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MATHEMATICAL SIMULATIONS FOR BETTER AQUIFER MANAGEMENT*

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ABSTRACT

Manipulation of groundwater storage in conjunction with surface water supplies often requires simultaneous consideration of varied aquifer hydraulic and geometric characteristics, highly variable pumping patterns in time and space, imperfectly connected bodies of surface water and non-deterministic natural recharge. In addition, legal, economic and social conditions impose constraints which must be considered in aquifer operation and management. This paper describes mathematical modeling and computer analysis techniques that allow consideration of the many varied and changing factors mentioned above. Application is made to specific aquifer management problems.

RÉSUMÉ

Les manipulations de l'emmagasinement d'eaux souterraines en conjonction avec des disponibilités d'eau de surface exige souvent la prise en considération simultanée de caractéristiques hydrauliques et géométriques de la nappe, données de pompages hautement variables dans le temps et l'espace, volumes d'eau de surface imparfaitement reliés et rechanges naturels peu prévisibles. De plus, des conditions légales, économiques et sociales imposent des conditions qui doivent être prises en considération dans l'aménagement et l'utilisation de la nappe. La contribution décrit les techniques de l'utilisation de modèles mathématiques et de l'analyse par conjusteurs, qui permettent la prise en considération des nombreux facteurs variables mentionnés ci-dessus. L'application de ces principes est faite aux problèmes d'aménagement d'une nappe donnée.

Increasing demands upon limited water supplies make it imperative that water officials efficiently manage the total water supplies of an area. Groundwater aquifer management should normally be coordinated and integrated with surface water management in order to achieve maximum beneficial use. This paper discusses mathematical and computer techniques developed for simulating complex groundwater-surface water systems so as to achieve better water management.

SELECTION OF SIMULATION TECHNIQUE

Adequate modeling of complex systems depends upon the detail and accuracy required of the results. As pressures on water supplies increase, there is an increasing need for more sophisiticated simulation and analysis techniques for projection into the future and development of plans for optimum management. The authors have sought a technique to meet the following requirements:

1) Simulate non-steady conditions;
2) Simulate at least two space dimensions;
3) Simulate physiographical influences (impermeable, semipermeable, and hydraulic boundaries) accurately without undue idealization;

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4) Simulate non-linear flow conditions caused by a varying transmissibility in unconfined aquifers;
5) Simulate in both time and space, the simultaneous effects of many influences on the system, such as precipitation, seepage from irrigation activities, withdrawals by wells and phreatophytes, etc.;
6) Be easily modified (physical and hydraulic characteristics) during verification stage, and convenient for studying effects of many different operational conditions in projecting into the future;
7) Be flexible and readily adaptable to different study areas by merely introducing new geologic and hydrologic data;
8) Provide immediately usable output of results;
9) Utilize equipment and personnel readily available to water administrators; and,
10) Provide rapid analyses at reasonable cost.

It is recognized that many of the above requirements are somewhat subjective. For example, a reasonable cost of modeling one system may be completely unreasonable for a similar system at another location. However, after study and consulation, the approach using a mathematical model and digital computer solution was chosen as meeting the above requirements under most conditions. A literature review indicates that in recent years the petroleum industry ([1,2,3,4]) and the California Department of Water Resources ([5,6]) have begun to use mathematical simulation and digital computer solutions for reservoir management.

Mathematical basis for model

The non-linear partial differential equation describing transient two-dimensional flow in a saturated porous medium may be derived from the mass continuity equation and Darcy's law and can be written as,

$$\frac{\partial}{\partial x}\left(Kh\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial y}\left(Kh\frac{\partial H}{\partial y}\right) = S\frac{\partial H}{\partial t} + \frac{Q}{\delta x \delta y}, \tag{1}$$

where

K = hydraulic conductivity,
h = saturated thickness of aquifer,
H = potential, referred to an established datum,
S = storage coefficient or effective porosity,
Q = net groundwater withdrawal,
x, y = space dimensions, and
t = time dimension.

This equation is subject to the Dupuit-Forchheimer assumptions and should be used with these limitations in mind.

Finite Difference Equation

Equation (1) has no general solution, therefore a finite difference approximation is utilized to allow a numerical solution with a digital computer. Application of the finite difference approach requires subdivision of the study area into a system of finite grids. Approximating the differentials of equation (1) by first-order finite difference expressions leads to the following general expression describing the potentials at time level $t+1$.

$$\beta^{t+1}_{i-\frac{1}{2},j}\left[\frac{Kh_{i-\frac{1}{2},j}(H^{t+1}_{i-1,j}-H^{t+1}_{i,j})}{(\Delta x)^2}\right]+\beta^{t}_{i-\frac{1}{2},j}\left[\frac{Kh_{i-\frac{1}{2},j}(H^{t}_{i-1,j}-H^{t}_{i,j})}{(\Delta x)^2}\right]$$

$$+\beta^{t+1}_{i+\frac{1}{2},j}\left[\frac{Kh_{i+\frac{1}{2},j}(H^{t+1}_{i+1,j}-H^{t+1}_{i,j})}{(\Delta x)^2}\right]+\beta^{t}_{i+\frac{1}{2},j}\left[\frac{Kh_{i+\frac{1}{2},j}(H^{t}_{i+1,j}-H^{t}_{i,j})}{(\Delta x)^2}\right]$$

$$+\beta^{t+1}_{i,j-\frac{1}{2}}\left[\frac{Kh_{i,j-\frac{1}{2}}(H^{t+1}_{i,j-1}-H^{t+1}_{i,j})}{(\Delta y)^2}\right]+\beta^{t}_{i,j-\frac{1}{2}}\left[\frac{Kh_{i,j-\frac{1}{2}}(H^{t}_{i,j-1}-H^{t}_{i,j})}{(\Delta y)^2}\right]$$

$$+\beta^{t+1}_{i,j+\frac{1}{2}}\left[\frac{Kh_{i,j+\frac{1}{2}}(H^{t+1}_{i,j+1}-H^{t+1}_{i,j})}{(\Delta y)^2}\right]+\beta^{t}_{i,j+\frac{1}{2}}\left[\frac{Kh_{i,j+\frac{1}{2}}(H^{t}_{i,j+1}-H^{t}_{i,j})}{(\Delta y)^2}\right]$$

$$= S\frac{(H^{t+1}_{i,j}-H^{t}_{i,j})}{\Delta t}+\frac{Q^{t+\frac{1}{2}}_{i,j}}{\Delta x\Delta y} \qquad (2)$$

The β's are flux weighting factors, the values of which depend upon the computational technique used. The i, j notation refers to the grid point for which a particular equation is written and the superscripts represent the time level of computation. Figure 1 illustrates the physical significance of each of the flux terms.

Further discussion of equation (2) and computational techniques applied to petroleum reservoirs and heat conduction problems is presented by Quon *et al.* (7,8).

Computational Techniques

Many computational techniques have been considered, five of which are briefly discussed here. These techniques differ only in the method by which the average potential gradients are approximated.

FDE. The Forward Difference Explicit technique is derived from equation (2) by setting all β^{t+1} weighting factors equal to zero and the β^{t} coefficients equal to 1.0. Thus, the differentials at each time step are assumed to be extrapolative. Computations are easily performed by this method, since only one unknown exists at time level $t+1$ in each equation. Depending upon the size of Δt, Δx and Δy, computations by this method may result in stability problems.

ADEP. The Alternating Direction Explicit Procedure was developed by dividing the left-hand side of equation (2) into two parts. Each resulting equation includes two terms at the t-level (one in the x- and one in the y-direction) and two terms at the $t+1$ level in reverse x and y directions. In this procedure one time step, Δt, is equal to $2t$. By proper choice of the starting point with relation to boundary conditions, this method results in only one unknown at each grid point. Credit for development of this procedure has generally been given to Saul'yev (9) and Larkin (10).

CNI. The Crank-Nicholson Implicit method places equal weight ($\beta = 0.5$) on all terms of equation (2). This method gives a sound representation of the physical situation, but poses considerable computational problems. If the model is represented by N and M grids in the x- and y-directions respectively, this method requires solution of an $NM \times NM$ matrix for each time step.

BDI. In the Backward Difference Implicit method, all weight is placed at the $t+1$ level (all β^{t} coefficients are set equal to zero). This interpolative method, like the Crank-Nicholson technique, results in five unknowns at each grid point and requires the simultaneous solution of $N \times M$ equations.

ADIP. The Alternating Direction Implicit Procedure, like ADEP, requires two computational steps, with two flux terms at each of the two time levels. In the first step, flux terms are weighted in the x-direction at $t+1$ and in the y-direction at t, ($\Delta t = 2t$); and in the second step, flux terms are weighted in perpendicular directions to those of the first step. This procedure results in an $N \times N$ matrix solved M times, a much easier computation than the $NM \times NM$ solution required of the two previous methods. Early developmental work on this procedure was reported by Peaceman and Rachford ([11]) and Douglas ([12]).

It should be noted that in all methods the transmissibility (Kh) term is held constant during each time step. Approximation of the original non-linear equation is obtained by adjusting the value of Kh after each computation. If the change in Kh is small during each Δt, this procedure produces acceptable results.

Techniques such as iterative and explicit procedures, not discussed herein, may prove advantageous for the solution of three-dimensional and very large two-dimensional problems.

It is not the purpose of this paper to compare the various solution techniques for accuracy and efficiency of computation; rather it is to present a mathematical simulation technique fulfilling the requirements set forth at the beginning. The following presentation of simulation procedures is equally applicable to any of the computational techniques.

Simulation of the prototype system

Scaling the Model

Selection of the space and time dimensions (Δx, Δy, Δt) is dependent upon the availability of geologic and hydrologic data and the desired accuracy and detail of analysis. The accuracy of the solution is enhanced as the dimensions of Δx, Δy and Δt are decreased, providing the availability of geologic and hydrologic data justifies the additional computational problems. Space dimensions should be small enough that

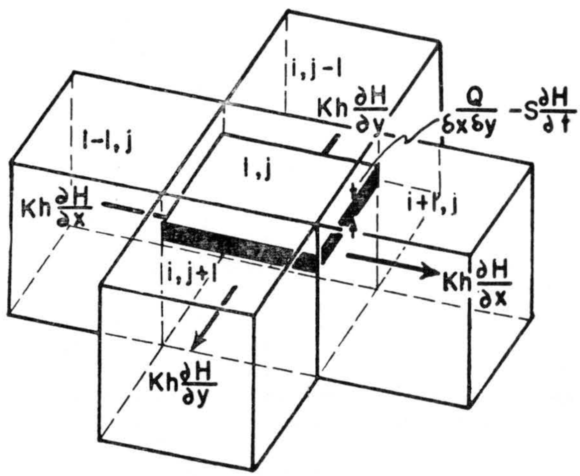


Fig. 1 — Physical Significance of Flux Terms in equation (2). Applied to Grid *i, j*.

the geologic and hydrologic conditions may be reasonably assumed uniform over the entire grid.

The maximum size of time increment, Δt, which will provide adequate accuracy should be used. Solution of sample problems with representative data can be used to determine the optimum value of Δt for the selected grid dimensions. Smaller grid dimensions may require shorter time increments.

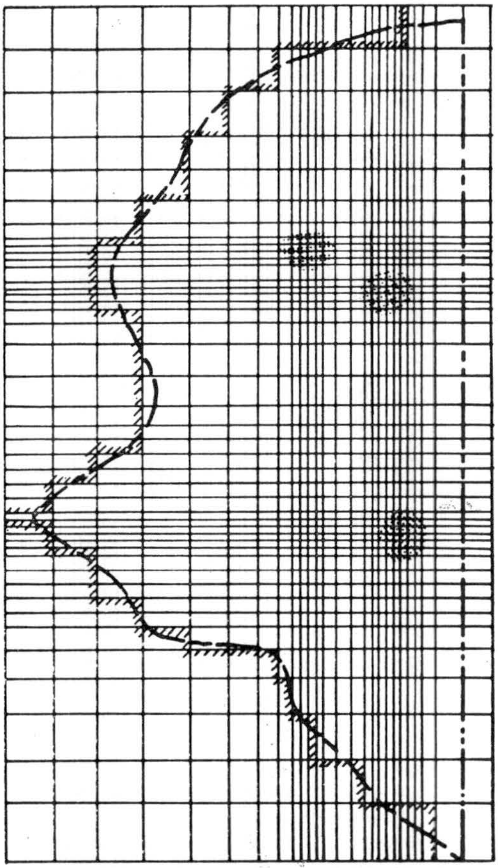
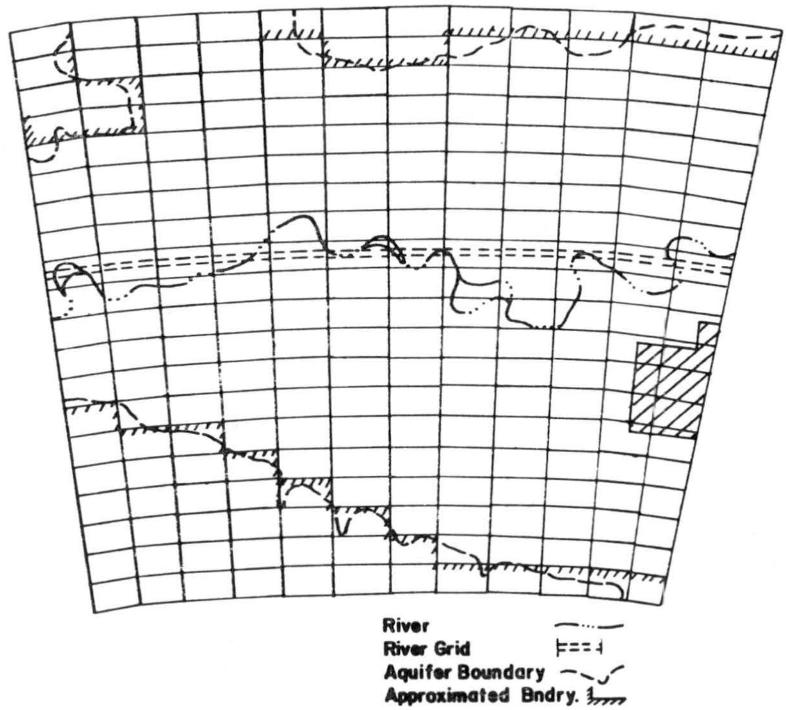


Fig. 2 — Grid Network "Focused" on Areas of Interest.

The solution techniques described in this paper require only that grid cells be tetragonal and that common sides of adjacent cells be coincident. The grid may be entirely uniform, but if available data and desired detail of results so justify, the grid dimensions may be modified as shown in figure 2. This "focusing" is of particular value in a large aquifer having concentrated areas of development. A curvilinear grid as shown in figure 3. is particularly adaptable to a sinuous alluvial valley. This curved grid allows close simulation of physical and hydrologic conditions and minimizes the size of matrix required. Focusing may also be utlized in the curvilinear grid as shown in figure 3. for better simulation of the hydraulic relationship between the river and aquifer. Also, in this illustration the grid dimensions perpendicular to the river are shorter than those parallel, allowing for greater detail perpendicular to the river.



Fig, 3 — Curvilinear Grid Network.

Simulating Boundary Conditions

Boundary conditions due to geologic and hydrologic influences include (*a*) impermeable or no-flow boundaries, (*b*) constant head or hydraulic boundaries, (*c*) boundaries of vertical inflow, and (*d*) boundaries of horizontal underflow.

No-flow boundaries. Simulation of an impermeable boundary such as the edge of an aquifer is most easily accomplished by assigning a zero hydraulic conductivity to all grid cells outside the aquifer. The aquifer boundaries are approximated as closely as possible by full grid cells.

Hydraulic boundary. Hydraulically connected surface water sources such as a river or lake may be simulated by programming a time-varying or constant water surface elevation in the appropriate grids. The time varying conditions are used to model a system where surface water levels fluctuate widely, affecting the exchange of groundwater and surface water. The differences in potential between the surface water sources and groundwater in adjacent grids may be used to calculate the volumetric exchange between the two. Conditions, such as infiltration restrictions and insufficient surface water to supply influent flows, may be simulated as special hydraulic boundary conditions.

Vertical-inflow boundaries. In a two-dimensional model, simulation of the influence of vertical flow due to a leaky aquifer may be incorporated as a function of the potential heads in the two aquifers and the hydraulic conductivity of the separating aquitard.

Horizontal-inflow boundaries. Groundwater underflow boundaries may be encountered when it is desirable to analyze only a portion of an aquifer or where computer limitations require segmentation of the problem. Such boundaries represent truncation of an aquifer as shown in figure 3. To account for the underflow, the gradient existing between each pair of grids perpendicularly adjacent to the underflow boundary may be considered as that which controls the groundwater flow into or out of the model. These gradients are adjusted after each time step. The model should be extended beyond the area of immediate interest to give a "buffer zone" for reduction of errors introduced by this extrapolation of heads to the underflow boundaries.

Data Input

The mathematical model has data requirements similar to those for other simulation techniques. Each grid cell is assigned a value for hydraulic conductivity, storage coefficient, elevation of the base of the aquifer, and an initial water table elevation (for confined aquifers this would be aquifer thickness and piezometric head). In addition, the time distribution of net withdrawals of water from each cell are programmed. Net withdrawals may include one or more contributions such as precipitation and applied irrigation water, as well as withdrawals by pumps and evapotranspiration. Hydrologic evaluations should be made to determine the portions of these factors which affect the water table.

Simplification of data preparation can be accomplished by computer subprograms. For instance, point data for hydraulic conductivity obtained from well tests can be distributed throughout the model by interpolation using a surface-fitting subprogram.

Modification of hydrologic input data for simulation of proposed management schemes can be readily introduced. The effect of proposed artificial recharge, redistribution of pumping, changes in irrigation practices or the transfer of water rights may be easily simulated by changing the net withdrawals in the appropriate grid cells.

Output of Results

Results generally desired in an aquifer management study include the prediction of changes in water levels and of volumetric exchange between the groundwater and surface water with time. Because of the type of analyses, these data are computed for each cell and can be obtained in tabular form. For rapid visual analyses or verification of the model the results can be reduced to maps and graphs by existing computer equipment. These may include hydrographs comparing historic and computed water levels and graphs of volumetric exchanges of groundwater and surface water as they vary in time and space. Figure 4. shows the output from a computer line-printer in which the symbols represent various ranges in calculated values. Similar maps, often referred to as shadowgraphs, readily display changes in water levels and comparisons of computed and historic water levels. The shadowgraph is also invaluable as a means of spotting anomalies or errors.

Output from the digital computer can easily be stored on magnetic tape or on punched cards to be used in later analyses, either hydrologic or socio-economic.

Verification of the Model

Prior to use for prediction, it must be made certain that the model accurately simulates the prototype system. For verification purposes, the model is programmed to simulate an historic period for which both groundwater level and stream flow records are available. If the computed results do not match the historical data, then scientific judgment is required to decide the input data that must be adjusted to achieve verification. Experience and knowledge of the study area are invaluable in making realistic modifications. Consideration must be given to the accuracy of the data used in the model and checking and updating should be continued as additional information becomes available or hydrologic conditions change.

Another important step in checking and verification is a continuous volumetric accounting of water within the model. This accounting, easily incorporated in the digital computer simulation, may be obtained at desired intervals to insure a balanced water budget and to check for unrealistic values of the budget components.

Use of model for management decisions

The mathematical simulation described above may be applied to many aquifer management problems and conditions. Upon development and verification of a model for a particular area, water administration officials possess a valuable tool for planning and analyzing proposed management and operation schemes. Legal and economic constraints may be programmed into the model so as to arrive at sound decisions based upon these factors as well as upon physical feasibility. Where specific objectives can be stated, operations research techniques may be used to determine optimum management plans.

A particular example of problems being studied using a mathematical model and digital computer is the "stream-aquifer" problem described by Eshett and Bittinger ([13]). In brief, complex management problems occur in many irrigated river valleys of the Western United States because during the initial period of development of surface water, and later groundwater, the hydraulic connection of the two supplies was often unknown or at least ignored. The supplies were developped, and legal rights perfected, as if the sources were separate and unrelated. Now, with full- or over-development, conflicts and confusion have arisen because use (or misuse) of water from each source affects the quantity and quality of water in the other source in both time and space. Such systems are often quite complex, with many points of recharge and discharge, each with individual patterns in time. Water rights and a multiplicity of interested political entities further complicate the management problem. The mathematical model and digital computer approach has proven to be very satisfactory for adequate simulation of such systems. Figure 3. shows a section of the Arkansas River Valley in Colorado with a curvilinear grid approximately paralleling the river and aquifer axis. The basic dimensions of each cell are 1320 feet by 2640 feet, the shorter side being at right angles to the river. Cells representing the river are 300 feet by 2640 feet. Calculations are made using a Δt of 10 days with changes in net withdrawals made at 30-day intervals. Management studies include development of plans for coordinated operation of the combined groundwater and surface water system. Specific problems are: (*a*) effects of out-of-priority diversions, (*b*) location of artificial recharge facilities to obtain efficient storage or desired flow patterns to the river, (*c*) effects of changes in water use practices and (*d*) equitable allocation of benefits and costs to those affected by proposed management operations.

In another application, the Wellton-Mohawk aquifer in Arizona, a grid size of 40 acres (1320 ft by 1320 ft) is being used. In this case severe salinity and drainage conditions are involved. These conditions, with the availability of adequate field data, justify using a finer grid network. Net withdrawals are changed at 10-day intervals allowing predictions of water table conditions to be made for rather short time periods under rapidly fluctuating conditions. The incorporation of a salt budget, much the same as a water budget, allows prediction of changes in salinity with time. Management thus includes both water table and salinity control to meet particular objectives.

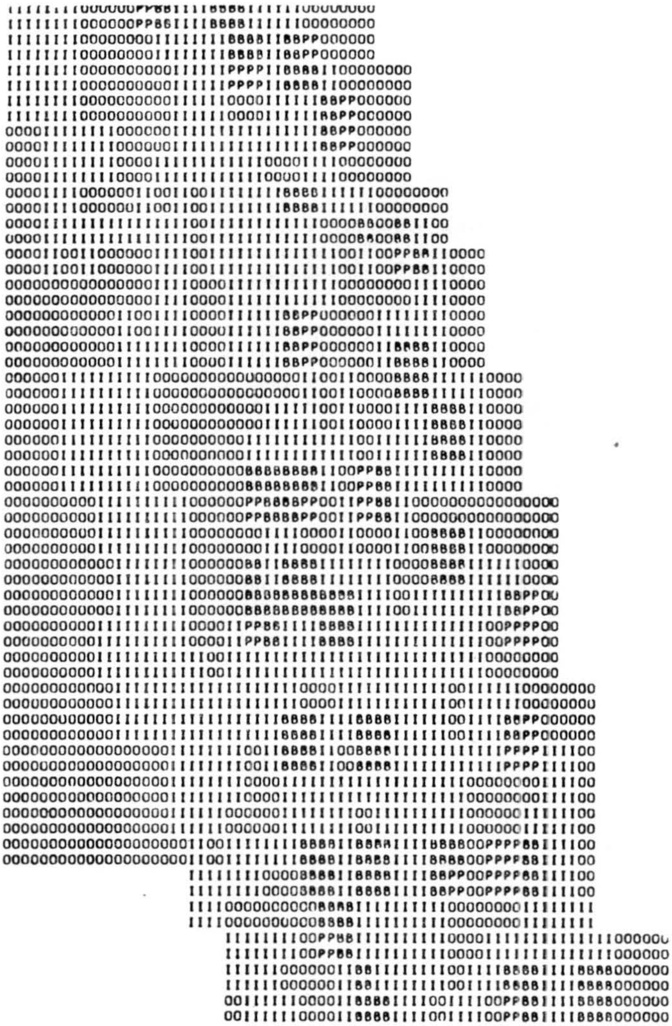


Fig. 4 — Example of a Map, or Shadowgraph, Printed Directly frcm Computer Results.

For many aquifers, especially in arid regions, the natural recharge is small compared to current or potential withdrawals. A major aquifer of this type is the Ogallala formation of the High Plains extending from Nebraska to Texas. The portion of this aquifer lying in the State of Colorado is shown in figure 2. with a finer grid network overlying areas having concentrated pumping development. Outer areas, with fewer high-capacity wells and less geologic information, are modeled more coarsely. Management studies in this type of area include (*a*) depletion problems under various projected pumping schemes, (*b*) optimum location of recharge facilities and determination of relative benefits for purposes of cost allocation, and (*c*) study of natural recharge characteristics by simulation of historical conditions.

Conclusions

Adequate simulations of complex groundwater systems for developing management criteria can be accomplished using digital computer solutions of a mathematical model. Such simulations, requiring data similar to that needed in other simulation techniques, are especially applicable to transient and non-linear conditions, such as for unconfined aquifers with varying transmissibility. This technique is readily adaptable from one system to another by simply changing the input data.

Output from the digital computer may be obtained as graphs, maps, and other readily visualized forms, which are immediately usable in making management decisions or in publications. Time-consuming manipulation and plotting of results is considerably reduced compared with other simulation techniques.

Verification and updating of the model is enhanced by the easy means of changing input data and the convenient forms of the output. Selected management criteria can also be introduced and evaluated.

The increasing availability of large, high-capacity computers makes the computer solution more easily accessible than many other simulation techniques requiring specialized equipment and highly trained personnel. Thus, the water administrator is provided a tool with which his own personnel may rapidly analyze the effects of management decisions. Presently available computers allow analysis of a year's response in an aquifer system, represented by several thousand grid cells, in a matter of minutes.

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