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TURBULENCE SPECTRA IN THE BUOYANCY SUBRANGE OF THERMALLY STRATIFIED SHEAR FLOWS

by

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ABSTRACT OF DISSERTATION

TURBULENCE SPECTRA IN THE BUOYANCY SUBRANGE OF THERMALLY STRATIFIED SHEAR FLOWS

A generalized eddy-viscosity approximation is used to study the turbulence spectra of thermally stratified shear flows. For a stationary process in the wave number range investigated--the buoyancy subrange--under the assumption of local homogeneity of the flow, two governing spectral equations with six unknowns are derived from the equations of motion and energy.

In order to reduce the number of unknowns to two so that the spectral equations can be solved, a generalized eddy-viscosity is used for expressing the integrated forms of the inertial transfers of energy and temperature inhomogeneity, the shear stress and vertical heat flux in terms of velocity spectrum $\phi(k)$ and temperature spectrum $\phi_{TT}(k)$.

Asymptotic solutions are obtained in the buoyancy subrange where the local production and local dissipation of turbulent energy is negligible as compared to the inertial transfer and vertical heat flux terms when the flow conditions satisfy the criterion

$$\varepsilon\left|\frac{d\overline{T}}{dz}\right| << N\frac{g}{\overline{T}} \quad \text{or} \quad \frac{g}{\overline{T}}\left|\frac{d\overline{T}}{dz}\right| << \frac{N}{\varepsilon}\left(\frac{g}{\overline{T}}\right)^2 .$$

In the buoyancy subrange of stably stratified turbulent flow, the power law for the velocity and temperature spectra is not universal but varies with the flow conditions in the way $\phi(k) \sim k^n$ and $\phi_{TT}(k) \sim k^m$ where $-\frac{11}{5} \geq n \geq -3$ and $-1 \geq m \geq -\frac{7}{5}$. According to the measurements of velocity spectra in the atmosphere (Pinus and

Schcherbakova, 1966; Myrup, 1968), the dependence of the power law on the flow conditions was confirmed. The solutions of Bolgiano (1959) and Lumley-Shur (1964) are only two particular cases of the present results under certain flow conditions.

In the case of the unstably stratified turbulent flow, the velocity spectrum exhibits a hump in the buoyancy subrange as a result of the energy input from the temperature field to the velocity field. On the left side of this hump the velocity spectrum approaches a +1 slope and the temperature spectrum shows a -3 slope. The measurements of the velocity spectra in the atmosphere (Ivanov and Ordanovich, 1967) confirms this tendency.

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LIST OF SYMBOLS

Symbol	Definition or Description	Dimension
$\vec{a}$	Position vector in Lagrangian sense	L
A_{ij}	$A_{ij} = \overline{(x_j - a_j)\, U_i}$	L^2T^{-1}
b	Constant	--
c_p, c_v	Specific heat at constant pressure, volume	QFT^2L^{-1}
$C_i(i=1, \ldots, 4)$	Parameter	--
$d_i(i=1, \ldots, 6)$	Constant	--
F	Inertial transfer of energy	L^3T^{-3}
F_{TT}	Inertial transfer of temperature inhomogeneity	LH^2T^{-1}
f_1, f_2	Numerical variables	--
g	Gravitational acceleration	LT^{-2}
$\vec{k}$	Wave number vector with components (k_1, k_2, k_3)	L^{-1}
k	Wave number $= \sqrt{k_1^2 + k_2^2 + k_3^2}$	L^{-1}
k_d	$k_d = \gamma_{s,1}^{1/2} \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$	L^{-1}
k_e	Wave number of energy-containing eddies	L^{-1}
k_o	$k_o = \gamma_{s,r}^{1/2}\ b^{3/4}\ N_*^{3/4}\ \varepsilon^{-5/4}\ \beta^{3/2}$	L^{-1}
ℓ_d	Kolmogorov length scale $\ell_d = \left(\frac{\varepsilon}{\nu^3}\right)^{-1/4}$	L
ℓ_e	Length scale of energy-containing eddies	L
ℓ_o	Obukhoff length scale $\ell_o = N^{-3/4}\ \varepsilon^{5/4}\ \beta^{-3/2}$	L
L_E	Eulerian space integral scale	L

Symbol	Definition or Description	Dimension
N	Total dissipation of temperature inhomogeneity smeared by thermal diffusivity	H^2T^{-1}
N_*	$N_* = 2N$	H^2T^{-1}
m	$m = \frac{d\bar{U}}{dz} \left\vert\frac{d\bar{U}}{dz}\right\vert^{C_2} \left(\frac{\nu}{\varepsilon}\right)^{1-\frac{C_2}{2}}$	--
m_T	$m_T = \left\vert\frac{d\bar{T}}{dz_*}\right\vert^{C_3} N_*^{\frac{C_4}{2}-1} \nu_T^{-\frac{C_4}{2}+1} \frac{d\bar{T}}{dz_*}$	--
P	Instantaneous pressure	FL^{-2}
$\bar{P}$	Mean pressure	FL^{-2}
p_1	Pressure fluctuation	FL^{-2}
$\vec{r}$	Difference between two position vectors	L
s,r	Constant	--
t	Time	T
T	Instantaneous temperature	H
$\bar{T}$	Mean temperature	H
U	Instantaneous velocity	LT^{-1}
$\bar{U}$	Mean velocity	LT^{-1}
u_1,u_2,u_3	Velocity fluctuations	LT^{-1}
Pr	$Pr = b\,\frac{\nu}{\nu_T}$	--
$\vec{x}$	Position vector	L
x_1,x_2,x_3	Cartesian coordinates	L
x	Dimensionless wave number, $x = \frac{k}{k_o}$, or $x = \frac{k}{k_d}$	--

Symbol	Definition or Description	Dimension
Y_s^2	$Y_s^2 = \int_0^x 2x^2\ \Phi\ (x)dx$	--
Z_s^2	$Z_s^2 = \int_0^x 2x^2\ \Phi_{TT}(x)dx$	--
α, α_ℓ	Kolmogorov's universal constants corresponding to three dimensional spectrum and structure function	--
α_-	Kolmogorov's universal constant corresponding to one dimensional spectrum	--
$\alpha_{\bar{-}}$	Obukhoff's universal constant corresponding to temperature spectrum	--
β	$\beta = \frac{g}{\overline{T}}$	$LT^{-2}H^{-1}$
β_1	$\beta_1 = b\beta\left\|\frac{d\overline{T}}{dz_*}\right\|^{C_3} \left(\frac{N_*}{\varepsilon_T}\right)^{\frac{C_4}{2}} \frac{\nu}{\varepsilon}$	--
$\gamma_{s,r}$	Constant	--
Γ	$\Gamma = \frac{d\overline{U}}{dz}\left\|\frac{d\overline{U}}{dz}\right\|^{C_1} b^{-1+\frac{C_2}{2}} N_*^{-1+\frac{C_2}{2}} \varepsilon^{1-\frac{C_2}{2}} \beta^{-2+C_2}$	--
Γ_1	$\Gamma_1 = N_*^{-1+C_4} \varepsilon^{1-C_4} \beta^{-1+C_4} \left\|\frac{d\overline{T}}{dz_*}\right\|^{C_3}$	--
Γ_T	$\Gamma_T = N_*^{-2+C_4} \varepsilon^{2-C_4} \beta^{-2+C_4} \left\|\frac{d\overline{T}}{dz_*}\right\|^{C_3} \frac{d\overline{T}}{dz_*}$	--
ε	Total dissipation of turbulent energy by viscosity	L^2T^{-3}
$\varepsilon(k)$	Inertial energy transfer flux	L^2T^{-3}
η_s	Generalized eddy-viscosity	L^2T^{-1}
θ	Temperature fluctuation	H
λ	Wave length of radio wave	L

Symbol	Definition or Description	Dimension
μ	Dynamic viscosity	$FL^{-2}T$
μ_T	Thermal conductivity	$QT^{-1}L^{-1}H^{-1}$
ν	Kinematic viscosity	L^2T^{-1}
ν_T	Thermal diffusivity	L^2T^{-1}
ρ	Instantaneous density	FT^2L^{-4}
$\overline{\rho}$	Mean density	FT^2L^{-4}
ρ_1	Density fluctuation	FT^2L^{-4}
$\phi(k)$	Velocity spectrum	L^3T^{-2}
$\phi_{TT}(k)$	Temperature spectrum	LH^2
$\phi_{uw}(k)$	Shear stress spectrum	L^3T^{-2}
$\phi_{wT}(k)$	Vertical heat flux spectrum	$L^2T^{-1}H$
$\Phi(x)$	Dimensionless velocity spectrum	--
$\Phi_{TT}(x)$	Dimensionless temperature spectrum	--
$\Phi_{uw}(x)$	Dimensionless shear stress spectrum	--
$\Phi_{wT}(x)$	Dimensionless vertical heat flux spectrum	--
ϕ_d	$\phi_d = \gamma_{s,1}^{-3/2} (\varepsilon/\nu^5)^{1/4}$	L^3T^{-2}
ϕ_{TT}^d	$\phi_{TT}^d = \gamma_{s,1}^{-3/2} \frac{N_*}{\nu_T} \left(\frac{\nu^3}{\varepsilon}\right)^{3/4}$	LH^2
ϕ_{uw}^d	$\phi_{uw}^d = \gamma_{s,1}^{-1/2} \varepsilon^{\frac{C_2}{2} - \frac{1}{4}} \nu^{\frac{7}{4} - \frac{C_2}{2}} \left\lvert\frac{d\overline{U}}{dz}\right\rvert^{C_2}$	L^3T^{-2}
ϕ_{wT}^d	$\phi_{wT}^d = b\gamma_{s,1}^{-1/2} \varepsilon^{-1/4} N_*^{\frac{C_4}{2}} \nu^{\frac{7}{4}} \nu_T^{-\frac{C_4}{2}} \left\lvert\frac{d\overline{T}}{dz_*}\right\rvert^{C_3}$	$L^2T^{-1}H$

Symbol	Definition or Description	Dimension
ϕ_o	$\phi_o = \gamma_{s,r}^{-3/2}\, b^{-5/4}\, N_*^{-5/4}\, \varepsilon^{11/4}\, \beta^{-5/2}$	L^3T^{-2}
ϕ^o_{TT}	$\phi^o_{TT} = \gamma_{s,r}^{-3/2}\, b^{-9/4}\, N_*^{-1/4}\, \varepsilon^{7/4}\, \beta^{-5/2}$	LH^2
ϕ^o_{uw}	$\phi^o_{uw} = \gamma_{s,r}^{-1/2}\, b^{-\frac{7}{4}+\frac{C_2}{2}}\, N_*^{-\frac{7}{4}+\frac{C_2}{2}}\, \varepsilon^{\frac{13}{4}-\frac{C_2}{2}}\, \beta^{-\frac{7}{2}+C_2} \left\lvert\frac{d\overline{U}}{dz}\right\rvert^{C_1}$	L^3T^{-2}
ϕ^o_{wT}	$\phi^o_{wT} = \gamma_{s,r}^{-1/2}\, b^{-3/4}\, N_*^{-\frac{7}{4}+C_4}\, \varepsilon^{\frac{13}{4}-C_4}\, \beta^{-\frac{7}{2}+C_4} \left\lvert\frac{d\overline{T}}{dz_*}\right\rvert^{C_3}$	$L^2T^{-1}H$

** F: force, L: length, T: time, H: temperature, Q: heat

Chapter I

INTRODUCTION

Recent studies of locally isotropic turbulence of homogeneous fluids in the inertial subrange have been helpful in providing solutions to several engineering problems which are related to air pollution, the long distance propagation of ultra-high frequency radio waves by scattering in the ionosphere, and the safe structural design of high speed aircraft.

When smoke or radio-active material is dispersed by turbulent diffusion, Tchen (1959) has shown that the dispersion from a point source can be related to a function of some power of time for non-stratified fluids. Especially, when the -5/3 law holds in the inertial subrange of locally isotropic turbulence, the dispersion of particles is proportional to t^3 where t indicates time.

In the second problem, if it is assumed that scattering wave numbers fall in the inertial subrange of locally isotropic turbulence, the scattering cross section exhibits a $\lambda^{11/3}$ dependence where λ is the wave length of the radio wave (Bolgiano, 1959).

As to the last problem, the vibration of aircraft due to the atmospheric turbulence can cause fatigue of aircraft material and may even cause the aircraft to crash if the critical frequency of vibration with respect to the aircraft is induced. Of course, a better understanding of the energy spectrum of atmospheric turbulence can give criteria for safe design of high speed aircraft.

However, in addition to the complexity of turbulence, the atmosphere itself presents complications, i.e., the atmosphere is usually thermally stratified and in a state of shear. The turbulence changes its spectrum of energy in the magnitude or scale since the conversion of potential energy into or from the kinetic energy of the flow can increase or decrease the kinetic energy of turbulence depending upon whether the stratification is unstable or stable. Moreover, the spectrum of energy will change its shape and form because the stratification can cause some anisotropic effects on the turbulence. Thus, the turbulence will have directional properties and local isotropy can never exist in the wave number range where buoyancy is an influencing factor.

Based on the assumption that the energy spectrum depends only on the total dissipation of density fluctuation by molecular effects in the buoyancy subrange of the equilibrium range of turbulence, Bolgiano (1959) reached a solution of the energy spectrum being proportional to $k^{-11/5}$ where k is the wave number.

However, according to another hypothesis, Lumley (1964) obtained a different spectral form in this buoyancy subrange, since he postulated that the energy spectrum $\phi(k)$ and the buoyancy flux spectrum $\phi_{wT}(k)$ are functions of the local energy transfer flux $\varepsilon(k)$ and the local wave number k and that the spectrum of the buoyancy flux in a stably stratified flow is proportional to the mean temperature gradient. In this way, Lumley obtained a -3 power law of the wave number in the buoyancy subrange if Kolmogorov's hypothesis can be extended to this subrange, i.e., energy spectrum is determined by k and $\varepsilon(k)$ alone.

It is clear that from the above statements, Bolgiano's and Lumley's results seem to be mutually exclusive at first sight. Hence, the author's motivation will be not only to determine the discrepancy in their basic assumptions but to search for the basic mechanism of turbulence in a thermally stratified turbulent shear flow.

Chapter II

LITERATURE REVIEW

In this chapter, previous studies of turbulence spectra in the inertial subrange are briefly reviewed. Two hypotheses given by Bolgiano and Lumley-Shur to study the turbulence spectra of a stably stratified flow are described. Recent works of Monin, Gisina, and Pao are stated, and some measurements of turbulence spectra are reviewed.

2.1 Locally Isotropic Turbulence--Kolmogorov Hypotheses

From the definition, turbulence is characterized as an irregular condition of fluid flow in which fluid properties such as vorticity components are distributed randomly in space and time. Beyond its irregularity, turbulence as a result of nonlinear interaction shows turbulent energy transfer through motion of the eddies. This idea of turbulent energy transfer is characterized by L. F. Richardson's rhyme: "Big whirls have little whirls, that feed on their velocity, and little whirls have lesser whirls, and so on to viscosity." This idea may be stated in a clearer form -- the turbulent flow contains eddies of various sizes characterized by the nonlinear interactions between eddies. In other words, the turbulent energy is transferred from large eddies to smaller eddies until it is dissipated into heat because of viscosity.

In the case of a flow of high Reynolds number, Kolmogorov (1941) postulated that small eddies of turbulence are statistically steady, locally isotropic, and independent of the structure of large eddies of turbulence from which the small eddies are generated. He further postulated that the statistical characteristics of small eddies of locally isotropic turbulence can uniquely be described by parameters ν the kinematic viscosity and ε the total dissipation of turbulent energy by viscosity.

In his second hypothesis, Kolmogorov postulated that in the universal equilibrium range where small eddies lie, there exists a subrange in which the viscosity effects are negligible and only the parameter ε determines the turbulence structure. Thus, based on dimensional arguments, the velocity structure function, i.e., the averaged square of the difference of velocities at two points separated by a distance $\vec{r}$, is

$$\overline{|U(\vec{x}+\vec{r},t)-U(\vec{x},t)|^2} = \alpha_\ell\, \varepsilon^{\frac{2}{3}}\, r^{\frac{2}{3}} \quad , \quad \ell_d << r << \ell_e \tag{2.1}$$

and equivalently, the three dimensional energy spectrum has the form in terms of wave number k

$$\phi(k,t) = \alpha\, \varepsilon^{\frac{2}{3}}\, k^{-\frac{5}{3}} \quad , \quad \ell_e^{-1} = k_e << k << \ell_d^{-1} \tag{2.2}$$

in which α and α_ℓ are universal constants, $\ell_d = (\nu^3/\varepsilon)^{1/4}$ is the Kolmogorov length scale that characterizes a cut-off length scale below which viscosity affects the turbulence structure essentially, and ℓ_e is the length scale of energy-containing eddies. Historically, the above stated universal function was reached independently by Onsager (1945, 1949) and von Weizsäcker (1948).

Based on Kolmogorov's hypotheses, the turbulent motion of small scale in the inertial subrange can be predicted. Recent measurements of the turbulent energy spectrum of flow with high Reynolds number show that the one-dimensional energy spectra are proportional to $k_1^{-5/3}$ in the inertial subrange. The experimentally evaluated α_1 for one-dimensional spectra lies in the range 0.48 ± 0.055, (Pond et al., 1966) where k_1 is an orthogonal component of $\vec{k}$ in the streamwise direction and $\alpha_1 = \frac{18}{55}\alpha$ derived from the assumption of isotropy. Examples of those measurements are listed chronologically, Gurvich (1960) measured in a wind over land, Grant and his colleagues (1962a, 1962b) in a tidal channel, Pond and his co-workers (1963, 1966) in wind over water waves, Gibson (1963) in a round jet, and Payne and Lumley (1966) in an atmospheric surface layer by an airborne hot-wire anemometer.

In case the temperature field is considered, Obukhoff (1949) and Corrsin (1951) extended the Kolmogorov's hypotheses to the temperature spectrum, i.e., in the inertial convective subrange the temperature spectrum also follows the -5/3 law and has the form

$$\phi_{TT}(k,t) = \alpha_T N \varepsilon^{-\frac{1}{3}} k^{-\frac{5}{3}} , \quad k_e \ll k \ll \ell_d^{-1} \tag{2.3}$$

where α_T is a universal constant, N is the total dissipation of temperature fluctuation by molecular transport. Experimentally, Gibson and Schwarz (1963) showed the existence of the -5/3 law in the inertial convective subrange of the temperature and concentration spectra measured behind grids in a water tunnel. Tsvang (1960) also found the -5/3 law of temperature spectra in the atmospheric surface layer.

Although Eqs. 2.2 and 2.3 give good prediction of spectra in the inertial subrange of the equilibrium range, a detailed study of the spectra covering the whole equilibrium range must take ν into consideration as k approaches to and beyond k_d. To reach this point, an additional assumption on the mechanism of the turbulent energy transfer must be proposed.

Historically, Obukhoff (1941) first gave the assumption that the energy transfer across the wave number k is analogous to the process of the production of turbulent energy due to the work of Reynolds stress against the mean motion. From the other point of view, Heisenberg (1948) considered that the eddies with wave numbers larger than k act as turbulent eddy viscosity on the eddies with wave numbers less than k. From the eddies of wave number less than k energy is transferred to the smaller eddies with wave number greater than k. Assuming the local property of the energy transfer function, i.e., the energy transfer is only a function of wave number k and the energy spectrum at this local wave number k, Kovasznay (1948) obtained some solution also. Using a different approach--cascade process approximation, Pao (1965) obtained some solution for the locally isotropic turbulence at high wave numbers.

For the purpose of generalizing the problem, Stewart and Townsend (1951) gave the assumption of generalized eddy-viscosity which is actually expressed in the form of a series. Due to the difficulties involved in arriving at a closed form for the energy spectrum when a series form of generalized eddy-viscosity approximation is used, Panchev (1967) used only one term of this series and obtained some results for locally isotropic turbulent flow. In particular, one thing must be noted that Heisenberg's and Kovasznay's approximations can be deduced from Panchev's approach, and moreover, Pao's method can also be reached if some special nondimensional parameter is introduced. Panchev's book (1967) must be referred to for the details of these relationships.

Stimulated by Panchev's work, the author tried to extend the generalized eddy-viscosity approximation to the thermally stratified turbulent shear flow which will be investigated intensively in the next chapter.

Before we study the eddy-viscosity approximation, some limitation of this approximation must be described. Batchelor (1953) objected that introduction of the eddy viscosity implies that the smaller eddies must be statistically independent of the larger eddies. But as k approaches to k_d, this statistical independence does not exist (Hinze, 1959). Thus, at high wave numbers the eddy-viscosity approximation cannot be valid as, on the other hand, indicated by the fact that -7 law at high wave numbers implied by the eddy-viscosity approximation is unrealistic because -7 law will mean the discontinuity of velocity derivatives. However, according to Kolmogorov's hypothesis, this statistical independence may be assumed in the inertial subrange $k_e \ll k \ll \ell_d^{-1}$, and thus the validity of the eddy-viscosity approximation will be assumed.

2.2 Bolgiano's and Lumley-Shur's Hypotheses on Stably Stratified Turbulent Flow

As described above, the Kolmogorov's hypotheses shed some light on turbulent structure of flow without any thermal effects. In the atmosphere the flow is not only compressible but thermally stratified.

In case free convection occurs, the atmospheric turbulence is excited. Even when the atmosphere is stably stratified, i.e., the lapse rate of temperature is less than the adiabatic one, there is convincing evidence, excepting when very strong stable stratification occurs, that there exists a random, irregular motion--turbulent motion in the atmosphere (Kellogg, 1956). Due to the existence of thermal stratification, the gravitational force must be introduced into the equations of motion, and the potential energy of the flow, as an evidence of the gravitational force, will affect the energy balance of the flow. Hence, it can be expected that any variation of the turbulent kinetic energy must be a function of atmospheric thermal stratification.

2.2.1 Mechanism of turbulence in thermally stratified flow - As a result of the introduction of gravitational effects due to the thermal stratification, the turbulent field becomes anisotropic since the vertical velocity fluctuation is suppressed if the flow is stably stratified and is excited if the flow is unstably stratified. It can be expected that in the absence of shear, the turbulent field as a first approximation tends to be axisymmetric with respect to the vertical axis to which the gravitational force is oppositely parallel. Incidentally, the anisotropic effect will appear in the turbulent energy spectrum since, in the range of wave numbers where damping or excitation of turbulence by buoyancy force occurs, a part of turbulent energy is abstracted from or into the turbulent velocity field and is converted into or from potential energy depending on whether the flow is stably or unstably stratified.

For a stably stratified flow, energy drained out of the turbulent velocity field may propagate away in the form of internal gravity wave disturbances of Long-Hines type (Long, 1953, 1955; Hines, 1960) or may cause the production of density or temperature inhomogeneity which is transferred inertially to smaller eddies and finally smeared out by the molecular effects. Thus, in the wave number range where the buoyancy force effects predominate, the turbulent energy transfer decreases with wave number, but the transfer of density or temperature inhomogeneities increases. Hereafter, the total turbulent energy dissipation by viscosity ε is reduced and the Kolmogorov wave number $(\varepsilon/\nu^3)^{1/4}$ decreases accordingly. In other words, the turbulent scale at which the viscous cut-off occurs will indirectly increase through the effects of stable stratification.

On the other hand, in the case of unstably stratified turbulent flow, the potential energy of the temperature or density field is converted to the turbulent velocity field and the velocity spectrum may exhibit a hump in the buoyancy subrange where the gravitational force affects essentially. Of course, the temperature or density spectrum in this buoyancy subrange may have a steeper slope as a result of energy export. It can also be expected that the total dissipation of energy increases and the wave number at which cut-off of the molecular effects occurs is increased.

Keeping the above described mechanism of stratified turbulent flow in mind, Bolgiano's and Lumley-Shur's hypotheses on the stably stratified flow are now introduced.

2.2.2 Bolgiano's hypothesis - If the Reynolds number of a stably stratified flow is sufficiently large, Bolgiano (1959) postulated that the equilibrium range of the turbulent energy spectrum can be divided into three distinct subranges:

(1) the buoyancy subrange in which the larger, anisotropic eddies are directly influenced by the density stratification,

(2) the inertial subrange in which the anisotropic effects due to buoyancy force decrease rapidly and the classical hypothesis of locally isotropic turbulence is applicable, and

(3) the dissipation subrange at high wave numbers where the molecular effects dominate.

It is obvious that the last two subranges fall into the category of the locally isotropic turbulence. But, the buoyancy subrange needs special analysis and great attention. As a lower limit of scale cut-off for the buoyancy subrange, Bolgiano introduced the Obukhoff length scale

$$\ell_o = N^{-\frac{3}{4}} \varepsilon^{\frac{5}{4}} \beta^{-\frac{3}{2}}$$

where N is the total dissipation of temperature fluctuation by molecular transport, ε is the total turbulent energy dissipation by viscosity, and $\beta = g/\overline{T}$ in which g is the acceleration of gravity and $\overline{T}$ the mean temperature.

Thus, Bolgiano further postulated that there exists a wide range of wave number between the scale of energy-containing eddies ℓ_e and the Obukhoff length ℓ_o , or equivalently, there exists a buoyancy subrange. He also assumed that in this wave number range ε is comparatively much smaller than $\varepsilon(k)$ the local rate of the inertial transfer of turbulent energy (energy transfer flux) and that the local dissipation in this subrange is so small that the statistical properties of turbulence such as velocity and temperature spectra are only a function of N, β and wave number k. Dimensional argument gives readily

$$\phi(k) \sim N^{\frac{2}{5}} \beta^{\frac{4}{5}} k^{-\frac{11}{5}} , \quad k_e \ll k \ll \ell_o^{-1} . \tag{2.4}$$

In the inertial subrange, energy transfer flux $\varepsilon(k)$ approaches to a constant ε ; the classical -5/3 law holds

$$\phi(k) \sim \varepsilon^{\frac{2}{3}} k^{-\frac{5}{3}} , \quad \ell_o^{-1} \ll k \ll \ell_d^{-1} . \tag{2.5}$$

Similarly the temperature spectrum can be worked out dimensionally as a form

$$\phi_{TT}(k) \sim N^{\frac{4}{5}} \beta^{-\frac{2}{5}} k^{-\frac{7}{5}} , \quad k_e \ll k \ll \ell_o^{-1} \tag{2.6}$$

and

$$\phi_{TT}(k) \sim N \varepsilon^{-\frac{1}{3}} k^{-\frac{5}{3}} , \quad \ell_o^{-1} \ll k \ll \ell_d^{-1} . \tag{2.7}$$

2.2.3 Lumley-Shur's hypothesis - In contrast to Bolgiano's hypothesis, Lumley (1964, 1965) developed a new hypothesis for the turbulence spectrum of a stably stratified flow. These two theories are mutually exclusive since they are based on entirely different physical backgrounds and, of course, lead to different predictions for the spectral forms of turbulent energy and temperature fluctuations in the "buoyancy" or "locally inertial" subrange.

In his paper, Lumley first extended the original Kolmogorov hypotheses in the inertial subrange into the "locally inertial" subrange. From Kolmogorov's hypothesis, the statistical properties of turbulence in the inertial subrange, if the inertial subrange exists, are characterized uniquely by ε and k as stated in section 2.1. In the inertial subrange, ε, the turbulent energy dissipation by viscosity, is in fact the energy transfer flux through wave numbers. Thus, Lumley postulated that the statistical properties of turbulence of a stably stratified flow in the inertial-buoyancy subrange, such as energy spectrum $\phi(k)$ and heat flux spectrum $\phi_{WT}(k)$, are determined by the wave number k and the local energy transfer flux $\varepsilon(k)$ at this wave number k.

In addition to the above hypothesis as an extension of Kolmogorov's hypothesis, Lumley postulated further that the temperature fluctuation field is determined solely by the velocity field. From the temperature fluctuation equation of flow with high Reynolds and Peclet number

$$\frac{\partial\theta}{\partial t} + U_i\left(\frac{\partial\theta}{\partial x_i} + \frac{\partial\overline{T}}{\partial x_i}\right) = 0 \quad , \quad \sqrt{2}\ \frac{\partial^2\overline{T}}{\partial x_i^2}\ L_E \Big/ \frac{\partial\overline{T}}{\partial x_i} << 1 \tag{2.8}$$

in which x_i are the Cartesian coordinates, x_1 is in the streamwise direction, x_2 lateral and x_3 vertical; θ is the fluctuating temperature; U_i are the instantaneous velocities; $\overline{T}$ is the mean temperature; the repeated index denotes summation and L_E is the Eulerian space integral scale, Lumley derived that

$$\theta(\vec{x}, t) = -\frac{\partial \overline{T}}{\partial x_j} [x_j - a_j(\vec{x}, t)] \tag{2.9}$$

where $\vec{a}(\vec{x}, t)$ is the position at $t=0$ of a particle which will reach $\vec{x}$ at time t . Thus,

$$\overline{\theta(\vec{x}, t)U_i(\vec{x}, t)} = -\frac{\partial \overline{T}}{\partial x_j} \overline{(x_j - a_j)U_i(\vec{x}, t)}$$

$$= -\frac{\partial \overline{T}}{\partial x_j} A_{ij} \tag{2.10}$$

where A_{ij} is a second rank tensor characterized by the velocity field completely.

Now, for simplicity, assume the mean temperature gradient exists only in the vertical direction, i.e., $\partial\overline{T}/\partial x_1 = \partial\overline{T}/\partial x_2 = 0$. Then the heat flux has only the form $\overline{\theta U_3}$. If $\phi_{wT}(\vec{k})$ the spectrum of the vertical heat flux $\overline{\theta U_3}$ is integrated spherically, the directional information can be missed for simplifying derivations. Thus,

$$\phi_{wT}(k) = \oint \phi_{wT}(\vec{k})\, d\sigma = -\frac{d\overline{T}}{dz} \oint \phi_{A_{33}}(\vec{k})\, d\sigma$$

in which $\phi_{A_{33}}(\vec{k})$ is the spectrum form of A_{33}, $d\sigma$ is the surface element of a sphere with radius $k = |\vec{k}|$, and $d\overline{T}/dz = d\overline{T}/dx_3$.

Finally, based on the above derivation, Lumley postulated that the spherically averaged spectrum of the vertical heat flux $\phi_{wT}(k)$ is proportional to the mean temperature gradient in vertical direction $d\overline{T}/dz$ for a stably stratified flow. So, from the dimensional reasoning,

$$\phi_{wT}(k) = - a \frac{d\overline{T}}{dz} \varepsilon^{\frac{1}{3}}(k) k^{-\frac{7}{3}} \tag{2.11}$$

in which a is constant.

Now, if the production of turbulence is very weak in the wave number range considered, we have

$$\begin{aligned} \frac{\partial \varepsilon(k)}{\partial k} &= \phi_{wT}(k) \frac{g}{\overline{T}} \\ &= - a \frac{d\overline{T}}{dz} \frac{g}{\overline{T}} \varepsilon^{\frac{1}{3}}(k) k^{-\frac{7}{3}} \end{aligned} \tag{2.12}$$

which results in

$$\begin{aligned} \varepsilon^{\frac{2}{3}}(k) &= \varepsilon^{\frac{2}{3}} + \frac{g}{\overline{T}} \frac{d\overline{T}}{dz} \frac{a}{2} k^{-\frac{4}{3}} \\ &= \varepsilon^{\frac{2}{3}} [1 + \frac{a}{2} (\frac{k}{k_b})^{-\frac{4}{3}}] \end{aligned} \tag{2.13}$$

where

$$k_b = \left(\frac{g}{\overline{T}} \frac{d\overline{T}}{dz} \varepsilon^{-\frac{2}{3}} \right)^{\frac{3}{4}} . \tag{2.14}$$

After inserting $\varepsilon(k)$ into the generalized Kolmogorov's spectrum

$$\phi(k) = \alpha \varepsilon^{\frac{2}{3}}(k) k^{-\frac{5}{3}} \quad , \quad k_e << k << \ell_d^{-1} \quad , \tag{2.15}$$

we have thus

$$\phi(k) = \alpha \varepsilon^{\frac{2}{3}} [1 + \frac{a}{2} (\frac{k}{k_b})^{-\frac{4}{3}}] k^{-\frac{5}{3}} \quad , \quad k_e << k << \ell_d^{-1} \quad . \tag{2.16}$$

It is clear that in case $k << k_b$, $\phi(k) \sim k^{-3}$.

Based on the same argument described above, Lumley-Shur's theory gives the temperature spectrum in a form

$$\phi_{TT}(k) = \left[N + \frac{\overline{T}\,\frac{d\overline{T}}{dz}}{g}\,\varepsilon \left\{ 1 - \left[1 + \frac{a}{2}\left(\frac{k}{k_b}\right)^{-\frac{4}{3}} \right]^{\frac{3}{2}} \right\} \right] \varepsilon^{-\frac{4}{3}}$$

$$\cdot \left[1 + \frac{a}{2}\left(\frac{k}{k_b}\right)^{-\frac{4}{3}} \right]^{-\frac{1}{2}} k^{-\frac{5}{3}} \qquad (2.17)$$

for $k_e << k << \ell_d^{-1}$.

Defining

$$k_b^* = \frac{1}{\left(\frac{g^2}{\overline{T}^2} N \right)^{1/2}} \left(\frac{g}{\overline{T}} \frac{d\overline{T}}{dz} \right)^{\frac{5}{4}} ,$$

thus, in case $k_b >> k >> k_b^*$, $\phi_{TT}(k) \sim k^{-1}$. In order to make the approximation meaningful, it must be $k_b >> k_b^*$ or equivalently

$$\left(\frac{\varepsilon \overline{T} \frac{d\overline{T}}{dz}}{gN} \right)^{\frac{1}{2}} << 1 \ . \qquad (2.18)$$

Now, the discrepancies between Bolgiano's and Lumley-Shur's hypotheses are apparent when Eqs. 2.4, 2.6, 2.16 and 2.17 are reviewed. Their differences in physical background can readily be seen from their hypotheses. The details will be discussed and compared after the generalized eddy-viscosity approximation is introduced in Chapter III.

2.3 Monin's, Gisina's, and Pao's Works

Besides the above stated Bolgiano's and Lumley-Shur's hypotheses, Monin (1962) and Gisina (1966) used Heisenberg's eddy-viscosity approximation to study the turbulence spectrum in a stably stratified flow. Monin obtained Bolgiano's solution in the buoyancy subrange for the stably stratified flow, and found some humps in the spectra in the case of unstable stratification. Gisina considered more complicated conditions of stably stratified flow:

(1) weak interaction of velocity and temperature fields,

(2) strong interaction of velocity fields and temperature fields,

(3) strong interaction of temperature fields and weak interaction of velocity fields.

The results for Gisina's first two conditions are trivial since Tchen's arguments (1953) on the shear flow can be applied to the thermally stratified shear flow, and as can be expected, the spectra of velocity and temperature are proportional to $k^{-5/3}$ for the first flow condition, and a -1 law is obtained for both spectra of velocity and temperature in case of the second flow condition. As to the last flow condition, Gisina obtained Bolgiano's solution in case the relationship between parameters

$$N \frac{g}{\overline{T}} >> b \, \varepsilon \frac{d\overline{T}}{dz} \,, \tag{2.19}$$

in which b is a numerical constant corresponding to the ratio of kinematic eddy viscosity of momentum to kinematic eddy conductivity of heat flux, can be fulfilled. As we can see later, Monin's and Gisina's solutions are only some special conditions of the generalized eddy-viscosity approximation considered in this present study, thus, their works will not be reviewed in detail.

In Pao's paper (1967) the cascade process is applied to solve a thermally stratified shear flow, however, he did not obtain any power law in the buoyancy subrange. Now, the situation is very clear, different approaches used by different people to study a problem--a thermally stratified turbulent shear flow problem result in different solutions. If they do not contradict one another, there must exist some way to solve this problem and to explain the discrepancies among them. In the following, the author investigates this problem by means of the generalized eddy-viscosity approximation. Before reaching this point, some turbulence spectral measurements in the atmosphere and in a wind tunnel will be reviewed.

2.4 Measurements of Turbulence Spectra in the Atmosphere

Turbulence spectra have been measured in the surface layer and in the free atmosphere by several authors. Different stratifications of flows were involved in these measurements. In case of neutral stratification, the - 5/3 law holds for a wide range of wave numbers as can be expected from Kolmogorov's hypotheses for locally isotropic turbulent flow of neutrally stratified fluids.

Figure 1a displays a spectral density curve of longitudinal velocity component taken at 500 m above the ground when the lapse rate of temperature from ground to 1000 m is 1°C per 100 m, i.e., the adiabatic lapse rate, and the mean velocity gradient is 0.36 m per sec per 100 m (Pinus and Shcherbakova, 1966).

In the surface layer the - 5/3 law was also observed for the energy spectrum, but the lower limit of the inertial subrange is related to stratification or Richardson number. Generally speaking, the stable stratification shrinks the inertial subrange and the unstable stratification extends it (Zubkovskii, 1962, Gurvich, 1960).

In the free atmosphere the spectra has a more complicated form because of buoyancy effects. In the following, two categories of flows are described--stable and unstable stratifications--but the details of the discussion will be in Chapter IV where the numerical solutions of the present study are given.

2.4.1 Stable stratification - As described in section 2.2.1, some energy will be abstracted from the velocity field and fed into the temperature or density field. It can be expected that the velocity spectrum in the buoyancy subrange will present a steeper slope than $-5/3$. Shur (1962) first showed the existence of the buoyancy subrange from his measurements. Later, Pinus and Shcherbakova (1966) measured the velocity spectrum in the atmospheric layer from 400 to 4500 m. In case of stable stratification, the slope of measured velocity spectra in the buoyancy subrange increased with height for roughly the same mean temperature gradient. The exponent n of the velocity spectrum k^{-n} in the buoyancy subrange varies from 2.0 to 3.5. For the sake of interest, the following Table 1 digested from Pinus and Shcherbakova (1966) is listed. For better understanding, three typical energy spectra of stably stratified flows from their measurements are demonstrated in Figs. 1c, 1d and 1e.

In 1963, Pinus measured the spectral density of the horizontal velocity component at heights of 6-12 km. Fig. 2b displays one plot of his results which appears steeper slope than $-5/3$ in a certain wave number range of spectrum. Another measurement by Vinnichenko (1966) as displayed in Fig. 2a indicates the existence of the buoyancy subrange. Recently, Myrup (1968) found the buoyancy subrange in his measurements due to the fact that the steepened slope is close to -3 for the longitudinal velocity fluctuations and between - 11/5 and -3 for the vertical velocity fluctuation.

TABLE 1

Height, m	Vertical Temperature Gradient, deg/100 m	Vertical Gradient of the Mean Wind Velocity, m sec^{-1}/100 m	Number of Spectra	Range of Variation of n	$\bar{n}$
400-700	0.65	1.84	9	2.0-2.9	2.43
700-1200	0.61	0.78	17	2.0-2.9	2.50
1200-1700	0.76	0.86	6	2.2-3.5	2.83
1700-2500	0.46	0.45	6	2.3-3.3	2.70
2500-3500	----	----	1	2.8	----
3500-4500	0.72	1.13	9	2.7-3.5	3.10

2.4.2 Unstable stratification - In this case, the velocity field absorbs energy from the temperature field as potential energy is converted into kinetic energy; some hump in the velocity spectrum can be expected. Fig. 1b shows a spectral density curve of longitudinal velocity component. (Pinus and Shcherbakova, 1966). In 1967, Ivanov and Ordanovich made a more detailed investigation of velocity spectra for unstable stratification in the low frequency range. In Fig. 3, some typical examples of the measured velocity spectra are presented. (Ivanov and Ordanovich, 1967).

As to the temperature spectrum, lesser information is available. Tsvang (1963) measured some spectral density curves of temperature for both stable and unstable stratifications. From his measurements, deviations from - 5/3 can be seen as the wave number is less than about 10^{-4} cm^{-1} for either stratification, however, no detailed discussion on these deviations from the theoretical point of view has been given in his paper.

2.5 Measurements of Turbulence Spectra in Wind Tunnel

Although the inertial subrange for locally isotropic turbulence has been confirmed as described in section 2.1, the buoyancy subrange has not been obtained in the laboratory since it is still difficult to generate turbulence of laboratory scale with buoyancy effects. Cermak and Chuang (1965) measured some vertical velocity spectra in thermally stratified shear flows, however, no buoyancy subrange was observed since the "buoyancy subrange" mentioned in their paper lies in the viscous dissipation subrange evidently. Also, Arya (1968) did not find any buoyancy subrange; however, the vertical velocity and temperature spectra, measured at close wall regions where both velocity and temperature gradients are great, present -1 slope at lower wave number range as predicted by Gisina (1966).

Chapter III

THEORETICAL STUDY

Most of the previous studies reviewed were made to find some asymptotic solutions for which some specific restrictions are assigned to the flow conditions. Thus, it would be helpful to retain every factor in the energy balance equation and to find some continuous spectra if the whole view of turbulence structure is to be obtained. In this chapter, the spectral equations of the turbulent energy and the temperature fluctuation are derived; the generalized eddy-viscosity approximation is introduced; and some analytical and asymptotic solutions will be given.

3.1 Derivation of the Spectral Equations of Turbulence Energy and Temperature Fluctuation

In an incompressible turbulent flow, the Navier-Stokes equation is assumed to be the equation governing the variation of the spatial distribution of the velocity with time. For simplifying the derivation, dynamic viscosity is assumed to be constant and Boussinesq's approximation is used. Thus, we have,

$$\rho_o \left(\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} \right) = - \frac{\partial P}{\partial x_i} - \rho g_i + \mu \frac{\partial}{\partial x_j} \left(\frac{\partial U_i}{\partial x_j} \right) \tag{3.1}$$

and the incompressibility of the flow gives

$$\frac{\partial U_i}{\partial x_i} = 0 \quad , \tag{3.2}$$

where U_i is the velocity vector, ρ is the density, ρ_o is the mean density of the flow field, P is the pressure, and μ is the dynamic viscosity. The assumption for the incompressibility of the turbulent flow is unnecessary since the density can be replaced by the potential density if a compressible fluid like atmosphere is concerned. In that case, we need only assume the flow speed is small compared to the speed of sound, or equivalently to say that the Mach number is much less than 1. The detailed study can be referred to the papers by Long (1953b), Batchelor (1953b), Bolgiano (1962), Lumley and Panofsky (1964). However, for the sake of simplicity in our derivation, incompressibility is assumed for the turbulent flow.

In addition to the governing equations for the velocity field, an equation for the temperature field which causes the density fluctuation is required, i.e.,

$$\rho_o c_p \left(\frac{\partial T}{\partial t} + U_i \frac{\partial T}{\partial x_i} \right) = \mu_T \frac{\partial}{\partial x_i} \left(\frac{\partial T}{\partial x_i} \right) , \qquad (3.3)$$

where c_p is the specific heat capacity at constant pressure, T is the temperature and μ_T is the thermal conductivity. μ_T and c_p are assumed to be constant.

For a turbulent flow, the fluid properties can be split into two parts:

$$U_i = \overline{U}_i + u_i \quad , \quad \rho = \overline{\rho} + \rho_1 \quad , \quad P = \overline{P} + p_1 \quad ,$$

$$T = \overline{T} + \theta \quad ,$$

where the bar denotes the time average or ensemble average, and u_i, ρ_1, p_1, and θ are the fluctuations about their corresponding averages.

Inserting those quantities into Eqs. 3.1, 3.2 and 3.3, we have then

$$\rho_o\left(\frac{\partial\overline{U}_i}{\partial t} + \frac{\partial u_i}{\partial t} + \overline{U}_j\frac{\partial\overline{U}_i}{\partial x_j} + u_j\frac{\partial\overline{U}_i}{\partial x_j} + \overline{U}_j\frac{\partial u_i}{\partial x_j} + u_j\frac{\partial u_i}{\partial x_j}\right)$$

$$= -\frac{\partial(\overline{P} + p_1)}{\partial x_i} - (\overline{\rho} + \rho_1)g_i + \mu\frac{\partial^2\overline{U}_i}{\partial x_j\partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}$$

$$(3.4)$$

$$\frac{\partial\overline{U}_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} = 0 \qquad (3.5)$$

and

$$\rho_o c_p\left(\frac{\partial\overline{T}}{\partial t} + \frac{\partial\theta}{\partial t} + \overline{U}_i\frac{\partial\overline{T}}{\partial x_i} + u_i\frac{\partial\overline{T}}{\partial x_i} + \overline{U}_i\frac{\partial\theta}{\partial x_i} + u_i\frac{\partial\theta}{\partial x_i}\right)$$

$$= \mu_T\frac{\partial^2\overline{T}}{\partial x_j\partial x_j} + \mu_T\frac{\partial^2\theta}{\partial x_j\partial x_j} \quad . \qquad (3.6)$$

Assuming the mean flow is stationary and taking the average (either ensemble or time average) of Eqs. 3.4, 3.5, and 3.6, we have thus

$$\rho_o\left(\overline{U}_j\frac{\partial\overline{U}_i}{\partial x_j} + \overline{u_j\frac{\partial u_i}{\partial x_j}}\right) = -\frac{\partial\overline{P}}{\partial x_i} - \overline{\rho}\,g_i + \frac{\partial^2\overline{U}_i}{\partial x_j\partial x_j} \qquad (3.7)$$

$$\frac{\partial\overline{U}_i}{\partial x_i} = 0 \qquad (3.8)$$

$$\rho_o c_p\left(\overline{U}_i\frac{\partial\overline{T}}{\partial x_i} + \overline{u_i\frac{\partial\theta}{\partial x_i}}\right) = \mu_T\frac{\partial^2\overline{T}}{\partial x_j\partial x_j} \quad . \qquad (3.9)$$

Subtracting Eqs. 3.7, 3.8 and 3.9 from Eqs. 3.4, 3.5 and 3.6, respectively, the turbulent equations for velocity and temperature fields become

$$\rho_o \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial \overline{U}_i}{\partial x_j} + \overline{U}_j \frac{\partial u_i}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} - \overline{u_j \frac{\partial u_i}{\partial x_j}} \right)$$

$$= - \frac{\partial p_1}{\partial x_i} - \rho_1 g_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{3.10}$$

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{3.11}$$

$$\rho_o\, c_p \left(\frac{\partial \theta}{\partial t} + u_j \frac{\partial \overline{T}}{\partial x_j} + \overline{U}_j \frac{\partial \theta}{\partial x_j} + u_j \frac{\partial \theta}{\partial x_j} - \overline{u_j \frac{\partial \theta}{\partial x_j}} \right)$$

$$= \mu_T \frac{\partial^2 \theta}{\partial x_j \partial x_j} \quad . \tag{3.12}$$

Taking the advantage of incompressibility, Eqs. 3.10, 3.11, and 3.12 are rewritten as

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial \overline{U}_i}{\partial x_j} + \overline{U}_j \frac{\partial u_i}{\partial x_j} + \frac{\partial u_i u_j}{\partial x_j} - \frac{\partial \overline{u_i u_j}}{\partial x_j}$$

$$= - \frac{1}{\rho_o} \frac{\partial p_1}{\partial x_i} + \frac{\theta}{T_o} g_i + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \tag{3.13}$$

and

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \overline{T}}{\partial x_j} + \overline{U}_j \frac{\partial \theta}{\partial x_j} + \frac{\partial \theta u_j}{\partial x_j} - \frac{\partial \overline{\theta u_j}}{\partial x_j}$$

$$= \nu_T \frac{\partial^2 \theta}{\partial x_j \partial x_j} \tag{3.14}$$

in which $\nu = \mu/\rho_o$ is the kinematic viscosity, $\nu_T = \mu_T/\rho_o c_p$ is the thermal diffusivity, and $\theta/T_o = - \rho_1/\rho_o$ comes from the assumptions that the flow is incompressible and the gas law $P = (c_p - c_v)\rho T$ holds, where c_v is specific heat capacity at constant volume.

Multiplying Eqs. 3.13 and 3.14 by u'_k and θ' respectively, where the prime denotes that the quantities are measured at $\vec{x}'=\vec{x}+\vec{r}$, yield

$$u'_k \frac{\partial u_i}{\partial t} + u'_k u_j \frac{\partial \overline{U}_i}{\partial x_j} + u'_k \overline{U}_j \frac{\partial u_i}{\partial x_j} + u'_k \frac{\partial u_i u_j}{\partial x_j} - u'_k \frac{\partial \overline{u_i u_j}}{\partial x_j}$$

$$= -\frac{1}{\rho_o} u'_k \frac{\partial p_1}{\partial x_i} + \frac{u'_k \theta}{T_o} g_i + \nu u'_k \frac{\partial^2 u_i}{\partial x_i \partial x_j} \tag{3.15}$$

$$\theta' \frac{\partial \theta}{\partial t} + \theta' u_j \frac{\partial \overline{T}}{\partial x_j} + \theta' \overline{U}_j \frac{\partial \theta}{\partial x_j} + \theta' \frac{\partial \theta u_j}{\partial x_j} - \theta' \frac{\partial \overline{\theta u_j}}{\partial x_j}$$

$$= \nu_T \theta' \frac{\partial^2 \theta}{\partial x_j \partial x_j} \quad . \tag{3.16}$$

Similarly, the turbulent equations at $\vec{x}'$ are taken and multiplied by u_i and θ at $\vec{x}$, then we have

$$u_i \frac{\partial u'_k}{\partial t} + u_i u'_j \frac{\partial \overline{U}'_k}{\partial x'_j} + u_i \overline{U}'_j \frac{\partial u'_k}{\partial x'_j} + u_i \frac{\partial u'_k u'_j}{\partial x'_j}$$

$$- u_i \frac{\partial \overline{u'_k u'_j}}{\partial x'_j} = \frac{1}{\rho'_o} u_i \frac{\partial p'_1}{\partial x'_k} + \frac{u_i \theta'}{T'_o} g'_k + \nu' u_i \frac{\partial^2 u'_k}{\partial x'_j \partial x'_j} \tag{3.17}$$

and

$$\theta \frac{\partial \theta'}{\partial t} + \theta u'_j \frac{\partial \overline{T}'}{\partial x'_j} + \theta \overline{U}'_j \frac{\partial \theta'}{\partial x'_j} + \theta \frac{\partial \theta' u'_j}{\partial x'_j}$$

$$- \theta \frac{\partial \overline{\theta' u'_j}}{\partial x'_j} = \nu'_T \theta \frac{\partial^2 \theta'}{\partial x'_j \partial x'_j} \quad . \tag{3.18}$$

Adding Eqs. 3.15 and 3.17, and Eqs. 3.16 and 3.18 respectively and using the fact that the turbulent quantities at x are independent of the coordinates x' and vice versa result in

$$\frac{\partial}{\partial t} u_i u'_k + u'_k u_j \frac{\partial \overline{U}_i}{\partial x_j} + u_i u'_j \frac{\partial \overline{U}'_k}{\partial x'_j} + \overline{U}_j \frac{\partial U_i u'_k}{\partial x_j}$$

$$+ \overline{U}'_j \frac{\partial u'_k u_i}{\partial x'_j} + \frac{\partial u_i u_j u'_k}{\partial x_j} + \frac{\partial u'_k u'_j u_i}{\partial x'_j} - \frac{\partial \overline{u_i u_j u'_k}}{\partial x_j}$$

$$- \frac{\partial \overline{u'_k u'_j u_i}}{\partial x'_j} = - \left(\frac{1}{\rho_o} \frac{\partial p_1 u'_k}{\partial x_i} + \frac{1}{\rho'_o} \frac{\partial p'_1 u_i}{\partial x'_k} \right) + \frac{u'_k \theta}{T_o} g_i$$

$$+ \frac{u_i \theta'}{T'_o} g'_k + \left(\nu \frac{\partial^2}{\partial x_j \partial x_j} + \nu' \frac{\partial^2}{\partial x'_j \partial x'_j} \right) u'_k u_i \qquad (3.19)$$

and

$$\frac{\partial \theta' \theta}{\partial t} + \theta' u_j \frac{\partial \overline{T}}{\partial x_j} + \theta u'_j \frac{\partial \overline{T}'}{\partial x'_j} + \overline{U}_j \frac{\partial \theta' \theta}{\partial x_j} + \overline{U}'_j \frac{\partial \theta' \theta}{\partial x'_j}$$

$$+ \frac{\partial \theta u_j \theta'}{\partial x_j} + \frac{\partial \theta \theta' u'_j}{\partial x'_j} - \frac{\partial \overline{\theta u_j \theta'}}{\partial x_j} - \frac{\partial \overline{\theta' u'_j \theta}}{\partial x'_j}$$

$$= \left(\nu_T \frac{\partial^2}{\partial x_j \partial x_j} + \nu'_T \frac{\partial^2}{\partial x'_j \partial x'_j} \right) \theta' \theta \quad . \qquad (3.20)$$

From the transformations,

$$r_j = x'_j - x_j \qquad (3.21)$$

and

$$x''_j = \frac{1}{2} (x'_j + x_j) \quad , \qquad (3.22)$$

it results in

$$\frac{\partial}{\partial x_j} = \frac{1}{2} \frac{\partial}{\partial x''_j} - \frac{\partial}{\partial r_j} \qquad (3.23)$$

$$\frac{\partial}{\partial x'_j} = \frac{1}{2}\frac{\partial}{\partial x''_j} + \frac{\partial}{\partial r_j} \tag{3.24}$$

$$\frac{\partial^2}{\partial x_j \partial x_j} = \frac{1}{4}\left(\frac{\partial^2}{\partial x''_j \partial x''_j}\right) + \frac{\partial^2}{\partial r_j \partial r_j} - \frac{\partial}{\partial x''_j}\frac{\partial}{\partial r_j} \tag{3.25}$$

and

$$\frac{\partial^2}{\partial x'_j \partial x'_j} = \frac{1}{4}\left(\frac{\partial^2}{\partial x''_j \partial x''_j}\right) + \frac{\partial^2}{\partial r_j \partial r_j} + \frac{\partial}{\partial x''_j}\frac{\partial}{\partial x_j} \quad . \tag{3.26}$$

Using the above transformations 3.21--3.26 and taking average of Eqs. 3.19 and 3.20, we obtain

$$\frac{\partial}{\partial t}\overline{u_i u'_k} + \overline{u'_k u_j}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_i u'_j}\frac{\partial \overline{U}'_k}{\partial x'_j} + \frac{1}{2}(\overline{U}'_j + \overline{U}_j)\frac{\partial}{\partial x''_j}\overline{u'_k u_i}$$

$$+ (\overline{U}'_j - \overline{U}_j)\frac{\partial}{\partial r_j}\overline{u'_k u_i} + \frac{1}{2}\frac{\partial}{\partial x''_j}(\overline{u_i u_j u'_k} + \overline{u'_k u'_j u_i})$$

$$+ \frac{\partial}{\partial r_j}(\overline{u'_k u'_j u_i} - \overline{u_i u_j u'_k}) = \frac{\overline{u'_k \theta}}{T_o} g_i + \frac{\overline{u_i \theta'}}{T'_o} g'_k$$

$$- \frac{1}{2}\left(\frac{1}{\rho_o}\frac{\partial}{\partial x''_i}\overline{P_1 u'_k} + \frac{1}{\rho'_o}\frac{\partial}{\partial x''_k}\overline{P'_1 u_i}\right) - \left(\frac{1}{\rho'_o}\frac{\partial}{\partial r_k}\overline{P'_1 u_i} - \frac{1}{\rho_o}\frac{\partial}{\partial r_i}\overline{P_1 u'_k}\right)$$

$$+ \left[\frac{1}{4}(\nu+\nu')\frac{\partial^2}{\partial x''_j \partial x''_j} + (\nu+\nu')\frac{\partial^2}{\partial r_j \partial r_j} - \nu\frac{\partial}{\partial x''_j}\frac{\partial}{\partial r_j} + \nu'\frac{\partial}{\partial x''_j}\frac{\partial}{\partial r_j}\right]$$

$$\cdot \; \overline{u'_k u_i} \tag{3.27}$$

and

$$\frac{\partial}{\partial t}\overline{\theta'\theta} + \overline{\theta' u_j}\frac{\partial \overline{T}}{\partial x_j} + \overline{\theta u'_j}\frac{\partial \overline{T}'}{\partial x'_j} + \frac{1}{2}(\overline{U}'_j + \overline{U}_j)\frac{\partial}{\partial x''_j}\overline{\theta'\theta}$$

$$+ (\overline{U}'_j - \overline{U}_j)\frac{\partial}{\partial r_j}\overline{\theta'\theta} + \frac{1}{2}\frac{\partial}{\partial x''_j}(\overline{\theta u_j \theta'} + \overline{\theta\theta' u'_j})$$

$$+ \frac{\partial}{\partial r_j}(\overline{\theta\theta' u'_j} - \overline{\theta u_j \theta'}) = \left[\frac{1}{4}(\nu_T + \nu'_T)\frac{\partial}{\partial x''_j \partial x''_j} + (\nu_T + \nu'_T)\right.$$

$$\left.\cdot \frac{\partial}{\partial r_j \partial r_j} - \nu_T \frac{\partial}{\partial x''_j}\frac{\partial}{\partial r_j} + \nu'_T \frac{\partial}{\partial x''_j}\frac{\partial}{\partial r_j}\right]\overline{\theta'\theta} \quad . \tag{3.28}$$

In order to make Eqs. 3.27 and 3.28 simpler, the local homogeneity is assumed for the turbulent fields, i.e., the spatial variation of the averaged turbulent quantities is negligible over a distance $r << \ell_e$, where ℓ_e is the scale of energy-containing eddies. Thus,

$$\frac{\partial}{\partial x''_j} = 0 \quad \text{and} \quad \frac{\partial}{\partial x''_j \partial x''_j} = 0 \tag{3.29}$$

make Eqs. 3.27 and 3.28 become

$$\frac{\partial}{\partial t}\overline{u_i u'_k} + \overline{u'_k u_j}\frac{\partial \overline{U}_i}{\partial x_j} + \overline{u_i u'_j}\frac{\partial \overline{U}'_k}{\partial x'_j} + (\overline{U}'_j - \overline{U}_j)\frac{\partial}{\partial r_j}\overline{u'_k u_i}$$

$$+ \frac{\partial}{\partial r_j}(\overline{u'_k u'_j u_i} - \overline{u_i u_j u'_k}) = -\left(\frac{1}{\rho'_o}\frac{\partial}{\partial r_k}\overline{p'_1 u_i} - \frac{1}{\rho_o}\frac{\partial}{\partial r_i}\overline{P_1 u'_k}\right)$$

$$+ (\nu + \nu')\frac{\partial^2}{\partial r_j \partial r_j}\overline{u'_k u_i} + \frac{\overline{u'_k \theta}}{T_o} g_i + \frac{\overline{u_i \theta'}}{T'_o} g'_k \tag{3.30}$$

and

$$\frac{\partial}{\partial t}\overline{\theta'\theta} + \overline{\theta' u_j}\frac{\partial \overline{T}}{\partial r_j} + \overline{\theta u'_j}\frac{\partial \overline{T}'}{\partial x'_j} + (\overline{U}'_j - \overline{U}_j)\frac{\partial}{\partial r_j}\overline{\theta'\theta}$$

$$+ \frac{\partial}{\partial r_j}(\overline{\theta\theta' u'_j} - \overline{\theta u_j \theta'}) = (\nu_T + \nu'_T)\frac{\partial^2}{\partial r_j \partial r_j}\overline{\theta'\theta} \tag{3.31}$$

in which the averaged turbulent quantities are functions of r and t.

For further simplifying Eqs. 3.30 and 3.31 without loss of generality in a locally homogeneous turbulent shear flow of thermally stratified fluid, the following are assumed

$$\nu = \nu' \quad , \quad \nu_T = \nu'_T \quad , \quad g_3 = g'_3 = g \quad , \quad g_2 = g'_2 = g_1 = g'_1 = 0 \quad ,$$

$$\overline{U}_2 = \text{constant} \quad , \quad \overline{U}_3 = \text{constant} \quad , \quad \rho_o = \rho'_o$$

$$\overline{U}_1 = f(x_3) \quad , \quad \overline{T} = g(x_3) \quad ,$$

where x_1 is in the streamwise direction, x_2 is in the lateral direction, and x_3 is in the vertical direction.

Thus,

$$\frac{\partial}{\partial t}\overline{u_i u'_k} + (\delta_{i1}\overline{u'_k u_3} + \delta_{k1}\overline{u_i u'_3})\frac{d\overline{U}_1}{dx_3} + (\overline{U}'_1 - \overline{U}_1)\frac{\partial}{\partial r_1}\overline{u'_k u_i}$$

$$+ \frac{\partial}{\partial r_j}(\overline{u'_k u'_j u_i} - \overline{u_i u_j u'_k}) = -\frac{1}{\rho_o}\left(\frac{\partial}{\partial r_k}\overline{P'_1 u_i}\right.$$

$$\left. - \frac{\partial}{\partial r_i}\overline{P_1 u'_k}\right) + \frac{g}{T_o}(\delta_{i3}\overline{u'_k \theta} + \delta_{k3}\overline{u_i \theta'}) + 2\nu\frac{\partial^2}{\partial r_j \partial r_j}\overline{u'_k u_i} \tag{3.32}$$

and

$$\frac{\partial}{\partial t}\overline{\theta'\theta} + (\overline{\theta' u_3} + \overline{\theta u'_3})\frac{d\overline{T}}{dx_3} + (\overline{U'}_1 - \overline{U}_1)\frac{\partial}{\partial r_1}\overline{\theta'\theta}$$

$$+ \frac{\partial}{\partial r_j}(\overline{\theta\theta' u'_j} - \overline{\theta u_j \theta'}) = 2\nu_T \frac{\partial^2}{\partial r_j \partial r_j}\overline{\theta'\theta} \qquad (3.33)$$

where δ_{ij} is Kronecker's delta, $\delta_{ij} = 1$ if $i=j$; $\delta_{ij} = 0$ if $i \neq j$. Expanding $\overline{U'_1}$ into a Taylor series at x_3 gives

$$\overline{U'_1} - \overline{U}_1 = \frac{d\overline{U}_1}{dx_3}(x'_3 - x_3) + \frac{d^2\overline{U}_1}{dx_3^{\,2}} \cdot \frac{(x'_3 - x_3)^2}{2!} + \ldots\ldots$$

$$\simeq \frac{d\overline{U}_1}{dx_3}\, r_3 \qquad (3.34)$$

if we assume that $r_3 << (d\overline{U}_1/dx_3) \,/\, (d^2\overline{U}_1/dx^2{}_3)$.

Hence, substituting Eq. 3.34 into Eqs. 3.32 and 3.33 and contracting Eq. 3.32 gives

$$\frac{\partial}{\partial t}\overline{u_i u'_i} + (\delta_{i1}\overline{u'_i u_3} + \delta_{i1}\overline{u_i u'_3} + r_3\frac{\partial}{\partial r_1}\overline{u_i u'_i})\frac{d\overline{U}}{dx_3}$$

$$+ \frac{\partial}{\partial r_j}(\overline{u'_i u'_j u_i} - \overline{u_i u_j u'_i}) = \frac{g}{T_o}(\overline{u'_i \theta} + \overline{u_i \theta'})\delta_{i3}$$

$$+ 2\nu \frac{\partial^2}{\partial r_j \partial r_j}\overline{u_i u'_i} \qquad (3.35)$$

and

$$\frac{\partial}{\partial t}\overline{\theta'\theta} + (\overline{\theta' u_3} + \overline{\theta u'_3})\frac{d\overline{T}}{dx_3} + r_3\frac{\partial}{\partial r_1}\overline{\theta'\theta}\,\frac{d\overline{U}}{dx_3}$$

$$+ \frac{\partial}{\partial r_j}(\overline{\theta\theta' u'_j} - \overline{\theta u_j \theta'}) = 2\nu_T \frac{\partial^2}{\partial r_j \partial r_j}\overline{\theta'\theta} \qquad (3.36)$$

in which the pressure velocity correlation terms are eliminated in the contracted tensor form because of the incompressibility of flow, i.e.,

$$\frac{\partial}{\partial r_i} \overline{P'_1 u_i} = \frac{\partial}{\partial r_i} \overline{P_1 u'_i} = 0 \ .$$

In order to transform Eqs. 3.35 and 3.36 into wave number $\vec{k}$ space, the following definitions are given

$$\overline{u_i u'_k} = \int_{\vec{k}\text{ space}} E_{i,k}(\vec{k},t) e^{ik_\ell r_\ell} d\vec{k} \tag{3.37}$$

$$-\frac{\partial}{\partial r_j}(\overline{u'_k u'_j u_i} - \overline{u_i u_j u'_k}) = \int_{\vec{k}\text{ space}} F_{i,k}(\vec{k},t) e^{ik_\ell r_\ell} d\vec{k} \tag{3.38}$$

$$(\overline{u'_3 \theta} + \overline{u_3 \theta'}) = \int_{\vec{k}\text{ space}} E_{u_3\theta}(k,t) e^{ik_\ell r_\ell} d\vec{k} \tag{3.39}$$

$$\overline{\theta\theta'} = \int_{\vec{k}\text{ space}} E_T(\vec{k},t) e^{ik_\ell r_\ell} d\vec{k} \tag{3.40}$$

$$-\frac{\partial}{\partial r_j}(\overline{\theta\theta' u'_j} - \overline{\theta u_j \theta'}) = \int_{\vec{k}\text{ space}} F_T(\vec{k},t) e^{ik_\ell r_\ell} d\vec{k} \ . \tag{3.41}$$

Thus, Eqs. 3.35 and 3.36 yield

$$\frac{\partial E_{i,i}}{\partial t} + (2E_{1,3} - k_1 \frac{\partial E_{i,i}}{\partial k_3}) \frac{d\overline{U}_1}{dx_3} = F_{i,i} + \frac{g}{\overline{T}} E_{u_3\theta}$$

$$- 2\nu k^2 E_{i,i} \tag{3.42}$$

and

$$\frac{\partial E_T}{\partial t} + E_{u_3\theta} \frac{d\overline{T}}{dx_3} - k_1 \frac{\partial E_T}{\partial k_3} \frac{\partial \overline{U}_1}{\partial x_3} = F_T - 2\nu_T k^2 E_T \tag{3.43}$$

where $k = |\vec{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$, and the assumptions $\lim_{k_3 \to \infty} E_{i,i} = \lim_{k_3 \to -\infty} E_{i,i} = 0$ and $\lim_{k_3 \to \infty} E_T = \lim_{k_3 \to -\infty} E_T = 0$ are used. It is realized that $E_{1,3}$ and $E_{u_3\theta}$ are the cospectra of the Reynolds stress and vertical heat flux. Since the above two equations are expressed in $\vec{k}$ space, the directional information of the spectra is retained in both equations; needless to say, it is difficult to solve Eqs. 3.42 and 3.43. Now, if Eqs. 3.42 and 3.43 are averaged over a spherical shell then we have the equations which are only function of k , the magnitude of $\vec{k}$, and the directional information is lost. Thus,

$$\frac{\partial \phi(k,t)}{\partial t} + \phi_{uw}(k,t) \frac{d\overline{U}}{dz} - \left(k_1 \frac{\partial E_{i,i}}{\partial k_3} \right)_{sp.av.} \frac{d\overline{U}}{dz}$$

$$= F(k,t) + \beta \phi_{wT}(k,t) - 2\nu k^2 \phi(k,t) \tag{3.44}$$

and

$$\frac{\partial \phi_{TT}(k,t)}{\partial t} + \phi_{wT}(k,t) \frac{d\overline{T}}{dz_*} - \left(k_1 \frac{\partial E_T}{\partial k_3} \right)_{sp.av.} \frac{d\overline{U}}{dz}$$

$$= F_{TT}(k,t) - 2 \nu_T k^2 \phi_{TT}(k,t) \tag{3.45}$$

in which

$$\phi(k,t) = \frac{1}{2} \oint E_{i,i}(\vec{k},t)\,d\sigma(k)$$

$$\phi_{uw}(k,t) = \oint E_{1,3}(\vec{k},t)\,d\sigma(k)$$

$$\left(k_1 \frac{\partial E_{i,i}}{\partial k_3}\right)_{sp.\ av.} = \frac{1}{2} \oint k_1 \frac{\partial E_{i,i}(\vec{k},t)}{\partial k_3}\,d\sigma(k)$$

$$\phi_{TT}(k,t) = \oint E_T(\vec{k},t)\,d\sigma(k)$$

$$\phi_{wT}(k,t) = \frac{1}{2} \oint E_{u_3\theta}(\vec{k},t)\,d\sigma(k)$$

$$F(k,t) = \frac{1}{2} \oint F_{i,i}(\vec{k},t)\,d\sigma(k)$$

$$F_{TT}(k,t) = \oint F_T(\vec{k},t)\,d\sigma(k)$$

$$\left(k_1 \frac{\partial E_T}{\partial k_3}\right)_{sp.\ av.} = \oint k_1 \frac{\partial E_T(\vec{k},t)}{\partial k_3}\,d\sigma(k)$$

$$\beta = \frac{g}{\overline{T}}\ ,\quad \frac{d\overline{U}}{dz} = \frac{d\overline{U}_1}{dx_3}\ ,\quad \frac{d\overline{T}}{dz_*} = 2\,\frac{d\overline{T}}{dx_3} = 2\,\frac{d\overline{T}}{dz}\ , \qquad (3.46)$$

and $d\sigma(k)$ is the surface element on the spherical shell with radius k .

At this stage, it would be worth describing the physical sense of each term in the spectral Eqs. 3.44 and 3.45 before attempting to solve them. The first terms of Eqs. 3.44 and 3.45 are the rate of change of turbulent energy and temperature inhomogeneity in their corresponding spectral forms. In case of steady turbulent flow, both terms vanish. The second terms of Eqs. 3.44 and 3.45 represent the production of turbulent energy due to the work of the Reynolds stress against the mean shear and the production of temperature inhomogeneity transferred by the vertical heat flux against the mean temperature gradient. The integration of ϕ_{uw} and ϕ_{wT} with respect to k over the internal $(0, \infty)$ will give the Reynolds shear stress and the vertical heat flux respectively.

The third terms of Eqs. 3.44 and 3.45 are the turbulent energy transfer and the temperature inhomogeneity transfer respectively due to distortion by mean shear. Unlike the second terms of Eqs. 3.44 and 3.45, these terms do not contribute the total energy and temperature inhomogeneity balances but redistribute energy and temperature inhomogeneity by transfer through wave numbers. The same situation happens to the fourth terms of Eqs. 3.44 and 3.45, i.e., these are also the transfer terms, however, these transfer terms are due to distortion by the fluctuation gradients, or say, due to inertial processes. These inertial transfers from low wave numbers to high wave numbers are certainly the characteristics of turbulent flow.

As to the last terms of Eqs. 3.44 and 3.45, they are energy dissipation by viscosity and temperature inhomogeneity smeared out by thermal conductivity. Now, here comes the most important term $\beta\phi_{wT}(k,t)$ in Eq. 3.44 which reflects the effects on the turbulent spectra by buoyancy force due to stratifications. In the case of stable stratification, this term becomes a sink with respect to the turbulent energy, and on the other hand, it becomes an energy source in case of unstably stratified flow.

In the following, the turbulent flow is assumed to be steady state in the wave number range investigated. And also the third terms in Eqs. 3.44 and 3.45 are assumed to be negligible as compared to the fourth ones because it is believed that these shear transfer terms mainly affect the spectra in the lower wave number range than buoyancy subrange. Thus, the spectral equations for velocity and temperature fields are respectively the following:

$$F(k) - \phi_{uw}(k) \frac{d\overline{U}}{dz} + \beta\phi_{wT}(k) - 2\nu k^2\phi(k) = 0 \tag{3.47}$$

and

$$F_{TT}(k) - \phi_{wT}(k) \frac{d\overline{T}}{dz_*} - 2\nu_T k^2 \phi_{TT}(k) = 0 \tag{3.48}$$

whose integrated forms are

$$\varepsilon = 2\nu \int_0^k k^2\phi(k)\,dk - \frac{d\overline{U}}{dz} \int_k^\infty \phi_{uw}(k)\,dk + \int_k^\infty F(k)\,dk$$

$$+ \beta \int_k^\infty \phi_{wT}(k)\,dk \tag{3.49}$$

and

$$N_* = 2\nu_T \int_0^k k^2\phi_{TT}(k)\,dk - \frac{d\overline{T}}{dz_*} \int_k^\infty \phi_{wT}(k)\,dk + \int_k^\infty F_{TT}(k)\,dk \quad . \tag{3.50}$$

In Eqs. 3.49 and 3.50, ε the total dissipation of turbulent energy by viscosity and N_* twice of the total dissipation of temperature fluctuation by thermal conductivity are defined by

$$\varepsilon = 2\nu \int_0^\infty k^2\phi(k)\,dk \tag{3.51}$$

and

$$N_* = 2\nu_T \int_0^\infty k^2\phi_{TT}(k)\,dk = 2N \quad . \tag{3.52}$$

In the sections following the generalized eddy-viscosity approximation is introduced and closed forms of solutions of Eqs. 3.49 and 3.50 are analytically derived for different situations.

3.2 Generalized Eddy-Viscosity Approximation

As a consequence of the nonlinearity of the Navier-Stokes equations, the correlation equations like Eqs. 3.32 and 3.33 always contain one more unknown than the number of equations; in other words, a closure problem is involved. Thus, if the problem must be solved from the correlation equations or its corresponding spectral equations, additional assumptions have to be assigned to the turbulent energy transfer function if only the locally isotropic turbulence is considered. In case the temperature field and shear flow are introduced, other assumptions should be added. Thus, the six unknowns in Eqs. 3.49 and 3.50 are reduced to two unknowns and, of course, these equations are solvable.

In the following, the generalized eddy-viscosity of the form suggested by Panchev (1967),

$$\eta_s(k) = \gamma_s \left[\int_k^\infty \phi^{\frac{s}{2}}(k) k^{-\frac{s}{2} - 1} dk \right]^{\frac{1}{s}} , \quad s > 0 \tag{3.53}$$

will be introduced.

The physical sense of the generalized eddy-viscosity expressed by Eq. 3.53 is not difficult to be realized if Heisenberg's idea (1948) will be reviewed. His form to express the turbulent energy transfer function is listed as

$$\int_k^\infty F(k) dk = \eta_1(k) \int_o^k 2k^2 \phi(k) dk \tag{3.54}$$

and $\eta_1(k)$ called the kinematic eddy-viscosity is expressed as

$$\eta_1(k) = \gamma_1 \int_k^\infty \phi^{\frac{1}{2}}(k) k^{-\frac{3}{2}} dk \tag{3.55}$$

where γ_1 is a numerical constant.

The idea implied in the Eq. 3.54 is that the energy transfer from the wave number less than k to the wave number larger than k can be considered as eddy-viscosity working on the turbulent vorticity formed in the wave number interval $(0,k)$. This eddy-viscosity can be viewed as the integral effect of eddies with wave numbers larger than k on the eddies with wave numbers less than k. Thus, according to the dimensional arguments, $\eta_1(k)$ is expressed in a form expressed by Eq. 3.55. Now, we can see that the expression of the generalized eddy-viscosity $\eta_s(k)$ has certainly the dimension of the eddy-viscosity, moreover, the parameter s introduced in Eq. 3.53 can be interpreted as degrees of interaction between the motions of eddies, i.e., how the motions of eddies are interrelated to one another. Generally speaking, the introduction of s will not affect the spectral form but its magnitude as can be seen later.

As will be seen later, in case $s \to \infty$, the expression of Eq. 3.53 becomes only a function of the local wave number k and the associated energy spectrum. Of course, Heisenberg's expression for the eddy viscosity is only a special case of Eq. 3.53 when $s = 1$. Based on the same argument and for the purpose of further generalization, the turbulent energy transfer function is expressed as:

$$\int_k^\infty F(k)dk = \gamma_{s,r}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}dk\right]^{\frac{1}{s}}\left[\int_o^k 2\phi^r(k)k^{3r-1}dk\right]^{\frac{1}{r}} \tag{3.56}$$

which can evidently be derived from

$$\int_k^\infty F(k)dk = \text{const.}\left[\int_k^\infty \phi^{d_1}(k)k^{d_2}\,dk\right]^{d_3}\cdot\left[\int_o^k \phi^{d_4}(k)k^{d_5}\,dk\right]^{d_6} \tag{3.57}$$

given by Goldstein (1951), where the exponents are related from dimensional arguments as

$$d_3(d_2 + 1) + d_6(d_5 + 1) = \frac{5}{2} ,$$

and

$$d_3 d_1 + d_6 d_4 = \frac{3}{2} .$$

These two relations can reduce six parameters d_i in Eq. 3.57 into four arbitrary parameters; for further simplification, Eq. 3.56 is eventually derived. It can readily be seen that the first factor in Eq. 3.56 has the dimension of kinematic turbulent eddy viscosity and the second one has a dimension of turbulent vorticity. It is clear that the expression of Heisenberg's form of turbulent energy transfer Eq. 3.54 is a particular form of Eq. 3.56 in case $s = 1$ and $r = 1$.

Similarly, the transfer of turbulent temperature inhomogeneity $\int_k^\infty F_{TT}(k)dk$ can also be in a form of generalized eddy-viscosity, i.e.,

$$\int_k^\infty F_{TT}(k)dk = b\gamma_{s,r} \left[\int_k^\infty \phi^{\frac{s}{2}}(k) k^{-\frac{s}{2} - 1} dk \right]^{\frac{1}{s}} \cdot \left[\int_o^k 2\phi_{TT}^r(k) k^{3r-1} dk \right]^{\frac{1}{r}} \tag{3.58}$$

where b is a numerical constant of order 1 and is equivalent to the ratio of eddy thermal diffusivity to eddy kinematic viscosity.

Before expressing $\phi_{wT}(k)$ and $\phi_{uw}(k)$ in terms of the generalized eddy-viscosity, the validity and generality of Eq. 3.56 will be seen as follows.

Since

$$\lim_{s\to\infty}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}\,dk\right]^{\frac{1}{s}} = \phi^{\frac{1}{2}}(k)k^{-\frac{1}{2}} \tag{3.59}$$

and

$$\lim_{r\to\infty}\left[\int_o^k \phi^r(k)k^{3r-1}\,dk\right]^{\frac{1}{r}} = \phi(k)k^3 \quad ,$$

Eq. 3.56, in case $s \to \infty$ and $r \to \infty$, becomes

$$\int_k^\infty F_{\infty,\infty}(k)\,dk = \gamma_{\infty,\infty}\,\phi^{\frac{3}{2}}(k)k^{\frac{5}{2}} \quad . \tag{3.60}$$

It is obvious that Eq. 3.60 is nothing but Kovasznay's approximation, which is exactly the local limit of the nonlocal and generalized approximation of the form of Eq. 3.56. From the example illustrated above to indicate the usage of Eq. 3.56, we can say that by varying the values s and r , solutions corresponding to the different degrees of turbulent nonlinear interactions between eddies can be obtained. Obviously, the Kovasznay's approximation is a limit form since it means that the eddies interact themselves only. For the case $0 < s < \infty$ and $0 < r < \infty$, it may be interpreted that the eddies of wave number k interact with the other eddies with wave number $k \pm \Delta k$; of course, Δk is a function of the values of s and r , and Δk decreases with increasing s and r .

Keeping these ideas in mind, the generalized eddy-viscosity approximation can be extended to the spectra $\phi_{uw}(k)$ and $\phi_{wT}(k)$ without any trouble. In the following, $r = 1$ will be assigned since $[\int_o^k \phi(k)k^2dk]$ gives a clear physical sense--the spherically averaged square of the root-mean-square vorticity of turbulence in the range 0 to k. The exact local approximation when $s \to \infty$ and $r \to \infty$ will also be used.

In case $\phi_{uw}(k)$ and $\phi_{wT}(k)$ are concerned, we need consider not only the turbulent field but the interactions between the mean velocity and temperature fields and the turbulent field.

Thus, $\int_k^\infty \phi_{uw}(k)dk$ and $\int_k^\infty \phi_{wT}(k)dk$ are expressed as the following general forms

$$\int_k^\infty \phi_{uw}(k)\,dk = \mp \gamma_{s,r}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}\,dk\right]^{\frac{1}{s}} \left|\frac{d\overline{U}}{dz}\right|^{C_1}\left[\int_0^k 2\phi^r(k)^{3r-1}dk\right]^{\frac{C_2}{2r}} \tag{3.61}$$

where the upper sign indicates the case when $d\overline{U}/dz > 0$, and the lower sign for $d\overline{U}/dz < 0$, and

$$\int_k^\infty \phi_{wT}(k)\,dk = \mp b\gamma_{s,r}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}\,dk\right]^{\frac{1}{s}} \left|\frac{d\overline{T}}{dz_*}\right|^{C_3}$$

$$\cdot \left[\int_0^k 2\phi_{TT}^r(k)k^{3r-1}\,dk\right]^{\frac{C_4}{2r}} \tag{3.62}$$

where the upper sign denotes the stable stratification $d\overline{T}/dz_* > 0$, and the lower sign denotes the unstable one $d\overline{T}/dz_* < 0$ because the vertical heat flux is negative in case of stable stratification and is positive for the unstably stratified flow.

The parameters C_i in Eqs. 3.61 and 3.62 are related from dimensional argument as $C_1 + C_2 = 1$, and $C_3 + C_4 = 1$. C_1 and C_2 characterize the degrees of interaction between the mean velocity gradient and the turbulent vorticity; C_3 and C_4 denote the degrees of interaction between the mean temperature gradient and the turbulent temperature gradient. In case $C_1 = 0$, $C_2 = 1$, and $C_1 = 1$, $C_2 = 0$, Eq. 3.61 expresses the conditions of strong interaction and weak interaction of velocity field considered by Tchen (1953). Hence, increasing C_2 would mean that the interaction between the vorticity of main motion and turbulent vorticity becomes stronger, and according to Tchen (1953), the resonance between two motions is intensified. Thus, in case resonance is intensified, the inertial transfer process through eddies is interfered by the mean motion and energy is supplied to the eddies by means of the Reynolds stress working against the mean motion. In other words, the - 5/3 law is invalid and the energy spectrum has less steep slope. In the extreme case when $C_2 = 1$, the slope of the energy spectrum becomes -1.

As far as stratification is concerned, $C_3 = 0$, $C_4 = 1$, and $C_3 = 1$, $C_4 = 0$ are equivalent to the case of strong and weak interactions investigated by Gisina (1966). In the following sections, the significance of C_4 will be seen. Thus, it is clear that Eqs. 3.61 and 3.62 can give more general information on the structure of the thermally stratified turbulent shear flow since different values of C_i can be assigned to characterize different flow conditions.

In addition to the relationship between C_i described previously it seems to be helpful to let $C_1 > 0$ and $C_3 > 0$. Since $|d\overline{T}/dz_*| \to 0$ would mean that the flow becomes lesser stratification, the flow will be nonstratified when $|d\overline{T}/dz_*| = 0$, thus, it implies that there exists no vertical heat flux. But if $C_3 = 0$, this implication cannot be seen when $|d\overline{T}/dz_*| = 0$; in other words, if $C_3 = 0$ is assigned to Eq. 3.62, we always have the vertical heat flux even when $|d\overline{T}/dz_*| = 0$. However, once $|d\overline{T}/dz_*| \neq 0$, from the dimensional arguments C_3 can be less than or equal to 0 according to the relation $C_3 + C_4 = 1$. Thus, it may conclude that C_3 is a function of $|d\overline{T}/dz_*|$, i.e., in case $|d\overline{T}/dz_*| \to 0$, it is believed that C_3 must be greater than 0 as stated. Similar argument can be applied to C_1 if Eq. 3.61 is considered.

3.3 Solutions of the Spectral Equations of Thermally Stratified Turbulent Shear Flows

In this section, the analytical derivations of solutions based on the generalized eddy-viscosity approximation are described in detail. Nondimensionalized spectral equations are derived not only because they can give neat and concise forms but because some similarity theory of spectra can be made if suitable dimensionless variables are used.

3.3.1 Solutions of flows with molecular effects - The introduction of molecular effects does not mean that the spectra at high wave numbers can be studied by the present method as seen from the discussion in section 2.1. Thus, only the spectra in the buoyancy and inertial subrange are of interest.

Based on the generalized eddy viscosity approximation described in section 3.2, Eqs. 3.49 and 3.50 together with Eqs. 3.56, 3.58, 3.61 and 3.62 become

$$\varepsilon = 2\nu \int_0^k k^2\phi(k)\,dk + \gamma_{s,1}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)\,k^{-\frac{s}{2}-1}\,dk\right]^{\frac{1}{s}} \left\{\left|\frac{d\overline{U}}{dz}\right|^{1+C_1} \cdot \left[\int_0^k 2k^2\phi(k)\,dk\right]^{\frac{C_2}{2}} + \int_0^k 2k^2\phi(k)\,dk \mp b\beta \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} \left[\int_0^k 2k^2\phi_{TT}(k)\,dk\right]^{\frac{C_4}{2}}\right\} \tag{3.63}$$

and

$$N_* = 2\nu_T \int_0^k k^2 \phi_{TT}(k)\,dk + \gamma_{s,1}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)\,k^{-\frac{s}{2}-1}\,dk\right]^{\frac{1}{s}} \left\{ \pm\, b\left|\frac{d\overline{T}}{dz_*}\right|^{C_3} \right.$$

$$\left. \cdot\, \frac{d\overline{T}}{dz_*}\left[\int_0^k 2k^2\phi_{TT}(k)\,dk\right]^{\frac{C_4}{2}} + b\int_0^k 2k^2\phi_{TT}(k)\,dk \right\} , \qquad (3.64)$$

in which $r = 1$ is assigned to Eqs. 3.56, 3.61 and 3.62.

Using the nondimensional parameters

$$x = \frac{k}{k_d} , \quad \Phi = \frac{\phi}{\phi_d} , \quad \Phi_{TT} = \frac{\phi_{TT}}{\phi_{TT}^d}$$

$$m = \frac{d\overline{U}}{dz}\left|\frac{d\overline{U}}{dz}\right|^{C_1}\left(\frac{\nu}{\varepsilon}\right)^{1-\frac{C_2}{2}} , \quad \beta_1 = b\beta\left|\frac{d\overline{T}}{dz_*}\right|^{C_3}\left(\frac{N_*}{\nu_T}\right)^{\frac{C_4}{2}}\frac{\nu}{\varepsilon} ,$$

$$m_T = \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} N_*^{\frac{C_4}{2}-1}\,\nu_T^{-\frac{C_4}{2}+1}\,\frac{d\overline{T}}{dz_*} , \text{ and } Pr = b\,\frac{\nu}{\nu_T} , \qquad (3.65)$$

in which

$$k_d = \gamma_{s,1}^{\frac{1}{2}}\left(\frac{\varepsilon}{\nu^3}\right)^{\frac{1}{4}} , \quad \phi_d = \gamma_{s,1}^{-\frac{3}{2}}\,(\varepsilon\nu^5)^{\frac{1}{4}} , \text{ and}$$

$$\phi_{TT}^d = \gamma_{s,1}^{-\frac{3}{2}}\,\frac{N_*}{\nu_T}\left(\frac{\nu^3}{\varepsilon}\right)^{\frac{3}{4}} ,$$

Eqs. 3.63 and 3.64 are reduced to the dimensionless forms

$$1 = \int_0^x 2x^2\Phi(x)\,dx + \left[\int_x^\infty \Phi^{\frac{s}{2}}(x)\,x^{-\frac{s}{2}-1}\,ds\right]^{\frac{1}{s}}\left\{ |m|\left[\int_0^x 2x^2\Phi(x)\,dx\right]^{\frac{C_2}{2}} \right.$$

$$\left. + \int_0^x 2x^2\Phi(x)\,dx \mp \beta_1\left[\int_0^x 2x^2\Phi_{TT}(x)\,dx\right]^{\frac{C_4}{2}} \right\} \qquad (3.66)$$

and

$$1 = \int_0^x 2x^2\Phi_{TT}(x)\,dx + \left[\int_x^\infty \Phi^{\frac{s}{2}}(x)\,x^{-\frac{s}{2}-1}\,dx\right]^{\frac{1}{s}} \left\{\pm \Pr m_T \cdot \left[\int_0^x 2x^2\Phi_{TT}(x)\,dx\right]^{\frac{C_4}{2}} + \Pr \int_0^x 2x^2\Phi_{TT}(x)\,dx\right\} . \qquad (3.67)$$

Now let

$$Y_s^{\,2} = \int_0^x 2x^2\Phi(x)\,dx \quad \text{and} \quad Z_s^{\,2} = \int_0^x 2x^2\Phi_{TT}(x)\,dx \quad ,$$

thus, Eqs. 3.66 and 3.67 are simplified to

$$1 = Y_s^{\,2} + \left[\int_x^\infty (Y_s Y_s')^{\frac{s}{2}}\, x^{-\frac{3s}{2}-1}\,dx\right]^{\frac{1}{s}} \left\{|m|Y_s^{\,C_2} + Y_s^{\,2} \mp \beta_1 Z_s^{\,C_4}\right\} \qquad (3.68)$$

and

$$1 = Z_s^{\,2} + \left[\int_x^\infty (Y_s Y_s')^{\frac{s}{2}}\, x^{-\frac{3s}{2}-1}\,dx\right]^{\frac{1}{s}} \Pr \left\{|m_T|Z_s^{\,C_4} + Z_s^{\,2}\right\} \qquad (3.69)$$

where $Y_s' = dY_s/dx$; the upper sign of the last term in Eq. 3.68 denotes the stable stratification, while the lower sign indicates the unstable stratification.

Thus, from Eqs. 3.68 and 3.69 and from

$$Y_s^{\,2}(x) = \int_0^x 2x^2\Phi(x)\,dx \qquad (3.70)$$

and

$$Z_s^{\,2}\,(x) = \int_0^x 2x^2\Phi_{TT}(x)\,dx \quad , \qquad (3.71)$$

energy and temperature spectra can be evaluated by

$$\Phi(x) = \frac{dY_s^{\,2}(x)}{2x^2} = \frac{Y_s(x)Y_s'(x)}{x^2} \tag{3.72}$$

and

$$\Phi_{TT}(x) = \frac{Z_s(x)Z_s'(x)}{x^2} \quad . \tag{3.73}$$

As to the turbulent shear stress spectrum and the vertical heat flux spectrum, Eqs. 3.61 and 3.62 are transformed into non-dimensional forms. Using the dimensionless variables

$$\Phi_{uw} = \frac{\phi_{uw}}{\phi^d_{uw}} \quad , \quad \Phi_{wT} = \frac{\phi_{wT}}{\phi^d_{wT}} \quad , \quad \phi^d_{uw} = \gamma_{s,1}^{-\frac{1}{2}} \, \varepsilon^{\frac{C_2}{2}-\frac{1}{4}} \, \nu^{\frac{7}{4}-\frac{C_2}{2}} \left|\frac{d\overline{U}}{dz}\right|^{C_1}$$

$$\phi^d_{wT} = b\gamma_{s,1}^{-\frac{1}{2}} \, \varepsilon^{-\frac{1}{4}} \, N_*^{\frac{C_4}{2}} \, \nu^{\frac{7}{4}} \, \nu_T^{-\frac{C_4}{2}} \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} \quad , \quad \frac{d\overline{U}}{dz} > 0 \tag{3.74}$$

as well as the variables in Eq. 3.65, we have therefore

$$\int_x^\infty \Phi_{uw}(x)\,dx = - \left[\int_x^\infty \Phi^{\frac{s}{2}}(x)\,x^{-\frac{s}{2}-1}\,dx\right]^{\frac{1}{s}} \left[2\int_o^x x^2\Phi(x)\,dx\right]^{\frac{C_2}{2}} \tag{3.75}$$

and

$$\int_x^\infty \Phi_{wT}(x)\,dx = \mp \left[\int_x^\infty \Phi^{\frac{s}{2}}(x)\,x^{-\frac{s}{2}-1}\,dx\right]^{\frac{1}{s}} \left[2\int_o^x x^2\Phi_{TT}(x)\,dx\right]^{\frac{C_4}{2}} \tag{3.76}$$

which can be simplified as

$$\int_x^\infty \Phi_{uw}(x)\,dx = - \frac{1 - Y_s^2}{|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{\,C_4}} \; Y_s^{\,C_2}$$

and

$$\int_x^\infty \Phi_{wT}(x)\,dx = \mp \frac{1 - Y_s^{\,2}}{|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4}} \; Z_s^{C_4} \; .$$

Thus,

$$\Phi_{uw}(x) = \frac{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})\,[C_2 Y_s^{C_2-1} - (2+C_2) Y_s^{C_2+1}]\, Y_s'}{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})^2}$$

$$- \frac{Y_s^{C_2}\,(1 - Y_s^{\,2})(|m|Y_s' + 2Y_s Y_s' \mp \beta_1 C_4 Z_s^{C_4-1}\, Z_s')}{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})^2} \tag{3.77}$$

and

$$\Phi_{wT}(x) = \pm \left\{ \frac{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})\,[Z_s'(1 - Y_s^{\,2}) C_4 Z_s^{C_4-1} \quad 2 Z_s^{C_4} Y_s Y_s']}{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})^2} \right.$$

$$\left. - \frac{Z_s^{C_4}(1 - Y_s^{\,2})(|m|Y_s' + 2Y_s Y_s' \mp \beta_1 C_4 Z_s^{C_4-1}\, Z_s')}{(|m|Y_s + Y_s^{\,2} \mp \beta_1 Z_s^{C_4})^2} \right\} \; . \tag{3.78}$$

From Eqs. 3.68 and 3.69 for both stable and unstable stratifications, the spectral equations differ only in the turbulent energy spectral equation as indicated in Eq. 3.68. Thus the turbulent energy spectrum is explicitly influenced by thermal stratification while the temperature spectrum is implicitly affected due to the introduction of energy spectrum in a form of eddy viscosity as shown in Eq. 3.69.

Using Eqs. 3.68 and 3.69, a relationship between Y_s and Z_s can be deduced from

$$\frac{1 - Y_s^{\,2}}{|m|Y_s^{C_2} + Y_s^{\,2} \mp \beta_1 Z_s^{C_4}} = \frac{1 - Z_s^{\,2}}{Pr(Z_s^{\,2} + |m_T| Z_s^{C_4})} \tag{3.79}$$

Thus, Eq. 3.79 together with either Eq. 3.68 or Eq. 3.69, Y_s and Z_s can be solved explicitly or implicitly as functions of x, and consequently, $\Phi(x)$, $\Phi_{TT}(x)$, $\Phi_{uw}(x)$, $\Phi_{wT}(x)$ can be evaluated numerically by using Eqs. 3.72, 3.73, 3.77 and 3.78. However, in case $C_2=1$, $C_1=0$, some analytical closed forms can be obtained. For the sake of interest, the following analytical solution is derived. First, from Eq. 3.79 and $C_2=1$, we have

$$[(Pr-1)Z_s^2 + |m_T|PrZ_s^{C_4} + 1]Y_s^2 + |m|(1-Z_s^2)Y_s \mp \beta_1 Z_s^{C_4}(1-Z_s^2)$$

$$- Pr(Z_s^2 + |m_T|Z_s^{C_4}) = 0 \quad . \tag{3.80}$$

Now let

$$A(Z_s) = (Pr-1)Z_s^2 + |m_T|PrZ_s^{C_4} + 1 \tag{3.81}$$

$$B(Z_s) = |m|(1-Z_s^2) \tag{3.82}$$

$$C(Z_s) = \mp \beta_1 Z_s^{C_4}(1-Z_s^2) - Pr(Z_s^2 + |m_T|Z_s^{C_4}) , \tag{3.83}$$

then,

$$Y_s = \frac{-B + \sqrt{B^2-4AC}}{2A} , \tag{3.84}$$

the other root $Y_s = \dfrac{-B - \sqrt{B^2-4AC}}{2A}$ is omitted since $|Y_s|$ must be less than or equal to 1 and $Y_s \to 0$, $Z_s \to 0$ as $x \to 0$.

Squaring Eq. 3.84 yields

$$Y_s^2 = \frac{2B^2-4AC-2B\sqrt{B^2-4AC}}{4A^2} . \tag{3.85}$$

Differentiating Eq. 3.85 with respect to x gives

$$Y_s Y_s' = \frac{1}{2} \frac{dY_s^{\,2}}{dx}$$

$$= \frac{1}{8A^4} \left[A^2 \left\{ 4B \frac{dB}{dZ_s} Z_s' - 4A \frac{dc}{dZ_s} Z_s' - 4C \frac{dA}{dZ_s} Z_s' - 2 \frac{dB}{dZ_s} Z_s' \sqrt{B^2-4AC} \right.\right.$$

$$\left. - \frac{B}{\sqrt{B^2-4AC}} \left(2B \frac{dB}{dZ_s} Z_s' - 4A \frac{dc}{dZ_s} Z_s' - 4C \frac{dA}{dZ_s} Z_s'\right) \right\}$$

$$\left. - 2 \left(2B^2-4AC - 2B\sqrt{B^2-4AC} \right) A \frac{dA}{dZ_s} Z_s' \right]$$

$$= \frac{Z_s'}{8A^4} \left\{ A^2 \left[4 \left(B \frac{dB}{dZ_s} - A \frac{dC}{dZ_s} - C \frac{dA}{dZ_s} \right) - 2\sqrt{B^2-4AC} \frac{dB}{dZ_s} - \frac{B}{\sqrt{B^2-4AC}} \right.\right.$$

$$\left. \cdot \left(2B \frac{dB}{dZ_s} - 4A \frac{dC}{dZ_s} - 4C \frac{dA}{dZ_s} \right) \right] - 2 \left(2B^2-4AC - 2B\sqrt{B^2-4AC} \right) A \frac{dA}{dZ_s} \right\} , \tag{3.86}$$

in which

$$\frac{dA}{dZ_s} = 2(Pr-1)Z_s + |m_T| Pr\, C_4\, Z_s^{C_4-1}$$

$$\frac{dB}{dZ_s} = -2|m|Z_s$$

and

$$\frac{dC}{dZ_s} = \mp \beta_1 C_4 Z_s^{C_4-1} \pm \beta_1 (C_4+2) Z_s^{C_4+1} - 2PrZ_s - Pr|m_T| C_4 Z_s^{C_4-1} .$$

Also from differentiating Eq. 3.69 with respect x , we have

$$s\left[\frac{1-Z_s{}^2}{Pr(Z_s^2+|m_T|Z_s{}^{C_4})}\right]^{s-1}\frac{(Z_s^2+|m_T|Z_s{}^{C_4})(-2Z_s)-(1-Z_s^2)(2Z_s+C_4|m_T|Z_s{}^{C_4-1})}{Pr(Z_s^2+|m_T|Z_s{}^{C_4})^2}Z_s'$$

$$= -(Y_sY_s')^{\frac{s}{2}}x^{-\frac{3s}{2}-1} . \qquad (3.87)$$

Thus, from Eq. 3.86

$$s\left(\frac{1}{Pr}\right)^s\frac{(1-Z_s^2)^{s-1}\left[|m_T|(2-C_4)Z_s{}^{C_4+1}+2Z_s+C_4|m_T|Z_s{}^{C_4-1}\right]}{(Z_s{}^2+|m_T|Z_s{}^{C_4})^{s+1}}$$

$$= (Z_s')^{\frac{s}{2}}\left(\frac{1}{8A^4}\right)^{\frac{s}{2}}\left\{A^2\left[4\left(B\frac{dB}{dZ_s}-A\frac{dC}{dZ_s}-C\frac{dA}{dZ_s}\right)-2\sqrt{B^2-4AC}\,\frac{dB}{dZ_s}-\frac{B}{\sqrt{B\ \ -4AC}}\right.\right.$$

$$\left.\left.\cdot\left(2B\frac{dB}{dZ_s}-4A\frac{dC}{dZ_s}-4C\frac{dA}{dZ_s}\right)\right]-2\left[2B^2-4AC-2B\sqrt{B^2-4AC}\right]A\frac{dA}{dZ_s}\right\}^{\frac{s}{2}}x^{-\frac{3s}{2}-1}$$

(3.88)

$$Z_s' = x^{-\frac{3s-2}{2-s}} s^{\frac{2}{s-2}} (Pr)^{\frac{2s}{2-s}} (1-Z_s^2)^{-\frac{2s-2}{2-s}} (Z_s^2 + |m_T| Z_s^{C_4})^{\frac{2s+2}{2-s}}$$

$$\cdot \left[|m_T|(2-C_4) Z_s^{C_4+1} + 2Z_s + C_4 |m_T| Z_s^{C_4-1} \right]^{\frac{2}{s-2}} \left(\frac{1}{8A^4} \right)^{\frac{s}{2-s}}$$

$$\cdot \left\{ A^2 \left[4 \left(B \frac{dB}{dZ_s} - A \frac{dC}{dZ_s} - C \frac{dA}{dZ_s} \right) - 2\sqrt{B^2-4AC} \; \frac{dB}{dZ_s} - \frac{B}{\sqrt{B^2-4AC}} \right. \right.$$

$$\left. \left. \cdot \left(2B \frac{dB}{dZ_s} - 4A \frac{dC}{dZ_s} - 4C \frac{dA}{dZ_s} \right) \right] - 2 \left[2B^2-4AC - 2B\sqrt{B^2-4AC} \right] A \frac{dA}{dZ_s} \right\}^{\frac{s}{2-s}}$$

(3.89)

In case $s=2$, the abridged equation can be obtained from Eq. 3.88,

$$x^4 = \frac{Pr^2 (Z_s^2 + |m_T| Z_s^{C_4})^3}{16A^4 (1-Z_s^2) [|m_T|(2-C_4) Z_s^{C_4+1} + 2Z_s + C_4 |m_T| Z_s^{C_4-1}]}$$

$$\cdot \left\{ A^2 \left[4 \left(B \frac{dB}{dZ_s} - A \frac{dC}{dZ_s} - C \frac{dA}{dZ_s} \right) - 2\sqrt{B^2-4AC} \; \frac{dB}{dZ_s} \right. \right.$$

$$\left. - \frac{B}{\sqrt{B^2-4AC}} \left(2B \frac{dB}{dZ_s} - 4A \frac{dC}{dZ_s} - 4C \frac{dA}{dZ_s} \right) \right] - 2 \left[2B^2-4AC-2B\sqrt{B^2-4AC} \right.$$

$$\left. \cdot \; A \frac{dA}{dZ_s} \right\} \tag{3.90}$$

and in case $2 < s < \infty$, a closed integral form can be derived from Eq. 3.89,

$$
\begin{aligned}
\frac{s-2}{4s}\, x^{\frac{4s}{s-2}} = {} & s^{\frac{2}{2-s}} (Pr)^{\frac{2s}{s-2}} \int (1-Z_s{}^2)^{-\frac{2s-2}{s-2}} \left[|m_T| (2-C_4) Z_s{}^{C_4+1} + 2Z_s \right. \\
& \left. + C_4 |m_T| Z_s{}^{C_4-1} \right]^{-\frac{2}{s-2}} (Z_s{}^2 + |m_T| Z_s{}^{C_4})^{\frac{2s+2}{s-2}} \left(\frac{1}{8A^4} \right)^{\frac{s}{s-2}} \\
& \cdot \left\{ A^2 \left[4 \left(B \frac{dB}{dZ_s} - A \frac{dC}{dZ_s} - C \frac{dA}{dZ_s} \right) - 2 \sqrt{B^2-4AC}\, \frac{dB}{dZ_s} - \frac{B}{\sqrt{B^2-4AC}} \right. \right. \\
& \left. \left. \cdot \left(2B \frac{dB}{dZ_s} - 4A \frac{dC}{dZ_s} - 4C \frac{dA}{dZ_s} \right) \right] - 2 \left[2B^2-4AC-2B\sqrt{B^2-4AC} \right] A \frac{dA}{dZ_s} \right\}^{\frac{s}{s-2}} dZ_s + D_1
\end{aligned}
\tag{3.91}
$$

where the constant D_1 can be evaluated from the boundary condition $x = 0$, $Z_s = 0$, or $x = \infty$, $Z_s = 1$; the integral is an indefinite one. Therefore from Eqs. 3.72, 3.73, 3.77, and 3.78, spectra can be obtained as functions of s , Pr , m_T , C_4 , m , and β_1 .

3.3.2 <u>Solutions of flows with negligible molecular effects</u> - In the previous statement, molecular effects as represented by the kinematic viscosity and the thermal diffusivity are involved. However, in case the spectra at waves numbers far away from the Kolmogorov's wave number is considered, i.e., when $k << \ell_d^{-1}$, the local dissipation of turbulent energy and the local dissipation of temperature inhomogeneity in the range 0 to k are negligible as compared to ε and N , respectively. This will be the case when we consider the buoyancy subrange where the molecular effects are negligible. Thus, Eqs. 3.49 and 3.50 are reduced to

$$\varepsilon = -\frac{d\overline{U}}{dz} \int_k^\infty \phi_{uw}(k)\,dk + \int_k^\infty F(k)\,dk + \beta \int_k^\infty \phi_{wT}(k)\,dk \tag{3.92}$$

and

$$N_* = -\frac{d\overline{T}}{dz_*} \int_k^\infty \phi_{wT}(k)\,dk + \int_k^\infty F_{TT}(k)\,dk \quad . \tag{3.93}$$

Again, the generalized eddy-viscosity approximation is applied; thus, we have

$$\varepsilon = \gamma_{s,r}\left[\int_x^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}dk\right]^{\frac{1}{s}}\left\{\left|\frac{d\overline{U}}{dz}\right|^{1+C_1}\left[\int_o^k 2\phi^r(k)k^{3r-1}dk\right]^{\frac{C_2}{2r}}\right.$$

$$\left. + \left[\int_o^k 2\phi^r(k)k^{3r-1}dk\right]^{\frac{1}{r}} \mp b\beta\left[\int_o^k 2\phi^r_{TT}(k)k^{3r-1}dk\right]^{\frac{C_4}{2r}}\left|\frac{d\overline{T}}{dz_*}\right|^{C_3}\right\} \tag{3.94}$$

and

$$N_* = \gamma_{s,r}\left[\int_k^\infty \phi^{\frac{s}{2}}(k)k^{-\frac{s}{2}-1}dk\right]^{\frac{1}{s}}\left\{\pm b\frac{d\overline{T}}{dz_*}\left|\frac{d\overline{T}}{dz_*}\right|^{C_3}\left[\int_o^k 2\phi^r_{TT}(k)k^{3r-1}dk\right]^{\frac{C_4}{2r}}\right.$$

$$\left. + \left[\int_o^k 2\phi^r_{TT}(k)k^{3r-1}dk\right]^{\frac{1}{r}}\right\} \quad . \tag{3.95}$$

Now, with the nondimensional variables

$$x = \frac{k}{k_o} \quad , \quad \Phi = \frac{\phi}{\phi_o} \quad , \quad \Phi_{TT} = \frac{\phi_{TT}}{\phi^o_{TT}}$$

$$\Gamma = \frac{d\overline{U}}{dz} \left|\frac{d\overline{U}}{dz}\right|^{C_1} b^{-1+\frac{C_2}{2}} N_*^{-1+\frac{C_2}{2}} \varepsilon^{1-\frac{C_2}{2}} \beta^{-2+C_2}$$

$$\Gamma_1 = N_*^{-1+C_4} \varepsilon^{1-C_4} \beta^{-1+C_4} \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} \quad \text{and}$$

$$\Gamma_T = N_*^{-2+C_4} \varepsilon^{2-C_4} \beta^{-2+C_4} \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} \frac{d\overline{T}}{dz_*} \quad , \tag{3.96}$$

in which

$$k_o = \gamma_{s,r}^{\frac{1}{2}} b^{\frac{3}{4}} N_*^{\frac{3}{4}} \varepsilon^{-\frac{5}{4}} \beta^{\frac{3}{2}} \quad , \quad \phi_o = \gamma_{s,r}^{-\frac{3}{2}} b^{-\frac{5}{4}} N_*^{-\frac{5}{4}} \varepsilon^{\frac{11}{4}} \beta^{-\frac{5}{2}} \quad ,$$

and

$$\phi^o_{TT} = \gamma_{s,r}^{-\frac{3}{2}} b^{-\frac{9}{4}} N_*^{-\frac{1}{4}} \varepsilon^{\frac{7}{4}} \beta^{-\frac{5}{2}} \quad , \tag{3.97}$$

Eqs. 3.94 and 3.95 yield the dimensionless forms

$$1 = \left[\int_x^{\infty} \Phi^{\frac{s}{2}}(x) x^{-\frac{s}{2}-1} dx\right]^{\frac{1}{s}} \left\{ |\Gamma| \left[\int_0^x 2\Phi^r(x) x^{3r-1} dx\right]^{\frac{C_2}{2r}} + \left[\int_0^x 2\Phi^r(x) x^{3r-1} dx\right]^{\frac{1}{r}} \mp \Gamma_1 \left[\int_0^x 2\Phi^r_{TT}(x) x^{3r-1} dx\right]^{\frac{C_4}{2r}} \right\} \tag{3.98}$$

and

$$1 = \left[\int_{x}^{\infty} \Phi^{\frac{s}{2}}(x)\, x^{-\frac{s}{2}-1} dx\right]^{\frac{1}{s}} \left\{ |\Gamma_T| \left[\int_{o}^{x} 2\Phi_{TT}^{r}(x)\, x^{3r-1} dx\right]^{\frac{C_4}{2r}} \right.$$

$$\left. + \left[\int_{o}^{x} 2\Phi_{TT}^{r}(x)\, x^{3r-1} dx\right]^{\frac{1}{r}} \right\} \quad . \qquad (3.99)$$

Also letting

$$\Phi_{uw} = \frac{\phi_{uw}}{\phi_{uw}^{o}} \quad , \quad \Phi_{wT} = \frac{\phi_{wT}}{\phi_{wT}^{o}}$$

$$\phi_{uw} = \gamma_{s,r}^{-\frac{1}{2}}\, b^{-\frac{7}{4}+\frac{C_2}{2}}\, N_*^{-\frac{7}{4}+\frac{C_2}{2}}\, \varepsilon^{\frac{13}{4}-\frac{C_2}{2}}\, \beta^{-\frac{7}{2}+C_2} \left|\frac{d\overline{U}}{dz}\right|^{C_1} ,$$

and

$$\phi_{wT}^{o} = \gamma_{s,r}^{-\frac{1}{2}}\, b^{-\frac{3}{4}}\, N_*^{-\frac{7}{4}+C_4}\, \varepsilon^{\frac{13}{4}-C_4}\, \beta^{-\frac{7}{2}+C_4} \left|\frac{d\overline{T}}{dz_*}\right|^{C_3} , \qquad (3.100)$$

we have the dimensionless forms

$$\int_{x}^{\infty} \Phi_{uw}(x)\, dx = - \left[\int_{x}^{\infty} \Phi^{\frac{s}{2}}(x)\, x^{-\frac{s}{2}-1} dx\right]^{\frac{1}{s}} \left[\int_{o}^{x} 2\Phi^{r}(x)\, x^{3r-1} dx\right]^{\frac{C_2}{2r}} \qquad (3.101)$$

and

$$\int_{x}^{\infty} \Phi_{wT}(x)\, dx = \mp \left[\int_{x}^{\infty} \Phi^{\frac{s}{2}}(x)\, x^{-\frac{s}{2}-1} dx\right]^{\frac{1}{s}} \left[\int_{o}^{x} 2\Phi_{TT}^{r}(x)\, x^{3r-1} dx\right]^{\frac{C_4}{2r}} \quad . \qquad (3.102)$$

Thus, from Eqs. 3.98, 3.99, 3.101, and 3.102, $\Phi(x)$, $\Phi_{TT}(x)$, $\Phi_{uw}(x)$, and $\Phi_{wT}(x)$ can be evaluated numerically. However, for simplifying the analytical derivation of solutions, two sets of values of s and r are assigned.

(A) $2 \leq s < \infty$, $r = 1$

By letting

$$Y_s^2(x) = \int_0^x 2x^2\Phi(x)dx \quad , \text{ or } \quad \Phi(x) = \frac{Y_s(x)Y_s'(x)}{x^2} \tag{3.103}$$

and

$$Z_s^2(x) = \int_0^x 2x^2\Phi_{TT}(x)dx \quad , \text{ or } \quad \Phi_{TT}(x) = \frac{Z_s(x)Z_s'(x)}{x^2} \quad , \tag{3.104}$$

Eqs. 3.98 and 3.99 become

$$1 = \left[\int_x^\infty (Y_sY_s')^{\frac{s}{2}} x^{-\frac{3s}{2}-1} dx\right]^{\frac{1}{s}} \left[Y_s^2 + |\Gamma|Y_s^{C_2} \mp \Gamma_1 Z_s^{C_4}\right] \tag{3.105}$$

and

$$1 = \left[\int_x^\infty (Y_sY_s')^{\frac{s}{2}} x^{-\frac{3s}{2}-1} dx\right]^{\frac{1}{s}} \left[Z_s^2 + |\Gamma_T|Z_s^{C_4}\right] \quad . \tag{3.106}$$

Also from Eqs. 3.101 and 3.102, we have

$$\Phi_{uw}(x) = \frac{(Y_s^2 + |\Gamma|Y_s^{C_2} \mp \Gamma_1 Z_s^{C_4})\, C_2\, Y_s^{C_2-1}\, Y_s'}{(Y_s^2 + |\Gamma|Y_s^{C_2} \mp \Gamma_1 Z_s^{C_4})^2}$$

$$- \frac{Y_s^{C_2}(2Y_sY_s' + |\Gamma|C_2Y_s^{C_2-1}Y_s' \mp \Gamma_1 C_4 Z_s^{C_4-1} Z_s')}{(Y_s^2 + |\Gamma|Y_s^{C_2} \mp \Gamma_1 Z_s^{C_4})^2} \tag{3.107}$$

and

$$\Phi_{wT}(x) = \pm \frac{(Z_s^{\,2}+|\Gamma_T|Z_s^{\,C_4})C_4 Z_s^{\,C_4-1} Z_s' - Z_s^{\,C_4}(2Z_s Z_s' + |\Gamma_T| C_4 Z_s^{\,C_4-1} Z_s')}{(Z_s^{\,2}+ |\Gamma_T| Z_s^{\,C_4})^2} \quad . \tag{3.108}$$

Now, from Eqs. 3.105 and 3.106, we have

$$Y_s^{\,2} + |\Gamma| Y_s^{\,C_2} \mp \Gamma_1 Z_s^{\,C_4} = Z_s^{\,2} + |\Gamma_T| Z_s^{\,C_4} \quad , \tag{3.109}$$

and thus, $\Phi(x)$, $\Phi_{TT}(x)$, $\Phi_{uw}(x)$, and $\Phi_{wT}(x)$ can numerically be calculated for a given set of parameters Γ , Γ_T , Γ_1 , C_2 and C_4 since, by differentiating Eq. 3.105 with respect to x , x can be in terms of Y_s , Y_s' , Z_s , and Z_s' and Eq. 3.109 can offer the relationship among Y_s , Y_s' , Z_s , and Z_s' . However, if some analytically closed form is needed to be obtained, $C_2 = 1$, and $C_1 = 0$, can be assumed. Hence, Eq. 3.109 becomes

$$Y_s^{\,2} + |\Gamma| Y_s - Z_s^{\,2} - (|\Gamma_T| \pm \Gamma_1) Z_s^{\,C_4} = 0 \quad , \tag{3.110}$$

and so

$$Y_s = \frac{-|\Gamma| + \sqrt{\Gamma^2 + 4Z_s^{\,2} + 4(|\Gamma_T| \pm \Gamma_1) Z_s^{\,C_4}}}{2} \quad . \tag{3.111}$$

Squaring Eq. 3.111 results in

$$Y_s^{\,2} = \frac{2\Gamma^2 + 4Z_s^{\,2} + 4(|\Gamma_T| \pm \Gamma_1) Z_s^{\,C_4} - 2|\Gamma| \sqrt{\Gamma^2 + 4Z_s^2 + 4(|\Gamma_T| \pm \Gamma_1) Z_s^{\,C_4}}}{4} \tag{3.112}$$

Inserting Eq. 3.112 into Eq. 3.106 and differentiating the resultant equation with respect to x , we have thus

$$\frac{s(2Z_s+C_4|\Gamma_T|Z_s^{C_4-1})Z_s'}{(Z_s^2+|\Gamma_T|Z_s^{C_4})^{s+1}} = \left(\frac{1}{2}\right)^{\frac{s}{2}} \left[2Z_s + C_4(|\Gamma_T| \pm \Gamma_1)Z_s^{C_4-1}\right.$$

$$\left. - |\Gamma| \frac{2Z_s+C_4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4-1}}{\sqrt{\Gamma^2+4Z_s^2+4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4}}}\right]^{\frac{s}{2}} (Z_s')^{\frac{s}{2}} \quad x^{-\frac{3s}{2}-1} \qquad (3.113)$$

or

$$Z_s' = 2^{\frac{s}{s-2}} \, s^{\frac{2}{s-2}} \, (2Z_s + C_4|\Gamma_T|Z_s^{C_4-1})^{\frac{2}{s-2}} \, (Z_s{}^2 + |\Gamma_T|Z_s^{C_4})^{\frac{2s+2}{2-s}}$$

$$\cdot\left[2Z_s+C_4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4-1} - |\Gamma| \frac{2Z_s+C_4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4-1}}{\sqrt{\Gamma^2+4Z_s^2+4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4}}}\right]^{\frac{s}{2-s}} x^{\frac{3s+2}{s-2}} \; .$$

$$(3.114)$$

Consequently, we have then

$$\frac{s-2}{4s} \, x^{\frac{4s}{s-2}} = 2^{\frac{s}{2-s}} \, s^{\frac{2}{2-s}} \int_0^{Z_s} \left(2Z_s+C_4|\Gamma_T|Z_s^{C_4-1}\right)^{\frac{2}{2-s}}$$

$$\cdot \left(Z_s^2+|\Gamma_T|Z_s^{C_4}\right)^{\frac{2s+2}{-2+s}} \left[2Z_s+C_4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4-1}\right.$$

$$\left. - |\Gamma| \frac{2Z_s+C_4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4-1}}{\sqrt{\Gamma^2+4Z_s^2+4(|\Gamma_T|\pm\Gamma_1)Z_s^{C_4}}}\right]^{\frac{s}{s-2}} dZ_s \qquad (3.115)$$

which can numerically be integrated for $\infty > s > 2$. In case $s = 2$, an abridged form can be seen from Eq. 3.113, i.e.,

$$4x^4 = (Z_s^{\,2} + |\Gamma_T| Z_s^{C_4})^3 \, (2Z_s + C_4 |\Gamma_T| Z_s^{C_4 - 1})^{-1}$$

$$\cdot \left[2Z_s + C_4(|\Gamma_T| \pm \Gamma_1) Z_s^{C_4 - 1} - |\Gamma| \frac{2Z_s + C_4(|\Gamma_T| \pm \Gamma_1) Z_s^{C_4 - 1}}{\sqrt{\Gamma^2 + 4Z_s^2 + 4(|\Gamma_T| \pm \Gamma_1) Z_s^{C_4}}} \right] \quad . \qquad (3.116)$$

(B) $s \to \infty$, $r \to \infty$

In this case, an exact local limit of the generalized eddy-viscosity approximation can be formed and Eqs. 3.98 and 3.99 become

$$1 = |\Gamma| \; x^{-\frac{1}{2} + \frac{3C_2}{2}} \; \Phi^{\frac{1}{2} + \frac{C_2}{2}}(x) + x^{\frac{5}{2}} \; \Phi^{\frac{3}{2}}(x)$$

$$\mp \Gamma_1 \; x^{-\frac{1}{2} + \frac{3C_4}{2}} \; \Phi^{\frac{1}{2}}(x) \, \Phi_{TT}^{\frac{C_4}{2}}(x) \qquad (3.117)$$

and

$$1 = |\Gamma_T| x^{-\frac{1}{2} + \frac{3C_4}{2}} \; \Phi^{\frac{1}{2}}(x) \, \Phi_{TT}^{\frac{C_4}{2}}(x) + x^{\frac{5}{2}} \; \Phi^{\frac{1}{2}}(x) \, \Phi_{TT}(x) \qquad (3.118)$$

which are the simultaneous nonlinear equations of $\Phi(x)$ and $\Phi_{TT}(x)$ at a given dimensionless wave number x . Clearly the numerical solutions can be obtained as a function of x and the parameters $|\Gamma|$, Γ_1 , $|\Gamma_T|$, C_2 and C_4 . Since both parameters C_2 and C_4 are retained in Eqs. 3.117 and 3.118, and the spectral equations are presented in a clear and simpler form, Eqs. 3.117 and 3.118 will be investigated intensively. As to evaluating $\Phi_{uw}(x)$ and $\Phi_{wT}(x)$, Eqs. 3.101 and 3.102 will be used. Thus,

$$\Phi_{uw}(x) = \left(-\frac{1}{2} + \frac{3C_2}{2}\right) x^{-\frac{3}{2}+\frac{3C_2}{2}} \Phi^{\frac{1}{2}+\frac{C_2}{2}}(x)$$

$$+ \left(\frac{1}{2} + \frac{C_2}{2}\right) x^{-\frac{1}{2}+\frac{3C_2}{2}} \Phi^{-\frac{1}{2}+\frac{C_2}{2}}(x)\Phi'(x) \qquad (3.119)$$

and

$$\Phi_{wT}(x) = \pm\left[\left(-\frac{1}{2} + \frac{3C_4}{2}\right) x^{-\frac{1}{2}+\frac{3C_4}{2}} \Phi_{TT}^{\frac{1}{2}+\frac{C_4}{2}}(x)\right.$$

$$\left. + \left(\frac{1}{2} + \frac{C_4}{2}\right) x^{-\frac{1}{2}+\frac{3C_4}{2}} \Phi_{TT}^{-\frac{1}{2}+\frac{C_4}{2}}(x)\Phi'_{TT}(x)\right] \qquad (3.120)$$

where $\Phi'(x) = \dfrac{d\Phi(x)}{dx}$, and $\Phi'_{TT}(x) = \dfrac{d\Phi_{TT}(x)}{dx}$.

3.4 Asymptotic Solutions

In the previous sections, details have been given to solving the spectral equations of thermally stratified turbulent shear flows; of course, numerical solutions are available now. However, if we need examine the significances of all the parameters such as s , C_1 , C_2 , C_3 , and C_4 introduced in our generalized eddy-viscosity approximation, it would be helpful to investigate the asymptotic solutions under certain conditions. In order to avoid the tremendous complexity introduced by the consideration of molecular effects, we still prefer to consider the case when the molecular effects are negligible. Evidently in the buoyancy subrange of wave numbers this is the case.

3.4.1 Asymptotic solutions of stably stratified flow - First, consider the dimensionless wave number range $x >> 1$ or equivalently, $k >> \gamma_{s,r}^{1/2}\ b^{3/4}\ N_*^{3/4}\ \varepsilon^{-5/4}\ \beta^{3/2} = k_o$. In this wave number range, the buoyancy effect on the motions of eddies is negligible as compared to the inertial interaction among eddies as we can prove aposteriori in the next section. Thus, it will not be surprising that the classical -5/3 law holds in the dimensionless wave number range $x >> 1$. From Eq. 3.115, asymptotic solutions are available by assuming $|\Gamma| \approx 0$, and $Z_s^{\ 2} >> |\Gamma_T| Z_s^{\ C_4}$ as follows:

$$Z_s^{\ 2} = 3^{\frac{s-2}{3s}}\ 2^{\frac{4-s}{3s}}\ s^{\frac{2}{3s}}\ x^{\frac{4}{3}} \quad , \quad 2 < s < \infty \quad , \quad x >> 1 \ , \tag{3.121}$$

similarly, for the velocity field we have from Eq. 3.109

$$Y_s^{\ 2} = 3^{\frac{s-2}{3s}}\ 2^{\frac{4-s}{3s}}\ s^{\frac{2}{3s}}\ x^{\frac{4}{3}} \quad , \quad 2 < s < \infty \quad , \quad x >> 1 \tag{3.122}$$

Thus, equivalently, the corresponding velocity and temperature spectra are

$$\Phi(x) = 3^{\frac{s-2}{3s}}\ 2^{\frac{4-4s}{3s}}\ s^{\frac{2}{3s}}\ x^{-\frac{5}{3}} \tag{3.123}$$

and

$$\Phi_{TT}(x) = 3^{\frac{s-2}{3s}}\ 2^{\frac{4-4s}{3s}}\ s^{\frac{2}{3s}}\ x^{-\frac{5}{3}} \quad , \quad 2 < s < \infty \ , \quad x >> 1 \ . \tag{3.124}$$

As to $s = 2$, Eq. 3.116 or the limit form of Eqs. 3.123 and 3.124 as $s \to 2$ can be used, and we have then

$$\Phi(x) = 2^{-\frac{1}{3}}\ x^{-\frac{5}{3}} \tag{3.125}$$

$$\Phi_{TT}(x) = 2^{-\frac{1}{3}} x^{-\frac{5}{3}} \quad , \quad s = 2 \quad , \quad x >> 1 \quad . \tag{3.126}$$

In case $x << 1$, the situation is somehow more complicated since the buoyancy effects will distort the inertial interaction among eddies. Two cases will be considered in the dimensionless wave number range $x << 1$. In both cases, we still assume that the local production of turbulent energy in the wave number range considered is negligible. From the computational point of view, we can let $|\Gamma| \approx 0$ in Eq. 3.115. Thus, with the further assumption $|\Gamma_T| Z_s^{C_4} >> Z_s^{\,2}$, Eq. 3.115 gives:

$$Z_s^{C_4} = 2^{\frac{1}{3}} \left(\frac{3}{4}\right)^{\frac{s-2}{3s}} |\Gamma_T|^{-\frac{2}{3}} (|\Gamma_T|+\Gamma_1)^{-\frac{1}{3}} s^{\frac{2}{3s}} x^{\frac{4}{3}} \quad , \quad 2 \le s < \infty$$

and from Eq. 3.109,

$$Y_s^2 = 2^{\frac{1}{3}} \left(\frac{3}{4}\right)^{\frac{s-2}{3s}} |\Gamma_T|^{\frac{1}{3}} (|\Gamma_T|+\Gamma_1)^{-\frac{1}{3}} s^{\frac{2}{3s}} x^{\frac{4}{3}} \quad , \quad 2 \le s < \infty \quad . \tag{3.127}$$

Thus, the velocity and temperature spectra become in this case

$$\Phi(x) = 2^{-\frac{2}{3}} \left(\frac{3}{4}\right)^{\frac{s-2}{3s}} s^{\frac{2}{3s}} |\Gamma_T|^{\frac{1}{3}} (|\Gamma_T| + \Gamma_1)^{-\frac{1}{3}} x^{-\frac{5}{3}} \quad , \tag{3.128}$$

and

$$\Phi_{TT}(x) = 2^{\frac{2-3C_4}{3C_4}} \left(\frac{3}{4}\right)^{\frac{2s-4}{3sC_4}} |\Gamma_T|^{\frac{-4}{3C_4}} (|\Gamma_T|+\Gamma_1)^{-\frac{2}{3C_4}} s^{\frac{4}{3sC_4}} x^{\frac{8-9C_4}{3C_4}} \tag{3.129}$$

for $x << 1$, $C_4 > 0$, and $2 \le s < \infty$.

These asymptotic solutions can be considered as the spectra in the inertial and convective subrange where the inertial transfer process predominates the velocity field and the interaction between the main and turbulent temperature fields plays the principal role in development of temperature fluctuations.

Next, some asymptotic solutions in the buoyancy subrange will be investigated. In the buoyancy subrange, the local production of turbulent energy and turbulent temperature inhomogeneity is so small that the inertial transfer processes predominate turbulent motions. However, the buoyancy subrange differs from the inertial subrange in the fact that in the buoyancy subrange the buoyancy effects due to the vertical heat flux affect the energy balance. Thus, with the assumptions $|\Gamma| \approx 0$, $\Gamma_1 Z_s^{C_4} >> Z_s^2 >> |\Gamma_T| Z_s^{C_4}$, $\Gamma_1 >> |\Gamma_T|$, and

$$C_4(|\Gamma_T| + \Gamma_1) Z_s^{C_4 - 1} >> 2Z_s >> C_4 |\Gamma_T| Z_s^{C_4 - 1} ,$$

Eq. 3.115 gives the asymptotic solution as follows:

$$Z_s^{4+C_4} = \left(\frac{4+C_4}{4}\right)^{\frac{s-2}{s}} s^{\frac{2}{s}} \, 2^{\frac{s+2}{s}} [C_4(|\Gamma_T| + \Gamma_1)]^{-1} x^4 \tag{3.130}$$

and

$$Y_s^2 = \Gamma_1 \left(\frac{4+C_4}{4}\right)^{\frac{(s-2)C_4}{s(4+C_4)}} s^{\frac{2C_4}{s(4+C_4)}} \, 2^{\frac{C_4(s+2)}{s(4+C_4)}}$$

$$\cdot \left[C_4(|\Gamma_T| + \Gamma_1)\right]^{-\frac{C_4}{4+C_4}} x^{\frac{4C_4}{4+C_4}} , \quad \text{for } x_e << x << 1 ,$$

$$C_4 > 0 \quad \text{and} \quad \infty > s \geq 2 , \tag{3.131}$$

or equivalently we have the asymptotic temperature and velocity spectra

$$\Phi_{TT}(x) = \frac{1}{2}\left(\frac{4+C_4}{4}\right)^{\frac{2s-4}{s(4+C_4)}} s^{\frac{4}{s(4+C_4)}} 2^{\frac{2s+4}{s(4+C_4)}} [C_4(|\Gamma_T|+\Gamma_1)]^{-\frac{2}{4+C_4}} x^{\frac{-3C_4-4}{4+C_4}} \tag{3.132}$$

and

$$\Phi(x) = \frac{1}{2}\,\Gamma_1\left(\frac{4+C_4}{4}\right)^{\frac{C_4(s-2)}{s(4+C_4)}} s^{\frac{2C_4}{s(4+C_4)}} 2^{\frac{C_4(s+2)}{s(4+C_4)}}$$

$$\cdot\ [C_4(|\Gamma_T|+\Gamma_1)]^{-\frac{C_4}{4+C_4}} x^{\frac{C_4-12}{4+C_4}} \quad \text{for } x_e \ll x \ll 1 \ ,$$

$$C_4 > 0 \ , \text{ and } \infty > s \geq 2 \ . \tag{3.133}$$

Now the significance of C_4 can be seen clearly from Eqs. 3.132 and 3.133. In case $C_4 \to 0$ and $C_3 \to 1$, $\Phi(x) \sim x^{-3}$ and $\Phi_{TT}(x) \sim x^{-1}$ for $x_e \ll x \ll 1$, and this is exactly the solution predicted by Lumley (1964). If the expression (3.62) for the vertical heat flux spectrum is reviewed, $C_4 \to 0$ and $C_3 \to 1$ would mean that the vertical heat flux spectrum is determined by the velocity field implied by the eddy-viscosity $\gamma_{s,r}[\int_k^\infty \Phi^{s/2}(k)k^{-s/2-1}dk]^{1/s}$ in Eq. 3.62 and is proportional to $|d\overline{T}/dz_*|$. Thus, after reviewing Lumley-Shur's hypothesis in section 2.2.3, the generalized eddy-viscosity approximation considered at $C_4 \to 0$, $C_3 \to 1$, $|\Gamma| \approx 0$ and $\Gamma_1 \gg |\Gamma_T|$ is equivalent to Lumley-Shur's hypothesis. However, as we can see later, because of the generality of the generalized eddy-viscosity approximation implied by varying C_1, C_2, C_3, C_4, $|\Gamma_T|$, $|\Gamma|$ and Γ_1, the present results are more fruitful. And if $C_4 = 1$, $C_3 = 0$ and $s = 2$ are assigned for Eqs. 3.132 and 3.133, then $\Phi(x) \sim x^{-11/5}$ and $\Phi_{TT}(x) \sim x^{-7/5}$ are obtained. This is actually the case considered by Monin (1962) and Gisina (1966).

3.4.2 Asymptotic solutions of unstably stratified flow - In case the unstable stratification is concerned, it is better to study Eqs. 3.117 and 3.118 instead of Eq. 3.113. For the same reason described above, let $|\Gamma| \approx 0$ and $\Gamma_1 >> |\Gamma_T|$ in the buoyancy subrange. Thus, we have

$$1 = x^{\frac{3}{2}} \Phi^{\frac{3}{2}}(x) + \Gamma_1 x^{-\frac{1}{2}+\frac{3C_4}{2}} \Phi^{\frac{1}{2}}(x) \Phi_{TT}^{\frac{C_4}{2}}(x) \tag{3.126}$$

and

$$1 = x^{\frac{5}{2}} \Phi^{\frac{1}{2}}(x) \Phi_{TT}(x) \quad . \tag{3.127}$$

In the wave number range $x >> 1$, the inertial transfer process dominates the turbulent structure, it can be expected that the - 5/3 law holds for both velocity and temperature spectra. However, where $x << 1$ and is still far away from the range x_e in which the production of the turbulent energy and temperature inhomogeneity predominates, we have the buoyancy subrange for an unstably stratified turbulent flow, and the asymptotic solutions would be

$$\Phi(x) \sim x$$

and

$$\Phi_{TT}(x) \sim x^{-3} \quad \text{for} \quad x_e << x << 1 \quad . \tag{3.128}$$

This shows that, in an unstably stratified turbulent flow, the velocity field absorbs energy converted from the temperature field in the buoyancy subrange, and presents a hump in the velocity spectrum. This is actually what we expected in our search for the basic mechanism of unstably stratified turbulent flow in section 2.2.1.

3.5 On Bolgiano's and Lumley-Shur's Hypotheses and the Modified Hypotheses

After introducing asymptotic solutions by the generalized eddy-viscosity approximation as presented in section 3.4, it would be worth comparing Bolgiano's and Lumley-Shur's hypotheses for stably stratified flow. Phillips (1965) made comparisons between these two hypotheses. In his paper, Phillips classified these hypotheses as follows:

Lumley-Shur's hypotheses---

1. "The statistical properties of the components of the turbulence with wave number k in the inertia-buoyancy subrange, including the energy spectrum and the buoyancy flux spectrum, are determined by the spectral kinetic energy flux $\varepsilon(k)$ at this wave number and not at distant wave number in either direction,"

2. "The spectrum of the buoyancy flux in physical space of stably stratified environment is proportional to the mean buoyancy gradient."

Bolgiano's hypothesis---

1. In the buoyancy subrange, the statistical properties of the motion are determined by N , $g/\overline{T}$ and wave number k alone.

Phillips agreed with Lumley on Lumley-Shur's second hypothesis, but did not agree with the first since it is still doubtful if the Kolmogorov's hypothesis can be extended to the buoyancy subrange. As to Bolgiano's hypothesis, Phillips stated that the quantity N is certainly a property of the turbulence; it is a local property of the inertial subrange and an integrated property of the buoyancy and energy-containing ranges, but it is not a local property of the buoyancy subrange itself as can be seen from the equation

$$\varepsilon(k) + \left(\frac{\frac{d\overline{T}}{dz}}{\frac{g}{\overline{T}}}\right)^{-1} N(k) = \varepsilon + \left(\frac{\frac{d\overline{T}}{dz}}{\frac{g}{\overline{T}}}\right)^{-1} N \tag{3.134}$$

which is derived according to the assumption that the production of turbulent energy is less important in the buoyancy and inertial subranges. (Phillips, 1965).

Certainly Phillips' argument is right for a general stratified flow. However, as stated in sections 2.2 and 2.3, Lumley's solution would be meaningful only when

$$\left(\frac{\varepsilon\overline{T}\,\frac{d\overline{T}}{dz}}{gN}\right)^{\frac{1}{2}} << 1 \quad . \tag{2.19}$$

Gisina obtained the same restriction for the existence of Bolgiano's solution as expressed in Eq. 2.20. Also in the present study, asymptotic solutions for the buoyancy subrange are obtained with the assumption $|\Gamma| \approx 0$ and $\Gamma_1 >> |\Gamma_T|$ as described in section 3.4. After we review the definitions of Γ_1 and Γ_T as expressed in Eq. 3.96, $\Gamma_1 >> |\Gamma_T|$ is certainly equivalent to

$$\frac{\varepsilon\left|\frac{d\overline{T}}{dz_*}\right|}{N_*\,\frac{g}{\overline{T}}} << 1 \quad , \quad \text{or} \quad \frac{g}{\overline{T}}\left|\frac{d\overline{T}}{dz}\right| << \frac{N}{\varepsilon}\left(\frac{g}{\overline{T}}\right)^2 \tag{3.135}$$

for $N_* = 2N \quad , \quad \frac{d\overline{T}}{dz_*} = 2\,\frac{d\overline{T}}{dz} \quad ,$

which means that the internal frequency $(N/\varepsilon)^{1/2}\, g/\overline{T}$ of turbulence of stably stratified flow must be greater than Brunt-Väisälä frequency of the external flow $(g/\overline{T})^{1/2}\,(d\overline{T}/dz)^{1/2}$ and which is exactly the same expression as Eq. 2.19 derived by Lumley and Eq. 2.20 obtained by Gisina.

By combining Eqs. 3.134 and 3.135, we will see what Eq. 3.135 will really imply in the buoyancy subrange. Thus, rewriting Eq. 3.134 as

$$\frac{\varepsilon \frac{d\overline{T}}{dz}}{\frac{Ng}{\overline{T}}} \frac{\varepsilon(k)}{\varepsilon} + \frac{N(k)}{N} = \frac{\varepsilon \frac{d\overline{T}}{dz}}{\frac{Ng}{\overline{T}}} + 1 \tag{3.136}$$

and assuming $\varepsilon(k)/\varepsilon$ to be finite, Eqs. 3.135 and 3.136 will imply $N(k) \approx N$ in the buoyancy subrange. In other words, the objection from Phillips to Bolgiano's hypothesis can be released if the inequality of Eq. 2.19 or Eq. 3.135 can be accepted as the basic requirement for the existence of buoyancy subrange. As we can see later, in case Eq. 3.135 does not exist for a certain flow condition, or equivalently, the condition $\Gamma_1 >> |\Gamma_T|$ cannot be fulfilled, there exists no buoyancy subrange. Thus, one word can be added that if there exists any buoyancy subrange, $N(k) \approx N$ is still a local property. Of course, in the energy-containing range, this argument breaks down without any doubt.

Before the author elaborates on the modified hypothesis proposed for the stably stratified turbulent flow, it is worthwhile to review Lumley's idea concerning the production of turbulent energy. In his paper, Lumley (1965) considered the effect of the turbulent energy production on the turbulence spectra by expressing the shear stress spectrum as

$$\phi_{uw}(k) = a\,\frac{d\bar{U}}{dz}\;\varepsilon^{\frac{1}{3}}(k)\,k^{-\frac{7}{3}} \quad . \tag{3.137}$$

Thus, it results in

$$\frac{\partial\varepsilon(k)}{dk} = a\left(\frac{d\bar{U}}{dz}\right)^2 (1-R_f)\,\varepsilon^{\frac{1}{3}}(k)\,k^{-\frac{7}{3}} \tag{3.138}$$

where R_f is the spectral flux Richardson number.

As stated by Lumley, in Eq. 3.138, both production and buoyancy spectra have the same form; thus, there is no range of wave number in which production is unimportant. However, if Eqs. 3.61 and 3.62 are used and if the corresponding values C_1 , C_2 , C_3 , and C_4 are known for a given flow condition such as stratification, mean velocity gradient, etc., the difficulties induced by Lumley's model can be removed. That is, by varying C_i , there may exist a range of wave number in which production is unimportant and the buoyancy force certainly dominates the flow. Now, if Eqs. 3.137 and 3.61 are compared, the former corresponds to the latter when $C_1 \rightarrow 1$, $C_2 \rightarrow 0$ i.e., when the shear stress spectrum is proportional to $d\bar{U}/dz$ the mean strain rate. This condition can only be fulfilled when the mean strain rate is small compared to the eddy strain rate (Tchen, 1953). Thus, the validity and the generality of Eqs. 3.61 and 3.62 certainly offer a better opportunity for the study of buoyancy subrange of a thermally stratified flow.

Consequently, the modified hypotheses for a stably stratified turbulent flow are given as:

(1) The buoyancy subrange of a stably stratified turbulent flow exists when the local production and local dissipation of

turbulent energy is not important in this wave number range and when the flow conditions satisfy the criterion

$$\frac{\varepsilon \left|\frac{d\overline{T}}{dz}\right|}{N \frac{g}{\overline{T}}} << 1 \quad , \text{ or } \quad \frac{g}{\overline{T}} \left|\frac{d\overline{T}}{dz}\right| << \frac{N}{\varepsilon} \left(\frac{g}{\overline{T}}\right)^2 \quad . \tag{3.139}$$

(2) In the buoyancy subrange, the spectra of turbulent energy and temperature fluctuation are determined by N , ε , $g/\overline{T}$, $d\overline{T}/dz$, k , and C_4 completely and are expressed in forms

$$\phi(k) = N^{\frac{-2C_4+4}{4+C_4}} \varepsilon^{\frac{-4+4C_4}{4+C_4}} \left(\frac{g}{\overline{T}}\right)^{\frac{8-4C_4}{4+C_4}} f_1(s,\Gamma_T,\Gamma_1,C_4) k^{\frac{C_4-12}{4+C_4}} \tag{3.140}$$

$$\phi_{TT}(k) = N^{\frac{2+2C_4}{4+C_4}} \varepsilon^{\frac{2-2C_4}{4+C_4}} \left(\frac{g}{\overline{T}}\right)^{\frac{-4+2C_4}{4+C_4}} f_2(s,\Gamma_T,\Gamma_1,C_4) k^{\frac{-3C_4-4}{4+C_4}} \tag{3.141}$$

where

$$\Gamma_1 = N_*^{-1+C_4} \varepsilon^{1-C_4} \left(\frac{g}{\overline{T}}\right)^{-1+C_4} \left|\frac{d\overline{T}}{dz_*}\right|^{1-C_4} ,$$

$$\Gamma_T = N_*^{-2+C_4} \varepsilon^{2-C_4} \left(\frac{g}{\overline{T}}\right)^{-2+C_4} \left|\frac{d\overline{T}}{dz_*}\right|^{1-C_4} \frac{d\overline{T}}{dz_*} ,$$

$$C_4 > 0 \quad , \quad \infty > s \geq 2 \quad , \quad N_* = 2N \quad , \quad \frac{d\overline{T}}{dz_*} = 2\,\frac{d\overline{T}}{dz} ,$$

and f_1 and f_2 are numerical variables as function of s , Γ_T , Γ_1 and C_4 .

In case $C_4 = 1$, thus $\Gamma_1 = 1$, then Eqs. 3.133 and 3.134 become the Bolgiano's solutions as expressed in Eqs. 2.4 and 2.7.

Now, from Eq. 3.140, we can see that the reason why the parameter ε is dropped from the Bolgiano's hypothesis is not because $\varepsilon \ll \varepsilon(k)$ as interpreted by Bolgiano (1959) but because $C_4 = 1$ makes zero exponents for ε in Eqs. 3.140 and 3.141. Accordingly, the parameter ε must be retained in the hypothesis as described in the hypothesis (2). As to the physical background on which the modified hypotheses are based, the section 4.4 in the next chapter must be reviewed.

Furthermore, if the upper limits of the buoyancy subrange for the velocity and temperature spectra are interested, the wave numbers k^* and k_T^* obtained by equating Eqs. 2.2 and 3.140 as well as Eqs. 2.3 and 3.141 respectively are expressed as

$$k^* \sim \alpha^{-\frac{3(4+C_4)}{8(2-C_4)}} f_1^{\frac{3(4+C_4)}{8(2-C_4)}} N^{\frac{3}{4}} \varepsilon^{-\frac{5}{4}} \left(\frac{g}{\overline{T}}\right)^{\frac{3}{2}} \tag{3.142}$$

and

$$k_T^* \sim \alpha_T^{\frac{3(4+C_4)}{8-4C_4}} f_2^{-\frac{3(4+C_4)}{8-4C_4}} N^{\frac{3}{4}} \varepsilon^{-\frac{5}{4}} \left(\frac{g}{\overline{T}}\right)^{\frac{3}{2}} \quad . \tag{3.143}$$

Certainly, k^* and k_T^* are linearly proportional to k_o defined in Eq. 3.97. Hence, it has been proved aposteriori that k_o is a characteristic wave number to distinguish the buoyancy subrange from the inertial subrange in case $k_e \ll k_o$; of course, if $k_e \approx k_o$ or $k_e \gg k_o$, there exists no buoyancy subrange. Equations 3.142 and 3.143 also reveal another interesting thing, i.e., the upper limit of buoyancy subrange are a function of flow conditions such as $d\overline{T}/dz$, etc.

Chapter IV

RESULTS AND ANALYSIS

In this chapter, numerical solutions of turbulence spectra of a thermally stratified flow investigated in the last chapter will be given. The solutions studied in the previous chapter are mainly divided into two categories: one is for the solution of a flow with molecular effects, the other is for those of a flow with negligible molecular effects. The consideration of molecular effects in the flow of the first category does not mean that the spectra at high wave numbers beyond the molecular cut-off wave number can be studied by the present method--the generalized eddy-viscosity approximation, but is used to generate nondimensionalized spectra expressed with dimensionless variables containing the molecular parameters ν and ν_T .

Thus, for studying spectra in the buoyancy subrange, we need only consider the case when the molecular effects are negligible in the wave number range of interest. In other words, the solution given in section 3.3.1 will not be investigated extensively and only some typical spectral curves of a certain flow condition are displayed as Figs. 28, 29, and 30. However, for a better understanding of the spectral forms in the buoyancy subrange, the solutions developed in section 3.3.2 must be ascribed to; in particular the solution obtained in case $s \to \infty$ and $r \to \infty$ as presented in part B of section 3.3.2 will be studied extensively because of the clearer and simpler forms given by Eqs. 3.117 and 3.118.

4.1 Determination of Parameters Γ , Γ_1 , Γ_T , C_2 and C_4

In order to solve the nondimensionalized spectra such as $\Phi(x)$ and $\Phi_{TT}(x)$, etc. as functions of the dimensionless wave number x by using Eqs. 3.117, 3.118, 3.119 and 3.120, the parameters Γ , Γ_1 , Γ_T , C_2 and C_4 must be known. From Eq. 3.96, Γ , Γ_1 , and Γ_T can be related to only two parameters C_2 and C_4 if $d\bar{U}/dz$, $d\bar{T}/dz$, b , N , ε and β are known. Experimentally, C_2 and C_4 can be evaluated from the measured spectra at given flow conditions characterized by $d\bar{U}/dz$, $d\bar{T}/dz$, b , N , ε and β . Thus, the way to evaluate C_2 and C_4 would be equivalent to that of evaluating Kolmogorov's constant α in Eq. 2.2 for locally isotropic turbulent flow. For simplicity, the asymptotic forms of Eqs. 3.132 and 3.133 can be used for evaluating C_4 if there exists a wide buoyancy subrange and if the flow conditions upon which C_4 depends are known.

However, for the present study, these parameters can only be assumed before the numerical solutions are obtained. Generally speaking, $d\bar{U}/dz$, $d\bar{T}/dz$, and β can be found from the measurements of velocity and temperature profiles of the atmosphere, and b , N , and ε can be estimated from the measurements of heat flux and shear stress or from the energy balance budget equation or from the measured spectra. Thus, we can evaluate the maximal and minimal values of those flow characteristics, but how they are related to one another for given mean velocity and temperature gradients is unknown. Nevertheless, to the best knowledge of the author, there are no measurements of N and ε corresponding to the respective measurements of spectra in the free atmosphere. Therefore, in the present study, the values of Γ , Γ_1 , and Γ_T are assumed to facilitate the numerical solutions, and the significance of these three parameters can only be studied by varying their relative values. On the other hand, C_2 and C_4 are varied according to $0 < C_2$ and $0 < C_4$ in order to test how the spectral forms will be changed under a given set of values for Γ , Γ_1 , and Γ_T . Henceforth, we can obtain a general idea that the introduction of C_2 and C_4 gives more degrees of freedom to test the spectral forms, although more freedom means more complications are involved in the analysis. In the following, results and analysis are presented and classified according to the flow stratification.

4.2 Buoyancy Subrange of Stably Stratified Flow

4.2.1 Asymptotic solutions of stably stratified flow in the buoyancy Subrange by Varying C_4 - As studied in section 3.4, in the buoyancy subrange where the productions of energy and temperature inhomogeneity are negligible, that is equivalent to say that $|\Gamma| \approx 0$ and $\Gamma_1 \gg |\Gamma_T|$ in Eqs. 3.117 and 3.118 from the numerical point of view, there exist some asymptotic solutions of velocity and temperature spectra as functions of C_4 . Thus, it is proposed to solve Eqs. 3.117 and 3.118 numerically by letting $|\Gamma| = 0$, $|\Gamma_T| = 0$, and $|\Gamma_1|$ be finite. Figure 4 displays how the spectral forms vary as the parameter C_4 changes for stable stratification. In the derivation of Eqs. 3.132 and 3.133, C_4 is restricted to be $0 < C_4$, however, if Eqs. 3.117 and 3.118 are concerned, C_4 needs only satisfy the condition $C_4 < 1$ when $|d\overline{T}/dz|$ is very small as discussed in section 3.2, and no lower limit should be assigned to C_4 from the numerical point of view. Thus, for the sake of interest, in Fig. 4, some curves evaluated by Eqs. 3.117 and 3.118 in case $C_4 < 0$ are also presented. Of course, the power law derived in Eqs. 3.132 and 3.133 cannot be true in this case.

As displayed in Fig. 4, the spectral slopes in the buoyancy subrange show agreement with these derived in Eqs. 3.132 and 3.133, i.e., for $C_4 = 0.001$, $\Phi(x) \sim x^{-11.999/4.001} \sim x^{-3}$ and $\Phi_{TT}(x) \sim x^{-4.003/4.001} \sim x^{-1}$, and $C_4 = 0.9$, $\Phi(x) \sim x^{-11.1/4.9}$ and $\Phi_{TT}(x) \sim x^{-6.7/4.9}$. It is clear that with $C_4 = 0.001$ the asymptotic solution becomes very close to the prediction of Lumley-Shur, and with $C_4 = 0.9$ the slope approaches the predicted slope given by Bolgiano. For better understanding the behavior of the spectra, Fig. 5 is plotted by varying Γ_1 at given $C_4 = 0.001$. Thus, we can see that as Γ_1 increases the buoyancy effects penetrate gradually into higher wave numbers and the deviations from the - 5/3 law of locally isotropic flow are more apparent as the spectra become more anisotropic due to the buoyancy effects characterized by Γ_1. Figure 31 displays the asymptotic power law varied as C_4 for velocity and temperature spectra in the buoyancy subrange of stably stratified flow. m and n contained in $\phi(k) \sim k^n$ and $\phi_{TT}(k) \sim k^m$ are defined as $m = \dfrac{-3C_4-4}{4+C_4}$, $n = \dfrac{C_4-12}{4+C_4}$.

4.2.2 <u>The production effects of turbulence energy and temperature inhomogeneity on the spectra of stably stratified flow in the buoyancy subrange</u> - In section 4.2.1, it was assumed that the production of turbulence energy and temperature inhomogeneity is negligible in the buoyancy subrange. Without any doubt, in this buoyancy subrange, only the inertial transfer process and the buoyancy effects exist and the spectral forms must characterize the buoyancy effects. However, if $|\Gamma|$ and $|\Gamma_T|$ are large enough as compared to Γ_1 so that the production effects can penetrate into the buoyancy subrange, the spectral forms will be disturbed. Figure 6 shows two sets of spectral curves at different Γ_T , Γ , and Γ_1 with fixed C_2 and C_4 . The first set of curves is calculated with $\Gamma_T = 0.01$, $\Gamma = 0.1$, $\Gamma_1 = 0.1$, $C_2 = 1.0$, and $C_4 = 0.3$ and the second with $\Gamma_T = 0.001$, $\Gamma = 0.001$, $\Gamma_1 = 0.01$, $C_2 = 1.0$, and $C_4 = 0.3$. From the plotted curves, the temperature spectra present maximal points which show that some energy is converted from the velocity field into the temperature field. Near the wave number range in which the maximum of temperature spectra occurs, there appears a steeper slope than - 5/3 for the energy spectrum as a consequence of kinetic energy being transferred and converted into potential energy. It is clear that this wave number range is associated with the so-called buoyancy subrange in our previous investigation of asymptotic solutions.

According to the asymptotic solutions presented in section 3.4.1, the energy spectrum and the temperature spectrum must show the power law - 11.7/4.3 and -4.9/4.3 respectively if only the buoyancy effects are predominant in the buoyancy subrange, however, the power law of velocity spectrum of the first set of curves appearing in Fig. 6 is -2.4. This is not surprising after Fig. 7 is reviewed. Figure 7 displays the distributions of energy production, transfer and drainage by buoyancy force as represented by F_1 , F_2 , and F_3 respectively, and defined by

$$F_1 = |\Gamma| x^{-\frac{1}{2}+\frac{3C_2}{2}} \phi^{\frac{1}{2}+\frac{C_2}{2}}(x) \quad , \tag{4.1}$$

$$F_2 = x^{\frac{5}{2}} \Phi^{\frac{3}{2}}(x) \tag{4.2}$$

and

$$F_3 = - \Gamma_1 x^{-\frac{1}{2} + \frac{3C_4}{2}} \Phi^{\frac{1}{2}}(x) \Phi_{TT}^{\frac{C_4}{2}}(x) \quad . \tag{4.3}$$

The functions F_4 and F_5 are the production and inertial transfer of temperature inhomogeneity. Figure 7 shows that in the wave number range the - 5/3 law holds the inertial transfer presented as F_2 predominates the turbulent structure and tends to be a constant. In case the energy drainage by vertical heat flux becomes gradually important, F_2 increases as wave number decreases and consequently the buoyancy subrange is formed. As the wave number decreases down to the region where the energy production becomes important, F_2 decreases. Now, in the wave number range, say in the interval $0.03 \leq x \leq 0.15$, the energy drainage by buoyancy becomes important and the energy production becomes less negligible as well. In other words, the introduction of the energy production in this wave number range compensates the energy drainage by buoyancy and modifies the power law from - 11.7/4.3 to -2.4 . A similar situation can be seen from the second set of spectral curves which shows that the reduction in energy production can cause a wider buoyancy subrange and steeper power law. For a better understanding, Fig. 8 must be examined. In the interval $0.004 \leq x \leq 0.02$, the turbulent energy is mainly distributed by energy transfer F_2 and drained by buoyancy F_3 but the contribution from energy production F_1 on the energy distribution is less important. Hence, it will not be a surprise to have the maximum power law -2.5 for the energy spectrum curve B in the wave number range $0.004 \leq x \leq 0.02$.

Another interesting phenomenon also caused by the compensation of energy production F_1 on F_2 and F_3 can be observed in the wave number range $x < 0.001$ where the -1 law exists for the curve A and the -5/3 law for the curves B in Fig. 6. Now referring to Fig. 7, F_1 completely predominates the energy distribution in wave number range $x < 0.001$, the -1 slope for curve A can be predicted from Eq. 4.1 by inserting $C_2 = 1$. As to curve B, -5/3 slope in the wave number $x < 0.001$ results from the compensation among F_1 , F_2 , and F_3 such that F_2 tends to be constant over a wide range as can be seen from Fig. 8. Here one word must be added, the -5/3 slope in the wave number range $x < 0.001$ is obviously not a result of local isotropy. Thus, to the experimenters, the prediction for the local isotropy from the measured velocity spectra must be carefully worked out in case buoyancy effects exist. For example, if in the buoyancy subrange there exists any experimental error which causes data scattering, the -5/3 slope may be extended to low wave number without realizing the existence of the buoyancy subrange. In other words, the -5/3 slope appeared in low wave number $x < 0.001$ for curve B in Fig. 6 may incorrectedly be predicted as a result of local isotropy. The experimental data of Fig. 2a shows this situation very clearly since a -5/3 slope appears on the left side of the buoyancy subrange.

In order to test the local isotropy of the stably stratified turbulent shear flow, Fig. 9 is plotted with $x^{5/3}\Phi(x)$ and $x^{5/3}\Phi_{TT}(x)$ vs x . It is clear that the plot of Fig. 9 can present the tendency to local isotropy in a better way than the plots $\Phi(x)$

and $\Phi_{TT}(x)$ vs x, since the deviation from local isotropy can be detected easily without any ambiguity as stated in the last paragraph.

4.2.3 Validity of the generalized eddy-viscosity approximation- Up to this stage, the generalized eddy-viscosity approximation has been used to study turbulent energy and temperature spectra under the effects of buoyancy. Of course, the validity of the present study can be justified from the measured energy and temperature spectra, however, there is another way to test its validity by comparing the shear stress and vertical heat flux spectra predicted by the present study with respect to the measured ones. Since the present method--generalized eddy-viscosity approximation--rests on the assumptions for the integrated forms of energy transfer, temperature inhomogeneity transfer, shear stress, and vertical heat flux spectra presented as Eqs. 3.56, 3.58, 3.61, and 3.62, the validity of the present study must logically be checked by comparing those spectra with the measured ones although the measurements of these spectra are not easily performed.

Thus, for the above reasons, some spectra of shear stress and vertical heat flux predicted by the present study for given flow conditions are displayed in Figs. 10 and 11. In Fig. 11, the heat flux spectrum $\Phi_{wT}(x)$ of curve A shows a change in sign at wave number $x = 0.00315$. The same phenomenon also occurs in Fig. 12 which shows the Bolgiano's solution in case $C_4 = 1$. From these results the changing sign of $\Phi_{wT}(x)$ seems to be related to C_4.

For further understanding turbulence spectra of stably stratified flow, Figs. 13-18 are displayed systematically by varying one of the parameters Γ_T, Γ, Γ_1, C_2, C_4 when the others are fixed. In the spectra the buoyancy subrange certainly exists since for most the condition $\Gamma_1 >> \Gamma_T$ is satisfied. For example, in Fig. 13, when Γ_T is increased from 0.001 for curve A to 0.01 for curve B with the corresponding $\Gamma_1 = 0.5$, the buoyancy subrange is narrowed as expected.

In Fig. 14, the effects of energy production on the spectra are examined by varying Γ and keeping the other parameters fixed. Increasing Γ not only narrows the buoyancy subrange but even shrinks the inertial subrange. This situation can be seen from the curve A of velocity spectra. Near the region $x = 1$, the slope of velocity spectrum is -5/3 , but in the interval $0.1 < x < 1$, the slope is less than -5/3 , because the effects of energy production penetrate deeply into the region of high wave numbers such that the energy transfer decreases with decreasing wave number in this interval. While in the buoyancy subrange of stably stratified flow the energy transfer increases with decreasing wave number and the slope of the velocity spectrum is greater than the slope -5/3 . This situation reflects the case when the flow has great shear gradient.

Figure 15 displays a case when Γ_1 varies. From the plot, increasing Γ_1 would mean that the effects of buoyancy force intensify as indicated by a wider buoyancy subrange shown as curve C. Also as Γ_1 increases, the power law in the buoyancy subrange approaches to the asymptotic power law $(C_4-12)/(4+C_4)$. Another interesting thing should be noted from the plots in Fig. 16. For these spectra, C_2 is varied and we can see that when $C_2 < C_4$, the buoyancy subrange for the velocity spectrum $\Phi(x)$ disappears as C_2

decreases to 0.1 for curve A. Whether this phenomenon is realistic or not can only be determined by experiment.

In Fig. 17, C_4 is varied. As we can see that as C_4 varies from 0.5 for curve A to 1.0 for curve B, the hump in $\Phi_{TT}(x)$ disappears. Also since Γ is not negligible as compared to Γ_1 and Γ_T, the power law in the buoyancy subrange deviates greatly from the asymptotic power law which requires that Γ is negligible. In Fig. 18, C_4 is forced to be negative and a steeper slope of the velocity spectra in the buoyancy subrange is observed. In case $C_4 = -0.1$ the slope is close to the power law for $\Phi(x)$ given by $(C_4-12)/(4+C_4)$, but for $C_4 = -0.5$ the plotted curve has a steeper slope than $(C_4-12)/(4+C_4)$. This is not unexpected since the power law derived in Eq. 3.125 is valid only for $C_4 > 0$.

4.3 Buoyancy Subrange of Unstable Stratification

As described in Eqs. 3.127 and 3.128, the buoyancy subrange of unstably stratified turbulent flow exists when the production of turbulent energy is less important and $\Gamma_1 >> |\Gamma_T|$, and the velocity spectrum exhibits a hump and the temperature spectrum has a steeper slope than -5/3 in the buoyancy subrange. Figure 19 displays spectral curves for unstably stratified flow. Curves A, B and C show that buoyancy effects very clearly since $\Gamma_1 >> |\Gamma_T|$ is certainly fulfilled, however, curves D appear in a different way from curves A, B and C just because of the fact that $\Gamma_1 = \Gamma_T = 0.001$.

In Fig. 20, the effects of the production of turbulent energy due to Reynolds stress on the spectra are displayed by varying Γ when the other parameters are fixed. As can be seen from Fig. 20, the variation of Γ does not affect the spectra significantly, however, if C_2 is varied, i.e., the degrees of interaction between mean strain rate and the eddy strain rate are varied, the spectra appear differently as presented by Fig. 21. A plot of curves F_i (i 1,...,5) vs x is shown in Fig. 22 when F_i represents the terms on the right sides of Eqs. 3.117 and 3.118 respectively. F_2 decreases with decreasing wave number as a characteristic of buoyancy subrange of an unstably stratified flow. Note that in the buoyancy subrange of a stably stratified flow, F_2 the energy transfer flux increases with decreasing wave number as a consequence of compensating the energy drainage by vertical heat flux. For a better understanding of how the shear stress spectrum is distributed with respect to wave number in these cases of unstably stratified flow, Fig. 23 is displayed by plotting $x\Phi_{uw}(x)$ vs x on semilogrithmic paper. The interesting thing is the changing sign of $\Phi_{uw}(x)$; of course, the validity of the curves of $x\Phi_{uw}(x)$ plotted in Fig. 23 must be checked from experiments although it is certain that $\Phi_{uw}(x)$ need not necessarily be always positive or negative throughout all wave numbers. In addition, Fig. 24 is presented for the heat flux spectrum which does not show any change in sign.

The curves discussed above mainly correspond to the flow with $\Gamma_1 \geq |\Gamma_T|$. In Fig. 25, the case $|\Gamma_T| >> \Gamma_1$ is given by varying C_2 and is used for comparison with Fig. 21. Up to now, the plotted curves of temperature spectrum seem always to have steeper slope than -5/3 , however, Fig. 26 shows that this may not always be the case when some special values of the parameters are considered; of course, whether the curves shown in Fig. 26 are realistic or not, or say, whether the assigned values of parameters are reasonable or not must be determined by experiment.

In sections 4.2 and 4.3, great attention has been paid to the characteristics of the buoyancy subrange of both stable and unstable stratifications. However, very often there exists no buoyancy subrange at all because of the fact that the criterion $\Gamma_1 >> |\Gamma_T|$ for buoyancy subrange is not satisfied and that the effects of the production of turbulent energy penetrate greatly into high wave number range. For example, in case $C_2 = 1$, $C_4 = 1$, the -1 law exists for both velocity and temperature spectra as the productions of energy and temperature inhomogeneity are very large. Figure 30 shows this situation for both stratifications. Also in Fig. 27, the curves A show the existence of - 1/3 slope region for temperature spectrum when the velocity spectrum has extensive - 5/3 slope in this region. This region may be called the inertial and convective region as stated in Gisina's paper (1966) and as indicated by Eq. 3.129 when $C_4 = 1$.

Now in order to examine systematically how the parameters Γ , Γ_1 , Γ_T , C_2 , C_4 , s , and r vary for these figures, a brief table is listed as follows:

TABLE 2

Fig. No.	Curve	Stratification	Γ	Γ_1	Γ_T	C_2	C_4	s	r
4 & 11	A	stable	0.	0.01	0.	1.	0.9		
	B						0.001	∞	∞
	C						-0.5		
5	A	stable	0.	0.001	0.	1.	0.001		
	B			0.01				∞	∞
	C			0.1					
6 & 10	A	stable	0.1	0.1	0.01	1.	0.3	∞	∞
	B		0.001	0.01	0.001				
12		stable	0.01	0.1	0.001	1.	1.	∞	∞
13	A	stable	0.1	0.5	0.001	1.	1.	∞	∞
	B				0.01				
14	A	stable	1.0	0.1	0.01	1.	0.3		
	B		0.5					∞	∞
	C		0.1						
15	A	stable	0.5	0.1	0.01	1.	0.3		
	B			0.5				∞	∞
	C			1.0					
16	A	stable	0.1	0.1		0.1	0.3		
	B				0.01	0.5		∞	∞
	C					1.0			
17	A	stable	0.1	0.1	0.01	1.	0.5	∞	∞
	B						1.0		
18	A	stable	0.1	0.5	0.001	1.	-0.5	∞	∞
	B						-0.1		
19	A	unstable	0.0001	0.5	-0.001	1.	0.5		
	B			0.1				∞	∞
	C			0.01					
	D			0.001					
20	A	unstable	1.0	0.5	-0.1	1.	0.5		
	B		0.5					∞	∞
	C		0.01						
21	A	unstable	0.5	1.0	-1.0	0.1	1.0		
	B					0.5		∞	∞
	C					0.8			
25	A	unstable	2.	0.1	-1.	0.1	1.5		
	B					0.5		∞	∞
	C					0.8			
26	A	unstable	2.	0.5	-0.5	2.	0.1	∞	∞
	B					3.			
27	A	stable	0.01	1.0	0.75	1.0	1.0	∞	1
	B			0.1					

Now, the general characteristics of turbulence spectra of thermally stratified flows has been investigated very extensively for the buoyancy subrange by means of the generalized eddy-viscosity approximation, it would be worthwhile to make comparisons with measurements in the atmosphere and to test the validity of the generalized eddy-viscosity approximation.

4.4 Qualitative Comparison Between the Measurements of Spectra in the Atmosphere and the Results Obtained by the Generalized Eddy-Viscosity Approximation

Basically speaking, it is hard to make this comparison because of the fact that the experimental data normally relate to one-dimensional spectra whereas the theoretical consideration in the present study is concerned with three-dimensional spectra. As noted in the paper of Alkseev and Yaglom (1967), the one-dimensional spectrum always varies noticeably more smoothly than the three-dimensional spectrum. The wave number range in which a power law occurs in a one-dimensional spectrum will not be the same range in which the same power law occurs in a three-dimensional spectrum even if the local isotropy of the turbulence holds. Moreover, the turbulence spectra of a thermally stratified flow are certainly characterized by the anisotropy due to the buoyancy effects, and thus in our problem, the basic advantage of local isotropy to relate one-dimensional spectra to three-dimensional spectra is lost.

However, if only the approximate comparison is made to see how the parameters mentioned in the theoretical consideration are related to the experimental data, the difficulties as stated above can be relaxed. In particular, it is assumed that in the buoyancy subrange the deviation from the local isotropy may not be too great. In section 2.4, it is mentioned that the exponent n of the velocity spectrum k^{-n} in the buoyancy subrange of a stably stratified flow varies from 2.0 to 3.5. As we review Fig. 4, the values of n are associated with the parameter C_4 in the theoretical consideration, i.e., by varying C_4, some specific slope in the buoyancy subrange can be made. For example, to $n = 11/5$ there corresponds $C_4 = 1$, for $n = 3$ we have $C_4 \to 0$, and for higher values of n, a negative value can be assigned to C_4 as presented in Figs. 4 and 18. And in fact, the introduction of C_4 into the theoretical consideration is a great improvement in interpreting the spectra of a stably stratified flow.

At this stage, it would be worth mentioning the process by which the parameter C_4 was introduced into the theoretical consideration. In the inertial subrange of velocity spectrum, only ε the dissipation of turbulent energy is the parameter to characterize turbulence. Thus, the - 5/3 law is implied from the dimensional argument and this subrange is of universal equilibrium since the flow is independent of the parameters such as mean velocity and temperature gradients which in turn characterize the external flow conditions. As to the energy containing range, the flow is determined by the mean quantities and is influenced by the geometry which contains the flow or around which the flow passes. Of course, this range cannot be universally determined, and moreover, there exists no equilibrium state for turbulent flow in this range. Now, if we assume that there exists a certain subrange between the inertial subrange and the energy-containing range, then we can expect that the turbulence spectra are in a state of equilibrium but not of universal form. Consequently, it can be expected that in this subrange the turbulence is determined by the mean quantities and the turbulence parameters.

First, consider a subrange which is caused by the velocity field alone such as mean velocity. Certainly $d\overline{U}/dz$ and ε are the only parameters to determine the turbulence as can be seen from Tchen's solution (1953). Second, if this subrange is induced by the existence of thermal stratification, then based on the previous arguments, the turbulence in this "buoyancy subrange" must be determined by the mean quantities $d\overline{T}/dz$, $g/\overline{T}$, and the turbulent parameters ε and N . Note in the present arguments the significance of the second and higher order derivatives of $\overline{U}$ and $\overline{T}$ with respect to z are negligible as compared to $d\overline{U}/dz$ and $d\overline{T}/dz$ respectively. The following paragraph gives the reason why the parameter C_4 must be introduced.

In the buoyancy subrange the turbulence may be in equilibrium but not universal in character. Thus, for a given set of parameters, $d\overline{T}/dz$, $g/\overline{T}$, ε , and N of a certain turbulent flow, there exists a definite spectral form because of the equilibrium of the turbulence. However, the spectral form will vary from one turbulent flow to the other because the turbulence is not universal. Hence, we can see the necessity to introduce a new dimensionless parameter C_4 in order to characterize the spectral forms for varied flow conditions as related to the degree of interaction between the mean temperature field and temperature fluctuation field as described in Eq. 3.62. Of course, the same argument can be applied to C_2 introduced in Eq. 3.61. In fact, Tchen's solutions (1953) are only two particular cases implied by Eq. 3.61.

If C_4 is known for a given set of flow conditions, is it possible to find a similarity theory such that some characteristic variables can form a set of dimensionless variables and the turbulence becomes quasi-universal in this subrange? Here the term "quasi-universal" means that for different flow conditions which can build up the same values of dimensionless parameters there is a unique spectral form corresponding to the associated C_4. Certainly, the characteristics to be quasi-universal in the buoyancy subrange of a stably stratified flow was ignored and undetected by Bolgiano and Lumley since both of them devoted themselves to an effort to reach a universal solution for the buoyancy subrange. Thus, the reason why some dimensionless parameters are presented in Eqs. 3.65, 3.74, 3.96, 3.97 and 3.100 can be seen clearly. With these ideas in mind, it should not be surprising to get the hypothesis (2) for the modified hypotheses stated in section 3.5.

Now, it is time to clarify why the parameter $d\overline{U}/dz$ does not appear in the hypothesis (2). In Monin's paper (1965) it was noted that in the buoyancy subrange the dependence of spectra upon $d\overline{T}/dz$ only and not upon $d\overline{U}/dz$ seems unnatural. However, if $d\overline{U}/dz$ is introduced into hypothesis (2) in order to make this hypothesis complete, the simple solution as expressed in Eqs. 3.132 and 3.133 cannot be obtained. However, the situation will not be so pessimistic since the dependence of the spectra upon $d\overline{U}/dz$ can be investigated numerically, although not analytically. And in fact, the numerical investigation of the dependence of spectra upon $d\overline{U}/dz$ is certainly an improvement upon either Bolgiano's or Lumley-Shur's hypothesis. Curve A in Fig. 6 shows the effect of $d\overline{U}/dz$ on the spectra clearly. If Fig. 2b is reviewed, the less steep part of the curve on the left side of the buoyancy subrange reflects the effects of the production of turbulent energy due to Reynolds stress or flux divergence.

In case the unstable stratification is concerned, the velocity spectra shown in Fig. 3 show a hump at the frequency corresponding to the motion of Bénard cell in the thermal convection (Ivanov and Ordanovich, 1967). In the present study, this kind of hump due to unstable stratification can also be detected from Fig. 19, and it seems to the author that the left side of the hump of the spectra in Fig. 3 approaches to the +1 slope as shown in Fig. 19 from the theoretical consideration. Unfortunately, Ivanov and Ordanovich did not mention any measurements of the temperature spectra corresponding to the flow conditions under which the velocity spectra were taken. Hence, generally speaking, when compared to the above stated measurements of spectra in the atmosphere, the theoretical study by the present method--generalized eddy-viscosity approximation--can give better features of turbulence structure of thermally stratified flows than any of the previous hypotheses of Bolgiano and Lumley in the case of stable stratification and than Monin's results (1962) on the spectra of unstably stratified flow.

Chapter V

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this theoretical investigation, it was shown that the proposed generalized eddy-viscosity approximation can predict features of turbulence spectra of thermally stratified turbulent shear flows better than any previous hypotheses. The present method gives a general solution in the buoyancy subrange, and the effects of the turbulent energy production in the buoyancy subrange are investigated numerically. As a consequence of the application of the generalized eddy-viscosity approximation, modified hypotheses are established for the buoyancy subrange of a stably stratified flow as follows:

(1) The buoyancy subrange of a stably stratified turbulent flow exists when the local production and local dissipation of turbulent energy are not important and when the flow conditions satisfy the criterion

$$\frac{\varepsilon \left|\frac{d\overline{T}}{dz}\right|}{N \frac{g}{\overline{T}}} << 1 \quad , \quad \text{or} \quad \frac{g}{\overline{T}} \left|\frac{d\overline{T}}{dz}\right| << \frac{N}{\varepsilon} \left(\frac{g}{\overline{T}}\right)^2$$

which means that the internal frequency $(N/\varepsilon)^{1/2}\, g/\overline{T}$ is much greater than the Brunt-Väisälä frequency $\left(\frac{g}{\overline{T}} \frac{d\overline{T}}{dz}\right)^{1/2}$.

(2) In the buoyancy subrange, the spectra of turbulent energy and temperature fluctuation are determined by N , ε , $g/\overline{T}$, $d\overline{T}/dz$, k , and C_4 completely, and are expressed in forms

$$\phi(k) = N^{\frac{-2C_4+4}{4+C_4}} \varepsilon^{\frac{-4+4C_4}{4+C_4}} \left(\frac{g}{\overline{T}}\right)^{\frac{8-4C_4}{4+C_4}} f_1(s, \Gamma_T, \Gamma_1, C_4) k^{\frac{C_4-12}{4+C_4}}$$

and

$$\phi_{TT}(k) = N^{\frac{2+2C_4}{4+C_4}} \varepsilon^{\frac{2-2C_4}{4+C_4}} \left(\frac{g}{\bar{T}}\right)^{\frac{-4+2C_4}{c+C_4}} f_2(s,\Gamma_T,\Gamma_1,C_4) k^{\frac{-3C_4-4}{4+C_4}}$$

respectively. In these expressions,

$$\Gamma_T = N_*^{-2+C_4} \varepsilon^{2-C_4} \left(\frac{g}{\bar{T}}\right)^{-2+C_4} \left|\frac{d\bar{T}}{dz_*}\right|^{1-C_4} \frac{d\bar{T}}{dz_*}$$

$$\Gamma_1 = N_*^{-1+C_4} \varepsilon^{1-C_4} \left(\frac{g}{\bar{T}}\right)^{-1+C_4} \left|\frac{d\bar{T}}{dz_*}\right|^{1-C_4} , \quad C_4 > 0 \quad , \quad \infty > s \geq 2 ,$$

$$N_* = 2N \quad , \quad \frac{d\bar{T}}{dz_*} = 2 \frac{d\bar{T}}{dz} \quad ,$$

and f_1 and f_2 are numerical variables. Clearly, both the solution of Bolgiano and the solution of Lumley-Shur are contained in these forms when $C_4 = 1$ and $C_4 \to 0$ respectively.

As to the unstably stratified flow, the velocity spectrum exhibits hump in the buoyancy subrange. On the left side of this hump the velocity spectrum approaches a +1 slope and the temperature spectrum shows a -3 slope.

Before the generalized eddy-viscosity approximation is introduced, the basic assumption used to derive a simpler set of spectral equations is that the flow is locally homogeneous. In other words, in the derivation of the spectral equations the terms due to inhomogeneity of the flow field have been discarded. Thus, for better understanding of the effects due to the inhomogeneity it would be constructive to include these terms in the spectral equations. This may be important in the case of free convection and inversion layer flows because the divergences of energy flux and temperature inhomogeneity play an important role in the balance of the thermal turbulence energy budget equation. Thus, for further research on turbulence spectra of thermally stratified flows, these effects must be considered so that the spectral forms in the subrange where the production and diffusion of turbulent energy play the principal role in determining the turbulent structure can be investigated.

On the other hand, the fruitful results obtained up to now may induce some further advanced research in applied engineering problems such as turbulent diffusion related to air pollution, wave propagation in the atmosphere, and high speed aircraft design. For example, in his paper, Tchen (1959) did not consider the effects of stratification on the dispersion of smoke from a point source although the role of the Reynolds stresses in the momentum equations was studied extensively in order to clarify the internal interaction of the diffusing particles. In the classical theory of the scattering of sound waves (Tatarski, 1961), the vertical heat flux spectrum is assumed to be zero in order to simplify the problem; certainly the vertical heat flux spectrum is not zero in the case of stratified flows. As to the high speed aircraft design, the airplane frequency-response function can predict the associated response spectrum of airplane vibration caused by turbulence (Steiner and Pratt, 1967).

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FIGURES

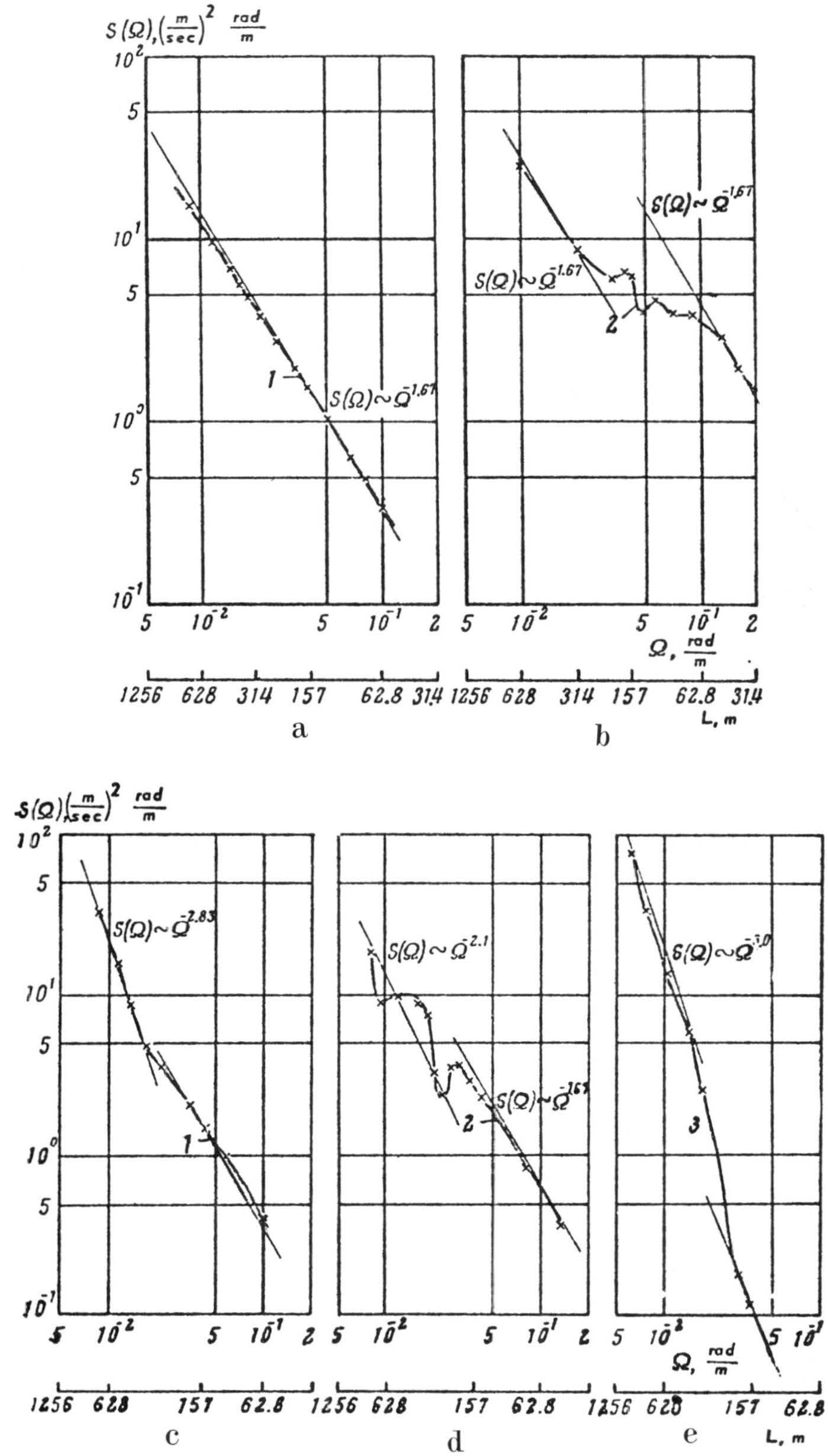

Fig. 1 Typical spectral density curves of longitudinal wind velocity component of thermally stratified flows (Pinus and Shcherbakova, 1966)

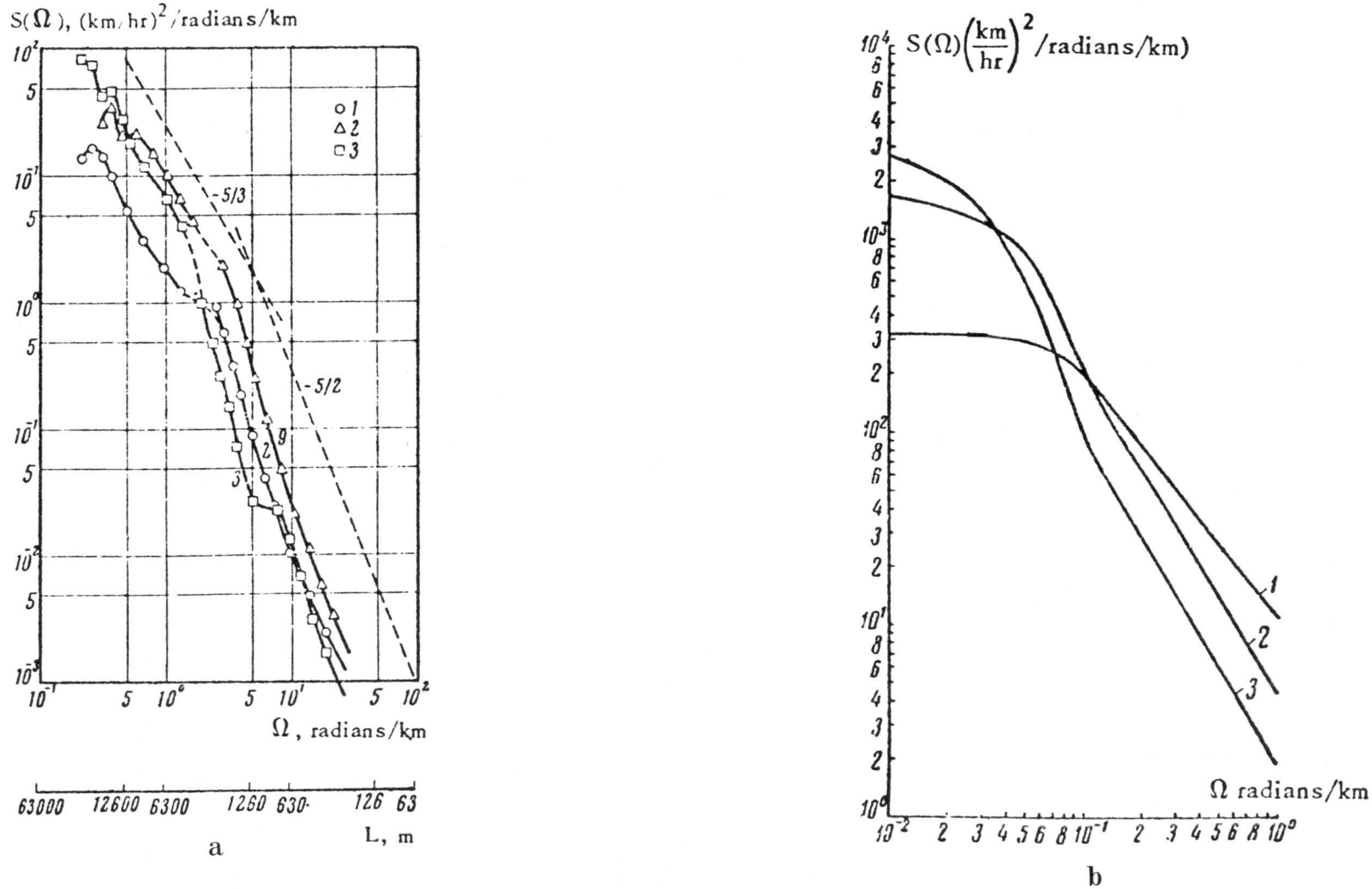

Fig. 2 Typical spectral density curves of the horizontal wind velocity component of stably stratified flows (a) Vinnichenko, 1966 (b) Pinus, 1963

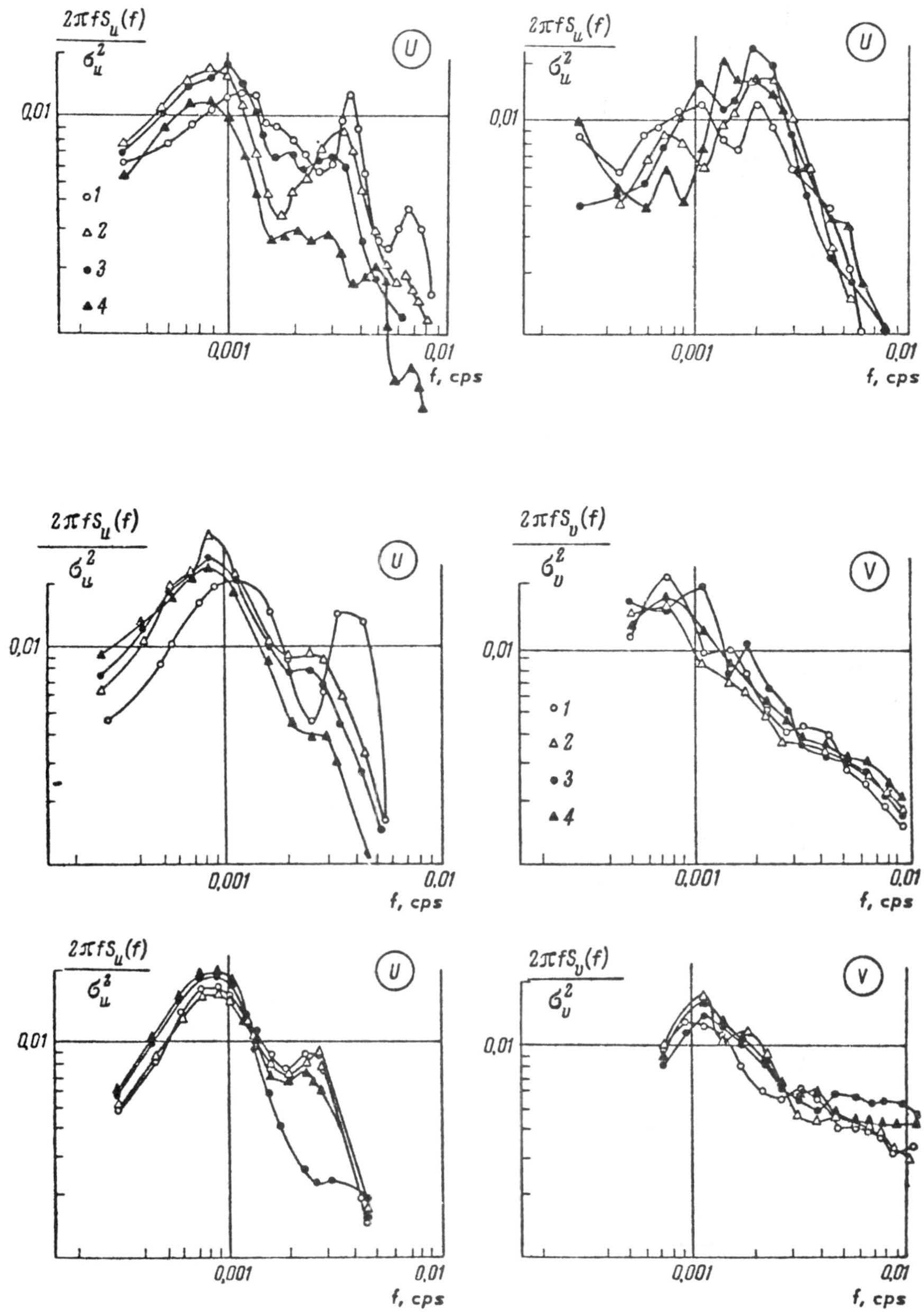

Fig. 3 Typical spectral densities of horizontal wind velocity components of unstably stratified flow at various heights (Ivanov and Ordanovich, 1967)

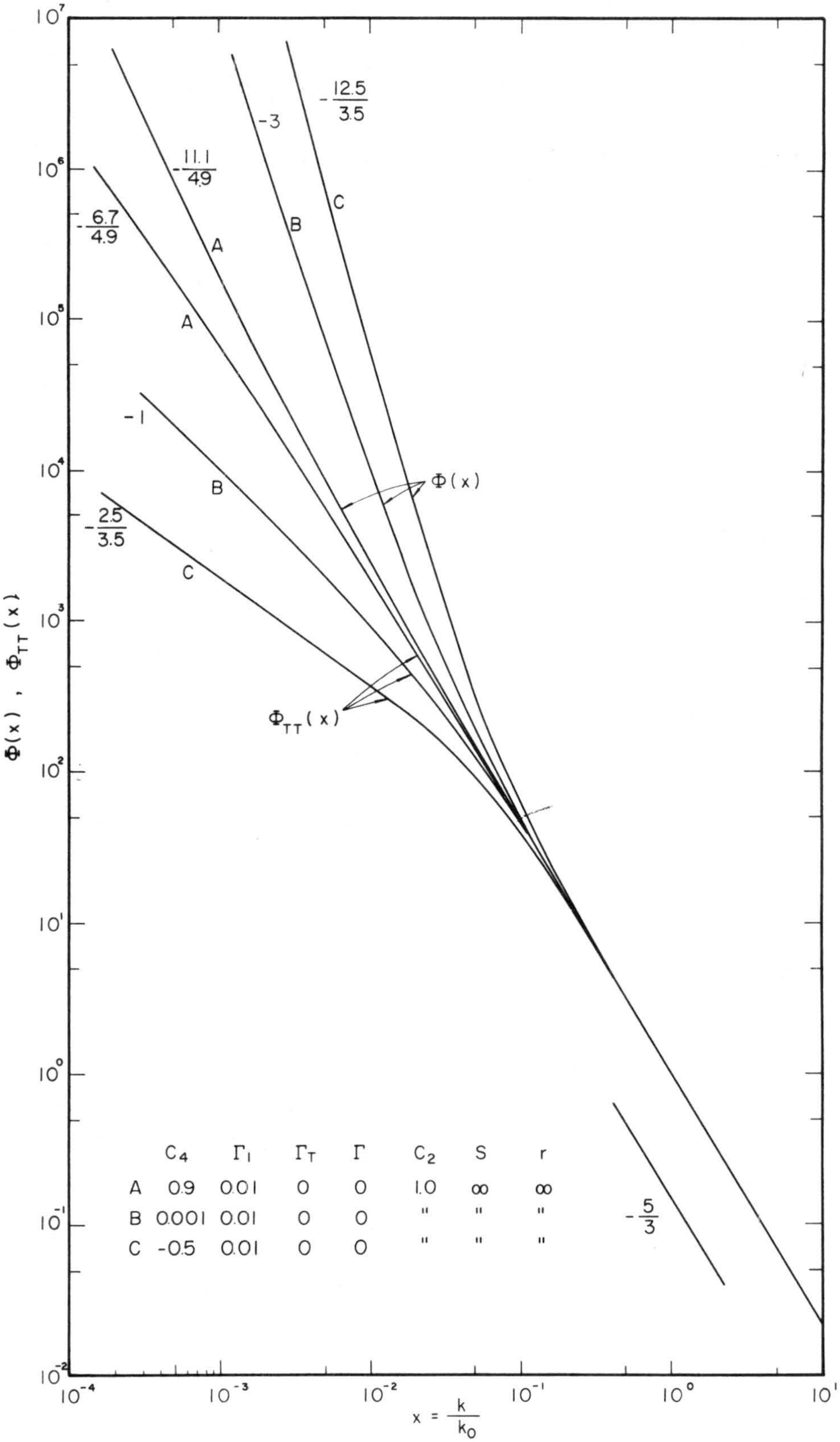

Fig. 4 Asymptotic velocity and temperature spectra of stably stratified flows varied as a function of C_4

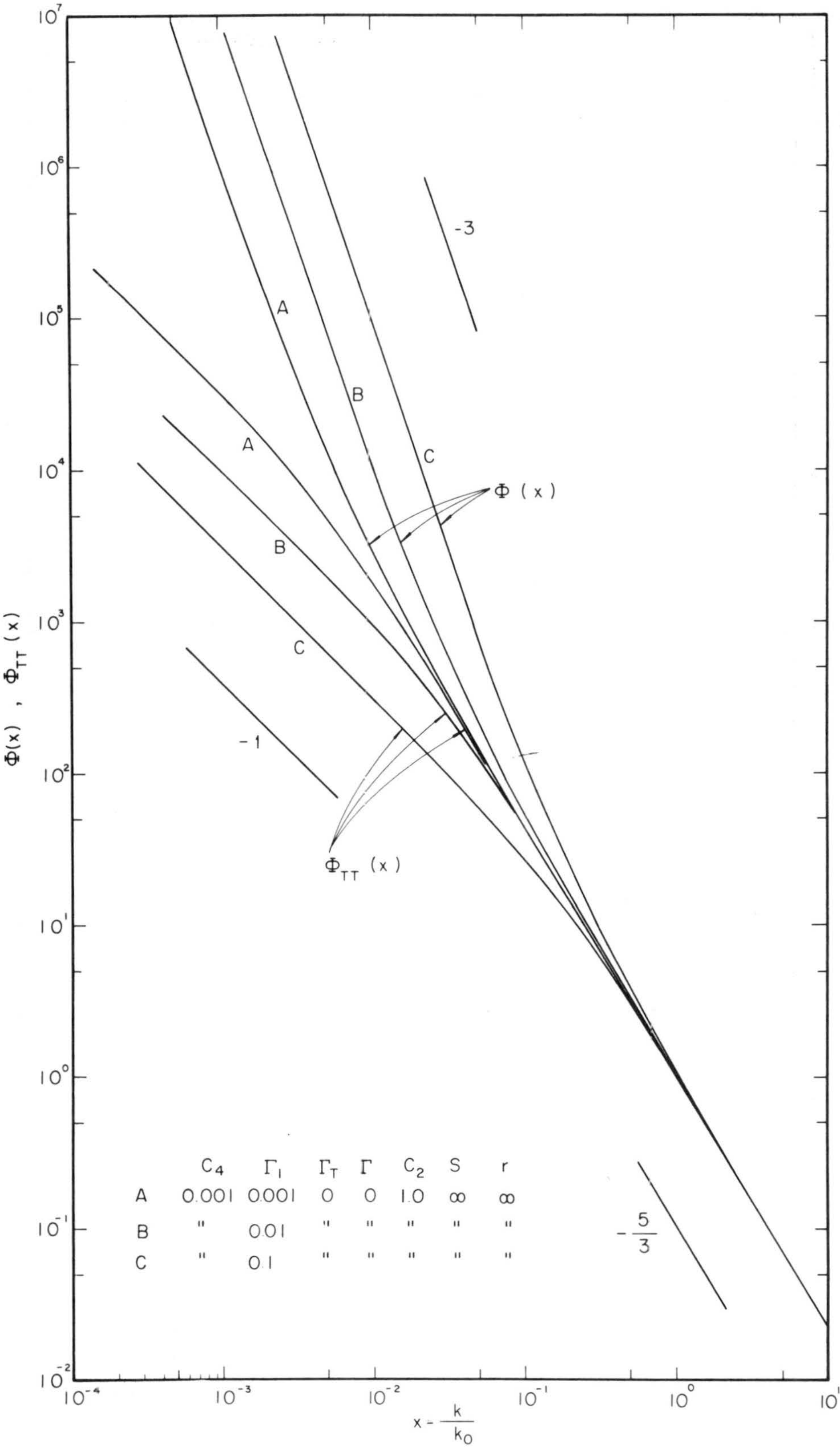

Fig. 5 Asymptotic velocity and temperature spectra of stably stratified flows varied as a function of Γ_1

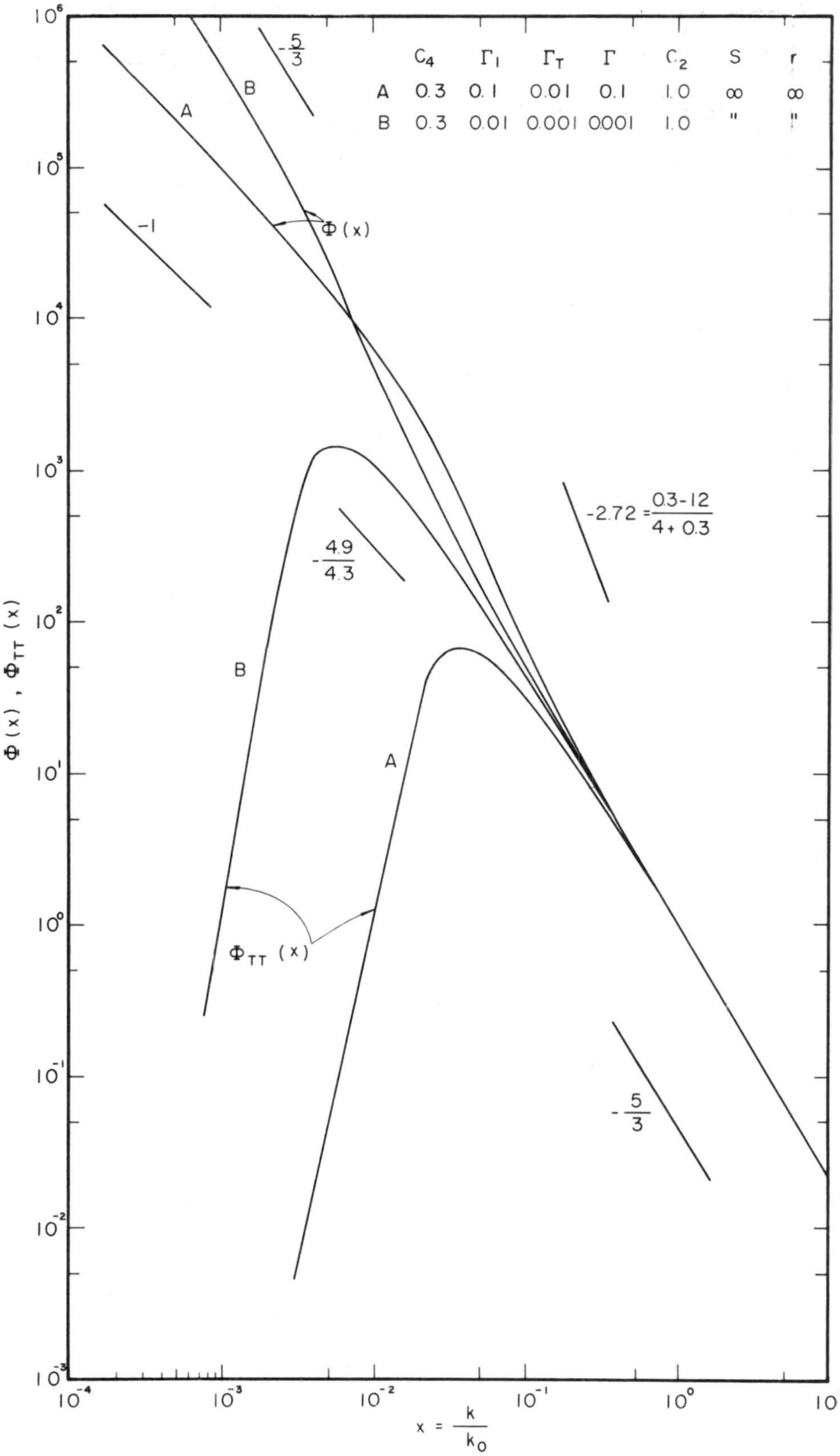

Fig. 6 Velocity and temperature spectra of stably stratified flows with the effects of the production of turbulence energy and temperature inhomogeneity

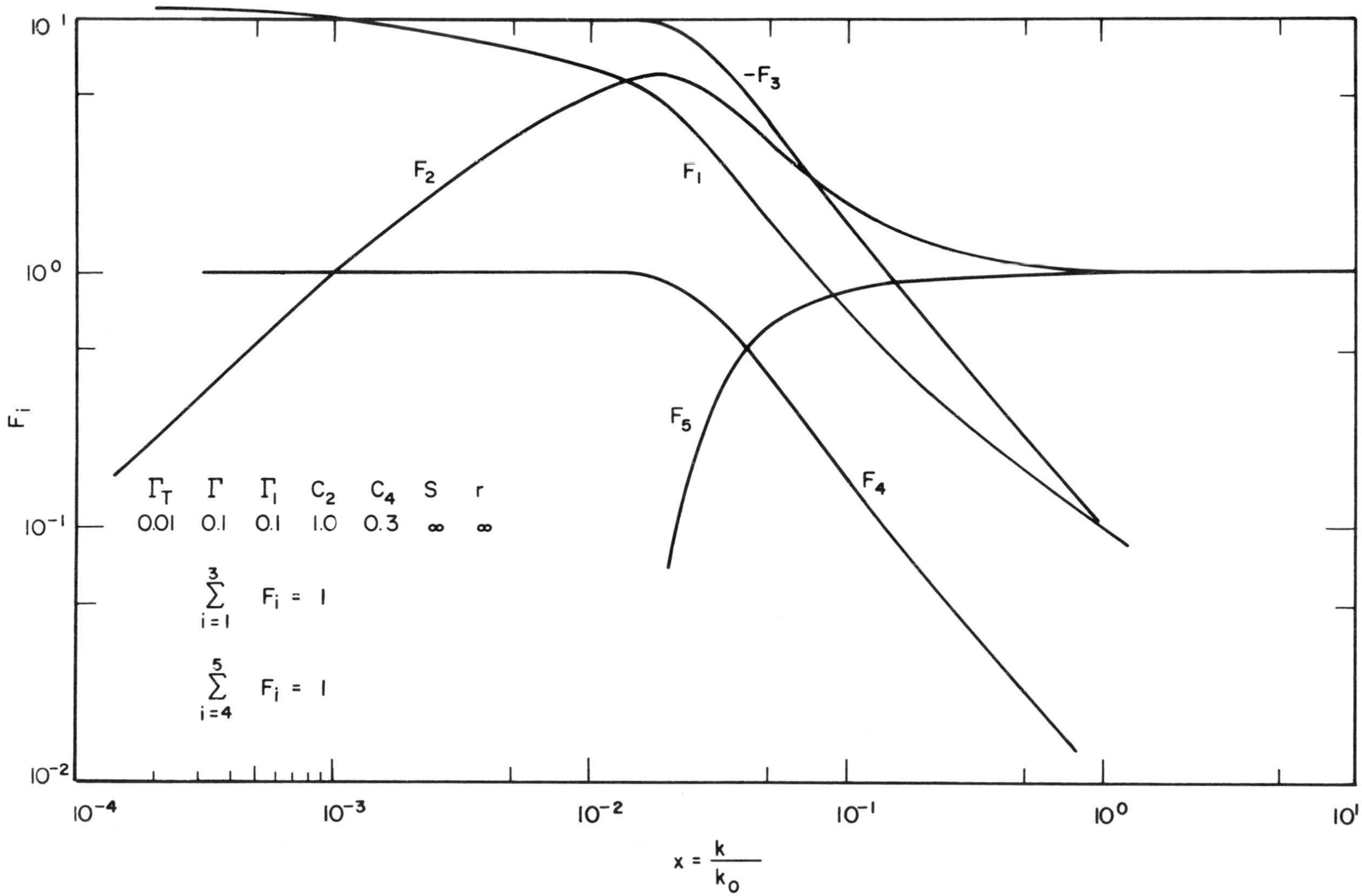

Fig. 7 Distributions of energy production F_1, transfer F_2, and drainage by buoyancy force F_3, and temperature inhomogeneity production F_4, and transfer F_5, of stably stratified flows across wave numbers

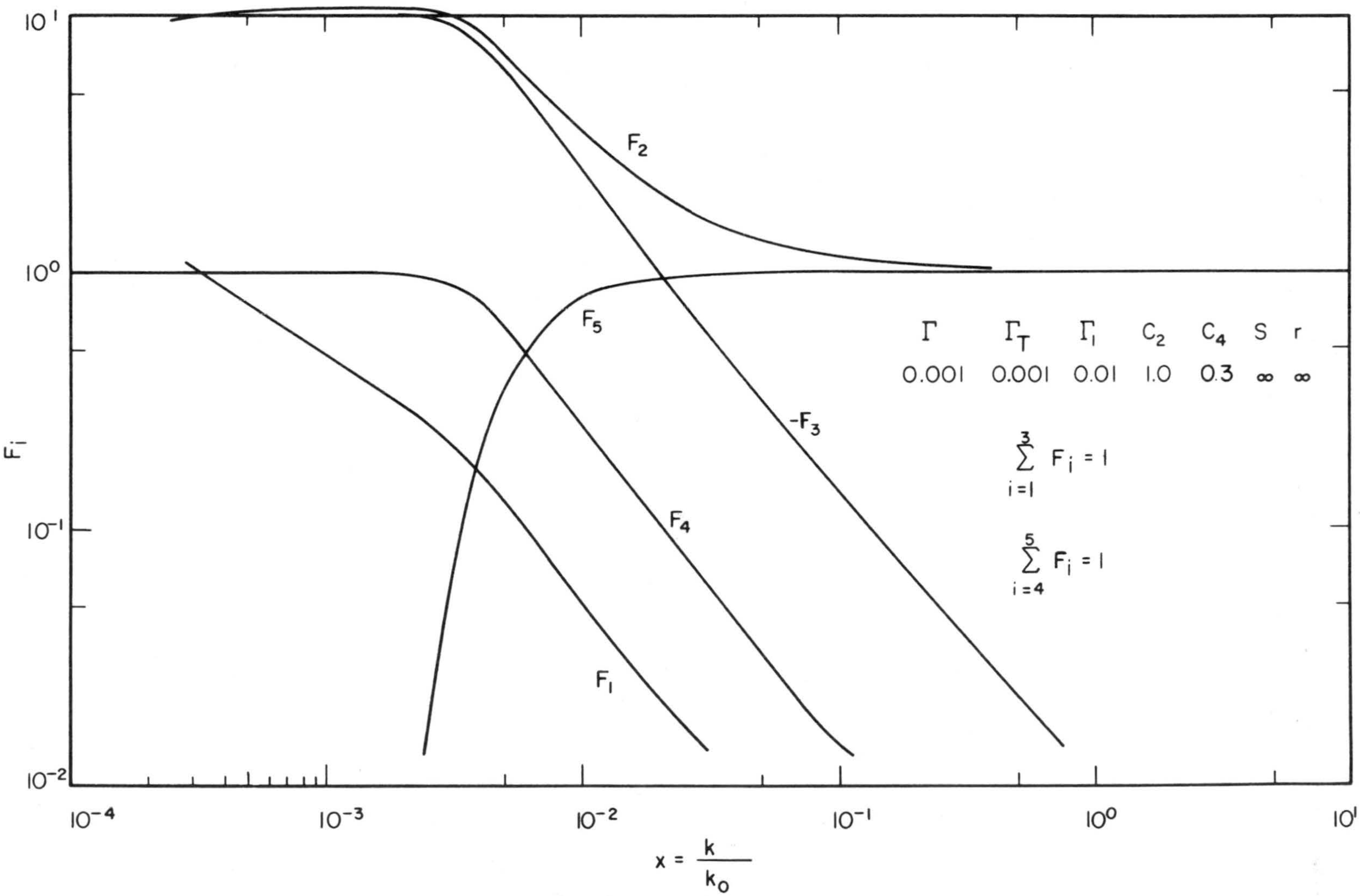

Fig. 8 Distributions of energy production F_1 , transfer F_2 , and drainage by buoyancy force F_3 , and temperature inhomogeneity production F_4 , and transfer F_5 , of stably stratified flows across wave numbers

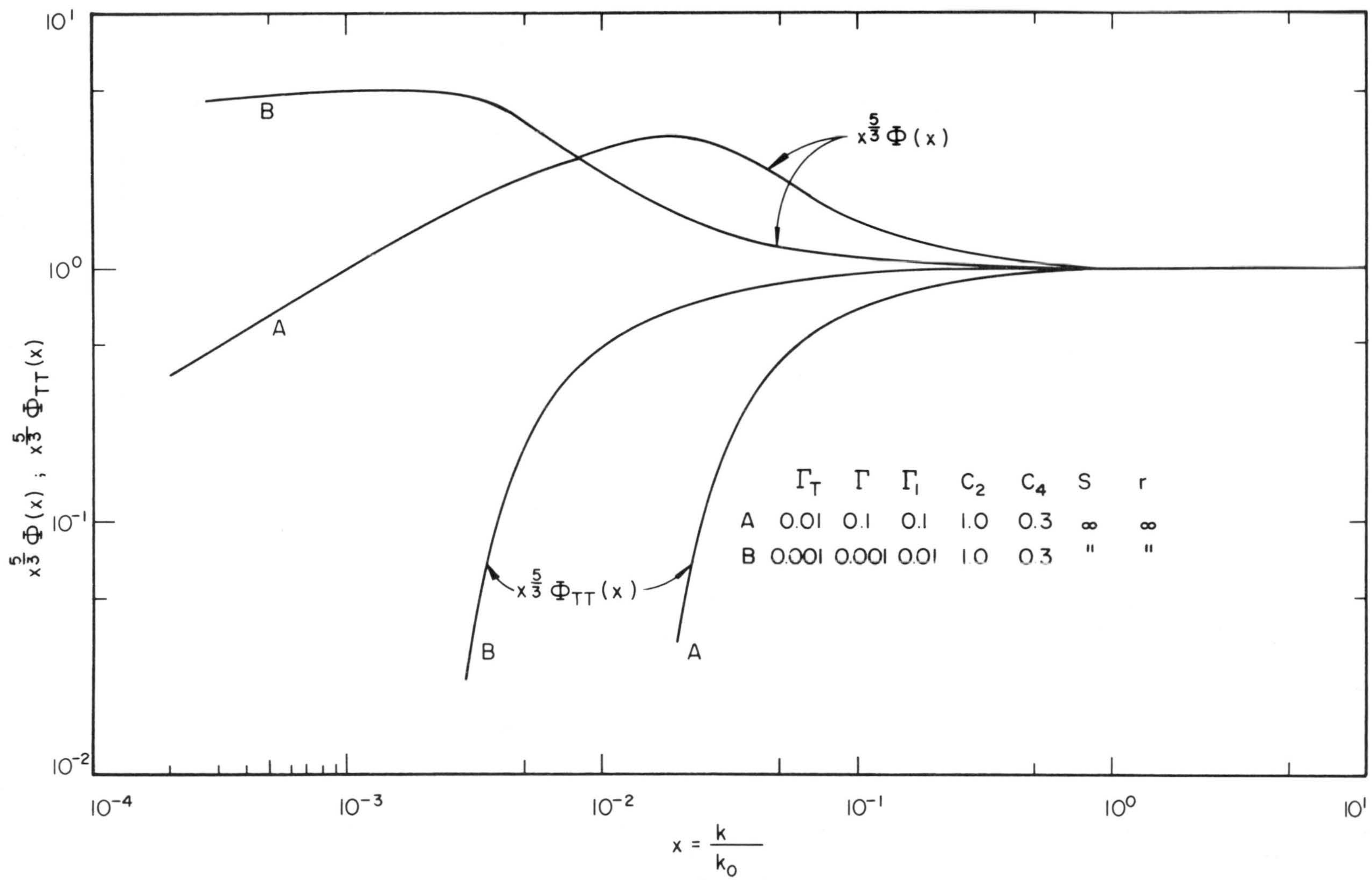

Fig. 9 Curves of $x^{\frac{5}{3}}\ \Phi(x)$ and $x^{\frac{5}{3}}\ \Phi_{TT}(x)$ vs dimensionless wave number x

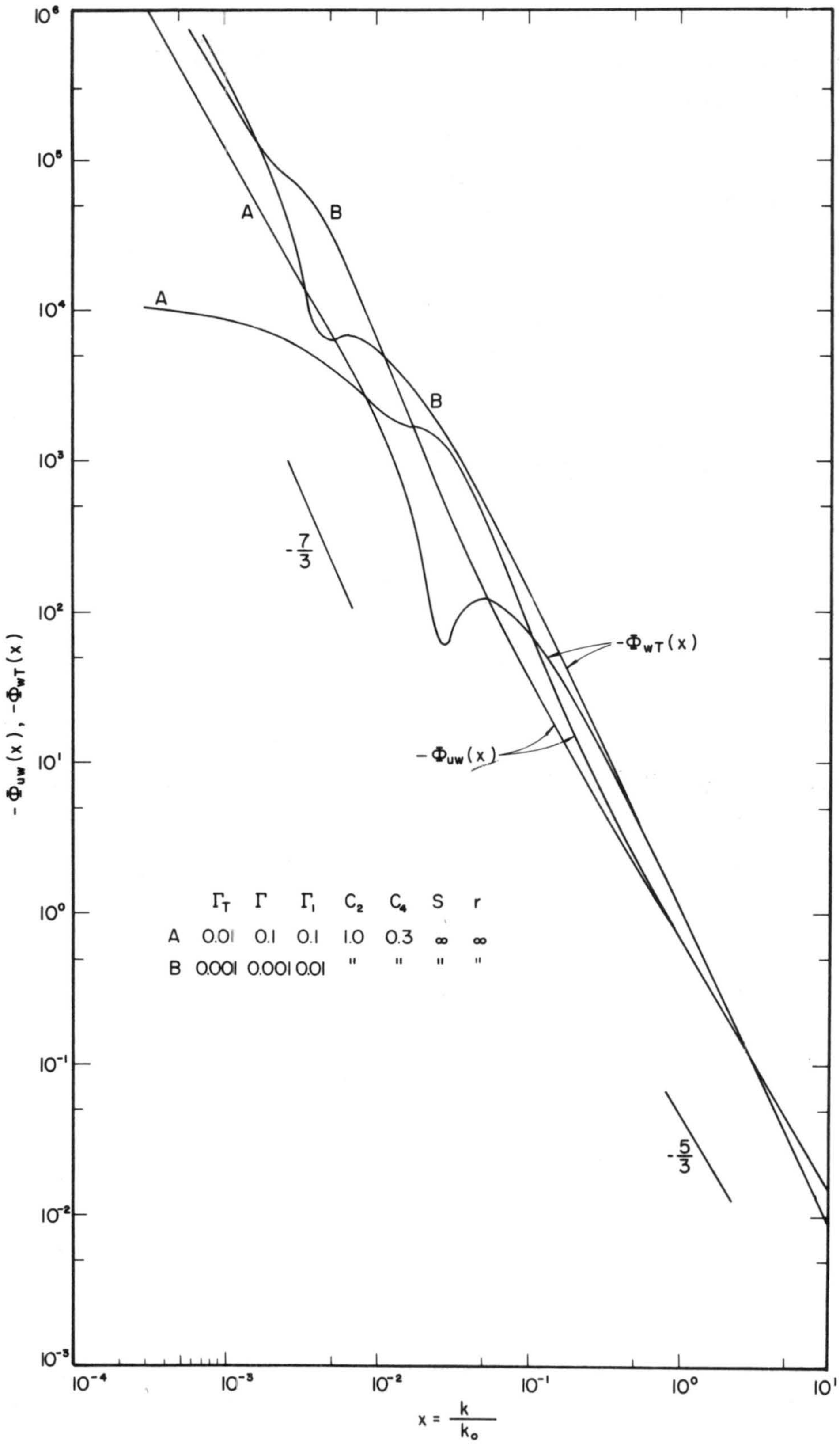

Fig. 10 Turbulence spectra of shear stress and vertical heat flux of stably stratified flows

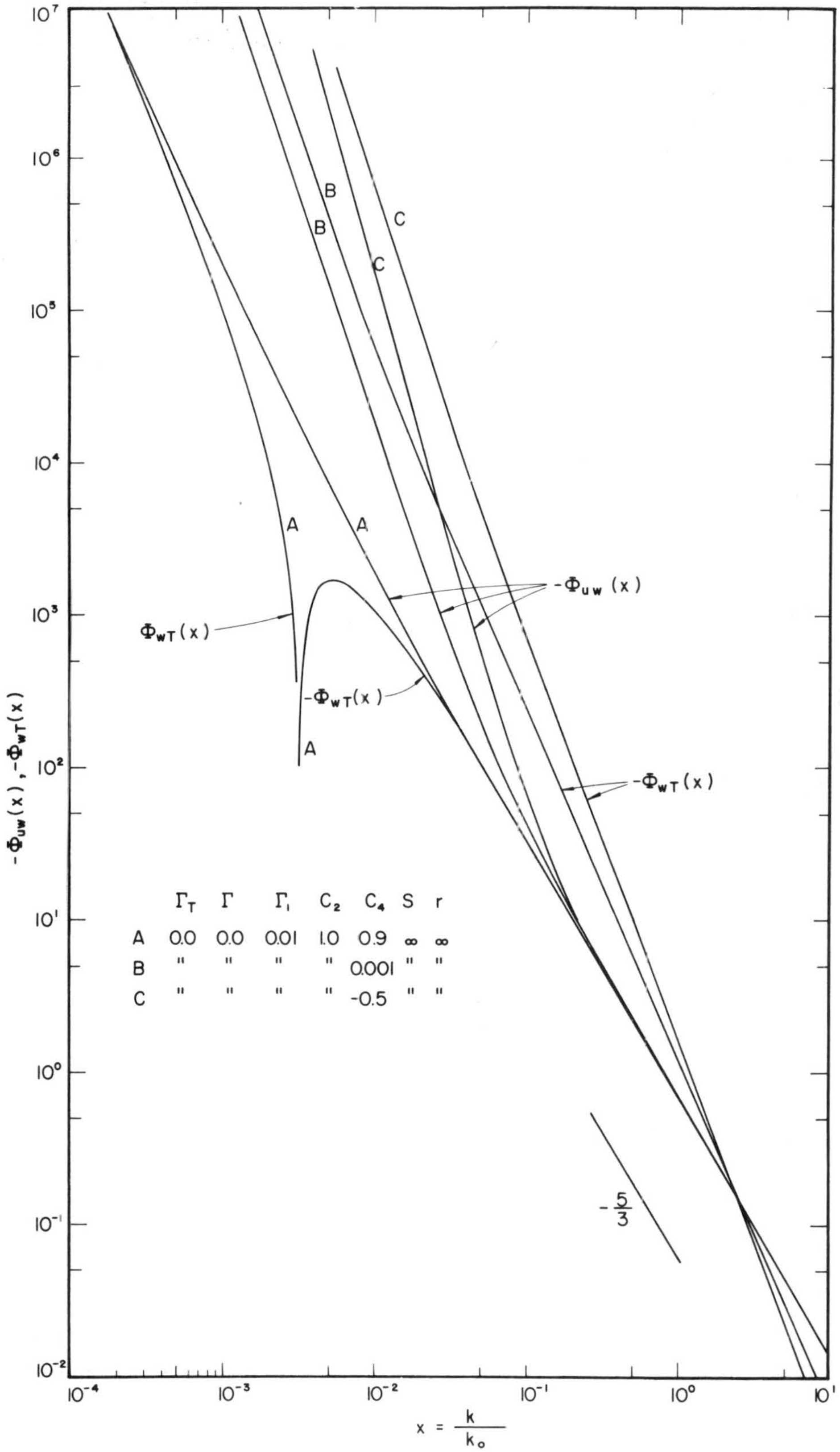

Fig. 11 Turbulence spectra of shear stress and vertical heat flux of stably stratified flows

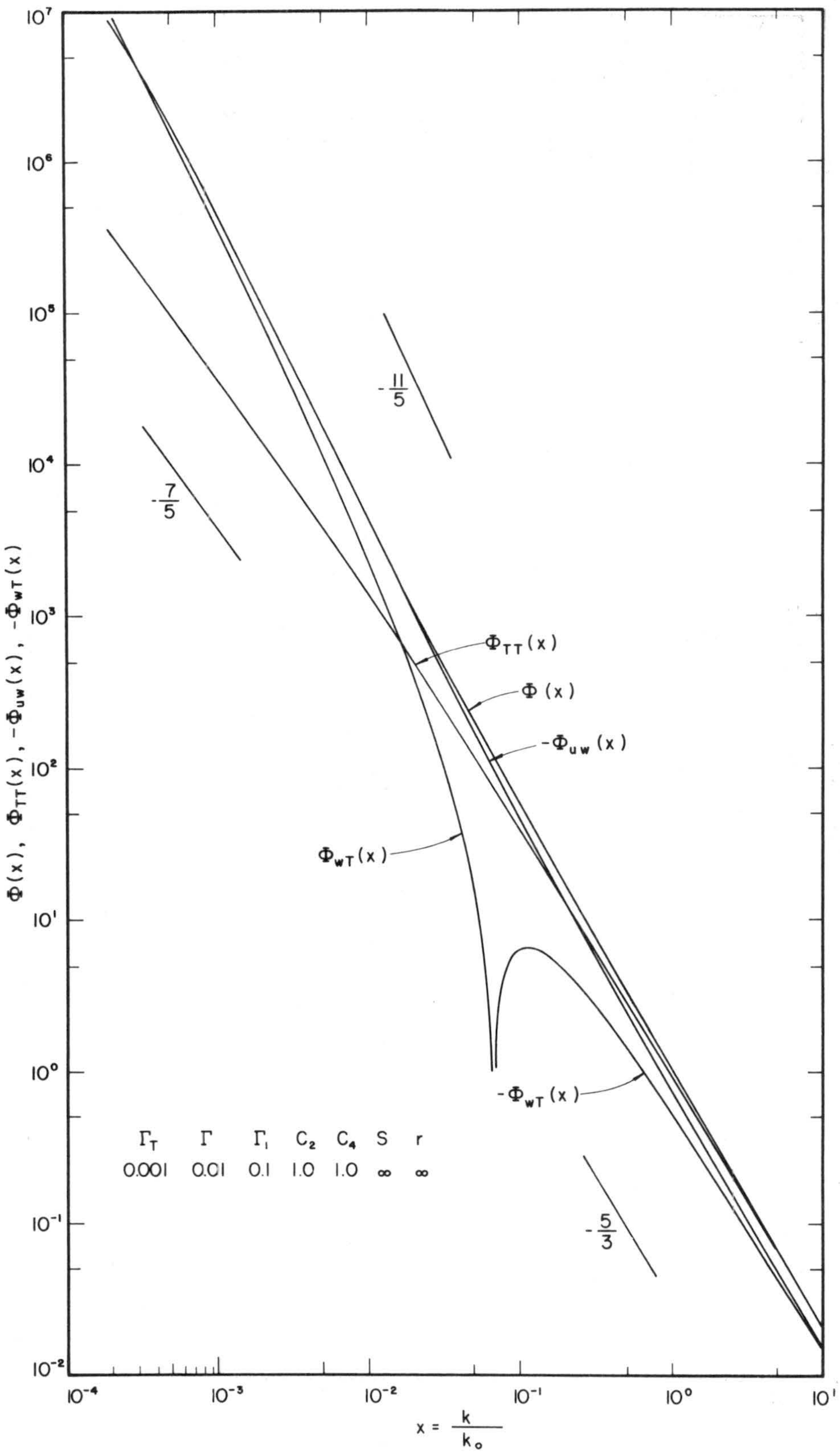

Fig. 12 Turbulence spectra $\Phi(x)$, $\Phi_{TT}(x)$, $\Phi_{uw}(x)$, $\Phi_{wT}(x)$ vs x

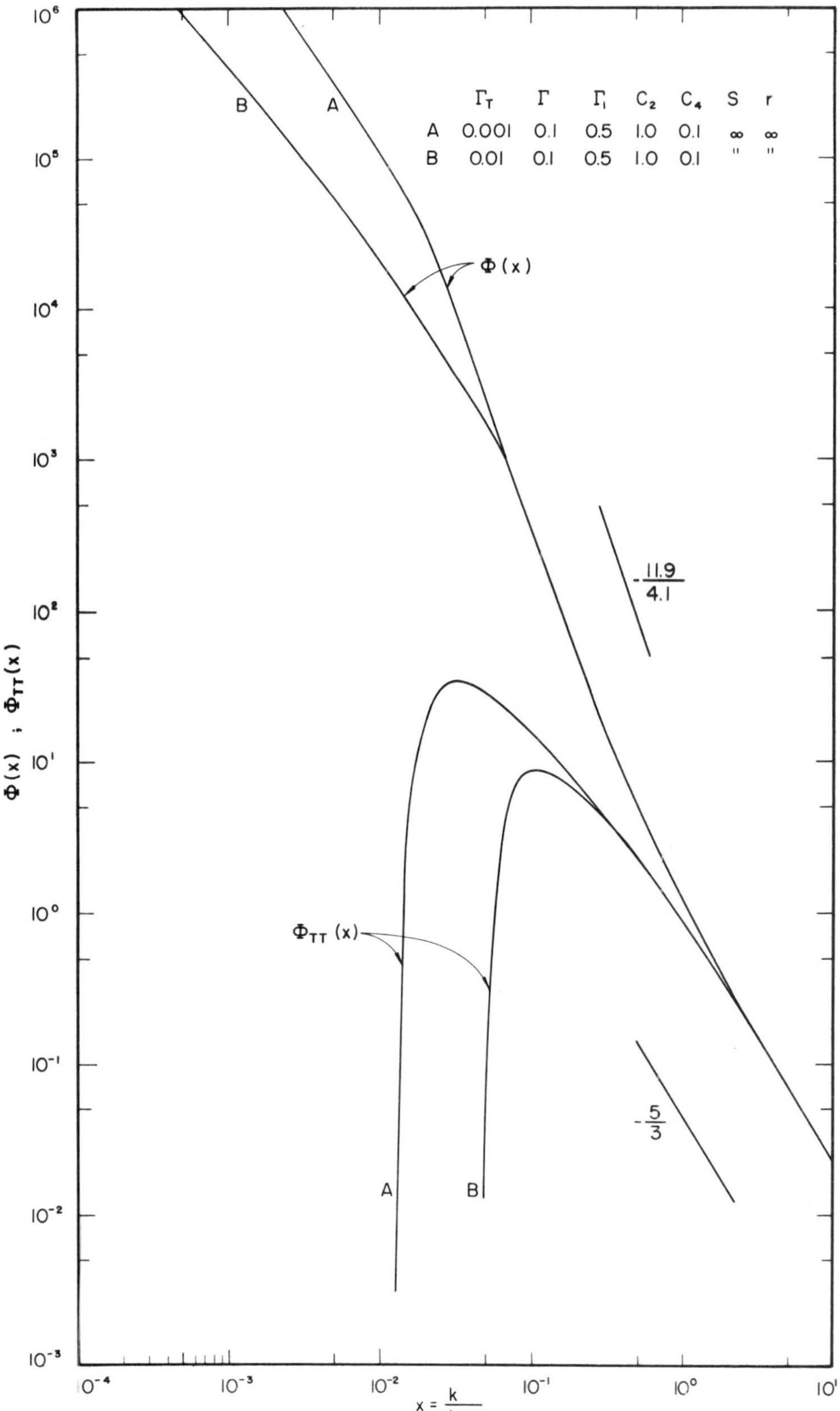

Fig. 13 Velocity and temperature spectra of stably stratified flows varied as a function of Γ_T

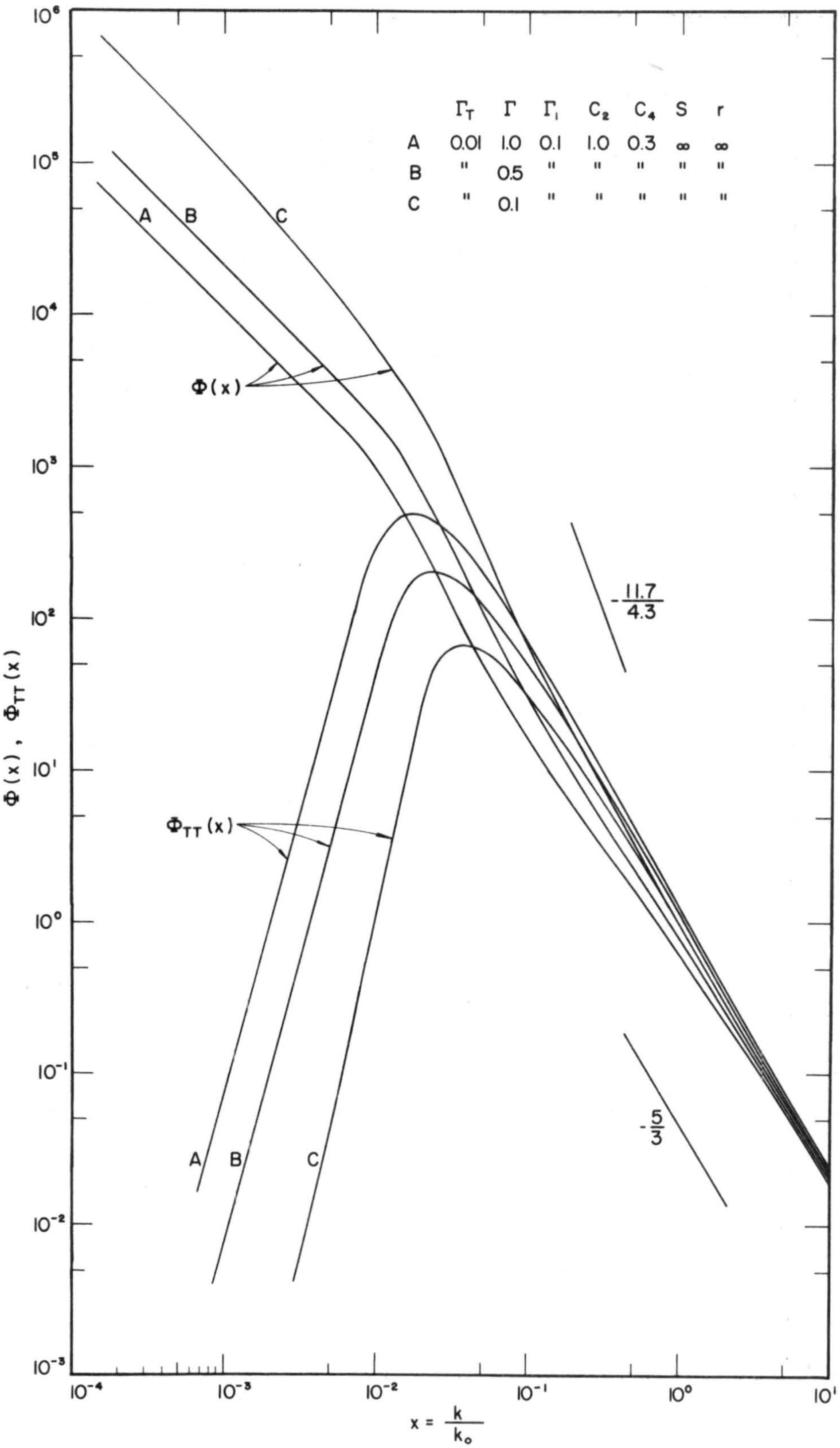

Fig. 14 Velocity and temperature spectra of stably stratified flows varied as a function of Γ

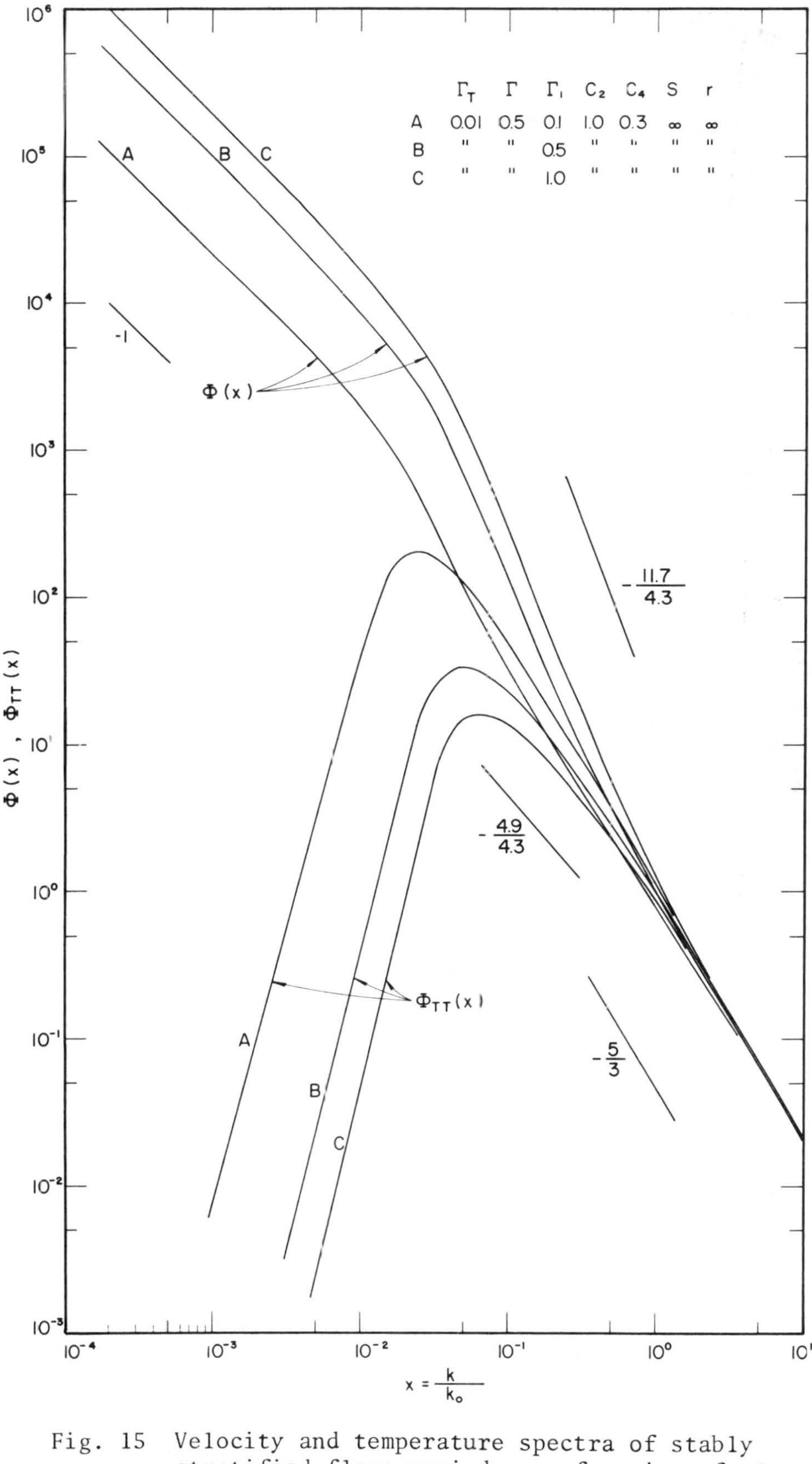

Fig. 15 Velocity and temperature spectra of stably stratified flows varied as a function of Γ_1

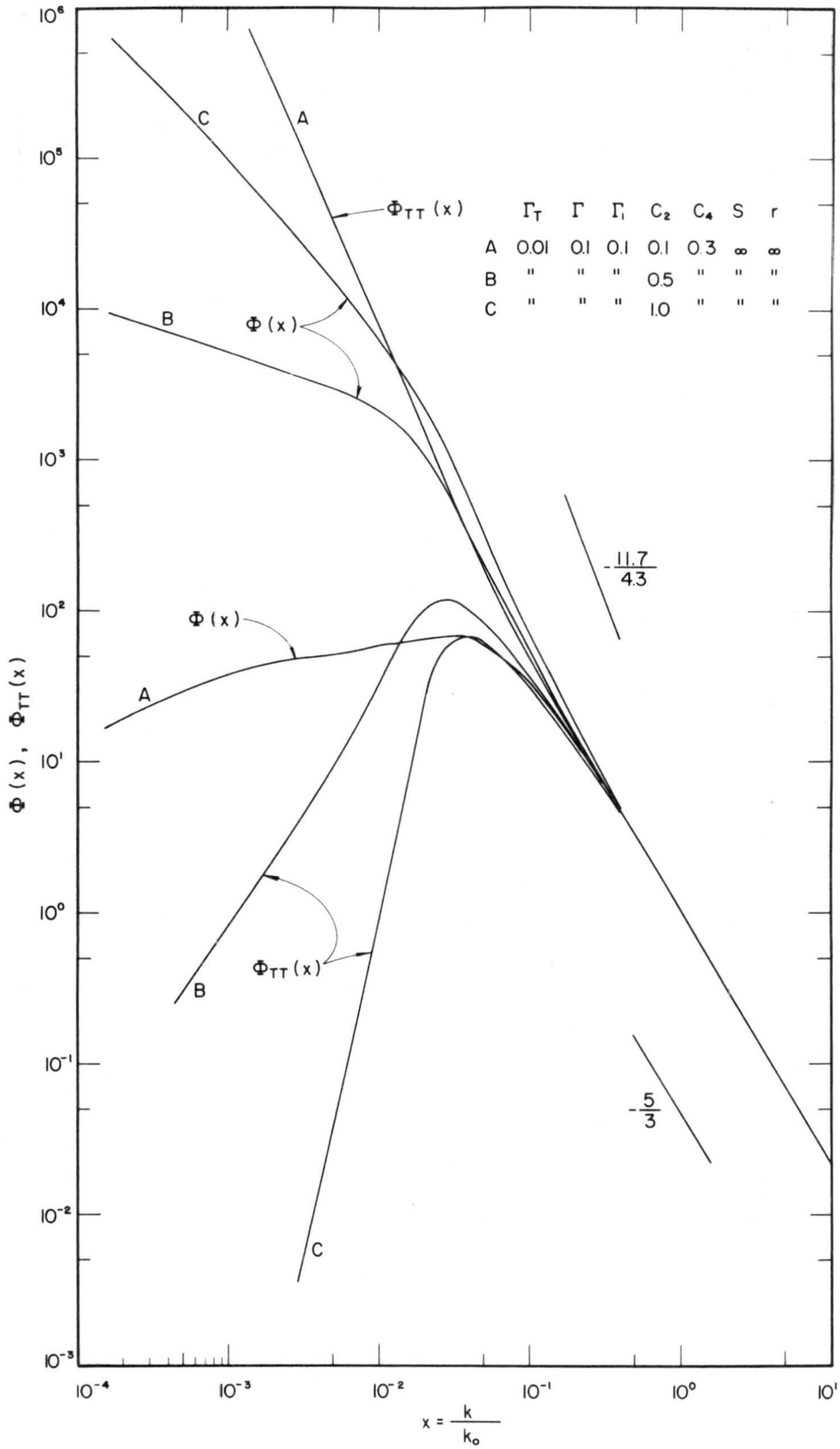

Fig. 16 Velocity and temperature spectra of stably stratified flows varied as a function of C_2

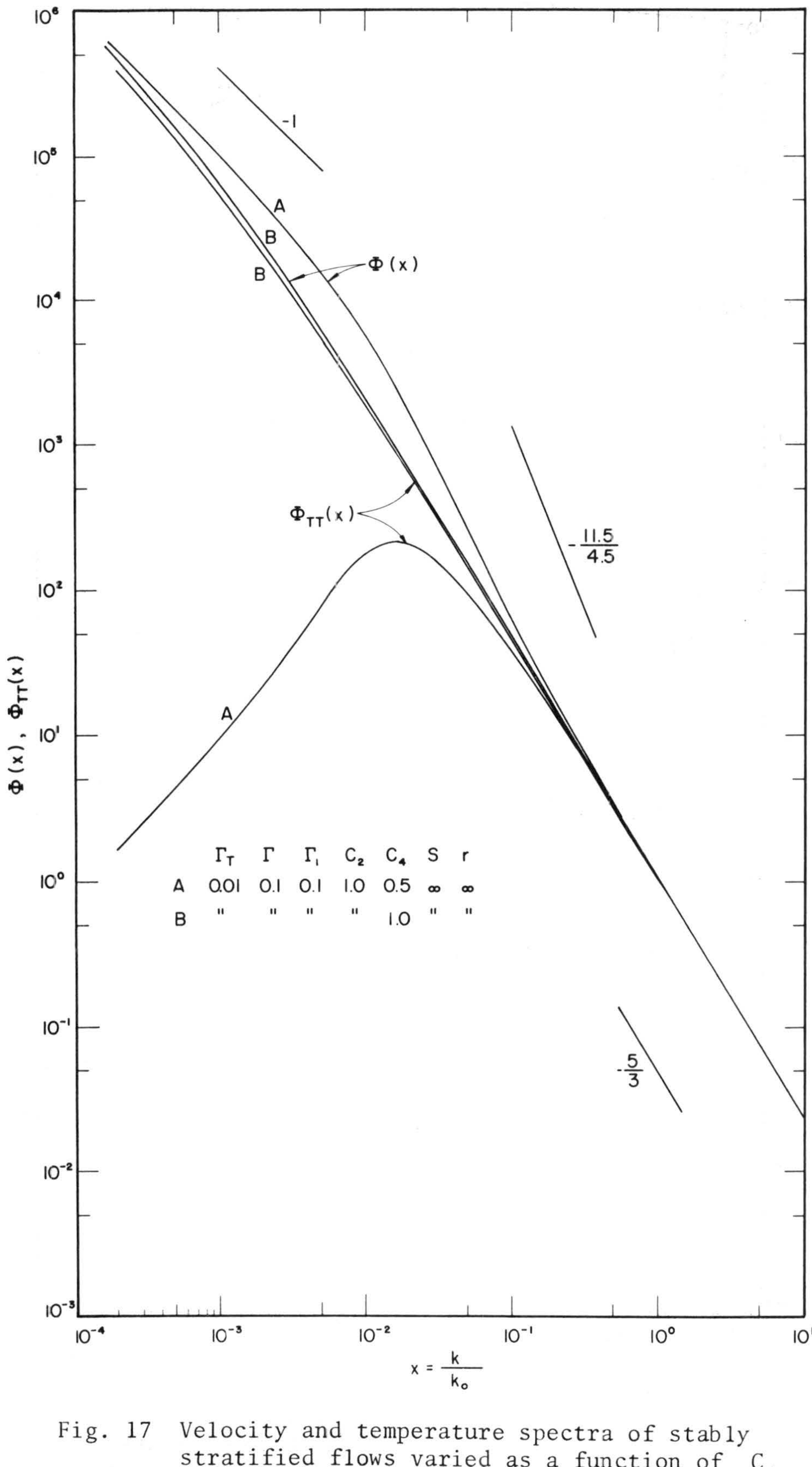

Fig. 17 Velocity and temperature spectra of stably stratified flows varied as a function of C_4

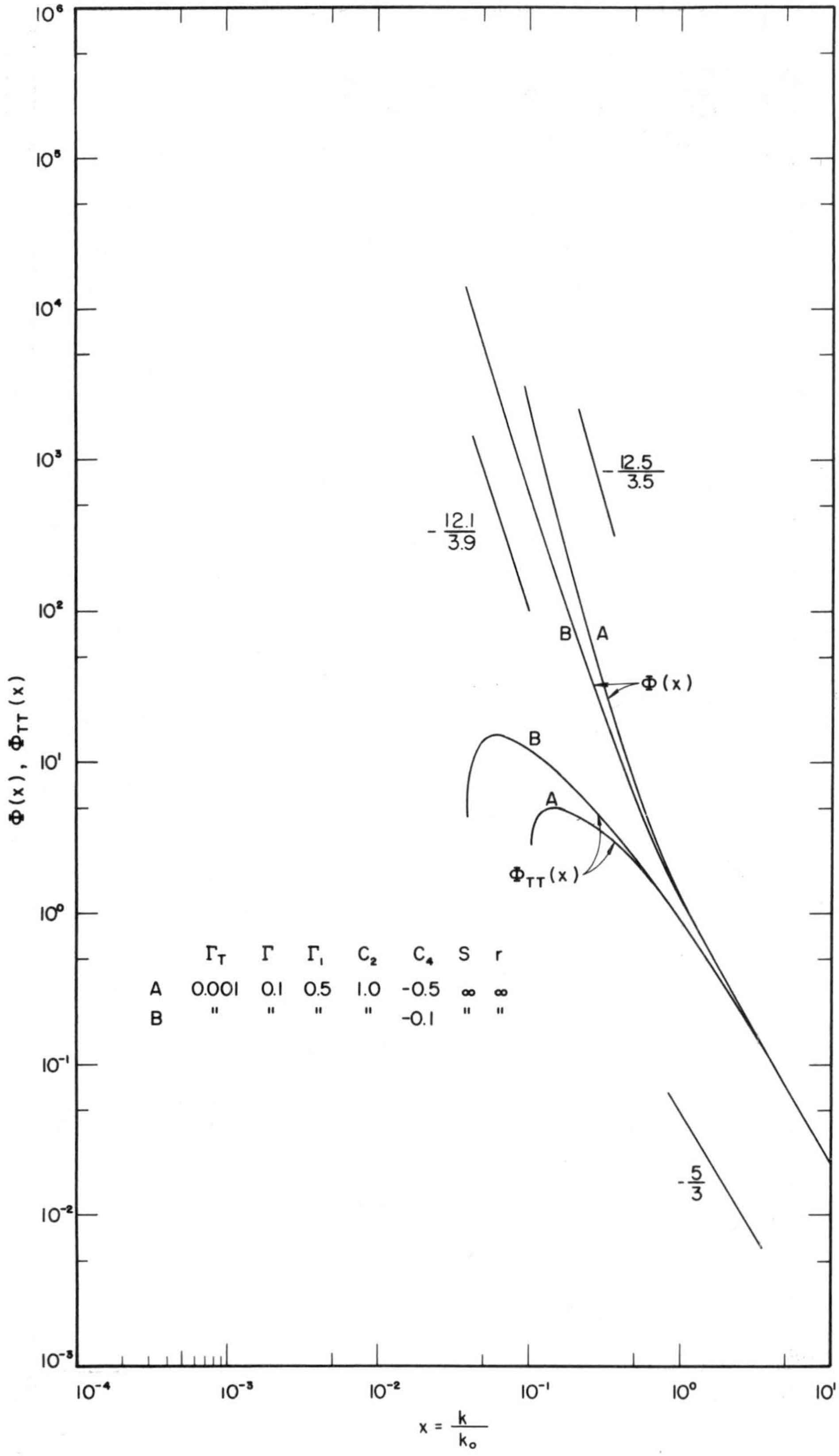

Fig. 18 Velocity and temperature spectra of stably stratified flows varied as a function of C_4

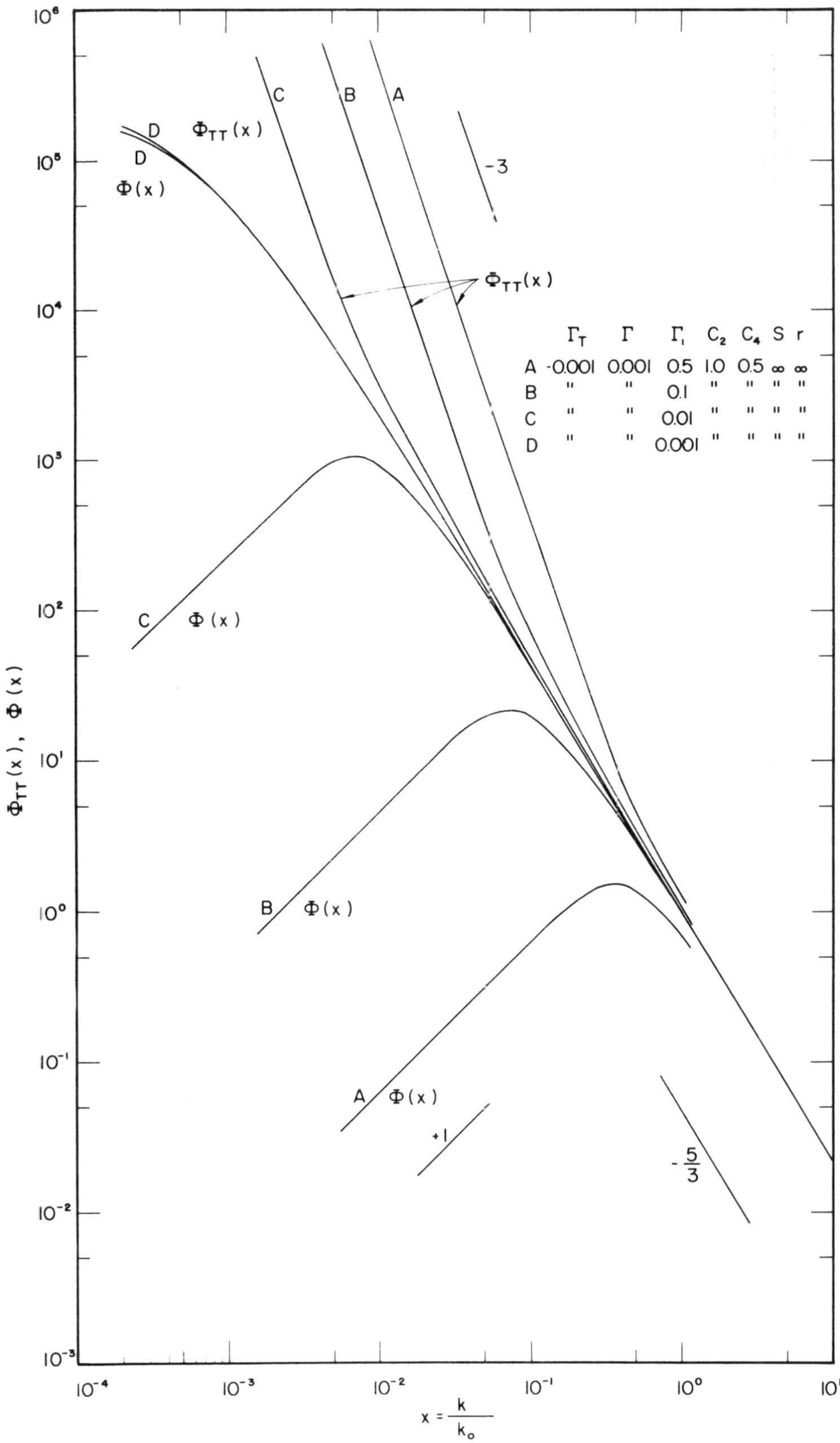

Fig. 19 Velocity and temperature spectra of unstably stratified flows varied as a function of Γ_1

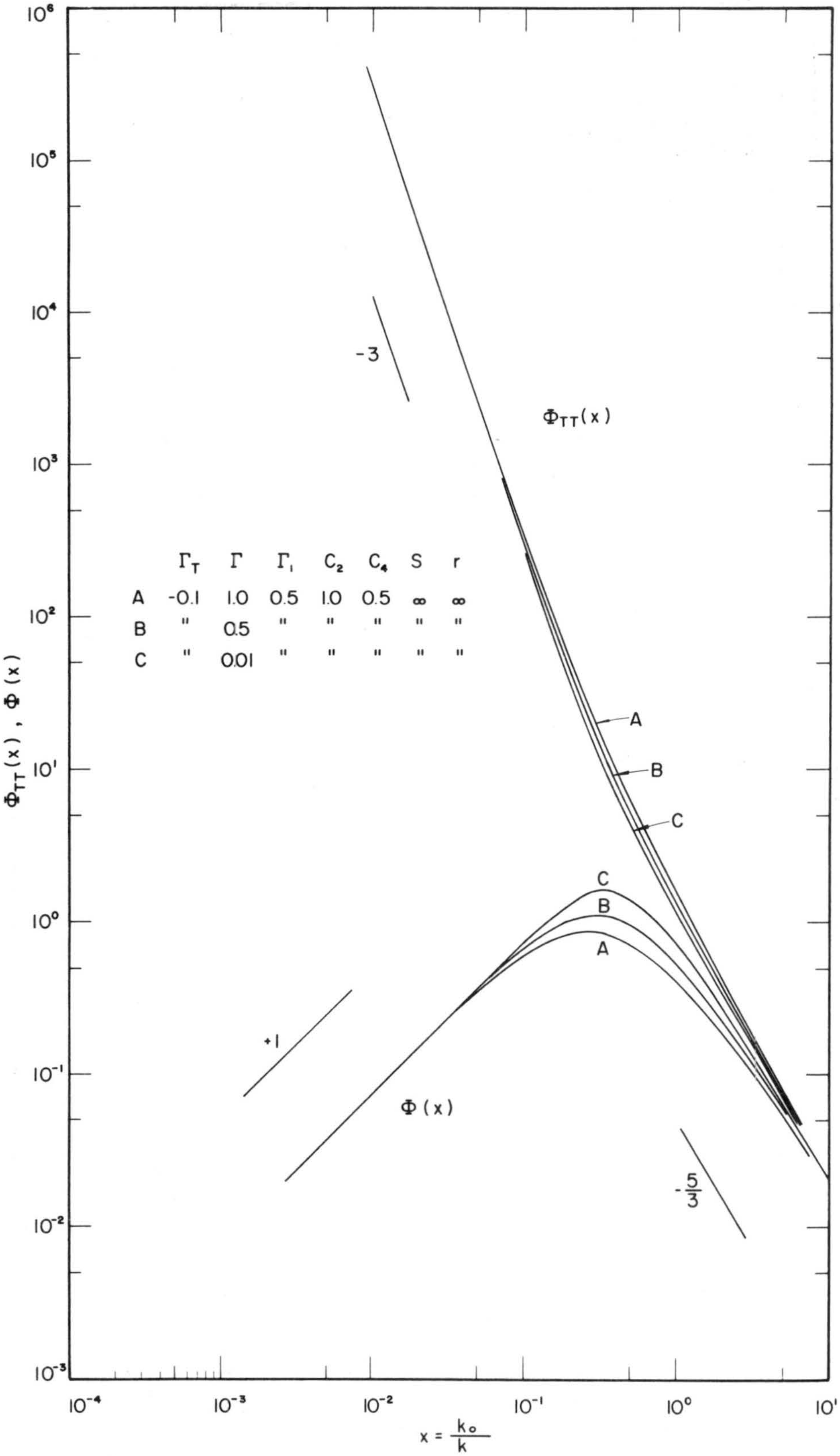

Fig. 20 Velocity and temperature spectra of unstably stratified flows varied as a function of Γ

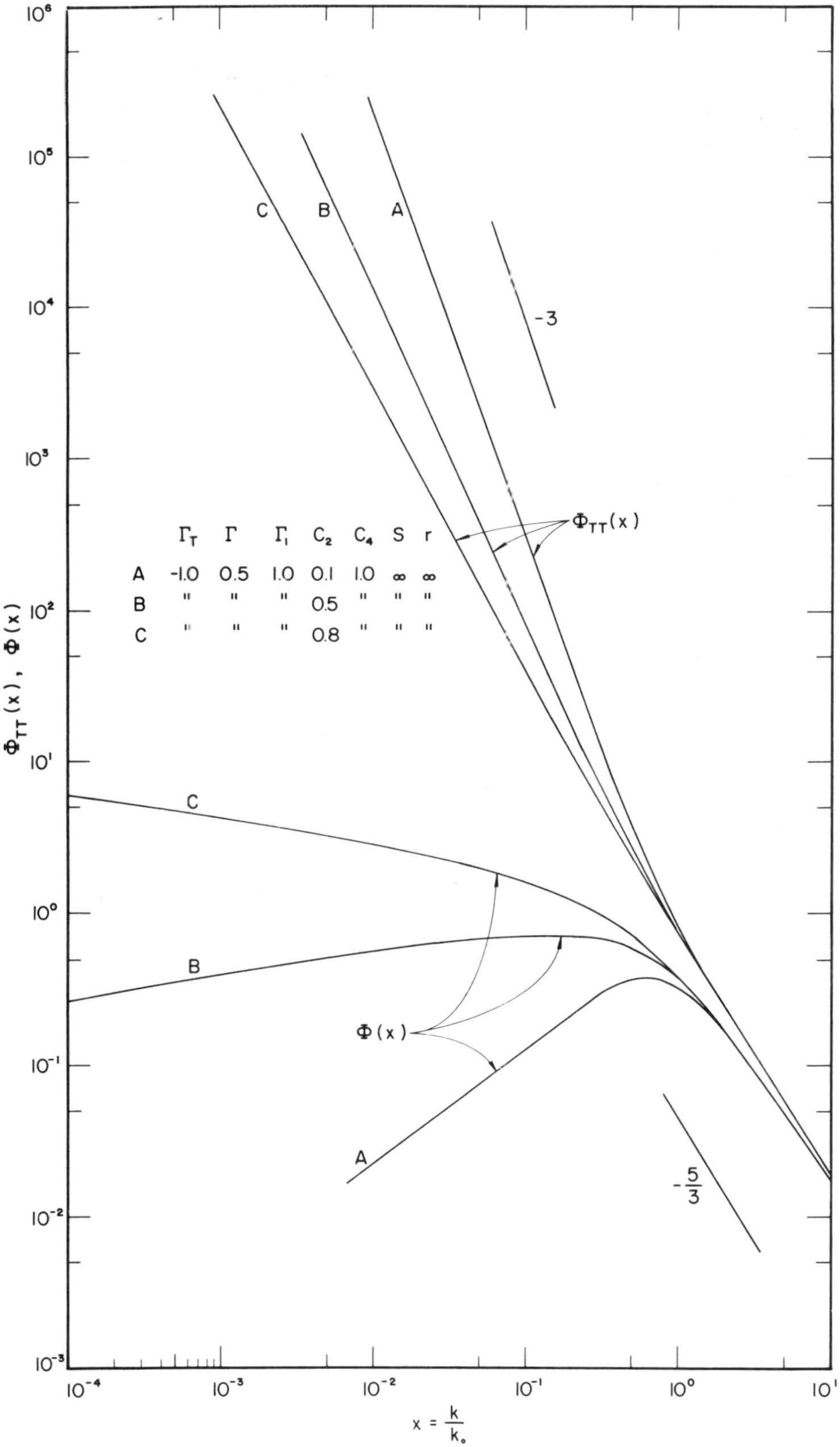

Fig. 21 Velocity and temperature spectra of unstably stratified flows varied as a function of C_2

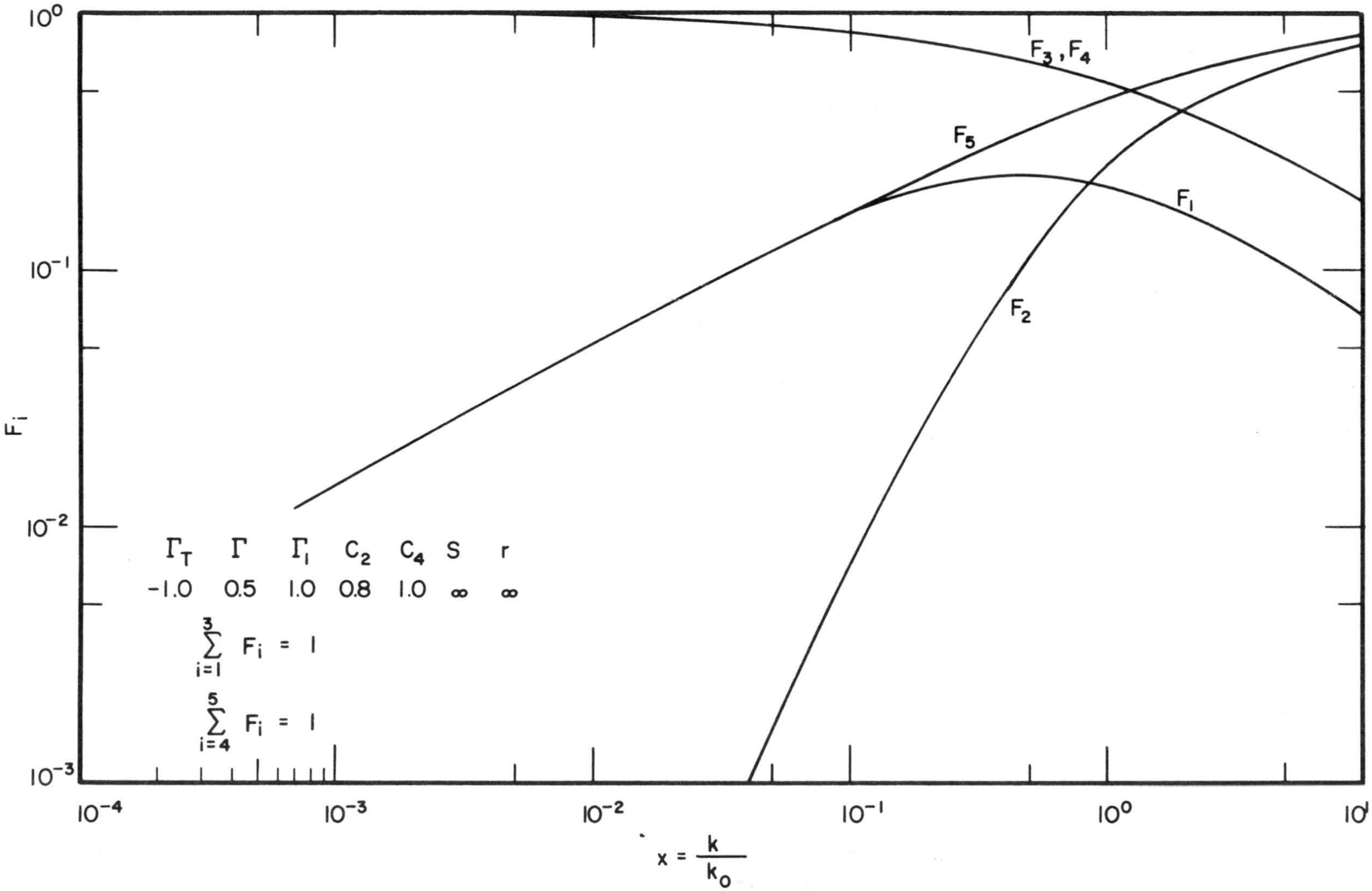

Fig. 22 Distributions of energy production, transfer, and input by buoyancy force, and temperature inhomogeneity production and transfer of unstably stratified flow across wave numbers

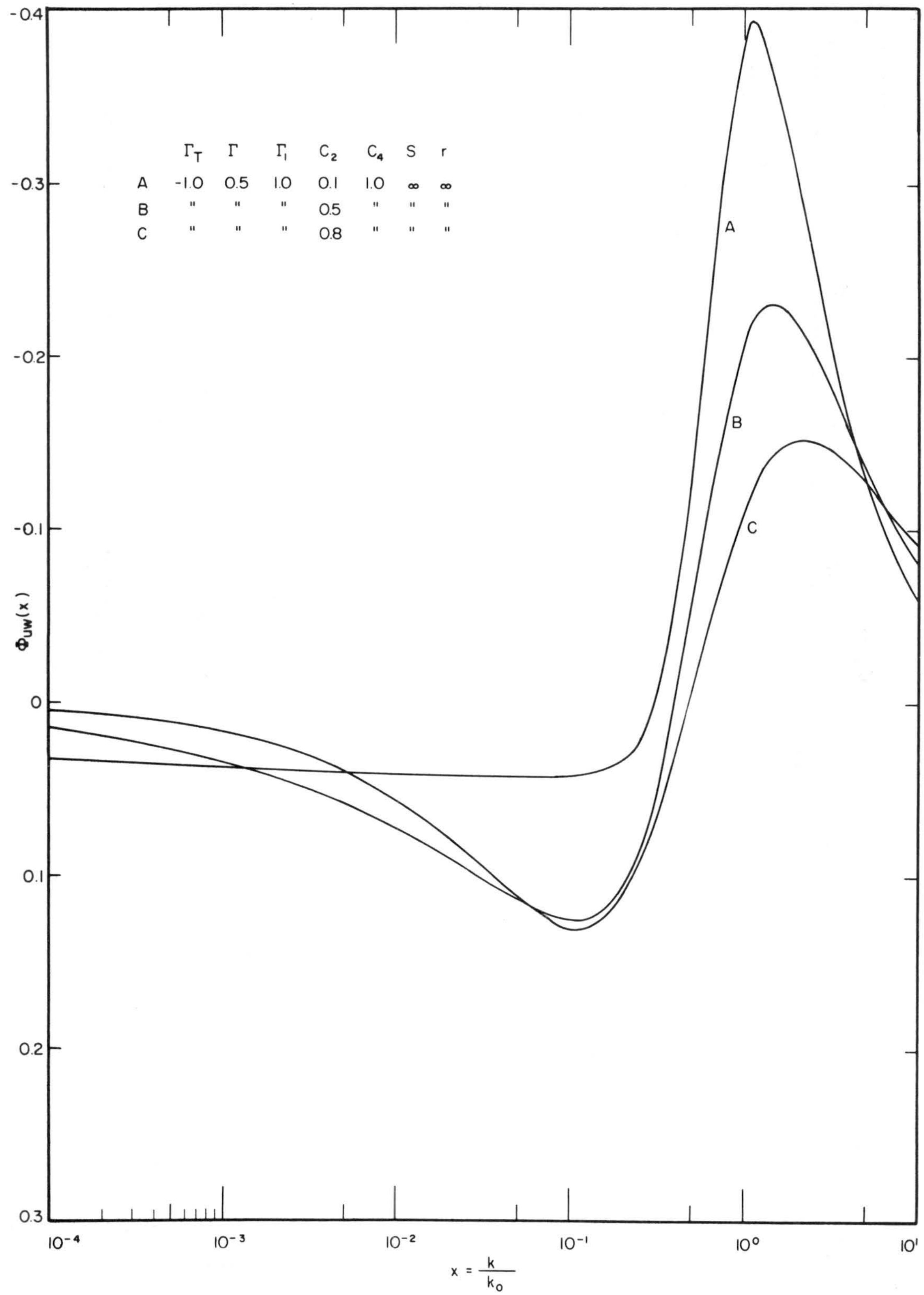

Fig. 23 $x\,\Phi_{uw}(x)$ vs x with varied C_2 for unstably stratified flows

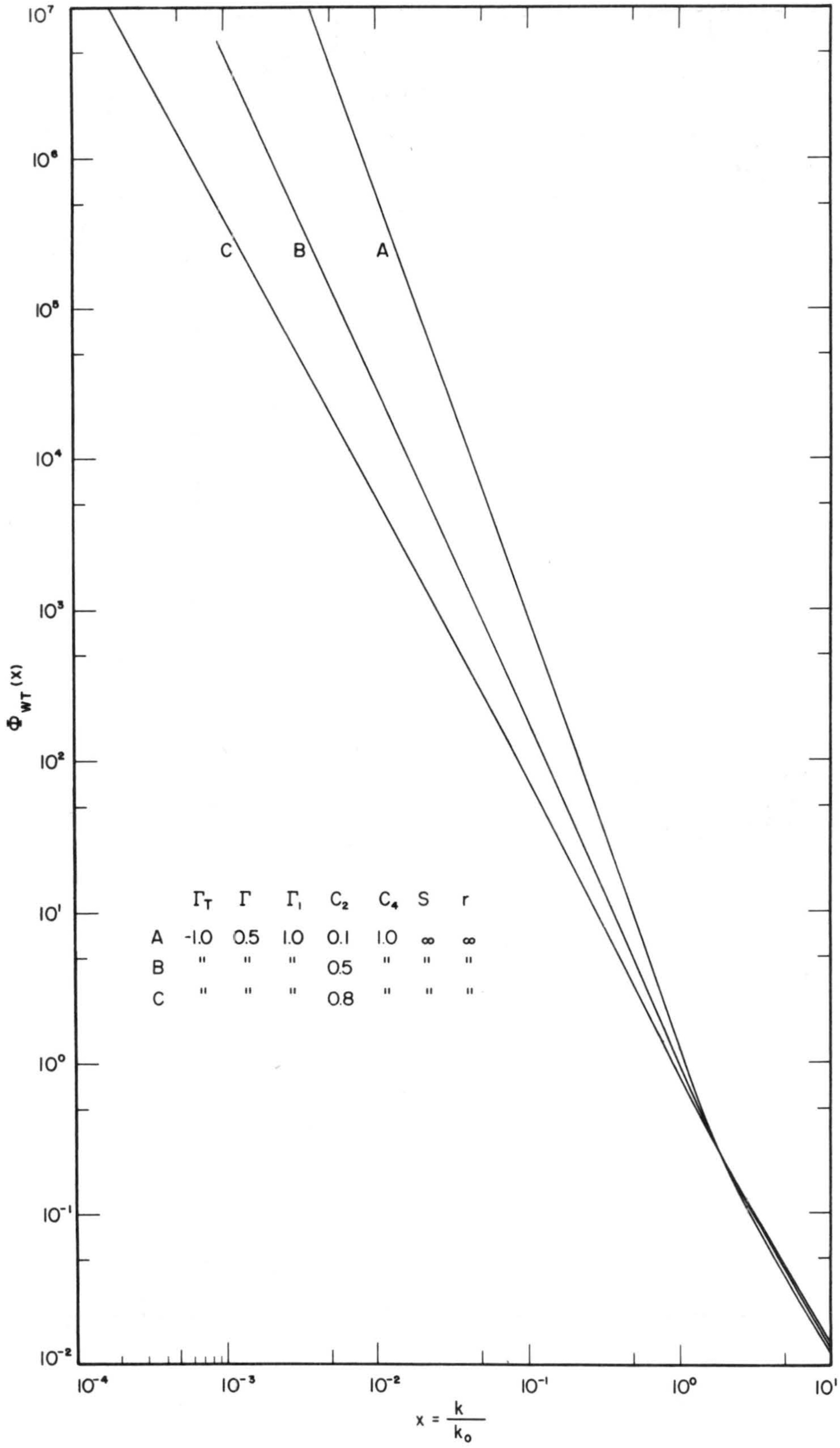

Fig. 24 Turbulence spectrum of vertical heat flux of unstably stratified flow varied as a function of C_2

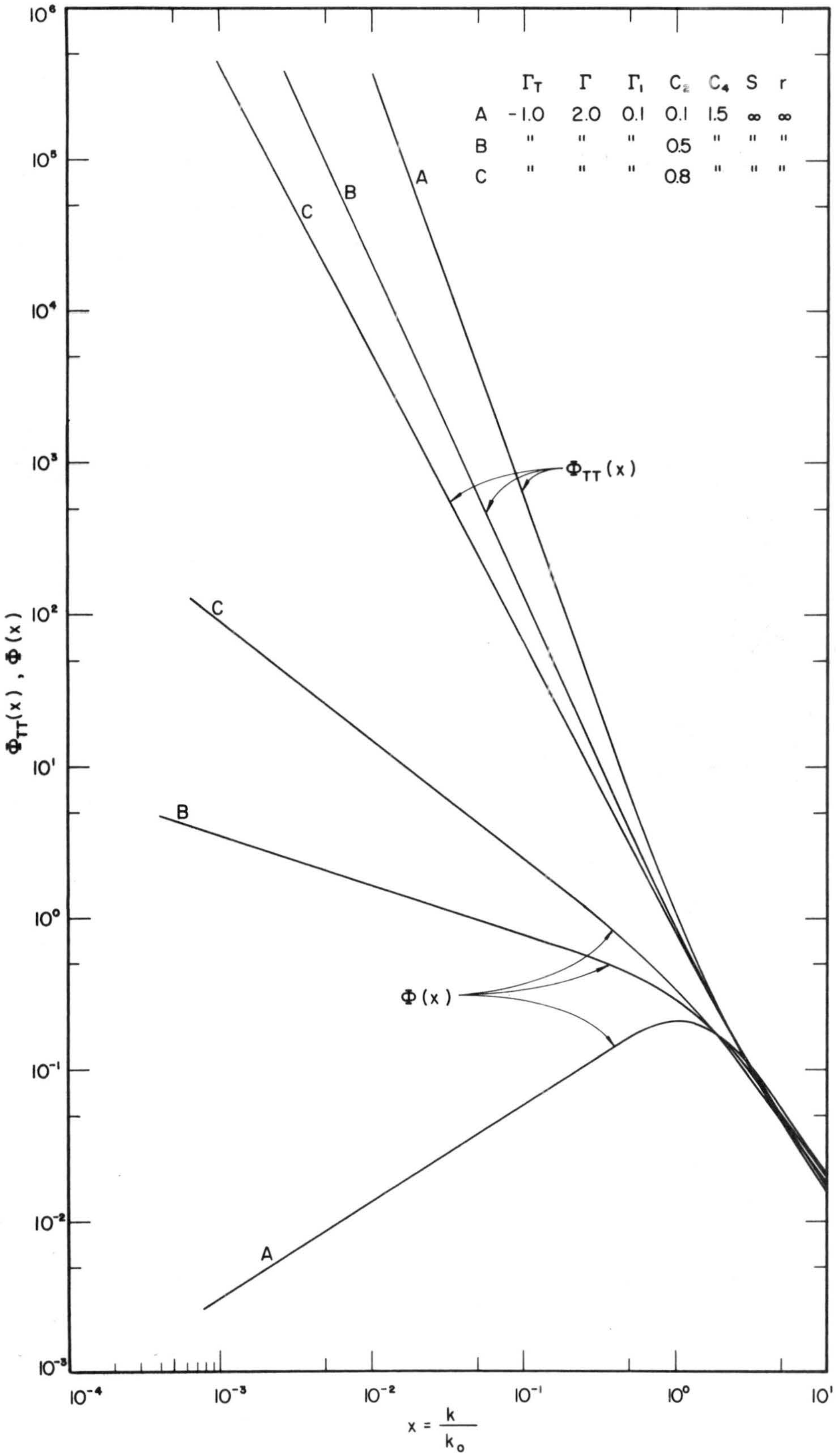

Fig. 25 Velocity and temperature spectra of unstably stratified flows varied as a function of C_2

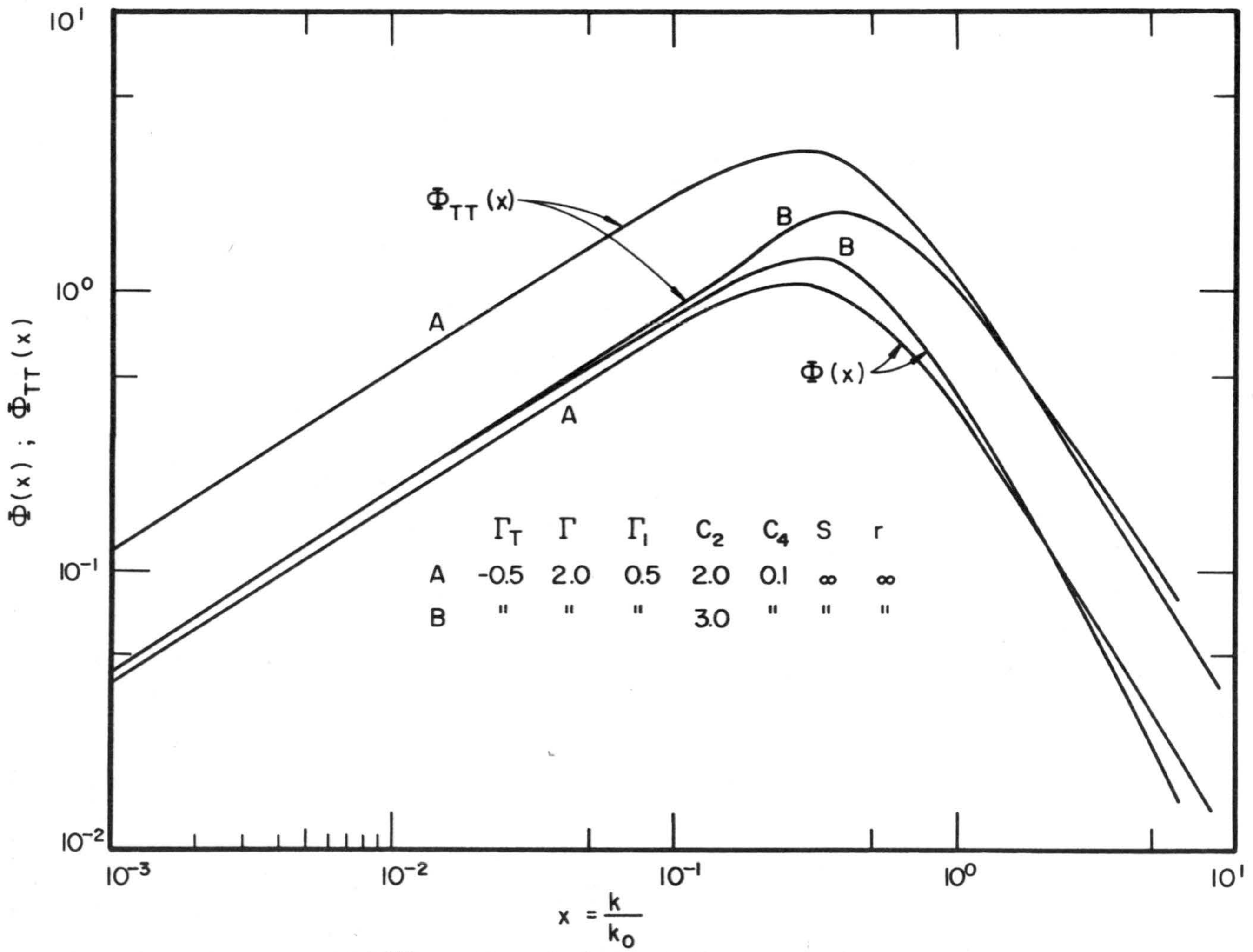

Fig. 26 Velocity and temperature spectra of unstably stratified flows varied as a function of C_2

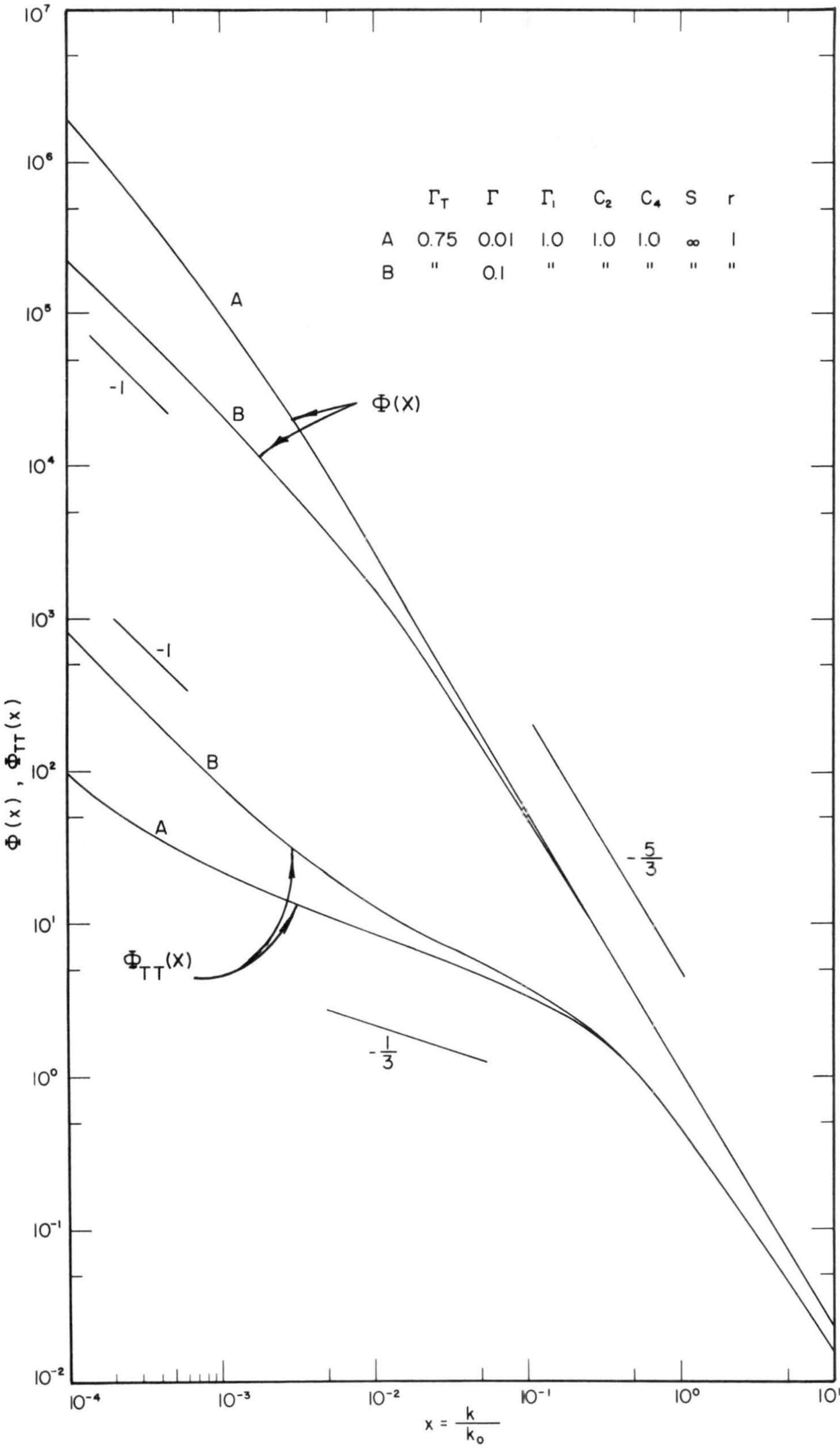

Fig. 27 Velocity and temperature spectra of stably stratified flows varied as a function of Γ

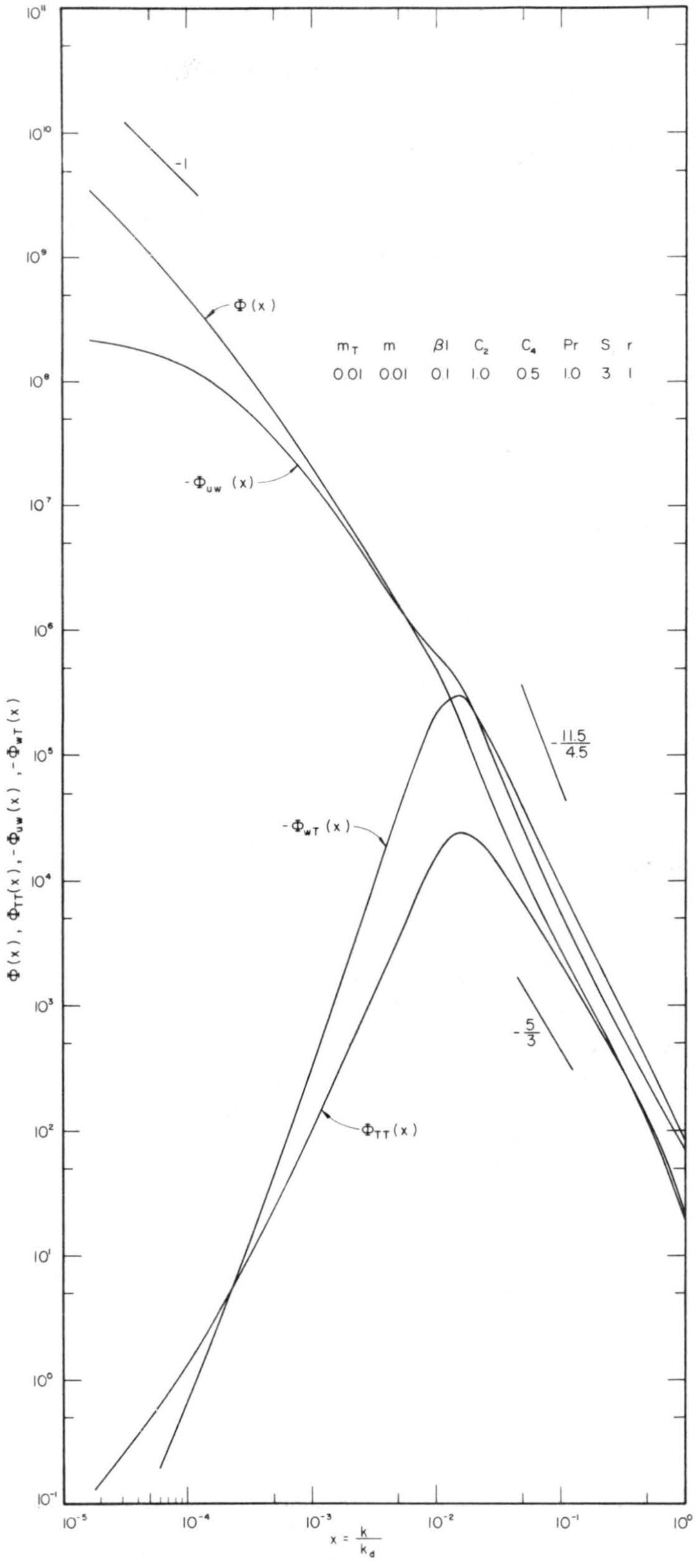

Fig. 28 Turbulence spectra of a stably stratified viscous shear flow

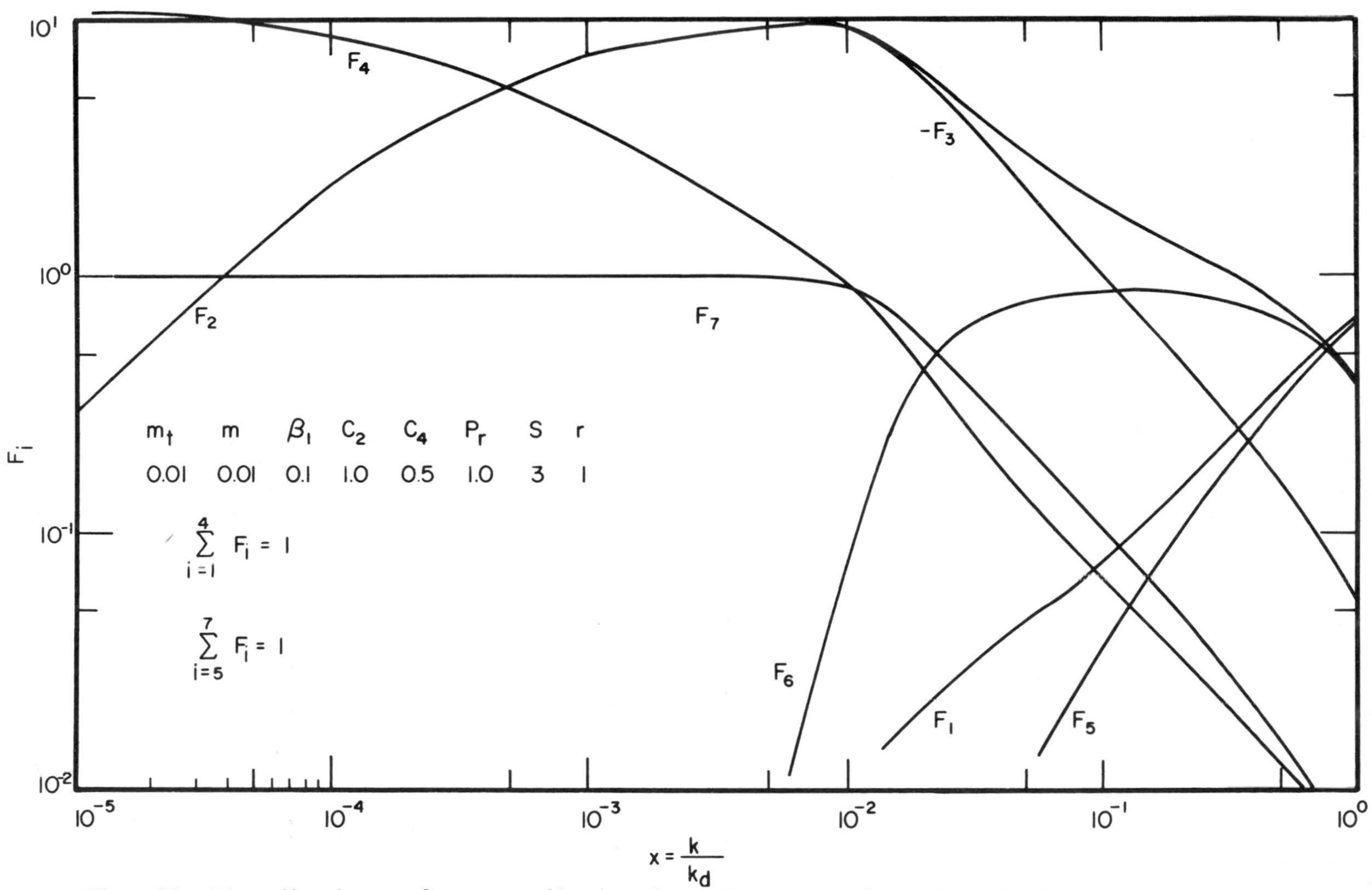

Fig. 29 Distributions of energy dissipation F_1 , transfer F_2 , drainage by buoyancy F_3 , and production F_4 , and temperature inhomogeneity dissipation F_5 , transfer F_6 , production F_7 , of a stably stratified viscous shear flow across wave numbers

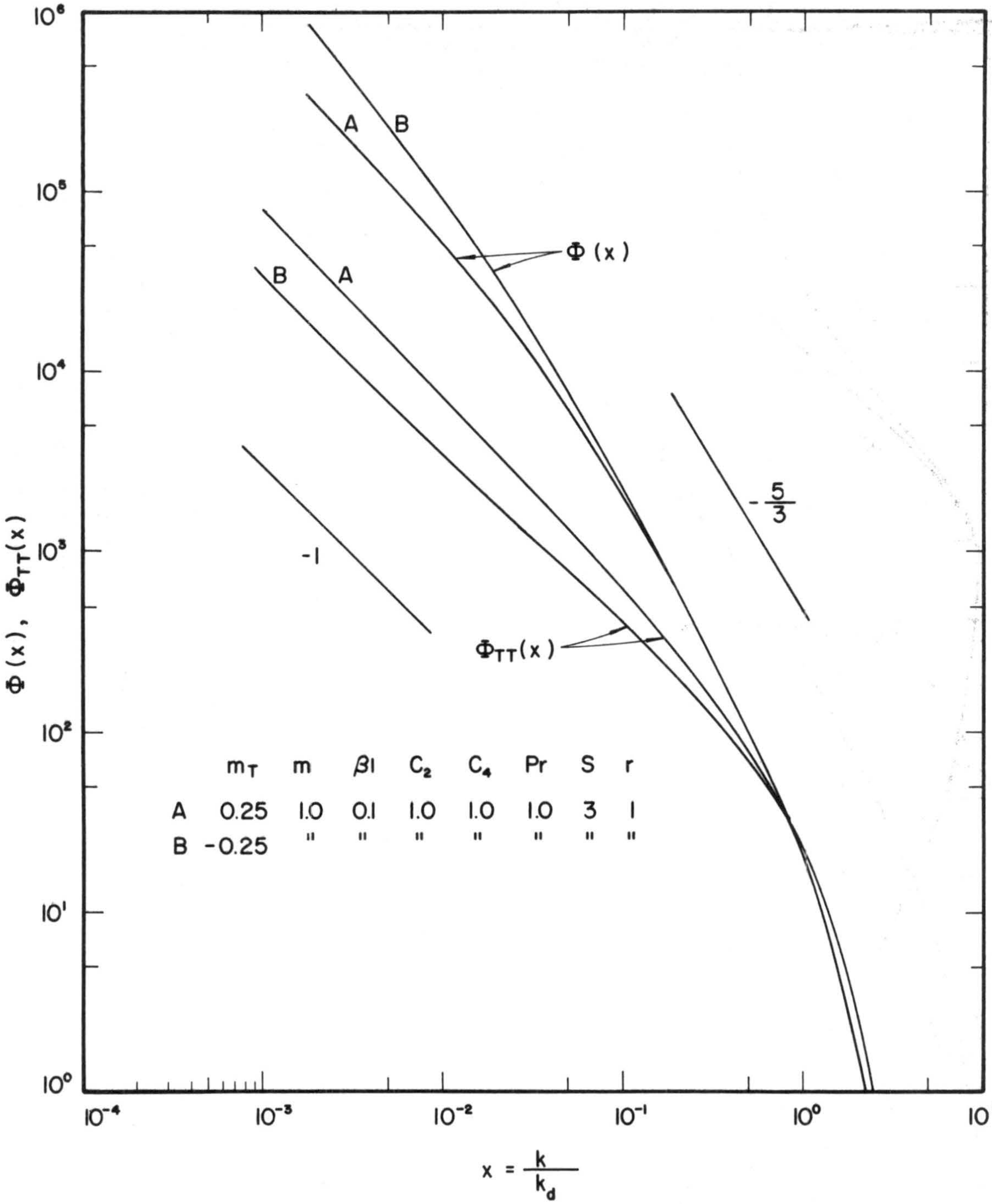

Fig. 30 Velocity and temperature spectra of thermally stratified viscous shear flows

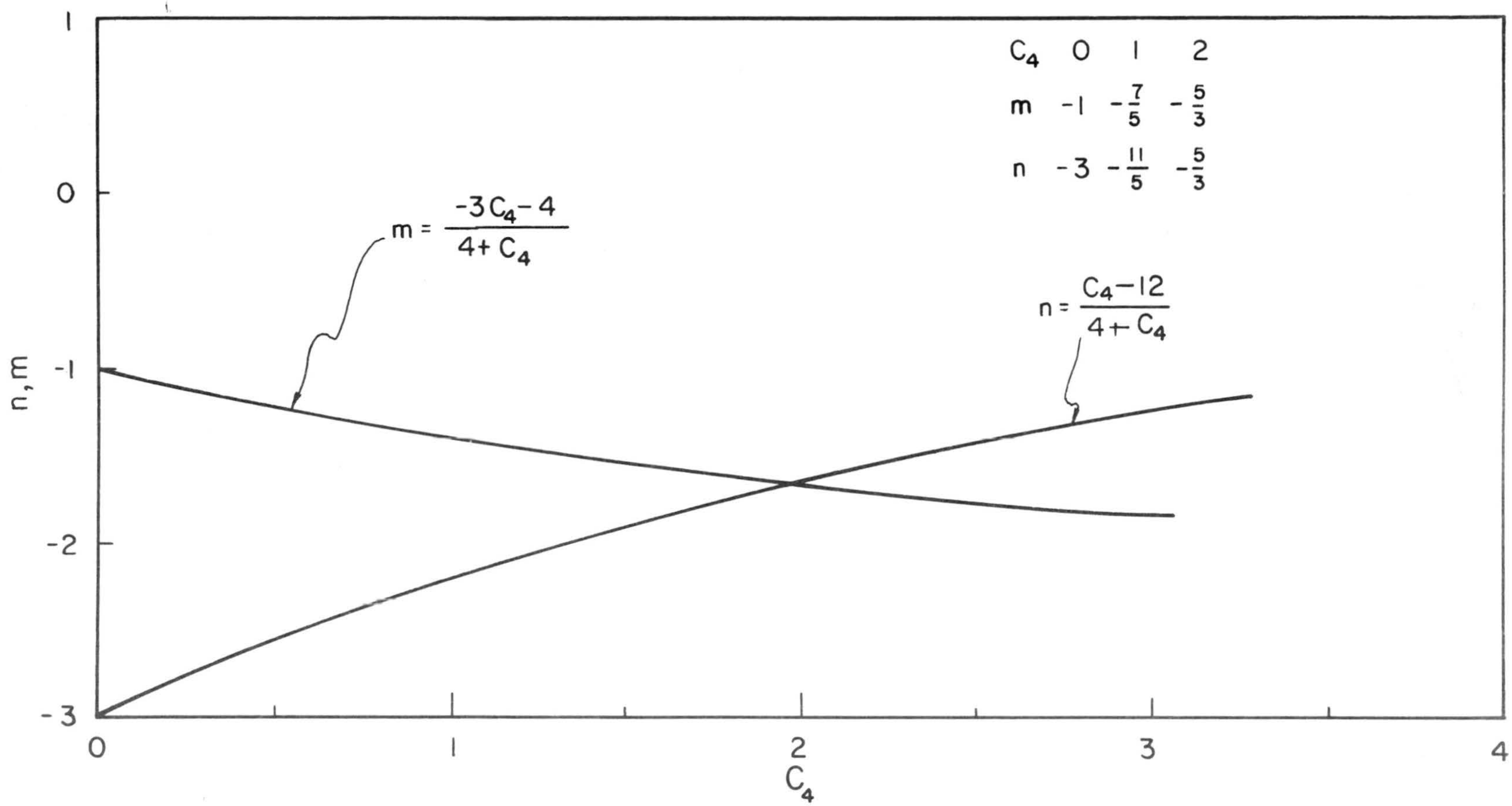

Fig. 31 Asymptotic power law varied as C_4 for velocity and temperature spectra in the buoyancy subrange of stably stratified flow

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A generalized eddy-viscosity approximation is used to study the turbulence spectra of thermally stratified shear flows. Numerical solutions of turbulence spectra are investigated.

Modified hypothesis in the buoyancy subrange of stably stratified flow is given.

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