}} - E^{i}{}_{\mathrm{clr,cld}} \\ & -\frac{\partial}{\partial z}(A^{i}{}_{\mathrm{clr}} \overline{M}^{i} - A^{i} M^{i}{}_{\mathrm{meso}}) + \frac{\partial}{\partial z}(M_{c}{}^{i} A^{i}{}_{\mathrm{clr}}) - E^{i} \frac{A^{i}{}_{\mathrm{clr}}}{A^{i}}\,, \end{aligned} \tag{73}$$

$$\begin{aligned} \frac{\partial}{\partial t}(m^{i} A^{i}{}_{\mathrm{cld}}) =& -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{nl}{}^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{cld}} + E^{i}{}_{\mathrm{clr,cld}} - E^{i}{}_{\mathrm{cld,clr}} \\ & -\frac{\partial}{\partial z}(A^{i}{}_{\mathrm{cld}} \overline{M}^{i} + A^{i} M^{i}{}_{\mathrm{meso}}) + \frac{\partial}{\partial z}(M_{c}{}^{i} A^{i}{}_{\mathrm{cld}}) - E^{i} \frac{A^{i}{}_{\mathrm{cld}}}{A^{i}} + D^{i}\,, \end{aligned} \tag{74}$$

$$0 = E^{i} - D^{i} - \frac{\partial}{\partial z}(M_{c}{}^{i} A^{i})\,, \tag{75}$$

$$\frac{\partial}{\partial t}(m^{i} A^{i}) = \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_{nl}{}^{i,i'} - \frac{\partial}{\partial z}(\overline{M}^{i} A^{i})\,. \tag{76}$$

The vertical mass flux divergence terms of (73)-(74) have been expanded to separately exhibit the contributions associated with the mean flow, with the mesoscale circulation, and with cumulus convection. Similarly, Eqs. (56) - (59) can be rewritten as:

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}) = E^i{}_{\mathrm{cld,clr}} h^i{}_{\mathrm{cld}} - E^i{}_{\mathrm{clr,cld}} h^i{}_{\mathrm{clr}} - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{clr}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\mathrm{clr}}$$

$$-\frac{\partial}{\partial z}\left[(A^i{}_{\mathrm{clr}} \overline{M}^i - A^i M^i{}_{\mathrm{meso}}) h^i{}_{\mathrm{clr}}\right] + (S_h)^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}} \tag{77}$$

$$+\frac{\partial}{\partial z}\left(M_c{}^i A^i{}_{\mathrm{clr}} h^i{}_{\mathrm{clr}}\right) - \frac{A^i{}_{\mathrm{clr}}}{A^i} E^i h^i{}_{\mathrm{clr}} \quad ,$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) = E^i{}_{\mathrm{clr,cld}} h^i{}_{\mathrm{clr}} - E^i{}_{\mathrm{cld,clr}} h^i{}_{\mathrm{cld}} - \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\mathrm{cld}}$$

$$-\frac{\partial}{\partial z}\left[(A^i{}_{\mathrm{cld}} \overline{M}^i + A^i M^i{}_{\mathrm{meso}}) h^i{}_{\mathrm{cld}}\right] + (S_h)^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} \tag{78}$$

$$+\frac{\partial}{\partial z}\left(M_c{}^i A^i{}_{\mathrm{cld}} h^i{}_{\mathrm{cld}}\right) - \frac{A^i{}_{\mathrm{cld}}}{A^i} E^i h^i{}_{\mathrm{cld}} + D^i h^i{}_{\mathrm{cu}} \quad ,$$

$$0 = E^i\left(\frac{A^i{}_{\mathrm{clr}}}{A^i} h^i{}_{\mathrm{clr}} + \frac{A^i{}_{\mathrm{cld}}}{A^i} h^i{}_{\mathrm{cld}}\right) - D^i h^i{}_{\mathrm{cu}}$$

$$-\frac{\partial}{\partial z}(M_c{}^i A^i h^i{}_{\mathrm{cu}}) + (S_h)^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}} \quad , \tag{79}$$

$$\frac{\partial}{\partial t}(m^i \overline{h}^i A^i) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \hat{h}^{i,i'} + G^{i,i'}(h) - \frac{\partial}{\partial z}(A^i \overline{M}^i \overline{h}^i)$$

$$-\frac{\partial}{\partial z}\{A^i[\overline{M}^i \overline{h}^i + M^i{}_{\mathrm{meso}}(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}})]\} + [(S_h)^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} + (S_h)^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}}] \tag{80}$$

$$-\frac{\partial}{\partial z}[A^i M^i{}_c(h^i{}_{\mathrm{cu}} - \overline{h}^i)] + (S_h)^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}} \quad .$$

In (80), we have expanded the source/sink terms to show the clear, cloudy, and cumulus contributions separately. In each of (77), (78), and (80), we have written the cumulus terms at the end. Eqs. (75) and (79) are straightforward generalizations of corresponding equations proposed by Arakawa and Schubert (1974).[4]

As discussed by Arakawa and Schubert (1974), we can use the cumulus cloud model represented by (75) and (79) to write the cumulus terms of (80) in an alternative form, as follows:

$$
\begin{aligned}
\frac{\partial}{\partial t}(A^i m^i \bar{h}^i) &\sim -\frac{\partial}{\partial z}[A^i M_c{}^i (h^i{}_{\mathrm{cu}} - \bar{h}^i)] + A^i{}_{\mathrm{cu}}(S^i_h)_{\mathrm{cu}} \\
&= -\frac{\partial}{\partial z}(A^i M_c{}^i h^i{}_{\mathrm{cu}}) + A^i M_c{}^i \frac{\partial \bar{h}^i}{\partial z} + A^i \bar{h}^i \frac{\partial M_c{}^i}{\partial z} + A^i{}_{\mathrm{cu}}(S^i_h)_{\mathrm{cu}} \\
&= \left[D^i h^i{}_{\mathrm{cu}} - E^i\left(\frac{A^i{}_{\mathrm{clr}}}{A^i} h^i{}_{\mathrm{clr}} + \frac{A^i{}_{\mathrm{cld}}}{A^i} h^i{}_{\mathrm{cld}}\right) - (S_h)^i{}_{\mathrm{cu}} A^i{}_{\mathrm{cu}}\right] \\
&\qquad + A^i M_c{}^i \frac{\partial \bar{h}^i}{\partial z} + \bar{h}^i (E^i - D^i) + A^i{}_{\mathrm{cu}}(S^i_h)_{\mathrm{cu}} \\
&= A^i M_c{}^i \frac{\partial \bar{h}^i}{\partial z} + D^i (h^i{}_{\mathrm{cu}} - \bar{h}^i) + E^i\left[\frac{A^i{}_{\mathrm{clr}}}{A^i}(\bar{h}^i - h^i{}_{\mathrm{clr}}) + \frac{A^i{}_{\mathrm{cld}}}{A^i}(\bar{h}^i - h^i{}_{\mathrm{cld}})\right] \quad .
\end{aligned}
\tag{81}
$$

To obtain the second equality in (81), we have used (75) and (79). The entrainment terms on the last line of (81) did not appear in the corresponding equation of Arakawa and Schubert (1974), and represent a minor generalization of their results. These terms appear because of the differences between the cloudy and clear subregions.

The cumulus terms of (74) can also be rewritten in a more useful form, by using (75):

[4] Note, however, that the definitions of E and D used by Arakawa and Schubert (1974) included a normalization by the grid-cell area; no such normalization is used here, which is why the grid-cell area appears in the third term of (75).

$$\frac{\partial}{\partial t}(m^i A^i_{\text{cld}}) \sim \frac{\partial}{\partial z}(M_c{}^i A^i_{\text{cld}}) - E^i \frac{A^i_{\text{cld}}}{A^i} + D^i$$

$$= A^i_{\text{cld}} \frac{\partial M_c{}^i}{\partial z} + M_c{}^i \frac{\partial A^i_{\text{cld}}}{\partial z} - \left[D^i + \frac{\partial}{\partial z}(M_c{}^i A^i)\right]\frac{A^i_{\text{cld}}}{A^i} + D^i \tag{82}$$

$$= M_c{}^i \frac{\partial A^i_{\text{cld}}}{\partial z} + \left(1 - \frac{A^i_{\text{cld}}}{A^i}\right) D^i \ .$$

This form agrees with that proposed by Tiedtke (1993) and derived heuristically by Randall (1995). It is useful because it shows clearly that the detrainment term does not try to drive A^i_{cld} to values larger than one. Similarly, the cumulus terms of (73), (77) and (78) can be rewritten as follows:

$$\frac{\partial}{\partial t}(m^i A^i_{\text{clr}}) \sim \frac{\partial}{\partial z}(M_c{}^i A^i_{\text{clr}}) - E^i \frac{A^i_{\text{clr}}}{A^i}$$

$$= A^i_{\text{clr}} \frac{\partial M_c{}^i}{\partial z} + M_c{}^i \frac{\partial A^i_{\text{clr}}}{\partial z} - \left[D^i + \frac{\partial}{\partial z}(M_c{}^i A^i)\right]\frac{A^i_{\text{clr}}}{A^i} \tag{83}$$

$$= M_c{}^i \frac{\partial A^i_{\text{clr}}}{\partial z} - D^i \frac{A^i_{\text{clr}}}{A^i} \ ,$$

$$\frac{\partial}{\partial t}(m^i h^i_{\text{clr}} A^i_{\text{clr}}) \sim \frac{\partial}{\partial z}(M_c{}^i h^i_{\text{clr}} A^i_{\text{clr}}) - E^i h^i_{\text{clr}} \frac{A^i_{\text{clr}}}{A^i}$$

$$= h^i_{\text{clr}} A^i_{\text{clr}} \frac{\partial M_c{}^i}{\partial z} + M_c{}^i \frac{\partial}{\partial z}(h^i_{\text{clr}} A^i_{\text{clr}}) - \left[D^i + \frac{\partial}{\partial z}(M_c{}^i A^i)\right] h^i_{\text{clr}} \frac{A^i_{\text{clr}}}{A^i} \tag{84}$$

$$= M_c{}^i \frac{\partial}{\partial z}(h^i_{\text{clr}} A^i_{\text{clr}}) - D^i h^i_{\text{clr}} \frac{A^i_{\text{clr}}}{A^i} \ ,$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) \sim \frac{\partial}{\partial z}(M_c{}^i h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) - E^i h^i{}_{\mathrm{cld}} \frac{A^i{}_{\mathrm{cld}}}{A^i} + D^i h^i{}_{\mathrm{cu}}$$

$$= h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}} \frac{\partial M_c{}^i}{\partial z} + M_c{}^i \frac{\partial}{\partial z}(h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) - \left[D^i + \frac{\partial}{\partial z}(M_c{}^i A^i)\right] h^i{}_{\mathrm{cld}} \frac{A^i{}_{\mathrm{cld}}}{A^i} + D^i h^i{}_{\mathrm{cu}} \tag{85}$$

$$= M_c{}^i \frac{\partial}{\partial z}(h^i{}_{\mathrm{cld}} A^i{}_{\mathrm{cld}}) + D^i \left(h^i{}_{\mathrm{cu}} - h^i{}_{\mathrm{cld}} \frac{A^i{}_{\mathrm{cld}}}{A^i} \right) .$$

It may seem surprising that cumulus detrainment influences $\frac{\partial}{\partial t}(m^i A^i{}_{\mathrm{clr}})$ and $\frac{\partial}{\partial t}(m^i h^i{}_{\mathrm{clr}} A^i{}_{\mathrm{clr}})$, as indicated in (83) and (84), but the explanation is simple: detrainment decreases the clear area because it increases the area occupied by the stratiform cloud. In (83), the detrainment term decreases $A^i{}_{\mathrm{clr}}$, but because this term is proportional to $A^i{}_{\mathrm{clr}}$ it will never make $A^i{}_{\mathrm{clr}} < 0$. Similarly, the detrainment term of (82) tends to increase $A^i{}_{\mathrm{cld}}$, but it will never make $A^i{}_{\mathrm{cld}} > 1$ because it is proportional to $1 - \frac{A^i{}_{\mathrm{cld}}}{A^i}$.

4. Parameterization of intra-cell mass exchanges

4.1 The relationship between cloud perimeter and cloud area

The boundary between the cloudy air and the clear air is an obvious site for evaporation and/or sublimation to occur (hereafter we use the term "evaporation" as a shorthand for both evaporation and sublimation). The lateral mass exchanges between the clear and cloudy sub-regions of a cell occur across cloud lateral boundaries, so the rate of such mass exchange would be expected to vary in proportion to the cloud perimeter. From this point of view, $E^i{}_{\mathrm{cld,clr}}$ and $E^i{}_{\mathrm{clr,cld}}$ should depend in part on the amount of lateral "cloud edge," or the length of cloud perimeter, that exists within the cell.[5]

There are many possible relationships between the total cloud area and the total cloud perimeter. To illustrate this, we consider two possible ways in which the cloud area could increase. As a first example, suppose that a stratiform cloud is divided into n^2 identical square sub-clouds, each of area $a = N^2/n^2$ and each separated from the others by clear spaces of the same size as the clouds. This "checkerboard cloud" is illustrated in the left-hand panel of Fig. 3.

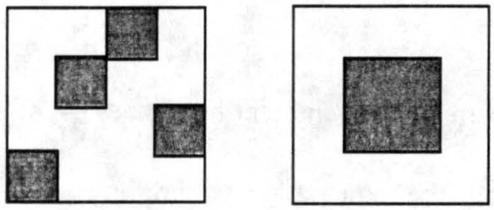

Figure 3: Two grid cells, each with a fractional cloudiness of 25%. In the cell on the left, the cloud has the form of a rectangular block. In the cell on the right, the cloud is divided into four blocks. The cloud perimeter is twice as large in the cell on the left. It could be made even larger by dividing the same cloud area into a larger number of smaller cells.

The perimeter of an individual cloud is $4N/n$, and the total perimeter of all clouds is $4nN$. Suppose that the cloud area increases through a process in which the areas of the individual clouds remain constant, while more clouds (of the same size as those already present) are added; in other words, some cells of the checkerboard are converted from clear to cloudy. Then if the cloud area is small, the total cloud perimeter increases in proportion to the number of clouds, which means that it increases in proportion to the total area occupied by the clouds:

$$\text{Perimeter} \sim \text{Area}. \tag{86}$$

This scenario is analyzed in more detail below.

[5] Evaporation and/or sublimation can still occur even if a grid cell contains no clear air, e.g. due to large-scale sinking motion, but such phase changes do not necessarily reduce the cloud area until the final and complete destruction of the cloud occurs.

As a second example, suppose that the cloudy portion of the grid cell is arranged in a single square block of area N^2, as illustrated in the right-hand panel of Fig. 3. In this case, the perimeter of the cloud is $4N$, so that the total cloud perimeter varies proportion to the square root of the area:

$$\text{perimeter} \sim \sqrt{\text{Area}}. \tag{87}$$

We now analyze the first possibility above, in some detail. Again, consider a checkerboard cloud field, consisting of an array of square cells, each with area a. Each cell can be either cloudy or clear. Suppose that a certain number of cloudy cells are randomly arranged in the field, while the remaining cells are clear. If we randomly choose a clear cell to be converted into a cloudy cell, what happens to the total area and total perimeter of the cloud field? The total cloud area increases by a, but the change in perimeter depends on whether or not the new cloudy cell has cloudy neighbors. If each of the four neighbors (across cell walls) is clear, then the perimeter increases by $4\sqrt{a}$. If one of the neighbors is already cloudy, then the net change in perimeter is $2\sqrt{a}$. If two neighbors are cloudy, there is no net change in the perimeter. If three are cloudy, the perimeter actually decreases; the net change in this case is $-2\sqrt{a}$. Finally, if all four neighbors are cloudy, the net change in the perimeter is $-4\sqrt{a}$. For a random arrangement of clouds with fractional area f, the probabilities of zero, one, two, three, and four cloudy neighbors are as indicated in Table 1 and Fig. 4. A simple calculation shows that the expected change in the total perimeter, when one

Number of cloudy neighbors	***Probability***	***Change in perimeter***
0	f^4	$4\sqrt{a}$
1	$4f^3(1-f)$	$2\sqrt{a}$
2	$6f^2(1-f)^2$	0
3	$4f(1-f)^3$	$-2\sqrt{a}$
4	$(1-f)^4$	$-4\sqrt{a}$

Table 1: Probability and incremental change in total cloud perimeter for an array of square cells. Here neighbors are counted only across cell walls. See Fig. 4.

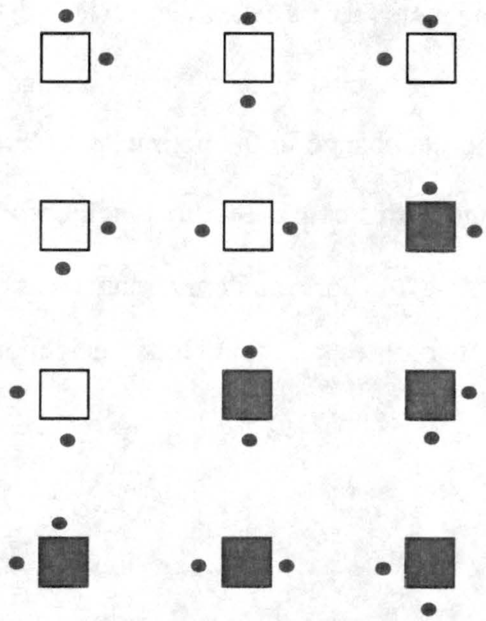

Figure 4: Here we display all possible combinations of square cells with two neighbors across cell walls; the positions of the neighbors are indicated by the dots. Duplicates are shaded; there are 6 unique arrangements. This is where the factor of 6 comes from in the third row of Table 1.

cloudy cell is added, is

$$d(\text{perimeter}) = 4\sqrt{a}(1-2f). \tag{88}$$

Integration gives

$$\text{perimeter}(f) = 4\sqrt{a}f(1-f). \tag{89}$$

The perimeter thus increases most rapidly with f when f is small, reaches a maximum when $f = 1/2$, and decreases symmetrically with further increases in f.

Obviously the preceding example, with its array of square clouds, is not very realistic. Real cloud fields do sometimes resemble arrays of hexagonal cells, however. In contrast to square arrays, hexagonal arrays have the nice property that every neighbor lies directly across a cell wall. For a hexagonal cloud field, we obtain the probabilities and perimeter changes shown in Table 2 and Fig. 5. The result is almost the same as before:

Number of cloudy neighbors	***Probability***	***Change in perimeter***
0	f^6	$6l$
1	$6f^5(1-f)$	$4l$
2	$15f^4(1-f)^2$	$2l$
3	$20f^3(1-f)^3$	0
4	$15f^2(1-f)^4$	$-2l$
5	$6f(1-f)^5$	$-4l$
6	$(1-f)^6$	$-6l$

Table 2: Probability and incremental change in total cloud perimeter for an array of hexagonal cells. Here l is the length of one segment of a hexagonal cell wall. See Fig. 5.

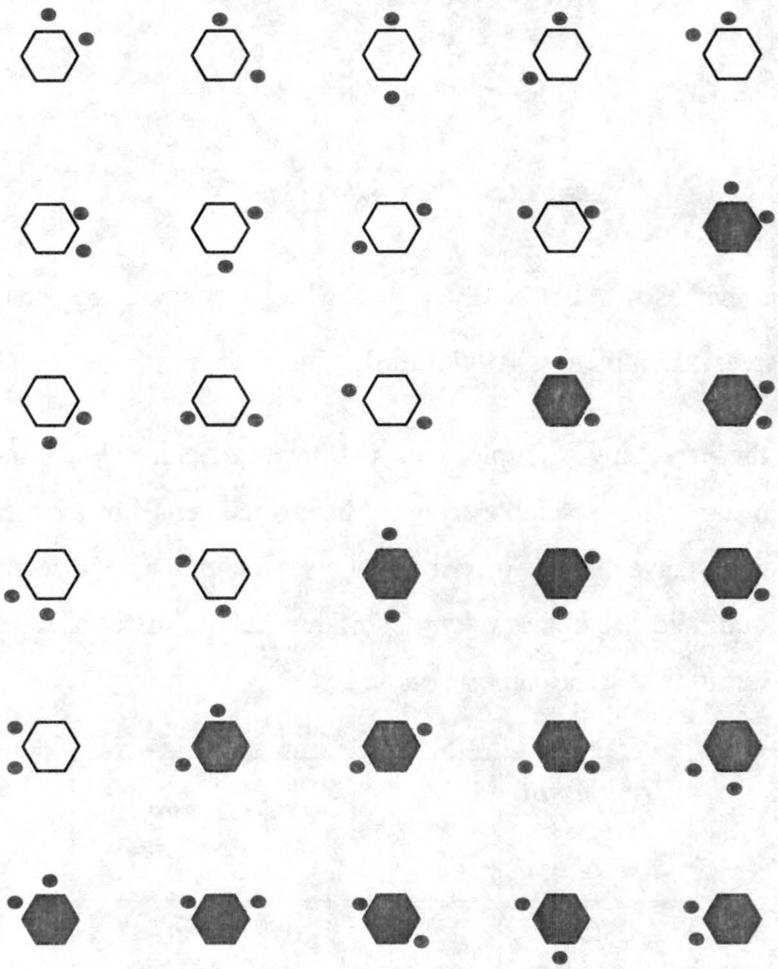

Figure 5: Here we display all possible combinations of hexagonal cells with two neighbors; the positions of the neighbors are indicated by the dots. Duplicates are shaded; there are 15 unique arrangements. This is where the factors of 15 come from in the third and fifth rows of Table 2. The factor of 20 in the fourth row can be obtained by requiring that the sum of all probabilities equals 1.

$$d(\text{perimeter}) = 6l(1-2f) \quad , \tag{90}$$

$$\text{perimeter}(f) = 6lf(1-f) \quad ; \tag{91}$$

here l is the length of one cell wall.

4.2 *Parameterization*

Assuming that the cloud field is divided into an array of cloudy cells, we expect the total perimeter of the cloud field to vary with the total area of the cloud field in accord with the $f(1-f)$ dependence shown in (89) and (91). This motivates the following assumptions:

$$E^i_{\text{cld,clr}} = -C_{\text{cld,clr}} m^i A^i \left[\frac{A^i_{\text{cld}} A^i_{\text{clr}}}{(A^i)^2}\right]\left(\frac{\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}}}{q^i_c + q^i_i}\right), \tag{92}$$

$$E^i_{\text{clr,cld}} = C_{\text{clr,cld}} m^i A^i \left[\frac{A^i_{\text{cld}} A^i_{\text{clr}}}{(A^i)^2}\right]\left[\frac{\dot{q}^i_{c\ \text{cond}} + \dot{q}^i_{i\ \text{cond}}}{(q^i_c + q^i_i)(1 - RH^i_{\text{clr}})}\right]. \tag{93}$$

Here $C_{\text{cld,clr}}$ and $C_{\text{clr,cld}}$ are nondimensional parameters, assumed for simplicity and in the absence of evidence to the contrary to be constants; $(\dot{q}_c)^i_{\text{evap}} \le 0$ and $(\dot{q}_i)^i_{\text{evap}} \le 0$ are the rates of change of q_c and q_i due to evaporation in the cloudy part of grid cell i; $(\dot{q}_c)^i_{\text{cond}} \ge 0$ and $(\dot{q}_i)^i_{\text{cond}} \ge 0$ are the corresponding rates of change of q_c and q_i due to condensation in the cloudy part of grid cell i; and $RH^i_{\text{clr}} \equiv (q^i_v)_{\text{clr}}/(q^i_*)_{\text{clr}}$ is the clear-air relative humidity of grid cell i. Both $E^i_{\text{cld,clr}}$ and $E^i_{\text{clr,cld}}$ are assumed to be proportional to $A^i_{\text{cld}} A^i_{\text{clr}}$, which, in turn, is assumed to be proportional to the total perimeter of the cloud, based on the arguments given above. According to (92), evaporation is associated with a transformation of cloudy air into clear air, which becomes very efficient when $q^i_c + q^i_i \to 0$. Similarly, (93) states that condensation is associated with a transformation of clear air into cloudy air, which becomes very efficient when $RH^i_{\text{clr}} \to 1$, i.e. when the relative humidity of the clear air approaches 100%. When the relative humidity of the clear air reaches 100%, the denominator of (93) goes to zero, which means that $E^{i}_{\text{clr,cld}}$ can become non-zero even if $A^{i}_{\text{cld}} = 0$; this means that we can create cloud directly from clear air as the water vapor mixing ratio of the clear air reaches saturation.

We can develop a further interpretation of (92), as follows. Begin by writing

$$\left\{\frac{\partial}{\partial t}[m^{i} A^{i}_{\text{cld}}(q^{i}_{c} + q^{i}_{i})]\right\}_{\text{evap}}$$

$$= m^{i} A_{\text{cld}}[(\dot{q}_{c})^{i}_{\text{evap}} + (\dot{q}_{i})^{i}_{\text{evap}}] + (q^{i}_{c} + q^{i}_{i})\left[\frac{\partial}{\partial t}(m^{i} A^{i}_{\text{cld}})\right]_{\text{evap}} \tag{94}$$

$$= m^{i} A_{\text{cld}}[(\dot{q}_{c})^{i}_{\text{evap}} + (\dot{q}_{i})^{i}_{\text{evap}}] - (q^{i}_{c} + q^{i}_{i})E^{i}_{\text{cld,clr}} .$$

The left-hand side of (94) represents the total rate of evaporation of cloud mass inside the grid cell. According to (94), this total consists of a part that comes from the reduction of the cloud water mixing ratio within the cloudy region, given by $m^{i} A_{\text{cld}}[(\dot{q}_{c})^{i}_{\text{evap}} + (\dot{q}_{i})^{i}_{\text{evap}}]$; and a part that comes from a reduction of the cloud area, given by $-(q^{i}_{c} + q^{i}_{i})E^{i}_{\text{cld,clr}}$. Let the ratio of these two contributions be denoted by B_{evap}, i.e.

$$B_{\text{evap}} \equiv \frac{\text{rate of cloud destruction by transformation of cloudy air to clear air}}{\text{rate of cloud destruction by reduction of the in-cloud condensate mixing ratio}}$$

$$= \frac{(q^{i}_{c} + q^{i}_{i})E^{i}_{\text{cld,clr}}}{m^{i} A_{\text{cld}}[(\dot{q}_{c})^{i}_{\text{evap}} + (\dot{q}_{i})^{i}_{\text{evap}}]} . \tag{95}$$

Rearranging (95) gives

$$E^{i}_{\text{cld,clr}} = -B_{\text{evap}} m^{i} A_{\text{cld}}\left(\frac{\dot{q}^{i}_{c\ \text{evap}} + \dot{q}^{i}_{i\ \text{evap}}}{q^{i}_{c} + q^{i}_{i}}\right). \tag{96}$$

Comparing (96) with (92), we see that

$$B_{\text{evap}} \equiv \frac{\text{rate of cloud destruction by transformation of cloudy air to clear air}}{\text{rate of cloud destruction by reduction of the in-cloud condensate mixing ratio}}$$
$$= C_{\text{cld,clr}} \frac{A^i_{\text{clr}}}{A^i} \ . \tag{97}$$

Recalling our assumption that $C_{\text{cld,clr}}$ is a constant, we see that (97) states that when A^i_{clr}/A^i is small, i.e. when there is very little clear air in the grid cell, the destruction of the cloud is mostly due to a reduction in the in-cloud condensate mixing ratio, and reduction in the cloud area is secondary. In effect, we are assuming that the transformation of cloudy air into clear air is favored by pre-existing clear air. This assumption implies that, if the grid cell is completely filed with cloud, clear air can be produced only by reducing the condensate mixing ratio to zero, at which point $E^i_{\text{cld,clr}} \to \infty$. As the area covered by clear air increases, the transformation of cloudy air to clear air becomes increasingly important, relative to the reduction of the in-cloud condensate mixing ratio.

To similarly interpret (93), begin with

$$\left\{\frac{\partial}{\partial t}[m^i A^i_{\text{cld}}(q^i_c + q^i_i)]\right\}_{\text{cond}}$$
$$= m^i A^i_{\text{cld}}[(\dot{q}_c)^i_{\text{cond}} + (\dot{q}_i)^i_{\text{cond}}] + (q^i_c + q^i_i)\left[\frac{\partial}{\partial t}(m^i A^i_{\text{cld}})\right]_{\text{cond}} \tag{98}$$
$$= m^i A^i_{\text{cld}}[(\dot{q}_c)^i_{\text{cond}} + (\dot{q}_i)^i_{\text{cond}}] - (q^i_c + q^i_i) E^i_{\text{clr,cld}} \ ,$$

and define

$$B_{\text{cond}} \equiv \frac{\text{cloud generation by transformation of clear air to cloudy air}}{\text{cloud generation by an increase of the in-cloud condensate mixing ratio}}$$

$$= \frac{(q^i_c + q^i_i)E^i_{\text{clr,cld}}}{m^i A_{\text{cld}}[(\dot{q}_c)^i_{\text{cond}} + (\dot{q}_i)^i_{\text{cond}}]} \,, \tag{99}$$

so that

$$E^i_{\text{clr,cld}} = B_{\text{cond}} m^i A_{\text{cld}} \left[\frac{(\dot{q}_c)^i_{\text{cond}} + (\dot{q}_i)^i_{\text{cond}}}{(q^i_c + q^i_i)} \right]. \tag{100}$$

Comparing (100) and (93), we see that

$$B_{\text{cond}} = \frac{\text{cloud generation by transformation of clear air to cloudy air}}{\text{cloud generation by an increase of the in-cloud condensate mixing ratio}}$$

$$= C_{\text{clr,cld}} \frac{A^i_{\text{clr}}}{A^i} \left(\frac{1}{1 - RH^i_{\text{clr}}} \right) . \tag{101}$$

An interpretation of (101) is that, for a given value of A^i_{clr}/A^i, as $RH^i_{\text{clr}} \rightarrow 1$ cloud generation by transformation of clear air to cloudy air dominates over cloud generation by an increase of the in-cloud condensate mixing ratio. In other words, transformation of clear air to cloudy air is favored if there is lots of humid clear air available. On the other hand, as the cloud amount approaches 100%, for a given value of RH^i_{clr}, cloud generation by an increase of the in-cloud mixing ratio dominates over cloud generation by transformation of clear air to cloudy air.

It is useful to compare (92) and (93) with the corresponding parameterizations of Tiedtke (1993), which are (using the notation of the present paper)

$$E^i_{\text{cld,clr}} = 0 \,, \tag{102}$$

$$E^{i}{}_{\text{clr,cld}} = \frac{A^{i}{}_{\text{clr}}}{A^{i}} \frac{m^{i}}{q^{i}{}_{*}} \left(\frac{1}{1 - RH^{i}{}_{\text{clr}}} \right) Max\left\{ -\frac{dq^{i}{}_{*}}{dt}, 0 \right\} . \tag{103}$$

Here $-\frac{dq^{i}{}_{*}}{dt}$ is (in Tiedtke's parameterization) proportional to the rate of production of cloud water and/or cloud ice by condensation. In contrast to Tiedtke (1993), we permit a non-zero "evaporative mass flux," $E^{i}{}_{\text{cld,clr}}$. On the other hand, our parameterization of the "condensation mass flux," $E^{i}{}_{\text{clr,cld}}$ is similar to Tiedtke's in that both are proportional to $m^{i} \frac{A^{i}{}_{\text{clr}}}{A^{i}} \left(\frac{1}{1 - RH^{i}{}_{\text{clr}}} \right)$ times a measure of the condensation rate.

5. Computational procedure

Using (81)-(85) and (92)-(93), we now rewrite (73)-(80) as follows:

$$\begin{aligned}
\frac{\partial}{\partial t}(m^{i} A^{i}{}_{\text{clr}}) &= -\sum_{i'} \hat{m}^{i,i'} v^{i,i'} {}_{n} l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\text{clr}}}{\hat{A}^{i,i'}} \\
& -m^{i} A^{i} \left[\frac{A^{i}{}_{\text{cld}} A^{i}{}_{\text{clr}}}{(A^{i})^{2}} \right] \left\{ C_{\text{cld,clr}} \left(\frac{\dot{q}^{i}{}_{c\ \text{evap}} + \dot{q}^{i}{}_{i\ \text{evap}}}{q^{i}{}_{c} + q^{i}{}_{i}} \right) + C_{\text{clr,cld}} \left[\frac{\dot{q}^{i}{}_{c\ \text{cond}} + \dot{q}^{i}{}_{i\ \text{cond}}}{(q^{i}{}_{c} + q^{i}{}_{i})(1 - RH^{i}{}_{\text{clr}})} \right] \right\} \\
& -\frac{\partial}{\partial z}(A^{i}{}_{\text{clr}} \overline{M}^{i} - A^{i} M^{i}{}_{\text{meso}}) + \left(M_{c}{}^{i} \frac{\partial A^{i}{}_{\text{clr}}}{\partial z} - D^{i} \frac{A^{i}{}_{\text{clr}}}{A^{i}} \right) ,
\end{aligned} \tag{104}$$

$$\frac{\partial}{\partial t}(m^i A^i{}_{\text{cld}}) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\text{cld}}}{\hat{A}^{i,i'}}$$

$$+ m^i A^i \left[\frac{A^i{}_{\text{cld}} A^i{}_{\text{clr}}}{(A^i)^2}\right] \left\{ C_{\text{clr,cld}} \left[\frac{\dot{q}^i_{c\ \text{cond}} + \dot{q}^i_{i\ \text{cond}}}{(q^i{}_c + q^i{}_i)(1 - RH^i{}_{\text{clr}})}\right] + C_{\text{cld,clr}} \left(\frac{\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}}}{q^i{}_c + q^i{}_i}\right)\right\} \tag{105}$$

$$- \frac{\partial}{\partial z}(A^i{}_{\text{cld}} \overline{M}^i + A^i M^i{}_{\text{meso}}) + \left[M_c{}^i \frac{\partial A^i{}_{\text{cld}}}{\partial z} + \left(1 - \frac{A^i{}_{\text{cld}}}{A^i}\right) D^i \right] ,$$

$$\frac{\partial}{\partial t}(m^i A^i) = \sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} - \frac{\partial}{\partial z}(\overline{M}^i A^i), \tag{106}$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\text{clr}} A^i{}_{\text{clr}}) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\text{clr}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\text{clr}}$$

$$+ m^i A^i \left[\frac{A^i{}_{\text{cld}} A^i{}_{\text{clr}}}{(A^i)^2}\right] \left\{ C_{\text{cld,clr}} \left(\frac{\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}}}{q^i{}_c + q^i{}_i}\right) - C_{\text{clr,cld}} \left[\frac{\dot{q}^i_{c\ \text{cond}} + \dot{q}^i_{i\ \text{cond}}}{(q^i{}_c + q^i{}_i)(1 - RH^i{}_{\text{clr}})}\right]\right\}$$

$$- \frac{\partial}{\partial z}\left[(A^i{}_{\text{clr}} \overline{M}^i - A^i M^i{}_{\text{meso}}) h^i{}_{\text{clr}}\right] + (S_h)^i{}_{\text{clr}} A^i{}_{\text{clr}} \tag{107}$$

$$+ \left[M_c{}^i \frac{\partial}{\partial z}(h^i{}_{\text{clr}} A^i{}_{\text{clr}}) - D^i h^i{}_{\text{clr}} \frac{A^i{}_{\text{clr}}}{A^i} \right] ,$$

$$\frac{\partial}{\partial t}(m^i h^i{}_{\text{cld}} A^i{}_{\text{cld}}) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\text{cld}}}{\hat{A}^{i,i'}} \hat{h}^{i,i'}{}_{\text{cld}}$$

$$+ m^i A^i \left[\frac{A^i{}_{\text{cld}} A^i{}_{\text{clr}}}{(A^i)^2}\right] \left\{ C_{\text{clr,cld}} \left[\frac{\dot{q}^i_{c\ \text{cond}} + \dot{q}^i_{i\ \text{cond}}}{(q^i{}_c + q^i{}_i)(1 - RH^i{}_{\text{clr}})}\right] - C_{\text{cld,clr}} \left(\frac{\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}}}{q^i{}_c + q^i{}_i}\right)\right\}$$

$$- \frac{\partial}{\partial z}\left[(A^i{}_{\text{cld}} \overline{M}^i + A^i M^i{}_{\text{meso}}) h^i{}_{\text{cld}}\right] + (S_h)^i{}_{\text{cld}} A^i{}_{\text{cld}} \tag{108}$$

$$+ \left[M_c{}^i \frac{\partial}{\partial z}(h^i{}_{\text{cld}} A^i{}_{\text{cld}}) + D^i \left(h^i{}_{\text{cu}} - h^i{}_{\text{cld}} \frac{A^i{}_{\text{cld}}}{A^i}\right)\right] ,$$

$$\frac{\partial}{\partial t}(m^i \bar{h}^i A^i) = -\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \hat{h}^{i,i'} + \sum_{i'} G^{i,i'}(h)$$

$$-\frac{\partial}{\partial z}\{A^i[\bar{M}^i \bar{h}^i + M^i{}_{\text{meso}}(h^i{}_{\text{cld}} - h^i{}_{\text{clr}})]\} + [A^i{}_{\text{cld}}(S^i_h)_{\text{cld}} + A^i{}_{\text{clr}}(S^i_h)_{\text{clr}}] \tag{109}$$

$$+ A^i M_c{}^i \frac{\partial \bar{h}^i}{\partial z} + D^i(h^i{}_{\text{cu}} - \bar{h}^i) + E^i\left[\frac{A^i{}_{\text{clr}}}{A^i}(\bar{h}^i - h^i{}_{\text{clr}}) + \frac{A^i{}_{\text{cld}}}{A^i}(\bar{h}^i - h^i{}_{\text{cld}})\right].$$

Following Tiedtke (1993), we prognostically determine $A^i{}_{\text{cld}}$. In view of (66), there is no need to predict $A^i{}_{\text{clr}}$ and $A^i{}_{\text{cld}}$ separately; prediction of $A^i{}_{\text{cld}}$ suffices, since (66) can then be used to diagnose $A^i{}_{\text{clr}}$.

Consider two possible ways to use equations (107)-(109):

Method A: Separately predict the properties of the clear and cloudy sub-regions. To determine $\bar{h}^i$, we could use (67). This approach is simple, and it is fine in principle, but practical and/or philosophical objections can be raised, along the following lines: Existing models predict $\bar{h}^i$. Rewriting a model to predict $h^i{}_{\text{clr}}$ and $h^i{}_{\text{cld}}$ would be a lot of work. Other current and/or future cloud parameterizations do/will not make use of the framework proposed in the present paper, so building this framework into a model at the "foundation level" seems unwise.

Method B: Continue to predict $\bar{h}^i$, as in current models, using (109). In addition, predict $A^i{}_{\text{cld}}$ and $h^i{}_{\text{cld}} - h^i{}_{\text{clr}}$, the latter using an equation to be derived below. From these predicted values, diagnose $h^i{}_{\text{cld}}$ and $h^i{}_{\text{clr}}$ using

$$h^i_{\text{cld}} = \bar{h}^i + \left(1 - \frac{A^i_{\text{cld}}}{A^i}\right)(h^i_{\text{cld}} - h^i_{\text{clr}}), \tag{110}$$

and

$$h^i_{\text{clr}} = \bar{h}^i - \frac{A^i_{\text{cld}}}{A^i}(h^i_{\text{cld}} - h^i_{\text{clr}}), \tag{111}$$

respectively.

We use Method B to predict the generalized moist static energy and the water vapor mixing ratio. We use Method A to predict the condensed water variables. Further discussion is given in Section 6.

For use with Method B, we need a prognostic equation for $h^i_{\text{cld}} - h^i_{\text{clr}}$, which can be derived as follows: By using (104) in (107), we can obtain the advective form corresponding to (107):

$$\begin{aligned}
\frac{\partial h^i_{\text{clr}}}{\partial t} = & -\frac{1}{m^i A^i_{\text{clr}}}\sum_{i'}\left(\hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\text{clr}}}{\hat{A}^{i,i'}}\right)(\hat{h}^{i,i'}{}_{\text{clr}} - h^i_{\text{clr}}) \\
& -C_{\text{cld,clr}}\left(\frac{A^i_{\text{cld}}}{A^i}\right)\left(\frac{\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}}}{q^i_c + q^i_i}\right)(h^i_{\text{cld}} - h^i_{\text{clr}}) \\
& -\left\{\frac{\overline{M}^i}{m^i} - \frac{1}{m^i}\left[\frac{A^i_{\text{cld}}}{A^i}(M^i_{\text{cld}} - M^i_{\text{clr}})\right]\right\}\frac{\partial h^i_{\text{clr}}}{\partial z} + \frac{(S_h)^i_{\text{clr}}}{m^i} + \frac{M_c{}^i}{m^i}\frac{\partial h^i_{clr}}{\partial z}\ .
\end{aligned} \tag{112}$$

Here we have used (45). Similarly, we can show that

$$\frac{\partial h^i_{\text{cld}}}{\partial t} = -\frac{1}{m^i A^i_{\text{cld}}} \sum_{i'} \left(\hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \frac{\hat{A}^{i,i'}_{\text{cld}}}{\hat{A}^{i,i'}} \right) (\hat{h}^{i,i'}_{\text{cld}} - h^i_{\text{cld}})$$

$$-C_{\text{clr,cld}} \left(\frac{A^i_{\text{clr}}}{A^i} \right) \left[\frac{\dot{q}_c{}^i_{\text{cond}} + \dot{q}_i{}^i_{\text{cond}}}{(q^i_c + q^i_i)(1 - RH^i_{\text{clr}})} \right] (h^i_{\text{cld}} - h^i_{\text{clr}}) \tag{113}$$

$$-\left\{ \frac{\overline{M}^i}{m^i} + \frac{1}{m^i} \left[\frac{A^i_{\text{clr}}}{A^i} (M^i_{\text{cld}} - M^i_{\text{clr}}) \right] \right\} \frac{\partial h^i_{\text{cld}}}{\partial z} + \frac{(S_h)^i_{\text{cld}}}{m^i} + \frac{M_c{}^i}{m^i} \frac{\partial h^i_{\text{cld}}}{\partial z} + \frac{D^i}{m^i A^i_{\text{cld}}} (h^i_{\text{cu}} - h^i_{\text{cld}}) \ .$$

In the horizontal advection term of (112) and in the horizontal advection and cumulus detrainment terms of (113), there is an apparent danger of division by zero when $A^i_{\text{clr}} \to 0$ or $A^i_{\text{cld}} \to 0$. The physical meaning is that h^i_{clr} can be instantly "re-set" by advection when $A^i_{\text{clr}} = 0$, and that h^i_{cld} can be instantly re-set by either advection or cumulus detrainment when $A^i_{\text{cld}} = 0$. The danger of division by zero can be eliminated through the use of suitable finite-difference methods. Details are discussed in Appendix B.

To obtain a prognostic equation for $h^i_{\text{cld}} - h^i_{\text{clr}}$, subtract (112) from (113):

$$
\begin{aligned}
\frac{\partial}{\partial t}(h^i_{\text{cld}} - h^i_{\text{clr}}) = & -\sum_{i'}\left(\frac{\hat{m}^{i,i'} v^{i,i'} {}_n l^{i,i'} \hat{A}^{i,i'}{}_{\text{cld}}}{m^i A^i_{\text{cld}}}\right)(\hat{h}^{i,i'}{}_{\text{cld}} - h^i_{\text{cld}}) \\
& + \sum_{i'}\left(\frac{\hat{m}^{i,i'} v^{i,i'} {}_n l^{i,i'} \hat{A}^{i,i'}{}_{\text{clr}}}{m^i A^i_{\text{clr}}}\right)(\hat{h}^{i,i'}{}_{\text{clr}} - h^i_{\text{clr}}) \\
& -\left(\frac{h^i_{\text{cld}} - h^i_{\text{clr}}}{q^i_c + q^i_i}\right)\left\{C_{\text{clr,cld}}\left(\frac{A^i_{\text{clr}}}{A^i}\right)\left(\frac{\dot{q}^i_{c\ \text{cond}} + \dot{q}^i_{i\ \text{cond}}}{1 - RH^i_{\text{clr}}}\right) - C_{\text{cld,clr}}\left(\frac{A^i_{\text{cld}}}{A^i}\right)(\dot{q}^i_{c\ \text{evap}} + \dot{q}^i_{i\ \text{evap}})\right\} \\
& - \frac{\overline{M}^i}{m^i}\frac{\partial}{\partial z}(h^i_{\text{cld}} - h^i_{\text{clr}}) - \left(\frac{M_{\text{cld}}{}^i}{m^i} - \frac{M_{\text{clr}}{}^i}{m^i}\right)\left(\frac{A^i_{\text{clr}}}{A^i}\frac{\partial h^i_{\text{cld}}}{\partial z} + \frac{A^i_{\text{cld}}}{A^i}\frac{\partial h^i_{\text{clr}}}{\partial z}\right) \\
& + \frac{(S_h)^i_{\text{cld}}}{m^i} - \frac{(S_h)^i_{\text{clr}}}{m^i} + \left[\frac{M_c{}^i}{m^i}\frac{\partial}{\partial z}(h^i_{\text{cld}} - h^i_{\text{clr}}) + \left(\frac{D^i}{m^i A^i_{\text{cld}}}\right)(h^i_{\text{cu}} - h^i_{\text{cld}})\right] .
\end{aligned}
\tag{114}
$$

The lateral mass exchange terms of (114) act to damp $(h^i_{\text{cld}} - h^i_{\text{clr}})$ towards zero.

In order to predict A^i_{cld}, $\bar{h}^i$ and $(h^i_{\text{cld}} - h^i_{\text{clr}})$, we must determine M^i_{meso}, $M_{\text{cld}}{}^i$, and $M_{\text{clr}}{}^i$. Methods to do so are discussed in Section 7.

6. Microphysical processes

The microphysical parameterizations of the model, which represent the microphysical processes occurring in the cloudy portion of the grid cell, basically follow those described by Fowler et al. (1996). We distinguish a total of five prognostic water species: water vapor, with mixing ratio q_v; cloud water, with mixing ratio q_c; cloud ice, with mixing ratio q_i; rain falling from stratiform clouds, with mixing ratio q_r; and snow falling from stratiform clouds, with mixing ratio q_s. Our bulk cloud microphysics parameterization also includes a prognostic equation for the generalized moist static energy, h, defined by Lord (1978) and discussed by Fowler et al. (1996). In addition, of course, we prognose A^i_{cld}.

Each of the six thermodynamic variables (h, q_v, q_c, q_i, q_r, and q_s) is assigned values for the cloudy and clear regions separately, although as discussed in Section 2 we *define* the clear sub-region to be one in which the mixing ratios of cloud water and cloud ice are zero. Table 3 summarizes the prognostic thermodynamic variables used by Eauliq NG. The model prognoses

Variable Definition	***Symbol***
Mean generalized moist static energy	$\bar{h}$
Generalized moist static energy difference between cloudy and clear air	$h_{\mathrm{cld}} - h_{\mathrm{clr}}$
Mean water vapor mixing ratio	$\overline{q_v}$
Water vapor mixing ratio difference between cloudy and clear air	$(q_v)_{\mathrm{cld}} - (q_v)_{\mathrm{clr}}$
Stratiform cloud area	A_{cld}
Cloud water mixing ratio in the cloudy sub-region	q_c; The subscript "cld" is omitted on the grounds that it would be redundant.
Cloud ice mixing ratio in the cloudy sub-region	q_i; The subscript "cld" is omitted on the grounds that it would be redundant.
Rain water mixing ratio in the cloudy sub-region	$(q_r)_{\mathrm{cld}}$
Rain water mixing ratio in the clear sub-region	$(q_r)_{\mathrm{clr}}$

Table 3: Prognostic thermodynamic variables used by Eauliq NG.

Variable Definition	*Symbol*
Snow mixing ratio in the cloudy sub-region	$(q_s)_{\mathrm{cld}}$
Snow mixing ratio in the clear sub-region	$(q_s)_{\mathrm{clr}}$

Table 3: Prognostic thermodynamic variables used by Eauliq NG.

(i.e. time-steps) the grid-cell averaged generalized moist static energy, and the difference in generalized moist static energy between the cloudy and clear portions of the box. Similarly, the model prognoses the grid-cell averaged water vapor mixing ratio, and the difference in water vapor mixing ratio between the cloudy and clear portions of the box. In the terminology of Section 5, we use Method B for the generalized moist static energy and the water vapor.

The model also prognoses the cloud water mixing ratio in the cloudy portion of the box (which can be considered to be the difference between the cloud water mixing ratio in the cloudy and clear portions of the cell). Similarly, the model prognoses the cloud ice mixing ratio in the cloudy portion of the box. There is no need to separately prognose the grid-cell averaged values of the cloud water or cloud ice, but we can diagnose them, if we wish, using (67). This means that Method A is used for the cloud water and cloud ice.

Precipitation obviously originates in the cloudy portions of grid cells, but it can fall into either the cloudy portion or the clear portion of a lower-level cell. We use Method A to predict the rain water and snow mixing ratios in the cloudy and clear sub-regions of each cell.

7. Mesoscale circulations

7.1 Background

As discussed in the Introduction, we assume that the stratiform cloud under consideration is and remains through time neutrally buoyant with respect to its clear environment. Neglecting (just temporarily) virtual temperature effects, this neutral buoyancy condition can be expressed by:

$$\overline{T} = T_{\text{cld}} = T_{\text{clr}}; \tag{115}$$

$$\overline{\Gamma} = \Gamma_{\text{cld}} = \Gamma_{\text{clr}}, \tag{116}$$

where $\Gamma \equiv -\frac{\partial T}{\partial z}$ is the lapse rate of temperature; and finally

$$\frac{d}{dt}\overline{T} = \frac{dT_{\text{cld}}}{dt} = \frac{dT_{\text{clr}}}{dt}. \tag{117}$$

We cannot simply assume that (115)-(117) are satisfied; a physical process must act to ensure that they are satisfied, and such a process can produce additional effects that go beyond (115)-(117). A candidate process is proposed below.

Consider a non-precipitating stratiform liquid water cloud that is caught up in dynamically imposed large-scale vertical motion, which could be either upward or downward. For simplicity, suppose that radiative cooling and horizontal advection are negligible, and neglect cumulus effects. We temporarily assume that within the cloud layer the cloud amount is 100%. As the cloudy air rises or sinks, it cools or warms along a moist adiabat, so that

$$\frac{DT}{Dt} = -\overline{w}\Gamma_m \text{ in cloud}, \tag{118}$$

where the Lagrangian derivative is defined to follow the mean motion:

$$\frac{D(\ )}{Dt} \equiv \frac{\partial}{\partial t}(\ ) + \overline{\mathbf{V}} \bullet \nabla(\ ) + \overline{w}\frac{\partial}{\partial z}(\ ). \tag{119}$$

In (119), which is (for purposes of this section only) the definition of the Lagrangian derivative, we include a term that represents horizontal advection, even though we are currently assuming that horizontal advection is negligible. We have used the mean vertical velocity, $\overline{w}$, in (119), with the understanding that because the cloud amount is 100% by assumption, $\overline{w} = w_{\text{cld}}$.

The dry air above cloud top follows a dry adiabat, so that

$$\frac{DT}{Dt} = -\overline{w}\Gamma_d \text{ in clear air.} \tag{120}$$

Here we have used the mean vertical velocity, with the understanding that because we are considering a level above the cloud top, $\overline{w} = w_{\text{clr}}$. Suppose that at a given moment the temperature varies continuously with height. For $\overline{w} < 0$, the sinking air will subsequently warm both above and below cloud top, but more rapidly (along the dry adiabat) above cloud top, and more slowly (along the moist adiabat) below cloud top, so that after some time there will be a discontinuous upward increase of temperature across the cloud top. This illustrates that large-scale sinking motion tends to produce sharp cloud-top inversions, which are of course often seen at real cloud tops. Although there are additional mechanisms, such as radiative cooling, that can promote the formation of inversions at stratiform cloud tops, we note here that large-scale sinking motion alone suffices.

For $\overline{w} > 0$, the rising air cools both above and below cloud top but more rapidly above, so that an initially continuous temperature profile tends to develop a statically unstable upward decrease of temperature across the cloud top. Of course, small-scale convection will very quickly develop so as to prevent such a statically unstable layer from actually being generated (Arakawa and Schubert, 1974).

Now allow the possibility of fractional cloudiness, denoted by $f = \frac{A_{\text{cld}}}{A}$. We continue to suppose for simplicity that radiative cooling, horizontal advection, and cumulus convection are negligible. Assume that the temperature distribution is initially horizontally uniform and vertically continuous. Consider the case of large-scale subsidence. As the sinking cloud evaporates, its temperature increases along a moist adiabat. If the environment at the same level as the cloud sinks at the same rate as the cloud, its temperature will increase more rapidly, along the dry adiabat. This means that after some time the cloud will be colder than its environment at the same level, so that (115)-(117) will be violated. We can avoid this predicament if we allow the vertical velocities experienced by the cloud and the environment to be different, i.e.

$$\frac{\partial T_{\text{cld}}}{\partial t} = w_{\text{cld}}(\Gamma_{\text{cld}} - \Gamma_m), \tag{121}$$

$$\frac{\partial T_{\text{clr}}}{\partial t} = w_{\text{clr}}(\Gamma_{\text{clr}} - \Gamma_d). \tag{122}$$

The large-scale vertical velocity is given by the area average of the vertical velocities of the cloud and its environment:

$$\bar{w} = f w_{\text{cld}} + (1 - f) w_{\text{clr}}. \tag{123}$$

Equations very similar to (121)-(123) were studied by Bjerknes (1938), although he did not use (115)-(117). Combining (115)-(117) and (121)-(123), we find that

$$w_{\text{clr}} - \bar{w} = -\left[\frac{f(\Gamma_d - \Gamma_m)}{f(\Gamma_d - \Gamma) - (1 - f)(\Gamma - \Gamma_m)}\right]\bar{w} \tag{124}$$

and

$$w_{\text{cld}} - \overline{w} = \left[\frac{(1-f)(\Gamma_d - \Gamma_m)}{f(\Gamma_d - \Gamma) - (1-f)(\Gamma - \Gamma_m)} \right] \overline{w} \tag{125}$$

are the mesoscale vertical velocities needed to keep the cloud and environment at the same temperature. From (124) and (125), we see that $w_{\text{clr}} = w_{\text{cld}} = 0$ if $\overline{w} = 0$; this is natural, because in the present simplified scenario it is *only* the large-scale vertical motion that is disturbing the equilibrium of the system. Suppose that $\Gamma = \Gamma_m$, which will be the case if turbulence keeps the interior of the cloud in a well-mixed state. Then (124) and (125) reduce to

$$w_{\text{clr}} = 0, \tag{126}$$

$$w_{\text{cld}} = \frac{\overline{w}}{f}. \tag{127}$$

These results mean that, within the current but temporary restrictive assumptions, *vertical motion occurs entirely within the cloudy region, and not at all in the environment.* The cloudy air rises, or the cloudy air sinks. The environment sits still. If a large-scale dynamical process imposes large-scale vertical motion on a region containing fractional cloudiness, the cloudy air carries out the required vertical movements, depending on the value of f, but the environment does not join in. Such differences in vertical motion between the cloud and its environment can be interpreted as consequences of an adjustment process, which counteracts the warming or cooling of the cloud relative to its environment so as to maintain a state of balance, with no horizontal pressure differences between the stratiform cloud and its clear environment.

As discussed earlier, the mesoscale circulations represented by (124)-(125) or (126)-(127) transport various quantities vertically. From (124) and (125), the mesoscale mass flux can be written as

$$w_{\mathrm{cld}} - w_{\mathrm{clr}} = \left[\frac{(\Gamma_d - \Gamma_m)}{f(\Gamma_d - \Gamma) - (1-f)(\Gamma - \Gamma_m)}\right]\overline{w}. \tag{128}$$

In case $\Gamma = \Gamma_m$, this reduces to

$$w_{\mathrm{cld}} - w_{\mathrm{clr}} = \frac{\overline{w}}{f}, \tag{129}$$

which implies that, for a given $\overline{w}$, a strong mesoscale mass flux is favored by a small cloud fractional area. If the cloud amount approaches one, the mesoscale mass flux goes to zero. Note that the mesoscale mass flux has the same sign as the large-scale vertical motion; in the case of large-scale subsidence, the mesoscale mass flux is downward.

7.2 Parameterization of the mesoscale mass flux

The preceding discussion is intended to show that the mesoscale circulations should act in such a way as to remove buoyancy differences between the stratiform cloudy and clear air. At present, we parameterize the mesoscale mass flux by introducing a simple prognostic equation for $w^i_{\mathrm{cld}} - w^i_{\mathrm{clr}}$, based on (114), which allows $w^i_{\mathrm{cld}} - w^i_{\mathrm{clr}}$ to increase due to mesoscale buoyancy forces:

$$
\begin{aligned}
\frac{\partial}{\partial t}(w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}) = & -\sum_{i'}\left(\frac{\hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{cld}}}{m^i A^i{}_{\mathrm{cld}}}\right)(\hat{w}^{i,i'}{}_{\mathrm{cld}} - w^i{}_{\mathrm{cld}}) \\
& + \sum_{i'}\left(\frac{\hat{m}^{i,i'} v^{i,i'}{}_{n} l^{i,i'} \hat{A}^{i,i'}{}_{\mathrm{clr}}}{m^i A^i{}_{\mathrm{clr}}}\right)(\hat{w}^{i,i'}{}_{\mathrm{clr}} - w^i{}_{\mathrm{clr}}) \\
& -\left(\frac{w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}}{q^i{}_c + q^i{}_i}\right)\left\{C_{\mathrm{clr,cld}}\left(\frac{A^i{}_{\mathrm{clr}}}{A^i}\right)\left(\frac{\dot{q}_c{}^i{}_{\mathrm{cond}} + \dot{q}_i{}^i{}_{\mathrm{cond}}}{1 - RH^i{}_{\mathrm{clr}}}\right) - C_{\mathrm{cld,clr}}\left(\frac{A^i{}_{\mathrm{cld}}}{A^i}\right)(\dot{q}_c{}^i{}_{\mathrm{evap}} + \dot{q}_i{}^i{}_{\mathrm{evap}})\right\} \\
& - \frac{\overline{M}^i}{m^i}\frac{\partial}{\partial z}(w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}) - \left(\frac{M^i{}_{\mathrm{cld}}}{m^i} - \frac{M^i{}_{\mathrm{clr}}}{m^i}\right)\left(\frac{A^i{}_{\mathrm{clr}}}{A^i}\frac{\partial w^i{}_{\mathrm{cld}}}{\partial z} + \frac{A^i{}_{\mathrm{cld}}}{A^i}\frac{\partial w^i{}_{\mathrm{clr}}}{\partial z}\right) \\
& - \left\{\left[\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]_{\mathrm{cld}} - \left[\frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\right]_{\mathrm{clr}}\right\} + \frac{g}{T}[(T_v)_{\mathrm{cld}} - (T_v)_{\mathrm{clr}}] \\
& + \left[\frac{M_c{}^i}{m^i}\frac{\partial}{\partial z}(w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}) + \left(\frac{D^i}{m^i A^i{}_{\mathrm{cld}}}\right)(w^i{}_{\mathrm{cu}} - w^i{}_{\mathrm{cld}})\right] .
\end{aligned}
\tag{130}
$$

Here T_v is the virtual temperature. We include condensate loading in the definition of T_v. The pressure terms of (130) must be parameterized or neglected; we currently neglect them. We expect that the effect of $w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}$, as predicted using (130), will be to limit $(T_v)_{\mathrm{cld}} - (T_v)_{\mathrm{clr}}$ to relatively small values -- relative, that is, to what might occur if $w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}} = 0$. In other words, we expect that the use of (130) will produce values of $w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}$ which act to keep the stratiform cloud neutrally buoyant with respect to its clear environment.

Once $w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}$ has been determined using (130), we can obtain $M^i{}_{\mathrm{meso}}$ from

$$
M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}} = m^i(w^i{}_{\mathrm{cld}} - w^i{}_{\mathrm{clr}}) \tag{131}
$$

with (45), and ${M_{\mathrm{cld}}}^i$ and ${M_{\mathrm{clr}}}^i$ from

$$
{M_{\mathrm{cld}}}^i = \overline{M}^i + \left(1 - \frac{A^i{}_{\mathrm{cld}}}{A^i}\right)(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}), \tag{132}
$$

and

$$
M^i{}_{\mathrm{clr}} = \overline{M}^i - \frac{A^i{}_{\mathrm{cld}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}), \tag{133}
$$

respectively. We assume that $\overline{M}^i$ is determined by the large-scale model.

8. Summary and conclusions

We have formulated a generalized version of Eauliq, the stratiform cloud parameterization developed by Fowler et al. (1996). The new version, called Eauliq NG, has the following enhancements:

- A prognostic stratiform cloudiness, similar to that of Tiedtke (1993).
- Separate prognosis of the thermodynamic properties of the cloudy and clear air.
- Fully consistent incorporation of the cumulus terms as they affect the microphysical variables and the cloud amount.
- Diagnosis of distinct vertical motion fields for the cloudy and clear portions of each grid cell, formulated in such a way that the stratiform cloud tends to remain neutrally buoyant through time. These distinct vertical motion fields are used consistently in the prediction of all thermodynamic fields.

We have not considered mesoscale fluctuations of the horizontal velocity, even though they are implied (through continuity) by the mesoscale fluctuations of the vertical velocity which we have included in our model. This deficiency should be remedied in the future.

This technical report is being published in order to record the preliminary formulation of Eauliq NG prior to its testing in a single column model and subsequently in a general circulation model. The final version of Eauliq NG may depart somewhat from the description given here, depending on the outcome of such tests.

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Appendix A

Analysis of Vertical Fluxes

We can rewrite (46) as follows:

$$A^i_{\text{clr}}M^i_{\text{clr}} = A^i\overline{M}^i - A^i_{\text{cld}}M^i_{\text{cld}} - A^i_{\text{cu}}M^i_{\text{cu}} \ , \tag{134}$$

or

$$A^iM^i_{\text{clr}} = A^i\overline{M}^i - A^i_{\text{cld}}(M^i_{\text{cld}} - M^i_{\text{clr}}) - A^i_{\text{cu}}(M^i_{\text{cu}} - M^i_{\text{clr}}), \tag{135}$$

or

$$A^i_{\text{clr}}M^i_{\text{clr}} = A^i_{\text{clr}}\overline{M}^i - \frac{A^i_{\text{cld}}A^i_{\text{clr}}}{A^i}(M^i_{\text{cld}} - M^i_{\text{clr}}) - \frac{A^i_{\text{clr}}A^i_{\text{cu}}}{A^i}(M^i_{\text{cu}} - M^i_{\text{clr}}). \tag{136}$$

Similarly, we can rewrite (47) as

$$A^i_{\text{cld}}M^i_{\text{cld}} = A^i\overline{M}^i - A^i_{\text{clr}}M^i_{\text{clr}} - A^i_{\text{cu}}M^i_{\text{cu}}, \tag{137}$$

or

$$A^iM^i_{\text{cld}} = A^i\overline{M}^i + A^i_{\text{clr}}(M^i_{\text{cld}} - M^i_{\text{clr}}) - A^i_{\text{cu}}(M^i_{\text{cu}} - M^i_{\text{cld}}), \tag{138}$$

or

$$A^i_{\text{cld}}M^i_{\text{cld}} = A^i_{\text{cld}}\overline{M}^i + \frac{A^i_{\text{cld}}A^i_{\text{clr}}}{A^i}(M^i_{\text{cld}} - M^i_{\text{clr}}) - \frac{A^i_{\text{cld}}A^i_{\text{cu}}}{A^i}(M^i_{\text{cu}} - M^i_{\text{cld}}). \tag{139}$$

The large-scale flux of h is given by

$$A^i \overline{mwh}^i = A^i{}_{\mathrm{clr}} M^i{}_{\mathrm{clr}} h^i{}_{\mathrm{clr}} + A^i{}_{\mathrm{cld}} M^i{}_{\mathrm{cld}} h^i{}_{\mathrm{cld}} + A^i{}_{\mathrm{cu}} M^i{}_{\mathrm{cu}} h^i{}_{\mathrm{cu}} \tag{140}$$

Substituting from (136) and (139), we find that

$$A^i\overline{mwh}^i = \left[A^i{}_{\mathrm{clr}}\overline{M}^i - \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}) - A^i{}_{\mathrm{clr}}\frac{A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{clr}})\right]h^i{}_{\mathrm{clr}} \tag{141}$$

$$+\left[A^i{}_{\mathrm{cld}}\overline{M}^i + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}}) - A^i{}_{\mathrm{cld}}\frac{A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{cld}})\right]h^i{}_{\mathrm{cld}} + A^i{}_{\mathrm{cu}}M^i{}_{\mathrm{cu}}h^i{}_{\mathrm{cu}}$$

$$= \overline{M}^i(A^i{}_{\mathrm{clr}}h^i{}_{\mathrm{clr}} + A^i{}_{\mathrm{cld}}h^i{}_{\mathrm{cld}}) + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}})$$

$$-A^i{}_{\mathrm{clr}}\frac{A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{clr}})h^i{}_{\mathrm{clr}} - A^i{}_{\mathrm{cld}}\frac{A^i{}_{\mathrm{cu}}}{A^i}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{cld}})h^i{}_{\mathrm{cld}} + A^i\frac{A^i{}_{\mathrm{cu}}}{A^i}M^i{}_{\mathrm{cu}}h^i{}_{\mathrm{cu}}$$

$$= \overline{M}^i(A^i{}_{\mathrm{clr}}h^i{}_{\mathrm{clr}} + A^i{}_{\mathrm{cld}}h^i{}_{\mathrm{cld}} + A^i{}_{\mathrm{cu}}h^i{}_{\mathrm{cu}}) + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}})$$

$$+\frac{A^i{}_{\mathrm{cu}}}{A^i}[-A^i{}_{\mathrm{clr}}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{clr}})h^i{}_{\mathrm{clr}} - A^i{}_{\mathrm{cld}}(M^i{}_{\mathrm{cu}} - M^i{}_{\mathrm{cld}})h^i{}_{\mathrm{cld}} + A^i(M^i{}_{\mathrm{cu}} - \overline{M}^i)h^i{}_{\mathrm{cu}}]$$

$$= A^i\overline{M}^i\overline{h}^i + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}})$$

$$+\frac{A^i{}_{\mathrm{cu}}}{A^i}\{M^i{}_{\mathrm{cu}}[-A^i{}_{\mathrm{clr}}h^i{}_{\mathrm{clr}} - A^i{}_{\mathrm{cld}}h^i{}_{\mathrm{cld}} + A^ih^i{}_{\mathrm{cu}}]$$

$$+A^i{}_{\mathrm{clr}}M^i{}_{\mathrm{clr}}h^i{}_{\mathrm{clr}} + A^i{}_{\mathrm{cld}}M^i{}_{\mathrm{cld}}h^i{}_{\mathrm{cld}} - A^i\overline{M}^ih^i{}_{\mathrm{cu}}\}$$

$$= A^i\overline{M}^i\overline{h}^i + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}})$$

$$+\frac{A^i{}_{\mathrm{cu}}}{A^i}\left\{M^i{}_{\mathrm{cu}}[-A^i\overline{h}^i + A^i{}_{\mathrm{cu}}h^i{}_{\mathrm{cu}} + A^ih^i{}_{\mathrm{cu}}] + (A^i\overline{M}^i\overline{h}^i - A^i{}_{\mathrm{cu}}M^i{}_{\mathrm{cu}}h^i{}_{\mathrm{cu}}) - A^i\overline{M}^ih^i{}_{\mathrm{cu}}\right\}$$

$$= A^i\overline{M}^i\overline{h}^i + \frac{A^i{}_{\mathrm{cld}}A^i{}_{\mathrm{clr}}}{A^i}(M^i{}_{\mathrm{cld}} - M^i{}_{\mathrm{clr}})(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}}) + A^i{}_{\mathrm{cu}}(M^i{}_{\mathrm{cu}} - \overline{M})(h^i{}_{\mathrm{cu}} - \overline{h}^i)$$

$$= A^i[\overline{M}^i\overline{h}^i + M^i{}_{\mathrm{meso}}(h^i{}_{\mathrm{cld}} - h^i{}_{\mathrm{clr}}) + M^i{}_{\mathrm{c}}(h^i{}_{\mathrm{cu}} - \overline{h}^i)] \quad .$$

Appendix B

Time-Differencing

The prognostic equation for the cloud amount, (105), is repeated here for convenience, with a slight expansion and rearrangement of the terms:

$$\frac{\partial}{\partial t}(m^i f^i) = \psi^i{}_{\mathrm{cld}} + m^i (f^i)(1 - f^i)\chi^i{}_{\mathrm{cld}} + (1 - f^i)\frac{D^i}{A^i}, \tag{142}$$

where for convenience we define

$$f^i \equiv \frac{A^i{}_{\mathrm{cld}}}{A^i}, \tag{143}$$

$$\chi^i{}_{\mathrm{cld}} \equiv \frac{1}{(q^i{}_c + q^i{}_i)}\left[C_{\mathrm{clr,cld}}\left(\frac{\dot{q}_c{}^i{}_{\mathrm{cond}} + \dot{q}_i{}^i{}_{\mathrm{cond}}}{1 - RH^i{}_{\mathrm{clr}}}\right) + C_{\mathrm{cld,clr}}(\dot{q}_c{}^i{}_{\mathrm{evap}} + \dot{q}_i{}^i{}_{\mathrm{evap}})\right], \tag{144}$$

and

$$\psi^i{}_{\mathrm{cld}} \equiv -\frac{1}{A^i}\sum_{i'} \hat{m}^{i,i'} v^{i,i'}{}_n l^{i,i'} \frac{\hat{A}^{i,i'}{}_{\mathrm{cld}}}{\hat{A}^{i,i'}} + M_c{}^i \frac{\partial f^i}{\partial z} - \frac{\partial}{\partial z}(f^i \overline{M}^i + M^i{}_{\mathrm{meso}}). \tag{145}$$

Note that $\chi^i{}_{\mathrm{cld}}$ can diverge to either plus or minus infinity. Here we discuss the time differencing of the lateral mass exchange and cumulus detrainment terms, which are shown explicitly in (142). We write

$$\frac{(m^i f^i)^{n+1} - (m^i f^i)^n}{\Delta t} = \psi^i{}_{\mathrm{cld}}$$

$$+ m^i (f^i)^{n+1} [1 - (f^i)^{n+1}] \chi^i{}_{\mathrm{cld}} + [1 - (f^i)^{n+1}] \frac{D^i}{A^i} \ . \tag{146}$$

Here superscripts n and $n+1$ denote successive time levels, and Δt is the time step. Eq. (146) can be solved as a quadratic equation for $(f^i)^{n+1}$:

$$[(f^i)^{n+1}]^2 m^i \chi^i{}_{\mathrm{cld}} \Delta t + (f^i)^{n+1} \left(1 - m^i \chi^i{}_{\mathrm{cld}} \Delta t + \frac{D^i}{A^i} \Delta t\right)$$

$$- \left[(m^i f^i)^n + \psi^i{}_{\mathrm{cld}} \Delta t + \frac{D^i}{A^i} \Delta t\right] = 0 \ . \tag{147}$$

The solution is

$$(f^i)^{n+1} = - \frac{1 - m^i \chi^i{}_{\mathrm{cld}} \Delta t + \frac{D^i}{A^i} \Delta t}{2 m^i \chi^i{}_{\mathrm{cld}} \Delta t}$$

$$\pm \frac{\sqrt{\left(1 - m^i \chi^i{}_{\mathrm{cld}} \Delta t + \frac{D^i}{A^i} \Delta t\right)^2 + 4(m^i \chi^i{}_{\mathrm{cld}} \Delta t)\left[(m^i f^i)^n + \psi^i{}_{\mathrm{cld}} \Delta t + \frac{D^i}{A^i} \Delta t\right]}}{2 m^i \chi^i{}_{\mathrm{cld}} \Delta t} \ . \tag{148}$$

Consider a simplified case in which lateral mass exchange is the only process. Then (148) reduces to

$$(f^i)^{n+1} = - \frac{1 - m^i \chi^i{}_{\mathrm{cld}} \Delta t}{2 m^i \chi^i{}_{\mathrm{cld}} \Delta t} \pm \frac{\sqrt{(1 - m^i \chi^i{}_{\mathrm{cld}} \Delta t)^2 + 4(m^i \chi^i{}_{\mathrm{cld}} \Delta t)(m^i f^i)^n}}{2 m^i \chi^i{}_{\mathrm{cld}} \Delta t} . \tag{149}$$

For the special case $|\chi^{i}{}_{\mathrm{cld}}| \rightarrow \infty$ this further simplifies to

$$(f^{i})^{n+1} = \frac{1}{2} \pm \frac{1}{2}. \tag{150}$$

If we choose the plus sign, we get $(f^{i})^{n+1} = 1$; if we choose the minus, we get $(f^{i})^{n+1} = 1$.

It thus appears that we should choose the plus sign for $\chi^{i}{}_{\mathrm{cld}} > 0$, and the minus for $\chi^{i}{}_{\mathrm{cld}} < 0$.

Now consider a second special case in which $\chi^{i}{}_{\mathrm{cld}} = 0$. Then (147) reduces to

$$(f^{i})^{n+1} = \frac{(m^{i} f^{i})^{n} + \psi^{i}{}_{\mathrm{cld}} \Delta t + \dfrac{D^{i}}{A^{i}} \Delta t}{1 + \dfrac{D^{i}}{A^{i}} \Delta t}. \tag{151}$$

For $D^{i} \rightarrow \infty$ we get $(f^{i})^{n+1} = 1$, as expected.

Next, consider (114), the prognostic equation for $h^{i}{}_{\mathrm{cld}} - h^{i}{}_{\mathrm{clr}}$:

$$\frac{\partial}{\partial t}(h^{i}{}_{\mathrm{cld}} - h^{i}{}_{\mathrm{clr}}) = J^{i}(h) - \mu^{i}(h^{i}{}_{\mathrm{cld}} - h^{i}{}_{\mathrm{clr}}) + \left(\frac{D^{i}}{m^{i} A^{i}{}_{\mathrm{cld}}}\right)(h^{i}{}_{\mathrm{cu}} - h^{i}{}_{\mathrm{cld}}). \tag{152}$$

Here for convenience we define

$$J^i(h) \equiv -\sum_{i'}\left(\frac{\hat{m}^{i,i'}v^{i,i'}{}_{n}l^{i,i'}\hat{A}^{i,i'}{}_{\text{cld}}}{m^i A^i{}_{\text{cld}}}\right)(\hat{h}^{i,i'}{}_{\text{cld}} - h^i{}_{\text{cld}})$$

$$+\sum_{i'}\left(\frac{\hat{m}^{i,i'}v^{i,i'}{}_{n}l^{i,i'}\hat{A}^{i,i'}{}_{\text{clr}}}{m^i A^i{}_{\text{clr}}}\right)(\hat{h}^{i,i'}{}_{\text{clr}} - h^i{}_{\text{clr}}) - \frac{\overline{M}^i}{m^i}\frac{\partial}{\partial z}(h^i{}_{\text{cld}} - h^i{}_{\text{clr}})$$

$$-\left(\frac{M_{\text{cld}}{}^i}{m^i} - \frac{M_{\text{clr}}{}^i}{m^i}\right)\left(\frac{A^i{}_{\text{clr}}}{A^i}\frac{\partial h^i{}_{\text{cld}}}{\partial z} + \frac{A^i{}_{\text{cld}}}{A^i}\frac{\partial h^i{}_{\text{clr}}}{\partial z}\right) + \frac{(S_h)^i{}_{\text{cld}}}{m^i} - \frac{(S_h)^i{}_{\text{clr}}}{m^i} \tag{153}$$

$$+\frac{M_c{}^i}{m^i}\frac{\partial}{\partial z}(h^i{}_{\text{cld}} - h^i{}_{\text{clr}}) \quad .$$

$$\mu^i \equiv \frac{1}{(q^i{}_c + q^i{}_i)}\left\{C_{\text{clr,cld}}\left(\frac{A^i{}_{\text{clr}}}{A^i}\right)\left(\frac{\dot{q}_c{}^i{}_{\text{cond}} + \dot{q}_i{}^i{}_{\text{cond}}}{1 - RH^i{}_{\text{clr}}}\right) - C_{\text{cld,clr}}\left(\frac{A^i{}_{\text{cld}}}{A^i}\right)(\dot{q}_c{}^i{}_{\text{evap}} + \dot{q}_i{}^i{}_{\text{evap}})\right\} \tag{154}$$

Note that

$$\mu^i \geq 0 . \tag{155}$$

We can rewrite (152) as

$$\frac{\partial}{\partial t}(h^i{}_{\text{cld}} - h^i{}_{\text{clr}}) = \left[J^i(h) + \left(\frac{D^i}{m^i A^i{}_{\text{cld}}}\right)(h^i{}_{\text{cu}} - h^i{}_{\text{clr}})\right] - \left(\mu^i + \frac{D^i}{m^i A^i{}_{\text{cld}}}\right)(h^i{}_{\text{cld}} - h^i{}_{\text{clr}}) . \tag{156}$$

We write

$$\frac{(h^i{}_{\text{cld}} - h^i{}_{\text{clr}})^{n+1} - (h^i{}_{\text{cld}} - h^i{}_{\text{clr}})^n}{\Delta t} = \left[J^i(h) + \left(\frac{D^i}{m^i A^i{}_{\text{cld}}}\right)(h^i{}_{\text{cu}} - h^i{}_{\text{clr}})\right]$$

$$-\left(\mu^i + \frac{D^i}{m^i A^i{}_{\text{cld}}}\right)(h^i{}_{\text{cld}} - h^i{}_{\text{clr}})^{n+1} \quad . \tag{157}$$

The solution is

$$(h^i_{\text{cld}} - h^i_{\text{clr}})^{n+1} = \frac{(h^i_{\text{cld}} - h^i_{\text{clr}})^n + \Delta t\left[J^i(h) + \left(\frac{D^i}{m^i A^i_{\text{cld}}}\right)(h^i_{\text{cu}} - h^i_{\text{clr}})\right]}{1 + \Delta t\left(\mu^i + \frac{D^i}{m^i A^i_{\text{cld}}}\right)}. \tag{158}$$

For $D^i \to \infty$, we get

$$(h^i_{\text{cld}} - h^i_{\text{clr}})^{n+1} = h^i_{\text{cu}} - h^i_{\text{clr}}. \tag{159}$$

For $\mu^i \to \infty$, we get

$$(h^i_{\text{cld}} - h^i_{\text{clr}})^{n+1} = 0. \tag{160}$$

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